Braids, polynomials and real algebraic links

Benjamin BODE (Osaka University)*

1. Introduction

In the last chapter of his seminal book [8] Milnor discusses the real analogue of links of isolated singularities of complex plane curves, later termed *real algebraic links*.

Definition 1.1. *A link L is real algebraic if there exists a polynomial $f:\mathbb{R}^4\to\mathbb{R}^2$ such that*

- $f((0,0,0,0))=(0,0)$,
- $\nabla f((0,0,0,0))=\left(\begin{smallmatrix}0&0&0&0\\0&0&0&0\end{smallmatrix}\right)$,
- *there is a neighbourhood B of $(0,0,0,0)$ such that $(0,0,0,0)$ is the only point in B where the rank of ∇f is not full,*
- $f^{-1}((0,0))\cap S^3_\rho=L$ *for all small enough radii ρ.*

The first three conditions state that f has an isolated singularity at the origin in $\mathbb{R}^4$.

Milnor himself points out that it is highly challenging to construct examples of such links that do not come from complex plane curves $f:\mathbb{C}^2\to\mathbb{C}$. Indeed, the family of links that is known to be real algebraic is still comparatively small [5, 7, 9, 10].

One difference between the complex and the real polynomials is that in general the argument of a real polynomial as in Definition 1.1 ($\arg f:S^3_\rho\to S^1$) is not a fibration. However, Milnor established that the following is still true.

Theorem 1.2 (Milnor [8])**.** *If a link L is real algebraic, then L is fibred.*

According to Benedetti and Shiota, this implication should be an equivalence.

Conjecture 1.3 (Benedetti-Shiota [2])**.** *A link L is real algebraic if and only if L is fibred.*

In this short note we discuss a construction of real polynomial maps with isolated singularities as in Definition 1.1, following [5]. Section 2 reviews the this construction in a quite general setting, while Section 3 focuses on the class of homogeneous braids.

Definition 1.4. *A braid B on s strands is called homogeneous if for every $i=1,2,\dots,s-1$ the generator σ_i appears in the word B if and only if σ_i^{-1} does not appear.*

The 3-strand braid $(\sigma_1\sigma_2^{-1})^2$ for example is homogeneous, because σ_1 always comes with a positive sign and σ_2 always comes with a negative sign. If we change one of the signs however, or if we consider the same braid word as a 4-strand braid, we obtain an inhomogeneous braid. Note that the homogeneous braids contain all alternating braids that do not close to split links.

Theorem 1.5 (Bode [5])**.** *Let B be a homogeneous braid. Then the closure of B^2 is real algebraic.*

This work was supported by JSPS KAKENHI Grant Number JP18F18751 and a JSPS International Postdoctoral Fellowshipship.
2000 Mathematics Subject Classification: Primary: 57M25; Secondary: 14J17, 32C05.
Keywords: homogeneous braid, singularity, algebraic link, Milnor fibration.
* e-mail: `ben.bode.2013@my.bristol.ac.uk`
web: `https://sites.google.com/my.bristol.ac.uk/benjaminbode/`

The proof that homogeneous braids close to fibred links is due to Stallings [12]. An important aspect of the proof in [5] is that for a loop in the space of complex polynomials, we can establish a close relation between the braid that is formed by the roots of the polynomials and the braid that is formed by their critical values. Here, we give more details on this relation and illustrate how this could be used to generalize the construction to a larger family of links.

Section 4 gives an example of the outlined construction.

2. Constructing real algebraic links

Let B be a braid on s strands. We denote by $\mathscr{C}$ the set of connected components of the closure of B or equivalently the set of cycles of the image of B under the permutation representation. For every $C \in \mathscr{C}$ we write s_C for the number of strands that make up the component C or equivalently the length of that corresponding cycle. Then the total number of strands s equals $\sum_{C\in\mathscr{C}} s_C$. Suppose we have a parametrisation of B in $\mathbb{C}\times[0,2\pi]$ given by

$$\bigcup_{C\in\mathscr{C}}\bigcup_{j=1}^{s_C}\left(F_C\left(\frac{t+2\pi j}{s_C}\right)+\mathrm{i}G_C\left(\frac{t+2\pi j}{s_C}\right),t\right),\quad t\in[0,2\pi], \tag{1}$$

where F_C, $G_C : [0,2\pi]\to\mathbb{R}$ are trigonometric polynomials. Such a parametrisation exists for every braid and in fact there are even some bounds on the Fourier degree of F_C and G_C [4].

Then we can define the polynomial $g_\lambda : \mathbb{C}\times[0,2\pi]\to\mathbb{C}$,

$$g_\lambda(u,t)=\prod_{C\in\mathscr{C}}\prod_{j=1}^{s_C}\left(u-\lambda\left(F_C\left(\frac{t+2\pi j}{s_C}\right)+\mathrm{i}G_C\left(\frac{t+2\pi j}{s_C}\right)\right)\right) \tag{2}$$

with $\lambda>0$ and the nodal set $g_\lambda^{-1}(0)$ is B for all values of λ.

Furthermore, expanding the product in Equation (2) results in a polynomial not only in the complex variable u, but also in $\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{-\mathrm{i}t}$.

We now replace every instance of $\mathrm{e}^{\mathrm{i}t}$ in the polynomial expression of g_λ by another complex variable v and every instance of $\mathrm{e}^{-\mathrm{i}t}$ by its conjugate $\overline{v}$. This identifies the variable t with the angular coordinate of v. We thus obtain a polynomial $f_\lambda : \mathbb{R}^4\cong\mathbb{C}^2\to\mathbb{C}\cong\mathbb{R}^2$ in u, v and $\overline{v}$. In general, f_λ does not have an isolated singularity. However, it is an easy calculation to show the following.

Proposition 2.1 (Bode [5])**.** *Let $k\geq(\deg f_\lambda)/(2s)$. Then for all small enough $\lambda>0$ the map $p_\lambda : \mathbb{R}^4\cong\mathbb{C}^2\to\mathbb{C}\cong\mathbb{R}^2$ given by*

$$p_\lambda(u,v)=(v\overline{v})^{sk}f_\lambda\left(\frac{u}{(v\overline{v})^k},\frac{v}{\sqrt{v\overline{v}}}\right), \tag{3}$$

or equivalently

$$p_\lambda(u,r\mathrm{e}^{\mathrm{i}t})=\begin{cases} r^{2sk}g_\lambda\left(\frac{u}{r^{2k}},t\right) & \text{if } r>0,\\ u^s & \text{if } r=0,\end{cases} \tag{4}$$

has an isolated singularity at the origin if and only if $\arg g_\lambda : (\mathbb{C}\times[0,2\pi])\backslash g_\lambda^{-1}(0)\to S^1$ *is a fibration. The link of the singularity is the closure of B.*

There are a couple of things to note here. Firstly, the condition on $\arg g_\lambda$ does not depend on λ. Secondly and very importantly, the map p_λ is in general not a polynomial because we have introduced square root terms. However if Equation (1) is a π-periodic parametrisation, then all of the exponents with non-zero coefficients in the trigonometric polynomials F_C and

G_C are even. This can be arranged if and only if B is a square, i.e., $B = A^2$ for some braid A. It follows that in this case all exponents of v and $\overline{v}$ in f_λ can be taken to be even, so that all square roots in Equation (3) cancel.

Proposition 2.2. *If a braid B can be parametrised as in Eq. (1) such that* $\arg g_\lambda$ *gives a fibration over S^1, then B^2 closes to a real algebraic link.*

We now want to investigate which braids satisfy the condition in Proposition 2.2 and study in particular the case of homogeneous braids.

3. Homogeneous real algebraic links

For every $t \in [0, 2\pi]$ the map $u \mapsto g_\lambda(u,t)$ is a monic complex polynomial. We denote by $v_i(t)$, $i = 1, 2, \ldots, s-1$, the critical values of $g_\lambda(u,t)$, i.e., the images $g_\lambda(c_i,t)$ of the critical points c_i of $u \mapsto g_\lambda(u,t)$, determined by the condition $\frac{\partial g_\lambda}{\partial u}(c_i,t) = 0$. It is again a simple calculation to show that $\arg g_\lambda$ is a fibration over S^1 if and only if for all $i = 1, 2, \ldots, s-1$ the derivative $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. This has a nice geometric interpretation in terms of the movement of the critical values in the complex plane as t varies. Since $g_\lambda(u,t)$ has distinct roots, the critical values $v_i(t)$ are always non-zero. The inequality states that no critical value $v_i(t)$ ever changes the orientation in which it twists around the origin $0 \in \mathbb{C}$ as t increases from 0 to 2π. Every critical value moves either always clockwise $\left(\frac{\partial \arg v_i(t)}{\partial t} < 0\right)$ or always counterclockwise $\left(\frac{\partial \arg v_i(t)}{\partial t} > 0\right)$.

Proposition 2.2 can therefore be updated and rewritten in terms of polynomials and critical values. Let $\tilde{X}_s$ be the space of monic complex polynomials of degree s with distinct roots. The fundamental theorem of algebra gives a straightforward identification of a polynomial $f = \prod_{j=1}^{s}(u - x_i) \in \tilde{X}_s$ with its unordered set of roots $\{x_1, x_2, \ldots, x_s\}$. This allows us to identify a loop f_t in $\tilde{X}_s$ with the (closed) braid that is formed by the roots of the polynomials

$$\bigcup_{j=1}^{s} (x_j(t), t) \subset \mathbb{C} \times S^1. \tag{5}$$

Proposition 3.1. *Let f_t, $t \in S^1$, be a loop in $\tilde{X}_s$ such that for all $i = 1, 2, \ldots, s-1$ the derivative $\frac{\partial \arg v_i(t)}{\partial t}$ never vanishes. Let B be the braid that is formed by the roots of f_t. Then the closure of B^2 is real algebraic.*

In Proposition 3.1 we do not require explicitly that the parametrisation of B is given in terms of trigonometric polynomials. Since these are C^1-dense in the space of 2π-periodic real C^1-functions, we can always approximate a parametrisation as in Proposition 3.1 without losing the property that $\frac{\partial \arg v_i(t)}{\partial t}$ does not vanish.

We can also assume that the critical values are distinct. This means that

$$(0,t) \cup \bigcup_{j=1}^{s-1} (v_j(t), t) \subset \mathbb{C} \times S^1 \tag{6}$$

forms a closed braid. Let $X_s \subset \tilde{X}_s$ be the space of those polynomials in X_s that have distinct critical values. Then the space of possible sets of critical values of a polynomial in X_s is given by

$$V_s = \{(v_1, v_2, \ldots, v_{s-1}) \in (\mathbb{C} \backslash \{0\})^{s-1} : v_i \neq v_j \text{ if } i \neq j\} / S_{s-1}, \tag{7}$$

where S_{s-1} is the permutation group on $s-1$ elements. Then the braid in Equation (6) can be interpreted as a loop in V_s. Another interpretation of Proposition 3.1 is therefore asking

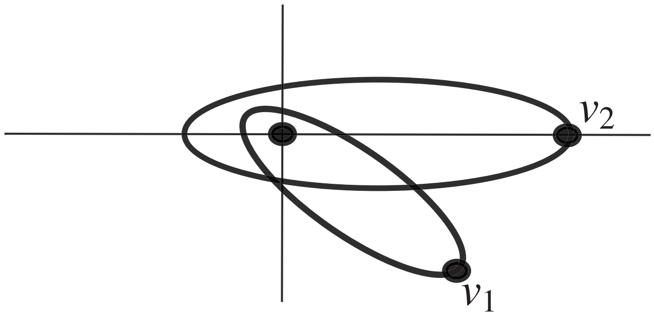

Figure 1: If the critical values $v_j(t)$ move on ellipses around the origin in the complex plane without ever changing direction (from clockwise to counterclockwise or vice versa) as t varies, the derivatives $\frac{\partial \arg v_j(t)}{\partial t}$ never vanishes.

which braids can be parametrized as loops in V_s satisfying the condition on $\frac{\partial \arg v_i(t)}{\partial t}$ and such that this loop is the image of a loop in X_s under the map that sends a polynomial to the set of its critical values.

The words

$$A_{i,j} = \sigma_j \sigma_{j-1} \dots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \dots \sigma_{j-1}^{-1} \sigma_j^{-1} \text{ with } i \leq j < s \tag{8}$$

are a set of generators for the braid group $\mathbb{B}_s$. The generator $A_{i,j}$ takes the strand $j+1$ and twists it around the strand i.

Lemma 3.2. *The s-strand braid* $\prod_{j=1}^n A_{1,i_j}^{\varepsilon_{i_j}}$ *can be parametrised as in Eq. (6) such that* $\frac{\partial \arg v_i(t)}{\partial t}$, $i = 1,2,\dots,s-1$, *never vanishes if for every* $k = 1,2,\dots,s-1$ *there is a* j *such that* $i_j = k$.

Proof. The parametrisation can be achieved if all $v_i(t)$ move on ellipses in the complex plane as t varies as indicated in Figure 1. Then for every j the generator $A_{1,j}$ can be parametrised such that $\frac{\partial \arg v_{j-1}(t)}{\partial t}$ does not vanish and all other strands are stationary. We can now concatenate the parametrisations for the $A_{1,j}$ to obtain the desired braid word.

The fact that the generator $A_{1,j}$ appears in the braid word if and only if $A_{1,j}^{-1}$ does not appear implies that none of the v_i ever turns around on its ellipse. The condition that for every $k = 1,2,\dots,s-1$ there is a j such that $i_j = k$ means that every v_i moves at some point. We can thus slightly perturb the parametrisation of each $A_{1,j}$ such that none of the v_i is non-stationary. For example, for the parametrisation of $A_{1,j}$ every v_i with $i \neq j-1$ moves an ε-amount on its ellipse. □

The braids in Lemma 3.2 allow for the desired kind of parametrisation. What we need to check now is that there is a loop in the space of polynomials X_s, whose critical values form that parametrisation. Then we want to know which braid is formed by the roots of these polynomials.

The following theorem is very useful for this. Recall that X_s is the space of monic polynomials with distinct roots and distinct critical values.

Theorem 3.3 (Beardon-Carne-Ng [1]). *Let* $X_s^0 \subset X_s$ *be the space of those polynomials in* X_s *that have constant term equal to 0. Then the map* $\theta_s : X_s^0 \to V_s$ *that sends a polynomial* $f \in X_s^0$ *to the set of its critical values* $(v_1, v_2, \dots, v_{s-1})$ *is a covering map of degree* s^{s-1}.

The map θ_s is well-defined even if the constant term is not equal to zero.

Corollary 3.4. *Let $\hat{X}_s \subset X_s$ be the space of those polynomials in X_s whose critical values are not equal to their constant terms, i.e. $v_i \neq f(0)$ for all $i = 1,2,\ldots,s-1$. Let γ be a loop in V_s and $\tilde{\gamma}$ a path in $\hat{X}_s$ such that $\theta_s(\tilde{\gamma}) = \gamma$. Then any homotopy of γ in V_s lifts to a homotopy of $\tilde{\gamma}$ in $\hat{X}_s$.*

The homotopy lifting property in Corollary 3.4 means that the properties that we want to check do not depend on the particular braid parametrisation, but only on the braid type. We are going to show that there is a braid that is conjugate to the braid in Lemma 3.2 for which it is relatively easy to determine that if it is interpreted as a loop in V_s, then it lifts to a loop in $\hat{X}_s$. Furthermore, determining the braid B that is formed by the roots of the corresponding polynomials is straightforward. Corollary 3.4 then implies that the braids in Lemma 3.2 also lift to a loop in $\hat{X}_s$ and the braid that corresponds to the roots of the polynomials is conjugate to B.

We start with a polynomial f, whose roots are real. Then all of its critical points and critical values are real as well. This is depicted in Figure 2. Between each pair of roots there is exactly one critical point. We label the critical points $\{c_i\}_{i=1,2,\ldots,s-1}$ such that c_i is the critical point between the i^{th} and $i+1^{\text{th}}$ smallest root. We set $v_i = f(c_i)$. We now consider a particular loop $\gamma_i : [0,2\pi] \to \mathbb{C}$ in the target complex plane based at the origin. The loop stays close to the real line and encircles the critical value v_i in a counterclockwise direction. The loop γ_i does not intersect any critical values. When it is about to encounter a critical value v_j, it avoids it by moving into the upper or the lower half plane. If $j > i$ and $i \equiv s+1 \bmod 2$ or if $j < i$ and $i \equiv s \bmod 2$, then γ_i moves into the upper half plane. Otherwise, it avoids v_j by moving into the lower half plane. At the moment this choice might seem arbitrary (and to some degree it is), but we will come back to why this turns out to be a good rule. An example of γ_1 is shown in Figure 2b).

Now we look at the preimage $f^{-1}(\gamma_i)$. These are $s-2$ distinct loops and two paths that exchange the ith and $i+1$th smallest root. The braid that is formed by $f^{-1}(\gamma_i)$ forms the generator σ_i if we choose the convention that the overpassing strand corresponds to the root with the smaller imaginary part. Note that this braid is given by $(f^{-1}(\gamma_i(t)),t) = ((f - \gamma_i(t))^{-1}(0),t) \subset \mathbb{C} \times [0,2\pi]$.

This gives us therefore a parametrisation of the generator σ_i as a loop in $\hat{X}_s$, whose critical values form the braid

$$(0,t) \cup \bigcup_{j=1}^{s-1} (v_i - \gamma_i(t),t) \subset \mathbb{C} \times [0,2\pi], \tag{9}$$

depicted in Figure 3. We can consider the path $\gamma = \prod_{j=1}^{\ell} \gamma_{i_j}^{\varepsilon_j}$, which is the concatenation of γ_i. The path $f - \gamma(t)$ is a loop in the space of polynomials $\hat{X}_s$, whose roots form the s-strand braid $B = \prod_{j=1}^{\ell} \sigma_{i_j}^{\varepsilon_j}$ and whose critical values form a braid that is easily seen to be conjugate to $\prod_{j=1}^{\ell} Y_{i_j}^{\varepsilon_j}$, where

$$Y_j = \begin{cases} A_{1,\frac{j+1}{2}} & \text{if } j \text{ is odd,} \\ A_{1,\frac{j}{2}+\lfloor\frac{s}{2}\rfloor} & \text{if } j \text{ is even.} \end{cases} \tag{10}$$

This is illustrated in an example in Figure 4.

We have shown the following.

Lemma 3.5. *For every s-strand braid $B = \prod_{j=1}^{\ell} \sigma_{i_j}^{\varepsilon_j}$ there is a conjugate of B that can be parametrised as a loop in $\hat{X}_s$ whose image in V_s corresponds to the braid $\prod_{j=1}^{\ell} Y_{i_j}^{\varepsilon_j}$. Equiv-*

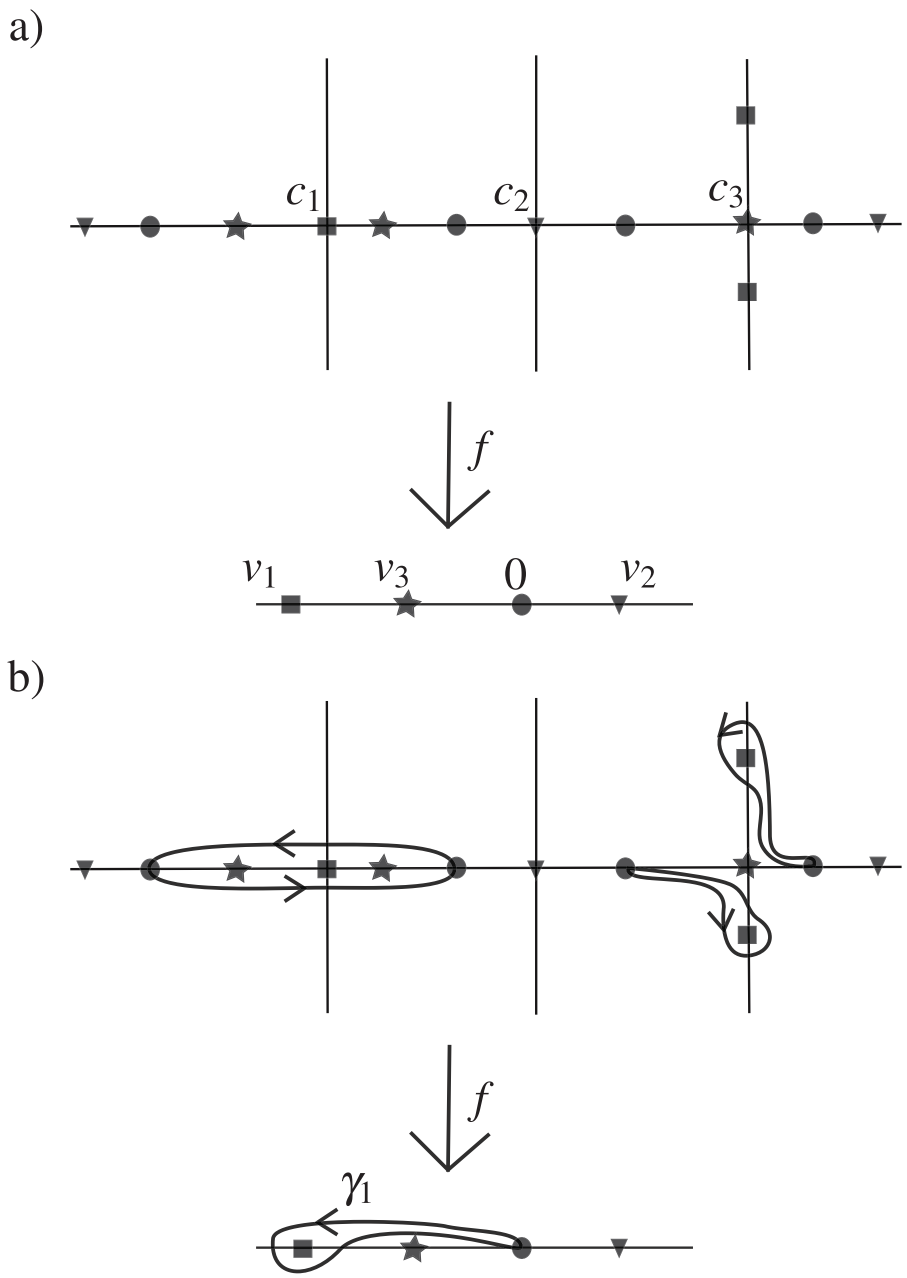

Figure 2: The definition of the loop γ_i. a) A real polynomial f has a critical point c_i between each pair of roots and the corresponding critical values $v_i = f(c_i)$ must be real too. Points with the same shape have the same image under f. E.g., the circles are the roots of f, all squares get mapped to v_1 and so on. The lines in the domain are the preimage set of the real line. b) The loop γ_1 encircles v_1. Its preimage set under f consists of two loops and two paths exchanging the first and second root.

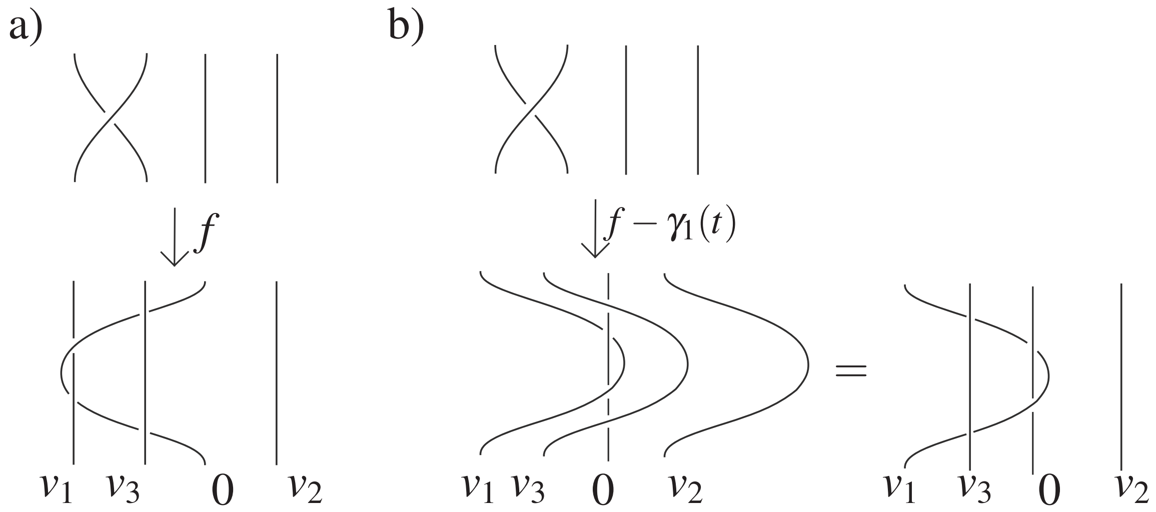

Figure 3: a) The relation between the braid of roots and the braid of critical values. a) The preimage set of $\gamma_1(t)$ under f forms the generator σ_1, while $\gamma_1(t)$ itself forms the braid $\sigma_2\sigma_1^2\sigma_2^{-1}$. (We read braid words from the bottom to the top.) b) We can interpret σ_1 as the roots of the loop of polynomials $f-\gamma_1(t)$. The braid of its critical values is isotopic to $\sigma_1^{-1}\sigma_2^2\sigma_1$.

alently, every parametrisation of any braid that is conjugate to $\prod_{j=1}^{\ell} Y_{i_j}^{\varepsilon_j}$ can be seen as a loop in V_s, which lifts to a loop in $\hat{X}_s$, which corresponds to a conjugate of B.

Note that if B is homogeneous, then the braid $\prod_{j=1}^{\ell} Y_{i_j}^{\varepsilon_j}$ is as in Lemma 3.2. Recall the rule in the construction of the loop γ_i that determines if γ_i avoids a critical value v_j by moving into the upper half of the complex plane or the lower half. This rule is quite arbitrary. The only thing it changes is which generator Y_k corresponds to which generator σ_j. With our rule we have the nice correspondence that Y_j is directly related to σ_j.

Corollary 3.6. *Every parametrisation of $\prod_{j=1}^{n} A_{1,i_j}^{\varepsilon_{i_j}}$ lifts to a loop in $\hat{X}_s$ and the corresponding braid is conjugate to a homogeneous braid. Conversely, for every homogeneous braid there is such a braid of critical values $\prod_{j=1}^{n} A_{1,i_j}^{\varepsilon_{i_j}}$.*

The construction that establishes this relation between the braid that is formed by the roots of a loop of polynomials and the braid that is formed by their critical values takes a lot of inspiration from work by Rudolph [11].

The main theorem now follows from Lemma 3.2 and Proposition 3.1:

Theorem 3.7. *If B is a homogeneous braid, then the closure of B^2 is real algebraic.*

It can also be shown that on 3-spheres of small radius $\arg p_\lambda$ is a fibration of the link complement of the circle, exactly as in the complex case [5].

4. Examples

In this section we give an example of the explicit construction of real polynomials with isolated singularities. If we wanted to strictly follow the procedure outlined in the proof of Theorem 1.5, we would first have to write down the parametrisation for the braid of critical values and then lift this loop in V_s to a loop in $\hat{X}_s$. This corresponds to solving a system of polynomial equations for every $t \in [0,2\pi]$. In practice, this will be done for a discrete set in $[0,2\pi]$ of sufficiently many data points. The resulting interpolating function gives a braid

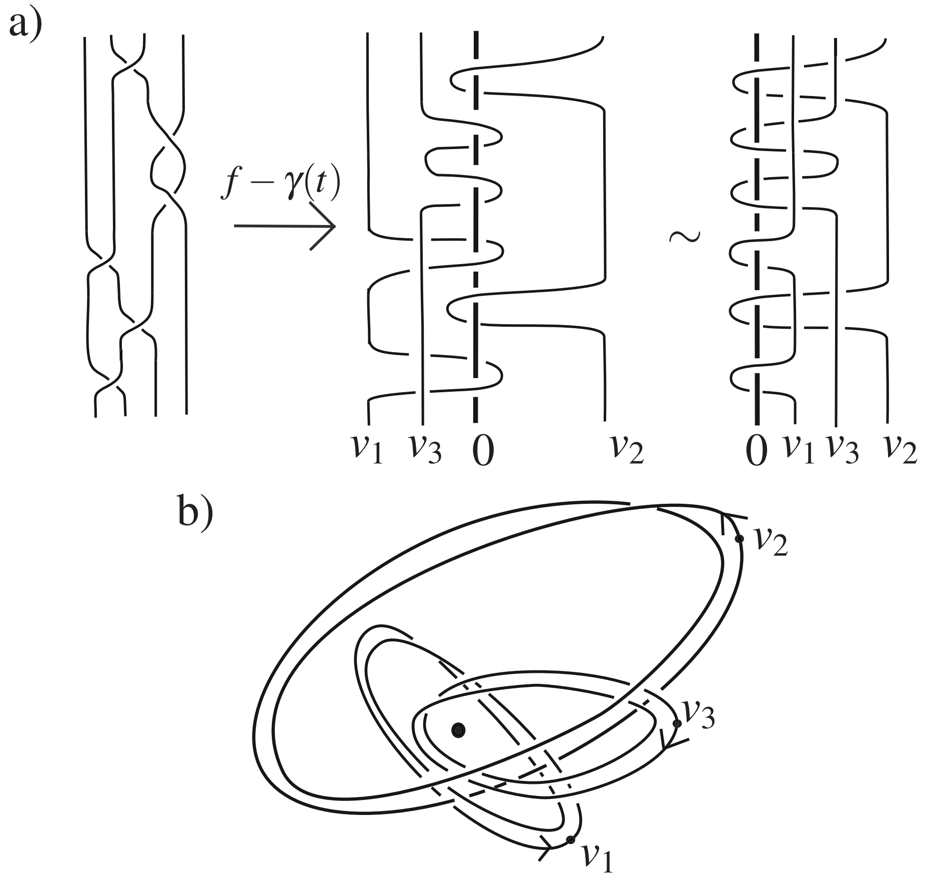

Figure 4: a) Concantenation of the loops γ_i allows us to construct any braid as the roots of a loop of polynomials in $\hat{X}_s$. Here we see the braid $\sigma_1\sigma_2\sigma_1\sigma_3^{-2}\sigma_2$ as the roots of $f-\gamma(t)$. The corresponding critical values form a braid that is conjugate to $Y_1Y_2Y_1Y_3^{-2}Y_2$. The generator Y_j describes a movement of the strand labelled by the critical value v_j around the 0-strand. b) The resulting braid $Y_1Y_2Y_1Y_3^{-2}Y_2$ can be parametrised such that $\frac{\partial \arg v_i}{\partial t}$ never vanishes. This is possible because $\sigma_1\sigma_2\sigma_1\sigma_3^{-2}\sigma_2$ is homogeneous.

parametrisation, which is then approximated by trigonometric polynomials, which we can use to define the functions g_λ and finally p_λ in Proposition 2.1.

Depending on the number of data points in $[0,2\pi]$ this can be a computationally quite extensive process. For our purposes it is easier to start with a trigonometric parametrisation and vary some parameters until the resulting g_λ leads to a fibration.

For all values of a, $b>0$ the parametric curves

$$\bigcup_{j=1}^{4}\left(a\cos\left(\frac{t+2\pi j}{4}\right)+bi\sin\left(\frac{3(t+2\pi j)}{4}\right),t\right) \tag{11}$$

form the braid $(\sigma_2\sigma_1^{-1}\sigma_3^{-1})$.

We set $a=1$ and $b=\frac{1}{2}$ and obtain

$$\begin{aligned} g_\lambda(u,t) =& u^4+\lambda^2u^2\left(\frac{1}{2}\mathrm{e}^{-\mathrm{i}t}-\frac{3}{8}-\frac{1}{2}\mathrm{e}^{\mathrm{i}t}\right)\\ &+\lambda^4\left(-\frac{1}{256}\mathrm{e}^{-3\mathrm{i}t}+\frac{1}{16}\mathrm{e}^{-2\mathrm{i}t}-\frac{1}{4}\mathrm{e}^{-\mathrm{i}t}+\frac{1}{128}-\frac{1}{16}\mathrm{e}^{\mathrm{i}t}-\frac{1}{256}\mathrm{e}^{3\mathrm{i}t}\right). \end{aligned} \tag{12}$$

Figure 5 shows a plot $\arg v_1(t)$, $\arg v_2(t)$ and $\arg v_3(t)$, the critical values of $g_1(u,t)$. We see immediately that there are no stationary points. Therefore $\arg g_\lambda(u,t)$ is a fibration and this means that $(u,t)\mapsto \arg g_\lambda(u,nt)$ is also fibration for all n. This justifies our choice of a and b. Note that the nodal set of $(u,t)\mapsto\arg g_\lambda(u,nt)$ is $(\sigma_2\sigma_1^{-1}\sigma_3^{-1})^n$. For even exponents we obtain the following polynomials:

$$\begin{aligned} f_\lambda(u,v) &= u^4+\lambda^2u^2\left(\frac{1}{2}\overline{v}^{2n}-\frac{3}{8}-\frac{1}{2}v^{2n}\right)\\ &\quad+\lambda^4\left(-\frac{1}{256}\overline{v}^{6n}+\frac{1}{16}\overline{v}^{4n}-\frac{1}{4}\overline{v}^{2n}+\frac{1}{128}-\frac{1}{16}v^{2n}-\frac{1}{256}v^{6n}\right),\\ p_\lambda(u,v) &= (v\overline{v})^{4\times 2n}f_\lambda\left(\frac{u}{(v\overline{v})^{2n}},\frac{v}{\sqrt{v\overline{v}}}\right)\\ &= u^4+\lambda^2u^2\left(\frac{1}{2}v^{7n}\overline{v}^{9n}-\frac{3}{8}(v\overline{v})^{8n}-\frac{1}{2}v^{9n}\overline{v}^{7n}\right)\\ &\quad+\lambda^4\left(-\frac{1}{256}v^{5n}\overline{v}^{11n}+\frac{1}{16}v^{6n}\overline{v}^{10n}-\frac{1}{4}v^{7n}\overline{v}^{9n}+\frac{1}{128}(v\overline{v})^{8n}-\frac{1}{16}v^{9n}\overline{v}^{7n}-\frac{1}{256}v^{11n}\overline{v}^{5n}\right). \end{aligned} \tag{13}$$

Here we have chosen $k=2n$, which is larger than $\deg f/s=6n/4$. Note that all exponents of v and $\overline{v}$ in f_λ are even, so that there are no square root terms in p_λ.

Therefore the braids $(\sigma_2\sigma_1^{-1}\sigma_3^{-1})^{2n}$ close to real algebraic links for all n. They belong to the family of lemniscate links which where studied in [3]. Note that the constructed functions are semiholomorphic, i.e., holomorphic in one complex variable u and $\deg_u p_\lambda=s$, the number of strands.

If we want to construct real algebraic links that are not in the family of closures of squares of homogeneous braids, it is important to recall that in our construction we only considered one lift of the braid of critical values $\prod_{j=1}^{\ell}A_{1,i_j}^{\varepsilon_{i_j}}$. If any of its other lifts in $\hat{X}_s$ are also loops, then the closure of the squares of the corresponding braids are also real algebraic. Recently, this was studied in more detail [6], but since it is challenging to check if a given link is the closure of a homogeneous braid it remains unclear if the constructed families in [6] lead to new real algebraic links.

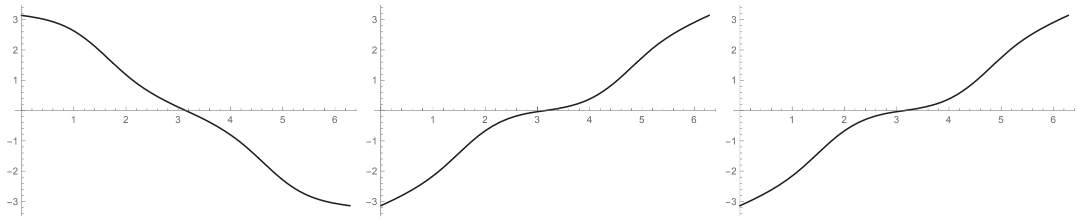

Figure 5: Plots of graphs of the arguments of the 3 critical values of g as functions of t. Because of symmetries of the braid parametrisations two of the three functions are identical. None of the functions have any stationary points.

References

[1] A. F. Beardon, T. K. Carne and T. W. Ng. *The critical values of a polynomial*. Constr. Approx. **18** (2002), 343–354.

[2] R. Benedetti and M. Shiota. *On real algebraic links on S^3*. Bolletino dell'Unione Mathematica Italiana, Serie 8 Volume 1B **3** (1998), 585–609.

[3] B. Bode, M. R. Dennis, D. Foster and R.P. King. *Knotted fields and explicit fibrations for lemniscate knots*. Proc. R. Soc. A **473** (2017), 20160829.

[4] B. Bode and M. R. Dennis. *Constructing a polynomial whose nodal set is any prescribed knot or link*. Journal of Knot Theory and its Ramifications **28**, no. 1 (2019), 1850082.

[5] B. Bode. *Constructing links of isolated singularities of real polynomials $\mathbb{R}^4 \to \mathbb{R}^2$*. Journal of Knot Theory and its Ramifications **28**, No. 1 (2019), 1950009.

[6] B. Bode. *Braid group actions on the n-adic integers*. arXiv:1903.06308 (2019).

[7] E. Looijenga. *A note on polynomial isolated singularities*. Ind. Math. (Proc.) **74** (1971), 418–421.

[8] J. W. Milnor. *Singular points of complex hypersurfaces*. Princeton University Press (1968).

[9] B. Perron. *Le nœud "huit" est algébrique réel*. Inv. Math. **65** (1982), 441–451.

[10] L. Rudolph. *Isolated critical points of mappings from $\mathbb{R}^4$ to $\mathbb{R}^2$ and a natural splitting of the Milnor number of a classical fibred link : 1. Basic theory and examples*. Comm. Math. Helv. **62** (1987), 630–645.

[11] L. Rudolph. *Some knot theory of complex plane curves*. arXiv:math/0106058 (2001).

[12] J. R. Stallings. *Constructions of fibred knots and links*. Algebraic and geometric topology, Proc. Sympos. Pure Math. **32**, Amer. Math. Soc., Providence, R. I (1978), 55–60.