The Language of Surfaces

J. Scott Carter (University of South Alabama)*

1. Introduction

A system of fonts that describes properly embedded surfaces in 3-space will be introduced via categorical considerations. The iconography considers dots along a line as 1-morphisms, Temperley-Lieb like diagrams as 2-morphisms, and transformations between these as 3-morphisms. The transformations will be described via icons that depict a simple closed curve that is being created or annihilated, two types of saddle transformations that occur, four possible types of cusps, and exchanges of critical events. As the iconography is developed, it will be easy to describe the 4-isomorphisms that generate isotopy.

Theorem 1.1. *The naturally monoidal, strictly* 2*-pivotal, weakly* 3*-pivotal, rotationally commutative, strictly* 3*-tortile* 3*-category with one object that is freely generated by a weakly self-invertible non-identity* 1*-morphism is equivalent to the* 3*-category of isotopy classes of properly embedded surfaces in* $\mathbb{R}^2 \times [0,1]$.

The result is known among experts. It is folklore, and this is the first printed version of which I know. The statement is one of the early cases of the cobordism hypothesis as formulated by Baez and Dolan [BD95], but there, the category is described as a 2-category. In that case, the non-trivial generating 1-morphism is considered to be a generating object, and the corresponding higher morphisms are also collapsed by one degree. An analogue is found in the difference between the classifying space for a group BG and its universal cover EG. Here EG is the analogue of the 3-category. There are two sources of inspiration: one is my desire to be more fluent in Asian languages and to develop a corresponding intuition about Hiragona and Kanji; the second is the web-based program GLOBULAR (See [BKV16] for some details). Meanwhile, much of the content here is a distillation of specific aspects that were presented in [CS98].

Here is an outline. We briefly describe categories and higher categories. Then we develop each descriptive term in the statement of the theorem. At the successive stages, we have 1-morphisms composed in a 1-dimensional manner, higher morphisms compose as rectangles, cubes, etc. that are stacked vertically. Horizontal juxtaposition is only allowed when one of the morphisms is an identity.

Two guiding principles govern this work: (1) different things may be naturally isomorphic, but they are not equal; (2) critical events occur at distinct instances. Let us proceed.

2. Categories and n-categories ($n \leq 4$)

A really small category[1] has a set of objects, and given objects a and b, the collection of arrows $b \xleftarrow{f} a$ from the *source* $a = s(f)$ to the *target* $b = t(f)$ is a set. If $c \xleftarrow{g} b$

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* e-mail: `carter@southalabama.edu`
web: `http://www.southalabama.edu/mathstat/personal_pages/carter/`

[1] The standard terminology is small and locally small. Since the higher morphisms are also categories, here we want every collection of morphisms to form a set.

and $b \xleftarrow{f} a$ are arrows so that $s(g) = t(f)$, then their composition is an arrow $c \xleftarrow{g \circ f} a$ with source $s(g \circ f) = a$ and target $t(g \circ f) = b$. Compositions of arrows is associative:

$$\left[d \xleftarrow{h} c \xleftarrow{g \circ f} a\right] = \left[d \xleftarrow{h \circ g} b \xleftarrow{f} a\right].$$

For any object a there is an arrow a—a that behaves as an identity under compositions:

$$\left(b \text{—} b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a\right) = \left(b \xleftarrow{f} a \text{—} a\right).$$

An overly simplistic definition of a *really small n-category* is that it is a category in which the set of morphisms between $(n-1)$-morphisms is a category. All n-categories that are defined here will be really small. The collection of objects is a set, and the category of k-morphisms between any pair of $(k-1)$-morphisms also forms a set, for $1 \leq k \leq 4$. A 0-morphism is called *an object.* A 1-morphism between a pair of objects is called a *(single) arrow.* In general, a k-morphism will be also called a *double, triple, or quadruple* arrow, for the obvious values of k. Composition of n-morphisms should be unital and associative.

In order to put an inductively defined category structure upon the set of n-morphism (or multiple arrows) between a pair of $(n-1)$-morphisms, we write

$$\begin{array}{ccc} b & \xleftarrow{g} & a \\ | & \Uparrow_F & | \\ b & \xleftarrow{f} & a \end{array}$$

for an n-morphism F whose source is the $(n-1)$-morphism $f = s(F)$ and whose target is the $(n-1)$-morphism $g = t(F)$. Note that $s(f) = s(g) = a$ while $t(f) = t(g) = b$. This figure is an accurate depiction when F is a 2-morphism. The case of a 3-morphism $\mathcal{R}$ with source F and target G is depicted in Fig. 1. A 4-morphism with source $\mathcal{R}$ and target $\mathcal{S}$ can be thought of as a symbol in the interior of a hypercube that connects two opposing cubical faces. The rectangular depiction, then, is a 2-dimensional projection of the double, triple, or quadruple arrow.

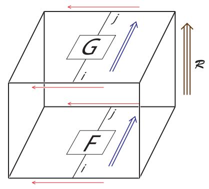


Figure 1: A 3-morphism or triple arrow

Since the sources and targets for f and g agree, it is customary to draw the 2-morphism F as on the left side of Fig. 2. The (vertical) composition of 2-morphisms is illustrated on the right of the same figure. Often authors define an ambiguous horizontal composition of 2-morphisms that resembles the central drawing in Fig. 3 and which is interpreted via either the left or right drawing. Interpreting these as equal, or allowing the 2-morphisms to occur at the same horizontal level, is to cheat our basic principles. Instead, we suppose that there is a natural 3-isomorphism connecting the two interpretations. We will return to the point later.

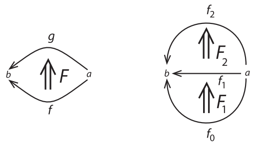


Figure 2: Composition of 2-morphims

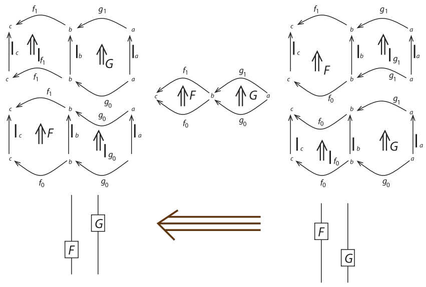


Figure 3: Horizontal composition is not well-defined

3. The axioms of the 3-category

Suppose there is a unique object x in a category, and there is a non-identity morphism $x \xleftarrow{\bullet} x$. Then define $(-\!\bullet\!-)^k$ inductively as the k-fold composition of $-\!\bullet\!-$ with itself: $(-\!\bullet\!-)^k = (-\!\bullet\!-)^{k-1} \circ -\!\bullet\!-$. Of course, $(-\!\bullet\!-)^0 = \text{—} = x \text{—} x$. The set of powers of $-\!\bullet\!-$ corresponds to the set of non-negative integers $\mathbb{N} = \{0, 1, 2, \ldots\}$, given in unary notation. This is the *free monoid on a single generator.*

The identity double arrow on — is $\square$. The identity 2-morphism on $-\!\bullet\!-$ is I. We define double arrows $\cap$ and $\cup$. The inclusion of bullets $(\bullet)$ will be slowly dropped from the notation.

Define $\mathsf{I}_1 = \mathsf{I}$, and more generally inductively define $\mathsf{I}_i = (\mathsf{I}_{i-1}) \otimes \mathsf{I}$ to be the identity on $(-\!\bullet\!-)^i$. Explicitly,

$$\mathsf{I}_i = \underbrace{\mathsf{I}\ \mathsf{I}\ \cdots\ \mathsf{I}}_{i}.$$

Let $\mathsf{U}^{i,j} = \mathsf{U}(i,j) = \mathsf{I}_i \otimes \mathsf{U} \otimes \mathsf{I}_j$ and $\cap_{i,j} = \cap(i,j) = \mathsf{I}_i \otimes \cap \otimes \mathsf{I}_j$ indicate the horizontal juxtaposition of either cup or cap with the identity on $(-\!\bullet\!-)^i$ on its left and the identity on $(-\!\bullet\!-)^j$ on its right. These are double arrows between non-negative integers with $s(\mathsf{U}(i,j)) = t(\cap(i,j)) = i + j$ while $t(\mathsf{U}(i,j)) = s(\cap(i,j)) = i + 2 + j$. The tensor $\otimes$ notation is dropped in favor of juxtaposition. The arrow $-\!\bullet\!-$ is *weakly self invertible* if (not necessarily invertible) double arrows $\cap$ and U exist.

Suppose that a pair of arbitrary double arrows $\overset{\mathsf{I}^i}{\underset{\mathsf{I}_k}{\boxed{F}}}$ and $\overset{\mathsf{I}^j}{\underset{\mathsf{I}_\ell}{\boxed{G}}}$ are given with

$i, j, k, \ell \in \mathbb{N}$. Then define

$$\begin{array}{c}|^{i}\\ \boxed{F}\\ |_{k}\end{array} \otimes \begin{array}{c}|^{j}\\ \boxed{G}\\ |_{\ell}\end{array} = \left(\begin{array}{c}|^{i}\\ \boxed{F}\\ |_{k}\end{array} \otimes \Big|_{j}\right) \circ_2 \left(\Big|_{i} \otimes \begin{array}{c}|^{j}\\ \boxed{G}\\ |_{\ell}\end{array}\right) = \left(\begin{array}{cc}|^{i} & |^{j}\\ \boxed{F} & |\\ | & \boxed{G}\\ |_{k} & |_{\ell}\end{array}\right).$$

The tensor product of any 2-morphism with the identity 2-morphism I_k is obtained by juxtaposing the identity horizontally.

A natural family of 3-isomorphisms is defined as follows:

$$\left(\Big|_{i} \otimes \begin{array}{c}|^{j}\\ \boxed{G}\\ |_{\ell}\end{array}\right) \circ_2 \left(\begin{array}{c}|^{i}\\ \boxed{F}\\ |_{k}\end{array} \otimes \Big|_{j}\right) \xleftarrow{\mathsf{X}} \left(\begin{array}{c}|^{i}\\ \boxed{F}\\ |_{k}\end{array} \otimes \Big|_{j}\right) \circ_2 \left(\Big|_{i} \otimes \begin{array}{c}|^{j}\\ \boxed{G}\\ |_{\ell}\end{array}\right)$$

Any one of these is called an *exchanger.* The notation is cumbersome when the source and target of an exchanger is specified.

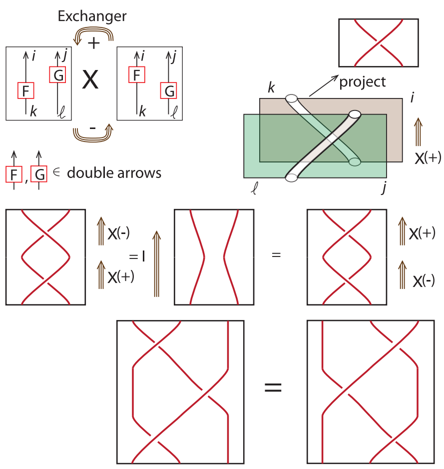
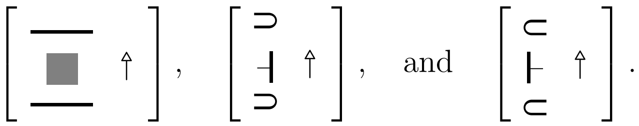


Figure 4: The invertible exchanger and naturality condition

The exchanger is indicated towards the upper left of Fig. 4, as a triple arrow that points leftward. A schematic diagram that indicates the exchanger as a kinematic process is drawn on the right. The translucent sheets that are labeled i, j, k and ℓ indicate parallel disks that may be folded along the tubes. via identities on U and $\mathsf{\cap}$. The exchanger $\mathsf{X}(+)$ is schematized as the positive crossing throughout the illustration.

The exchanger is invertible so that the equalities $\mathsf{X}(-) \circ_3 \mathsf{X}(+) = \mathsf{I}_G \otimes \mathsf{I}_F = \mathsf{X}(+) \circ_3 \mathsf{X}(-)$ hold for any pair of double arrows F and G. The final condition for the exchanger is that it is natural with respect to any other triple arrow. This implies, in particular, that the Yang-Baxter type relation holds. The naturality condition also implies that the exchanger commutes with any other triple arrows. The 3-category is *naturally monoidal* if a natural family (Fig. 5) of exchangers X exist.

Since the collection of triple arrows is also meant to be a category, the identity upon the identity double arrow I and the two non-trivial double arrows U, and $\mathsf{\cap}$ are defined as follows:



When necessary, *folds* which are the identities upon $\cup^{i,j}$ and $\cap_{i,j}$ can be adorned with double indices (i, j) to indicate the location (i sheets to to left [behind] and j sheets to the right [in front]) of them.

The following generating triple arrows are proposed.

Birth $[\smile]$: $(\cap) \circ_2 (\cup) \Lleftarrow \left(\overline{\underline{\square}} \right)$,

Death $[\frown]$: $\left(\overline{\underline{\square}} \right) \Lleftarrow (\cap) \circ_2 (\cup)$,

Saddle $[\dot{\cup}]$: $(\cup) \circ_2 (\cap) \Lleftarrow (\mathsf{I} \otimes \mathsf{I})$,

Crotch[2] $[\dot{\cap}]$: $(\mathsf{I} \otimes \mathsf{I}) \Lleftarrow (\cup) \circ_2 (\cap)$,

and 3-isomorphisms

Left cusp $[\curlyvee_L]$: $(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \leftrightarrows (\mathsf{I})$,

Right cusp $[\curlyvee_R]$: $(\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \leftrightarrows (\mathsf{I})$.

The names *birth, death, saddle, crotch, left cusp, and right cusp* are the names of the represented triple arrows, and one should also pronounce the associated icons in the same way. On the other hand, in the cases of the cusps, the leftward pointing triple arrows are called *left cusp down:* $\curlyvee_L$ and *right cusp down*: $\curlyvee_R$. We define *left cusp up*: $\curlywedge^L = \curlyvee_L^{-1}$ and *right cusp up*: $\curlywedge^R = \curlyvee_R^{-1}$. To say that $\curlywedge^L = \curlyvee_L^{-1}$ and $\curlywedge^R = \curlyvee_R^{-1}$ is to assert that the compositions

$$\mathsf{I} \overset{\curlywedge^L}{\Lleftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\Lleftarrow} \mathsf{I},$$

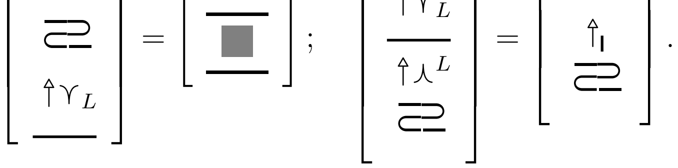
$$(\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup) \overset{\curlyvee_L}{\Lleftarrow} \mathsf{I} \overset{\curlywedge^L}{\Lleftarrow} (\cap \otimes \mathsf{I}) \circ_2 (\mathsf{I} \otimes \cup),$$

$$\mathsf{I} \overset{\curlywedge^R}{\Lleftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\Lleftarrow} \mathsf{I},$$

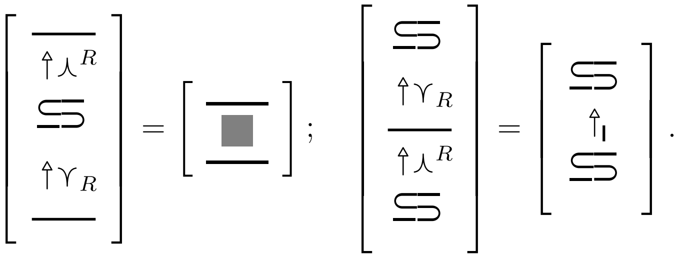
and

$$(\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I}) \overset{\curlyvee_R}{\Lleftarrow} \mathsf{I} \overset{\curlywedge^R}{\Lleftarrow} (\mathsf{I} \otimes \cap) \circ_2 (\cup \otimes \mathsf{I})$$

are the identity 3-morphisms on their (coincident) sources and targets. These relations are easier to imagine when written vertically:



[2] This mildly naughty term is meant to be used in the same way that a seamstress or tailor would use the word: as if it were the junction of the legs in a pair of pants.



The triple arrow $\Uparrow_{|}$ indicates the identity triple arrow on the zig-zagged compositions $(\cap\ |) \circ_2 (|\ \cup)$ and $(|\ \cap) \circ_2 (\cup\ |)$.

In addition, these identities can be represented as quadruple arrows that are, in turn, invertible as are all the higher order arrows derived therefrom.

A 3-category that has a weakly self-invertible arrow $-\!\!\bullet\!\!-$ is *strictly* 2-*pivotal* if there are 2-isomorphism, $\curlyvee_D$ and $\curlywedge^D$, for $D = L, R$ in the sense that is defined above.

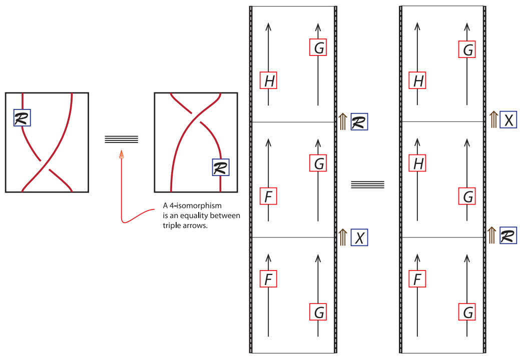


Figure 5: The naturality of the exchanger X

Fig. 5, depicts the naturality of the exchanger with respect to any triple arrow $\mathcal{R}$. For example, when $\mathcal{R}$ is one of X, $\smile$, $\frown$, $\cup$, $\cap$, $\curlyvee_D$, or $\curlywedge^D$ (for $D = R, L$), then half of the naturality identities are of the form

$$\left(\mathcal{R} \otimes |_F\right) \circ_3 \left(| \otimes \mathsf{X}(\pm)\right) \circ_3 \left(\mathsf{X}(\pm) \otimes |\right) = \left(|_F \otimes \mathcal{R}\right),$$

the others are of the form:

$$\left(| \otimes \mathsf{X}(\pm)\right) \circ_3 \left(\mathsf{X}(\pm) \otimes |\right) \circ_3 \left(|_F \otimes \mathcal{R}\right) = \left(\mathcal{R} \otimes |_F\right).$$

In a *naturally monoidal category*, (1) the exchanger is invertible in the sense that $\mathsf{X}(+) \circ_3 \mathsf{X}(-) = |_{G \otimes F} = \mathsf{X}(-) \circ_3 \mathsf{X}(+)$, and (2) the exchanger satisfies the naturality relations expressed in Fig. 5.

The adjoint relations that involve $\smile$ and $\dot{\cap}$ (or $\frown$ and $\dot{\cup}$) read as follows:

$$\left(\dashv \otimes \dot{\cap}\right) \circ_3 \left(\smile \otimes \dashv\right) = \dashv = \left(\dashv \otimes \frown\right) \circ \left(\dot{\cup} \otimes \dashv\right),$$

and

$$\left(\frown \otimes \vdash\right) \circ_3 \left(\vdash \otimes \dot{\cup}\right) = \vdash = \left(\vdash \otimes \dot{\cap}\right) \circ \left(\smile \otimes \vdash\right).$$

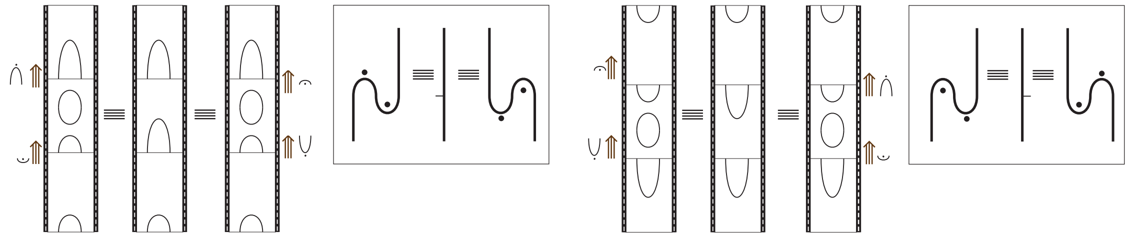


Figure 6: The adjoint relations

A 3-category which has a weakly self-invertible non-identity arrow is *weakly* 3-*pivotal* in case the non-invertible 3-morphisms $\dot{\smile}$, $\underset{\cdot}{\frown}$, $\underset{\cdot}{\cup}$, and $\dot{\cap}$ satisfy these adjoint relations.

There are four commutation relations between cusps and saddles or crotches. These are expressed in the following four equations. Here, in order to demonstrate the symmetries among these relations we denote $\curlywedge^R = \curlywedge^{\bullet}$, $\curlywedge^L = {}^{\bullet}\!\curlywedge$, $\curlyvee_R = \curlyvee_{\bullet}$, $\curlyvee_L = {}_{\bullet}\curlywedge$, and $\circ_3 = \circ$.

(i)
$$\left(\vdash \otimes \dot{\cap}\right) \circ \left(\curlyvee_{\bullet} \otimes \vdash\right) = \left({}^{\bullet}\!\curlywedge \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right)$$

(ii)
$$\left(\curlywedge^{\bullet} \otimes \vdash\right) \circ \left(\vdash \otimes \underset{\cdot}{\cup}\right) = \left(\vdash \otimes \dot{\cap}\right) \circ \left({}_{\bullet}\curlyvee \otimes \vdash\right)$$

(iii)
$$\left(\dashv \otimes \curlywedge^{\bullet}\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right) = \left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes {}_{\bullet}\curlyvee\right)$$

(iv)
$$\left(\dot{\cap} \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee_{\bullet}\right) = \left(\dashv \otimes {}^{\bullet}\!\curlywedge\right) \circ \left(\underset{\cdot}{\cup} \otimes \dashv\right)$$

For the sake of brevity we only illustrate two of these relations in Fig. 7. The other two can be obtained by turning the page upside down.

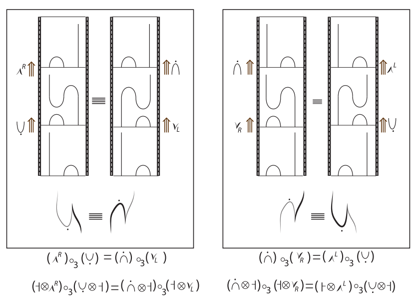


Figure 7: Commutations $\dot{\cap}$, $\curlyvee \leftrightharpoons \underset{\cdot}{\cup}$, $\curlywedge$ — part 1

Obeserve that equation (i) and (iv) are upside down versions of each other, as are (ii) and (iii). Furthermore, (ii) can be obtains from (i) by interchanging the left and right sides while also moving the indicators of the $\curlyvee$ and the $\curlywedge$ to the other side of the font. Equations (iii) and (iv) are similarly related. In an effort to obtain a concise version, we write

$$\boxed{\cap \circ \curlyvee = \curlywedge \circ \cup}$$

to encapsulate all four relations. A naturally monoidal, strictly 2-pivotal 3-category that has a weakly invertible non-identity 1-morphism is *rotationally commutative* if equations (i) through (iv) hold.

If, in addition, the following identities hold, then the 3-category is *strictly* 3-*tortile*.

$$\left(\curlywedge^L \otimes \dashv\right) \circ_3 \left(\vdash \otimes \mathsf{X}(+)\right) \circ \left(\curlyvee^L \otimes \dashv\right) = \dashv,$$

$$\dashv = \left(\curlywedge^R \otimes \dashv\right) \circ_3 \left(\vdash \otimes \mathsf{X}(-)\right) \circ \left(\curlyvee^L \otimes \dashv\right),$$

$$\left(\vdash \otimes \curlywedge^L\right) \circ_3 \left(\mathsf{X}(+) \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee^L\right) = \vdash,$$

and

$$\vdash = \left(\vdash \otimes \curlywedge^L\right) \circ_3 \left(\mathsf{X}(-) \otimes \dashv\right) \circ \left(\dashv \otimes \curlyvee^L\right).$$

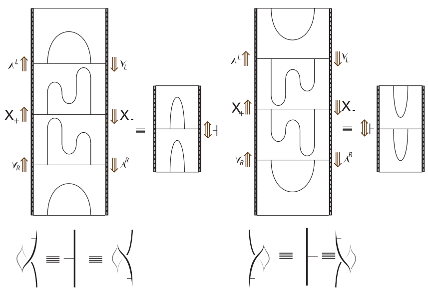


Figure 8: Swallowtail identities

As a final axiom, assert that the exchangers between 3-morphisms are also a natural family of isomorphisms.

4. Sketch of proof.

From a composition of triple arrows, a properly embedded surface in a 3-dimensional box can be obtained since the composition of the icons trace the folds of the projection of the surface onto a plane. In particular, the dots in the iconography for births, deaths, saddles, and crotches indicates the portion of the plane upon which more surface is projected. Similarly, the left/right distinction for cusps indicates which folds are closer to the plane of projection. Also the short segment along a fold indicates the side of the plane upon which a surface is folded. To finish creating the surface interpolate a cusp, birth, death, saddle, or crotch where these have their traditional meaning from singularity theory.

Given a properly embedded surface $S \subset \mathbb{R}^2 \times [0, 1]$, consider the following version of the fundamental groupoid as a higher category. There is a unique object that corresponds to the complement of the surface. The identity morphism is any arc that does not intersect the surface. An arc that intersects the surface transversely whose end-points are in the complement is a non-trivial 1-morphism. Any arc can be decomposed as a path product of arcs each of which intersects the surface S exactly once. Such 1-morphisms are composable if the initial point of one and the terminal point of the other are in the same component of the complement of S. In that case, connect the end points by an embedded arc that does not intersect S. A 2-morphism between arcs f and g is an embedded disk F whose boundary is decomposed as the union of the arcs f and g. Furthermore, the disk F should intersect the surface S transversely. A 3-morphism $\mathcal{R}$ is an embedded 3-ball whose top and bottom hemispheres are the disks representing the 2-morphisms F and G. In particular, the properly embedded surface $S \subset \mathbb{R}^2 \times [0,1]$ is a 3-morphism with source $\partial S_0 \subset \mathbb{R}^2 \times \{0\}$ and target $\partial S_1 \subset \mathbb{R}^2 \times \{1\}$. By setting up parametrizations and height functions in each interval, disk, or ball, one can then reconstruct the Us and $\cap$s in the disk F, and the $\dot{\smile}$s, $\underset{\cdot}{\frown}$s, $\underset{\cdot}{\cup}$s, $\dot{\cap}$s, $\curlyvee$s, and $\curlywedge$s in the 3-ball $\mathcal{R}$. In this way a functor between the topological 3-categories and the iconographic 3-category is constructed. Finally, isotopy moves are generated by geometric moves that correspond to the 4-isomorphisms defined above. These moves are critical cancelation (weakly 3-pivotal), lips and beal-to-beak singularities (strongly 2-pivotal), swallowtails (strictly 3-tortile), horizontal cusps (rotationally commutative), and exchanges of distant critical events (naturally monoidal)[3].

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[CS98] J. Scott Carter and Masahico Saito. *Knotted surfaces and their diagrams*, volume 55 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[3] Natural families of exchangers exist as 3 and 4-morphisms.