

日本数学会
2007年度年会

函数論分科会
講演アブストラクト

2007年3月
於 埼玉大学

第3日 3月29日(木)

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1

N- Fractional Calculus of Some Algebraic Functions

Katsuyuki Nishimoto

Descartes Press Co.

Abstract

In this article N- fractional calculus of the functions

$$(z^{2^n} \pm c^{2^n})^{-1} \quad (z^{2^n} \neq \pm c^{2^n}, n = 1, 2, 3)$$

are reported. A theorem obtained here is shown as follows for example.

Theorem. We have

$$\left(\frac{1}{z^4 - c^4}\right)_v = \frac{1}{2c^2} \left\{ g(z; \nu; c) \frac{1}{z^2 - c^2} - g(z; \nu; ic) \frac{1}{z^2 + c^2} \right\} \quad (z^2 \neq \pm c^2, c \neq 0),$$

where

$$g(z; \nu; c) = \frac{1}{2c} e^{-i\pi\nu} \Gamma(1 + \nu) \frac{(z+c)^{1+\nu} - (z-c)^{1+\nu}}{(z^2 - c^2)^\nu} \quad (z \neq \pm c, c \neq 0)$$

and $\nu \notin \mathbb{Z}^-$.

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Explicit Solutions of a Certain Class of Associated Legendre Equations by Means of Fractional Calculus

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Abstract

In recent years, many authors have demonstrated the usefulness of fractional calculus in the derivation of particular solutions of a number of linear ordinary and partial differential equations of the second and higher orders. The main object of the present paper is to show how several recent contributions on this subject, involving a certain class of associated Legendre equations, can be obtained (in a unified manner) by suitably applying some general theorems on particular solutions of a certain family of linear ordinary fractional differintegral equations.

Key Words and Phrases. Fractional calculus, Legendre equations, generalized Leibniz rule, analytic functions, differintegral equations, ordinary and partial differential equations, index law, linearity property, principal value, Bessel's equation, power-series solutions, Legendre polynomials.

3

Fractional Calculus Approaches to The Solutions of The Bessel Differential Equation of General Order

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Abstract

In a remarkably large number of recent works, one can find the emphasis upon (and demonstrations of) the usefulness of fractional calculus operators in the derivation of (explicit) particular solutions of significantly general families of linear ordinary and partial differential equations of the second and higher orders. The main object of this presentation is to survey some earlier investigations of this simple fractional-calculus approach to the solutions of the classical Bessel differential equation of general order and to show how it would lead naturally to several interesting consequences which include (for example) an alternative derivation of the complete power-series solutions obtainable usually by the Frobenius method. The underlying analysis presented here is based chiefly upon some of the general theorems on (explicit) particular solutions of a certain family of linear ordinary fractional differintegral equations with polynomial coefficients.

Key Words and Phrases. Operators of fractional calculus, Bessel differential equation, Fuchsian (and non-Fuchsian) differential equations, differintegral equations, (ordinary and partial) linear differential equations, polynomial coefficients, Frobenius method, power-series solutions, Bessel functions,

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A special fundamental solution base and its product

藤解和也 (金沢大学大学院自然科学研究科)

整函数 E と $n \in \mathbb{N}$ を与えて一次独立な n 個の整函数が成す系 $\{w_1, \dots, w_n\}$ で、Wronskian が $W(w_1, \dots, w_n) \equiv 1$ を満たし、かつその積は E に一致するものを構成する。この結果を、整函数を係数とする同次線型微分方程式の解の零点分布の研究に応用する。

問 1 *There are given a positive integer n , a domain Ω and a meromorphic function $E(z)$ on Ω . Find n functions w_1, \dots, w_n (which may not be meromorphic on Ω in general) such that the Wronski determinant and the product satisfy*

$$(1) \quad W(w_1, \dots, w_n) \equiv 1 \quad \text{and} \quad E(z) = \prod_{j=1}^n w_j(z) \quad (z \in \Omega).$$

本講演では、この問題に対するひとつの答として次の結果について述べる：

定理 1 *Define the functions w_j ($j = 1, \dots, n$) by*

$$(2) \quad w_j(z) = E(z)^{1/n} \exp\left(\frac{e^{2\pi i j}}{V_n} \int^z E(\zeta)^{-\frac{2}{n(n-1)}} d\zeta\right),$$

where V_n is the Vandermonde determinant of the $e^{2\pi i j/n}$ ($j = 1, \dots, n$), that is,

$$(3) \quad V_n := \prod_{\mu=1}^{n-1} \prod_{\nu=\mu}^{n-1} (e^{2\pi i(\nu+1)/n} - e^{2\pi i\nu/n}).$$

In the definition (2), the principal branch is chosen for $E^{1/n}$ and $E^{2/n(n-1)}$ respectively, and the integral is considered along a suitable curve in Ω . Then the system of the w_j satisfies (1).

次の疑問は自然であり、「一般には」正しいと考えるものの現時点で証明できていない：

問 2 *Is such a system of n functions w_j determined uniquely up to constant multiple c_j with $\prod_{j=1}^n c_j$?*

特殊な分解を許すなら否定的な回答も可能である。上記 (2) で与えられる w_j は、 $E(z) = z^3/2$ に対して多項式の 3 つ組 $\{1, z, z^2/2\}$, $W(1, z, z^2/2) \equiv 1$, ではなく平面上で定義される多価函数 $w_j := z^{1-c_j/(c-1)}/\sqrt[3]{2}$ ($j = 1, 2, 3$), $c = e^{2\pi i/3}$ を与える。対応する方程式 $w^{(3)} + \frac{1}{z^2}w' - \frac{1+\lambda^3}{z^3}w = 0$ は $z = 0$ に確定特異点を持ち、 $1, z, z^2/2$ に対応する方程式 $y^{(3)} = 0$ とは大きく異なる。

実際、証明では [2] のアイデアを元に与えられた E と W から n 階の同次線型微分方程式を構成し、その基本解として求める系を得る。

整函数を係数にもつ線型微分方程式の任意の解は超越整函数であるが、すべての係数が多項式であればその位数は正の有理数となる。また係数に超越函数が含まれていれば、位数無限大の解が存在する ([3])。これらの値分布についても係数の情報を用いて記述することが可能であり、本質的に零点以外の値分布は normal であることも判っている。講演では、解の零点分布について示されている定理と今回得られた結果との関連を述べる。例えば G. Frank [1] による多項式係数の方程式と 0 を除外値にもつ基本解系に関する結果

定理 A Consider a linear differential equation

$$(4) \quad L_n(w) := w^{(n)} + a_{n-1}(z)w^{(n-1)} + \cdots + a_0(z)w = 0,$$

where the $a_j(z)$ are polynomials, $a_0(z) \not\equiv 0$. It has a fundamental system with the Picard value 0, when there is a polynomial $q(z)$ such that the transformation

$$w(z) = \exp\{q(z)\}u(z)$$

reduces $L_n(w) = 0$ into a differential equation in $u(z)$ with constant coefficients.

や H. Wittich の定理などについて考察する。これは上述した「反例」とも関連する。また各係数の位数が有限で、いずれも零点の収束指数が有限であるような基本解系が存在すれば、それら基本解の積もまた有限位数をもつことが知られている。次の意味でその逆も成り立つ：

系 1 For a 'suitably' given entire function E of finite order, we find an n th order homogenous linear equation of the form

$$(5) \quad w^{(n)} + \sum_{j=0}^{n-2} A_j(z)w^{(j)} = 0,$$

and its fundamental system of solutions w_j satisfying:

1. every coefficient A_j is entire and of finite order;
2. the exponent of convergence of w_j is finite.

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5

Some notes on holomorphic curves extremal for the defect relation

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1. Introduction. (a) Let $f = [f_1, \dots, f_{n+1}]$ be a non-degenerate, transcendental holomorphic curve from \mathbf{C} into $P^n(\mathbf{C})$ with a reduced representation $(f_1, \dots, f_{n+1}) : \mathbf{C} \rightarrow \mathbf{C}^{n+1} - \{0\}$, where n is a positive integer.

For $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbf{C}^{n+1} - \{0\}$, we put

$$(\mathbf{a}, f(z)) = a_1 f_1(z) + \dots + a_{n+1} f_{n+1}(z), \quad (\mathbf{a}, f) = a_1 f_1 + \dots + a_{n+1} f_{n+1}.$$

When (\mathbf{a}, f) has at least one zero, we say that \mathbf{a} has multiplicity m if all the zeros of the function $(\mathbf{a}, f(z))$ have multiplicity at least m , while at least one zero has multiplicity m . When (\mathbf{a}, f) has no zero, we set $m = \infty$. We put

$$\mu_n(\mathbf{a}, f) = 1 - n / \max(m, n).$$

Then, $0 \leq \mu_n(\mathbf{a}, f) \leq \delta_n(\mathbf{a}, f) \leq 1$ and $\mu_n(\mathbf{a}, f) = 1$ if and only if $m = \infty$.

Let X be a subset of $\mathbf{C}^{n+1} - \{0\}$ in N -subgeneral position, where $N \geq n$ and

$$X(0) = \{\mathbf{a} = (a_1, \dots, a_{n+1}) \in X | a_{n+1} = 0\}, \quad M_n^1(X, f) = \{\mathbf{a} \in X | \mu_n(\mathbf{a}, f) = 1\}.$$

For a non-empty subset P of X , let $V(P)$ be the vector space spanned by $\{\mathbf{a} \in P\}$ and $d(P) = \dim V(P)$.

Defect Relation I ([1] ($N = n$), [3] ($N > n$). See [2]).

$$\sum_{\mathbf{a} \in X} \mu_n(\mathbf{a}, f) \leq \sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) \leq 2N - n + 1.$$

(b) We put for $\mathbf{a} \in \mathbf{C}^{n+1} - \{0\}$

$$\mathcal{T}(r, f) = \int_a^r \frac{T(t, f)}{t} dt; \quad \mathcal{N}(r, \mathbf{a}, f) = \int_a^r \frac{N(t, \mathbf{a}, f)}{t} dt; \quad \mathcal{M}(r, \varphi) = \int_a^r \frac{m(t, \varphi)}{t} dt,$$

where φ is a meromorphic function in $|z| < \infty$ and $a > 1$.

Further, we put $\tilde{\Delta}(\mathbf{a}, f) = 1 - \liminf_{r \rightarrow \infty} \mathcal{N}(r, \mathbf{a}, f) / \mathcal{T}(r, f)$. Then,

$$0 \leq \delta(\mathbf{a}, f) \leq \tilde{\Delta}(\mathbf{a}, f) \leq \Delta(\mathbf{a}, f) \leq 1.$$

Lemma 1. Let g be a transcendental meromorphic function in $|z| < \infty$. Then,

$$\mathcal{M}(r, g/g^{(k)}) \leq O\{\log \mathcal{T}(r, g)\} + O\{(\log r)^2\}.$$

(c) We put for $u(z) = \max_{1 \leq j \leq n} |f_j(z)|$

$$t(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \{\log u(re^{i\theta}) - \log u(e^{i\theta})\} d\theta, \quad t_o(r, f) = \int_a^r t(t, f) / t dt$$

and $\gamma = \liminf_{r \rightarrow \infty} t(r, f)/T(r, f)$, $\gamma_o = \liminf_{r \rightarrow \infty} t_o(r, f)/T(r, f)$. Then, $0 \leq \gamma \leq \gamma_o \leq 1$.

Let $w : X \rightarrow (0, 1]$ be the Nochka weight function for X and $\rho(f)$ be the order of f .

Defect relation II.

$$\sum_{\mathbf{a} \in X}^* \delta_n(\mathbf{a}, f) \leq 2N - n + 1 - \frac{N}{n}(n - d)(1 - \gamma_o),$$

where $d = \sum_{\mathbf{a} \in X(0)} w(\mathbf{a})$.

Note. γ_o can be replaced by γ when $\rho(f) < \infty$.

Lemma 2. Suppose that $\sum_{\mathbf{a} \in X} \mu_n(\mathbf{a}, f) = 2N - n + 1$. Then

$$\tilde{\Delta}(\mathbf{a}, f) = 0 \quad (\mathbf{a} \in X - M_n^1(X, f)).$$

Note. $\tilde{\Delta}(\mathbf{a}, f)$ can be replaced by $\Delta(\mathbf{a}, f)$ when $\rho(f) < \infty$.

2. Results. [I]. Suppose that $N > n \geq 2$ and that

$$\sum_{\mathbf{a} \in X} \delta_n(\mathbf{a}, f) = 2N - n + 1.$$

If $\gamma_o < 1$, then (a) $\#X(0) = N$; (b) there is a subset $P \subset X$ satisfying

$$\#P = N - n + 1, \quad d(P) = 1, \quad \delta_n(\mathbf{a}, f) = 1 \quad (\mathbf{a} \in P) \text{ and } X(0) \cap P = \phi.$$

[II]. Suppose that $N > n \geq 2$ and that $\gamma_o < 1$, then

$$\sum_{\mathbf{a} \in X} \mu_n(\mathbf{a}, f) < 2N - n + 1.$$

Note. In [I] and [II], γ_o can be replaced by γ when $\rho(f) < \infty$.

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6

双曲的ガウス写像の完全分岐値数について

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我々はこれまでユークリッド空間内の極小曲面のガウス写像の値分布論的性質について調べてきた。3次元ユークリッド空間 \mathbf{R}^3 内の極小曲面のガウス写像は開リーマン面上の有理型函数とみなすことができるので、除外値数やそれを拡張した完全分岐値数の評価式などの値分布論的性質を示すことができる。我々は論文 [KKM] において代数的極小曲面 (有限全曲率完備極小曲面のこと) やシャーク曲面など特殊な無限全曲率の場合を含む “擬代数的極小曲面” という完備極小曲面のクラスを定義して、そのクラスに属する曲面のガウス写像の完全分岐値数の最良の評価式を得ることができた。さらに \mathbf{R}^4 内の擬代数的極小曲面のガウス写像 ([Ka2] 参照) や \mathbf{R}^n 内の擬代数的極小曲面の一般化されたガウス写像 ([JR] 参照) の完全分岐値数についても最良の評価式を得ることができた。

ところで、上記のような性質は3次元双曲空間 \mathbf{H}^3 内の完備な平均曲率1の曲面 (以下、平均曲率1の曲面を “CMC-1” 曲面と呼ぶことにする) の双曲的ガウス写像に対しても成り立つことが期待される。実際、この曲面の双曲的ガウス写像は開リーマン面上の有理型函数とみなすことができ、除外値数の評価について極小曲面のガウス写像のときと同様の結果がユ先生 ([Yu]) やローゼンバーグ先生たち ([CHR]) によって知られている。そこで、我々はこの場合の完全分岐値数の評価について調べ、いくつか結果を得ることができた。まず完備な CMC-1 曲面の双曲的ガウス写像の完全分岐値数について、ユ先生の論文 [Yu] を改良することで次の結果を示すことができた。

主結果 1 ([Ka3]). $x: M \rightarrow \mathbf{H}^3$ を3次元双曲空間内の非平坦 (ガウス曲率が恒等的に0でない) な完備 CMC-1 曲面とし、 $G: M \rightarrow \hat{\mathbf{C}}$ をその双曲的ガウス写像とする。 D_G をその写像の除外値数、 ν_G を完全分岐値数とすると次の式が成り立つ。

$$D_G \leq \nu_G \leq 4. \tag{1}$$

実際、 $\nu_G = 4$ の例が存在するので、上の式の評価は最良である。

また完備な CMC-1 曲面の双曲的ガウス写像 G の次数が有限の場合 (この曲面のことを代数的 CMC-1 曲面と呼ぶことにする)、定義域のリーマン面はコンパクトリーマン面 (その種数を γ とする) から有限個 (k 個とする) の点を除いた穴あきリーマン面 $\overline{M}_\gamma \setminus \{p_1, \dots, p_k\}$ と等角同値になり、 G は穴で高々極しかとらないことがわかるが、このときの完全分岐値数に対して次の式が成り立つ。

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主結果 2 ([Ka3]). $x: \overline{M}_\gamma \setminus \{p_1, \dots, p_k\} \rightarrow \mathbf{H}^3$ を 3次元双曲空間内の非平坦な代数的 CMC-1 曲面とし, G をその双曲的ガウス写像とする. D_G をその写像の除外値数, ν_G を完全分岐値数とすると次の式が成り立つ.

$$D_G \leq \nu_G \leq 2 + \frac{2}{R}, \quad R = \frac{d}{\gamma - 1 + k/2} > 1. \quad (2)$$

特に $D_G \leq \nu_G < 4$ が成り立つので, この場合の双曲的ガウス写像の除外値数は高々3である

本講演ではここで述べたことの詳細や代数的極小曲面のガウス写像の性質との違い ($\nu_G = 2.5$ の例が発見できない!) について報告する予定である.

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7

Quasi-symmetry for the non-linear Green function on a tree

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Let $1 < p < \infty$. We consider p -harmonic functions on a tree. Let $\mathcal{T} = (V, E, r)$ be a locally finite connected tree with a resistance r , where $V = V(\mathcal{T})$ is the vertex set and $E = E(\mathcal{T})$ is the edge set. An edge $(x, y) \in E$ is an ordered pair of vertices such that $(x, y) \in E$ if and only if $(y, x) \in E$. If $(x, y) \in E$, then we say that x is adjacent to y and write $x \sim y$. A resistance r is a positive function on E such that $r(y, x) = r(x, y)$. We define the discrete derivative ∇u and the discrete p -Laplacian $\Delta_p u$ for a function u on V by

$$\begin{aligned}\nabla u(x, y) &= r(x, y)^{-1}(u(y) - u(x)), \\ \Delta_p u(x) &= \sum_{\substack{y \in V \\ y \sim x}} |\nabla u(x, y)|^{p-2} \nabla u(x, y).\end{aligned}$$

Let $D \subset V$. If $\Delta_p u = 0$ in D , then we say that u is p -harmonic in D . For these accounts see [1, 2, 3].

Let $x, y \in V$. A path joining x to y is a sequence $\{x = x_0, x_1, \dots, x_{l-1}, x_l = y\}$ of distinct vertices such that $x_0 \sim x_1 \sim \dots \sim x_{l-1} \sim x_l$. Since \mathcal{T} is a tree, it is uniquely determined and denoted by \overline{xy} . For $x \in V$ let $\deg(x)$ be the degree of x , i.e., $\deg(x) = \#\{y \in V; x \sim y\}$. Let $A \subset E$ and $x \in V$. We remove A from E , then we obtain some components. We denote by $\mathcal{S}(\mathcal{T}, A, x)$ the component which contains x .

We define the Dirichlet sum $D_p[u]$ of order p by

$$D_p[u] = \frac{1}{2} \sum_{(x,y) \in E} r(x, y) |\nabla u(x, y)|^p.$$

Denote by $\mathbf{D}^{(p)}(\mathcal{T})$ the set of functions on V with finite Dirichlet sum of order p . Then $\mathbf{D}^{(p)}(\mathcal{T})$ is a Banach space with the norm $\|u\|_p = (D_p[u] + |u(x_0)|)^{1/p}$, where x_0 is a fixed vertex. Let $L_0(\mathcal{T})$ be the set of functions on V with finite support. Also let $\mathbf{D}_0^{(p)}(\mathcal{T})$ be the closure of $L_0(\mathcal{T})$ in $\mathbf{D}^{(p)}(\mathcal{T})$ with respect to the norm $\|\cdot\|_p$. A tree \mathcal{T} is said to be of hyperbolic type of order p if $1 \notin \mathbf{D}_0^{(p)}(\mathcal{T})$; a tree \mathcal{T} is said to be of parabolic type of order p otherwise. Consider the discrete boundary value problem

$$\Delta_p u = -\delta_a, \quad u \in \mathbf{D}_0^{(p)}(\mathcal{T}), \quad (1)$$

where δ_a is the characteristic function of $\{a\}$, i.e., $\delta_a(x) = 1$ if $x = a$ and $\delta_a(x) = 0$ otherwise. The solution u to (1) uniquely exists if and only if the tree is of hyperbolic type of order p . We call the solution u the p -Green function with pole at a and denote it by g_a . For details see Yamasaki [2, 3].

Let

$$H(x, y) = \frac{g_x(y)}{g_y(x)} \quad \text{for } x, y \in V,$$
$$M(\mathcal{T}) = \sup_{x, y \in V} H(x, y)$$

It is well known that $M(\mathcal{T}) = 1$ for any tree \mathcal{T} if $p = 2$. However, if $p \neq 2$, then it is not known that $M(\mathcal{T}) = 1$, even whether $M(\mathcal{T})$ is finite or not. We consider the problem whether $M(\mathcal{T})$ is finite or not for $p \neq 2$. A tree \mathcal{T} is said to have a symmetric p -Green function if $M(\mathcal{T}) = 1$; a tree \mathcal{T} is said to have a quasi-symmetric p -Green function if $M(\mathcal{T})$ is finite.

Theorem 1. *Let $\mathcal{T} = (V, E, r)$ be a tree of hyperbolic type of order p . Let $(a_1, a_2) \in E$. Let $\mathcal{T}_1 = \mathcal{S}(\mathcal{T}, \{(a_1, a_2)\}, a_1)$ and $\mathcal{T}_2 = \mathcal{S}(\mathcal{T}, \{(a_1, a_2)\}, a_2)$. Then \mathcal{T} has a quasi-symmetric p -Green function if and only if each of \mathcal{T}_1 and \mathcal{T}_2 has a quasi-symmetric p -Green function.*

Theorem 2. *Let $p \neq 2$. Let (V, E, r) be a tree.*

- (1) *Suppose that there are only finitely many $x \in V$ such that $\deg(x) \geq 3$. Then (V, E, r) has a quasi-symmetric p -Green function whenever (V, E, r) is of hyperbolic type of order p .*
- (2) *Suppose that there are infinitely many $x \in V$ such that $\deg(x) \geq 3$. Then we find two resistances r_1 and r_2 with the following conditions.*
 - (a) *The tree (V, E, r_1) is of hyperbolic type of order p and has a quasi-symmetric p -Green function.*
 - (b) *The tree (V, E, r_2) is of hyperbolic type of order p and, however, does not have a quasi-symmetric p -Green function.*

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8

\mathbb{R}^n ($n \geq 3$) の回転不変計量に関する caloric morphism

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$\mathbb{R}^n \setminus \{0\}$ ($n \geq 3$) に, 回転不変なリーマン計量

$$\sum_{i,j=1}^n g_{ij} dx_i dx_j$$

を入れてリーマン多様体とする. 計量 g に関する gradient を ∇_g で, Laplacian を Δ_g でそれぞれ表す.

D を $\mathbb{R} \times M$ 内の領域とし, $f(t, x) = (f_0(t, x), f_1(t, x), \dots, f_n(t, x))$ を D から $\mathbb{R} \times M$ への C^∞ 級写像, φ を D 上の正値 C^∞ 級関数とする. 任意の領域 $E \subset f(D)$ 上で熱方程式

$$H_g u := \frac{\partial u}{\partial \tau} - \Delta_g u = 0$$

を満たす任意の関数 $u(\tau, y)$ に対して, $v(t, x) = \varphi(t, x)(u \circ f)(t, x)$ が $f^{-1}(E) \subset D$ 上で熱方程式

$$H_g v = 0$$

を満たす時, (f, φ) を **caloric morphism** と呼ぶ.

計量が回転不変ならば, x に関する回転は caloric morphism であるから, t に関する平行移動と合わせた

$$f(t, x) = (t + d, Rx), \quad \varphi(t, x) = 1, \quad (d \in \mathbb{R}, R \text{ は直交行列})$$

が, 自明な caloric morphism として, 常に存在する.

(f, φ) が caloric morphism であることと、次の (1) – (4) は同値である ([1]).

$$(1) \quad H_g \varphi = 0,$$

$$(2) \quad H_g f_i = 2g(\nabla_g \log \varphi, \nabla_g f_i) - [(\Delta_g y_i) \circ f] \frac{df_0}{dt}, \quad i = 1, \dots, n,$$

$$(3) \quad \nabla_g f_0 = 0,$$

$$(4) \quad g(\nabla_g f_i, \nabla_g f_j) = (g^{ij} \circ f) \frac{df_0}{dt}, \quad i, j = 1, \dots, n.$$

この講演では、3次元以上の場合について、回転不変な計量に関する caloric morphism (f, φ) および、その時の計量 g を具体的に決定する問題に関する結果を報告する。

特に

(1) どのような計量について自明でない caloric morphism が存在するか、

(2) その場合、caloric morphism を構成する基本的変換は何か、

を中心として報告する。

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9

Vanishing exponential integrability for Riesz potentials of functions in Orlicz classes

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本講演では, Adams-Hurri-Syrjänen [1], Edmunds-Gurka-Opic [2], Trudinger [6], Mizuta-Shimomura [4] の研究に関連して, リースポテンシャル $R_\alpha f$ の Vanishing exponential integrability について報告する.

α ($0 < \alpha < n$) 次のリースポテンシャル

$$R_\alpha f(x) = \int_{\mathbf{R}^n} |x-y|^{\alpha-n} f(y) dy$$

を考える. ここに, f は非負可測関数で $R_\alpha f \neq \infty$ と仮定する. C_{α, Φ_p} はリース容量とする.

条件

$$\int_{\mathbf{R}^n} \Phi_p(f(y)) dy < \infty$$

を考える:

(φ_1) $\Phi_p(r) = r^p \varphi(r)$, $1 < p < \infty$; φ は開区間 $(0, \infty)$ 上正値かつ単調. $\Phi_p(0) = 0$.

(φ_2) $c^{-1} \varphi(r) \leq \varphi(r^2) \leq c \varphi(r)$ ($r > 0$)

(φ_3)

$$\int_1^\infty \varphi(t)^{-1/(p-1)} t^{-1} dt = \infty.$$

ここでは, 簡単のため, 以下, $\varphi(r) = [\log(e + f(y))]^a [\log(e + \log(e + f(y)))]^b$, つまり,

$$\int_{\mathbf{R}^n} f(y)^p [\log(e + f(y))]^a [\log(e + \log(e + f(y)))]^b dy < \infty$$

を仮定する.

定理 1. $\alpha p = n$, $a < p - 1$, $\beta = p/(p - 1 - a)$, $\gamma = b/(p - 1 - a)$ とする. このとき, C_{α, Φ_p} -容量が零の集合 E が存在して, $\forall x_0 \in \mathbf{R}^n \setminus E$, $\forall A > 0$ に対して,

$$\lim_{r \rightarrow 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} \left\{ \exp(A |R_\alpha f(x) - R_\alpha f(x_0)|^\beta) \times (\log(e + |R_\alpha f(x) - R_\alpha f(x_0)|))^\gamma - 1 \right\} dx = 0 \quad (1)$$

$a = p - 1$ のとき, $\forall \beta > 0$ ($\forall \gamma > 0$) に対して (1) は成り立つ. $a > p - 1$ のとき, $R_\alpha f$ は \mathbf{R}^n 上連続である (Mizuta [3]). 文献 [1] で, Adams-Hurri-Syrjänen は, $0 \leq a < p - 1$, $b = 0$ のときについて論じている.

定理 2. $\alpha p = n$, $a = p - 1$, $b < p - 1$, $\beta = p/(p - 1 - b)$ とする. このとき, C_{α, Φ_p} -容量が零の集合 E が存在して, $\forall x_0 \in \mathbf{R}^n \setminus E$, $\forall A > 0$, $\forall B > 0$ に対して,

$$\lim_{r \rightarrow 0} \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} \{ \exp(A \exp(B |R_\alpha f(x) - R_\alpha f(x_0)|^\beta)) - e^A \} dx = 0 \quad (2)$$

$b > p - 1$ のとき, $R_\alpha f$ は \mathbf{R}^n 上連続であり (Mizuta [3]), $\forall x_0 \in \mathbf{R}^n$, $\forall \beta > 0$ に対して (2) は成り立つ.

本報告の結果は [5] による.

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優調和関数の平均連続性

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M を C^∞ 級の d 次元 ($d \geq 2$) 多様体で, コムパクトであってもなくてもかまわな
いが, 可符号かつ連結とする. M に C^∞ 級の弧要素 ds を

$$ds^2 = \sum_{1 \leq i, j \leq d} g_{ij}(x) dx_i dx_j \quad (x = (x_1, \dots, x_d))$$

で与えて M は Riemann 多様体であるとし, $g = \det(g_{ij})$, $(g^{ij}) = (g_{ij})^{-1}$ とおいて
 M 上の Schrödinger 方程式

$$(1) \quad \left(-\frac{1}{\sqrt{g}} \sum_{1 \leq i, j \leq d} \frac{\partial}{\partial x_i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x_j} \right) + \mu \right) u = 0$$

を考える (cf. e.g. [5]), 但しそのポテンシャル μ は M 上の Kato 測度, 即ち M 上
の Radon 測度で, M の任意の局所球 (V, x) に対して

$$\lim_{\epsilon \downarrow 0} \left(\sup_{|x| < \epsilon} \int_{|y| < \epsilon} N(x-y) d|\mu|(y) \right) = 0$$

をみたすものとする, 但し $N(t) = \log(1/|t|)$ ($d = 2$), $1/|t|^{d-2}$ ($d \geq 3$) で, $|\mu|$ は μ
の全変分測度とする. (1) の開集合 $U \subset M$ 上の連続超関数解 (それを U 上の μ 調
和関数と呼ぶ) の全体を $H_\mu(U)$ と記すとき, $H_\mu : U \mapsto H_\mu(U)$ を M 上の調和層
として (M, H_μ) は Brelot 調和空間となる (cf. [1]). よって各開集合 U 上の境界値
 f の Dirichlet 解 $(H_\mu)_f^U$ を考えることが出来る. そのとき M 上の下半連続関数 u
で, M 上常に $u > -\infty$ かつ M 上 $u \not\equiv +\infty$ となり, さらに harmonically concave
(即ち任意の十分小さな局所球 V 達に対して V 上 $u \geq (H_\mu)_u^V$) となるとき, u は
 M 上 μ 優調和であると言って, その全体を $S_\mu(M)$ と記す. $g_{ij} = \delta_{ij}$ (Kronecker
delta) かつ $\mu \equiv 0$ の場合には周知の基本的かつ初等的知識 (cf. e.g. [2]) に他なら
ない次の結果を報告する

定理. 任意の $u \in S_\mu(M)$ は M 上平均連続である, 即ち, すべての点 $a \in M$
に於いて

$$(2) \quad \lim_{r \downarrow 0} \frac{1}{\lambda(B(a, r))} \int_{B(a, r)} u(x) d\lambda(x) = u(a)$$

となる, 但し $B(a, r)$ は a 中心半径 r の測地球, $d\lambda(x) = \sqrt{g(x)}dx_1 \cdots dx_d$ ($x = (x_1, \dots, x_d)$) は M 上の体積要素とする.

証明は 2 段階の作業に分ける. 先ず $S_0(M)$ (即ち $\mu \equiv 0$ に対する 0 優調和関数族) に対して微分方程式論的議論に基づき (cf. [3], [4]) (2) を示す. 次に調和空間の攝動論的手法により $S_0(M)$ の結果を $S_\mu(M)$ へ移行させることにより $S_\mu(M)$ に対して (2) を示す. 上の定理の測地球 $B(a, r)$ は, a における測地座標による Euclid 球でおきかえることが出来る.

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特別講演

On asymptotically sharp inequalities for quasiconformal harmonic mappings

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Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and let $\text{Hom}^+(\mathbb{T})$ stand for the class of all sense-preserving homeomorphic self-mappings of \mathbb{T} . Given any $f \in \text{Hom}^+(\mathbb{T})$ we consider the Poisson extension $P[f]$ of f to \mathbb{D} , where

$$P[f](z) := \frac{1}{2\pi} \int_{\mathbb{T}} f(u) \operatorname{Re} \frac{u+z}{u-z} |du|, \quad z \in \mathbb{D}.$$

According to the famous Radó-Kneser-Choquet theorem (cf. e.g. [3], [5] and references quoted there), $P[f]$ is a harmonic and homeomorphic self-mapping of \mathbb{D} .

First in section 1 we assume that $P[f](0) = 0$ and that f admits a K -quasiconformal extension \tilde{F} to \mathbb{D} satisfying $\tilde{F}(0) = 0$. We then obtain a variant of Schwarz's lemma for $P[f]$, which gives an upper bound of $|P[f](z)|$ and which is asymptotically sharp as K tends to 1.

Next in section 2 assuming that $P[f]$ is a K -quasiconformal mapping and that $P[f](0) = 0$, we obtain a variant of Heinz's inequality. It gives lower bounds of $|\partial_x F|^2 + |\partial_y F|^2$ and $|\partial F|$ where $F = P[f]$, and they are asymptotically sharp as K tends to 1.

Finally on quasiconformality of $P[f]$, it is known that $P[f]$ is not quasiconformal in general even if f admits a quasiconformal extension to \mathbb{D} (see [10],[8] and [18]). In [11] and [12] we gave intrinsic characterizations of quasiconformality of $P[f]$ in terms of the Cauchy and Cauchy-Stieltjes singular integrals involving f . Subsequently and decisively in [17] M. Pavlović proved that if $F = P[f]$ is quasiconformal, then F is bi-Lipschitz with respect to the Euclidean metric. As an improvement of this, using the results given in sections 3 and 4, we find in section 5 explicit estimations of bi-Lipschitz constants for such mappings F that are expressed by means of the maximal

dilatation K of F and $|F^{-1}(0)|$. Under the additional assumption $F(0) = 0$ the estimations are asymptotically sharp as K tends to 1, so F behaves almost like a rotation for sufficiently small K .

This note is a summary of [13], [14], [15] and [16] which are joint works with D. Partyka (Catholic Univ. of Lublin, Poland).

1 A variant of Schwarz's lemma

Schwarz's lemma for harmonic mappings is stated as follows.

Theorem A. ([6],[2]) *Suppose that u is a complex-valued harmonic function on \mathbb{D} , $|u(z)| < 1$ on \mathbb{D} , and $u(0) = 0$. Then for every $z \in \mathbb{D}$,*

$$|u(z)| \leq \frac{4}{\pi} \arctan |z|.$$

We now recall that the Hersch-Pfluger distortion function Φ_K is defined for any $K > 0$ by the equalities

$$\Phi_K(r) := \mu^{-1}(\mu(r)/K), \quad 0 < r < 1; \quad \Phi_K(0) := 0, \quad \Phi_K(1) := 1,$$

where $\mu(r)$ stands for the module of the Grötzsch extremal domain $\mathbb{D} \setminus [0, r]$; cf. [7] and [9, p.53 and p.63].

By means of the distortion function $\Phi_{1/K}$ we obtain a variant of Schwarz's lemma for harmonic mappings.

Theorem 1. ([15, Th. 3.3 and Remark 3.4]) *Let $K \geq 1$ and let $F = P[f]$ for some $f \in \text{Hom}^+(\mathbb{T})$ which admits a K -quasiconformal extension \tilde{F} to \mathbb{D} satisfying $\tilde{F}(0) = 0$. Assume that $F(0) = 0$. Then for every $z \in \mathbb{D}$,*

$$\begin{aligned} |F(z)| &\leq \Lambda_K(|z|) \\ &:= \frac{4}{\pi} \arctan |z| - \frac{8|z|(1-|z|^2)}{\pi} \int_0^{\pi/2} \frac{(\cos t)\Phi_{1/K}(\sin(t/2))^2}{(1+|z|^2)^2 - 4|z|^2(\cos t)^2} dt \\ &= |z| + \frac{8|z|(1-|z|^2)}{\pi} \int_0^{\pi/2} \frac{(\cos t) [(\sin(t/2))^2 - \Phi_{1/K}(\sin(t/2))^2]}{(1+|z|^2)^2 - 4|z|^2(\cos t)^2} dt. \end{aligned}$$

Moreover, $\Lambda_1(|z|) = |z|$ and the following equalities hold:

$$\lim_{K \rightarrow 1^+} \limsup_{r \rightarrow 0^+} \frac{\Lambda_K(r)}{r} = 1, \quad \lim_{K \rightarrow 1^+} \liminf_{r \rightarrow 1^-} \frac{1 - \Lambda_K(r)}{1 - r} = 1.$$

2 A variant of Heinz's inequality

Heinz showed in [6] the following (see [4], too).

Theorem B. *Assume that F is a one-to-one harmonic mapping of \mathbb{D} onto itself normalized by $F(0) = 0$. Then for every $z = x + iy \in \mathbb{D}$,*

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2} .$$

For $f \in \text{Hom}^+(\mathbb{T})$ the limit

$$f'(z) := \lim_{u \rightarrow z} \frac{f(u) - f(z)}{u - z}$$

exists for a. e. $z \in \mathbb{T}$, and set

$$d_f := \text{ess inf}_{z \in \mathbb{T}} |f'(z)| . \quad (1)$$

For $K \geq 1$ set

$$L_K^* := \frac{2}{\pi} \int_0^{\Phi_{1/K}(1/\sqrt{2})^2} \frac{dt}{\Phi_K(\sqrt{t}) \Phi_{1/K}(\sqrt{1-t})} . \quad (2)$$

Then L_K^* is a strictly decreasing function of $K \geq 1$ such that

$$\lim_{K \rightarrow 1} L_K^* = L_1^* = 1 \quad \text{and} \quad \lim_{K \rightarrow +\infty} L_K^* = 0 \quad (\text{cf. [14, Lemma 1.4]}).$$

By means of L_K^* we obtain a lower bound of d_f and a variant of Heinz's inequality for quasiconformal harmonic mappings.

Theorem 2. ([14, Th. 2.1]) *Given $K \geq 1$ let F be a K -quasiconformal and harmonic self-mapping of \mathbb{D} satisfying $F(0) = 0$. If f is the boundary valued function of F , then*

$$d_f \geq \frac{1}{K} \max \left\{ \frac{2}{\pi}, L_K^* \right\} .$$

Theorem 3. ([14, Th. 2.2]) *Given $K \geq 1$ let F be a K -quasiconformal and harmonic self-mapping of \mathbb{D} satisfying $F(0) = 0$. Then the inequalities*

$$|\partial F(z)| \geq \frac{K+1}{2K} \max \left\{ \frac{2}{\pi}, L_K^* \right\} \quad (3)$$

and

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{1}{2} \left(1 + \frac{1}{K} \right)^2 \max \left\{ \frac{4}{\pi^2}, L_K^{*2} \right\} \quad (4)$$

hold for every $z \in \mathbb{D}$. Moreover, the right hand sides in (3) and (4) are decreasing and continuous functions of $K \geq 1$ with values in $(1/\pi, 1]$ and $(2/\pi^2, 2]$, respectively.

3 On boundary properties for quasiconformal self-mappings of the unit disk

We recall that the Cauchy singular integral $C_{\mathbb{T}}[f]$ of a function $f : \mathbb{T} \rightarrow \mathbb{C}$ Lebesgue integrable on \mathbb{T} is defined for every $z \in \mathbb{T}$ as follows:

$$C_{\mathbb{T}}[f](z) := \text{PV} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(u)}{u-z} \, d u := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{f(u)}{u-z} \, d u \quad (5)$$

whenever the limit exists and $C_{\mathbb{T}}[f](z) := 0$ otherwise, where $\mathbb{T}(e^{ix}, \varepsilon) := \{e^{it} \in \mathbb{T} : |t-x| < \varepsilon\}$. Here and subsequently, integration along any arc $I \subset \mathbb{T}$ is understood under counterclockwise orientation.

Lemma 1. ([16, Lemma 1.1]) *Suppose that $f \in \text{Hom}^+(\mathbb{T})$ is absolutely continuous on \mathbb{T} and that f is differentiable at a point $z \in \mathbb{T}$. Then both the following limits exist and*

$$\lim_{\varepsilon \rightarrow 0^+} \text{Re} \left[\frac{z \overline{f(z)}}{\pi i} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{f'(u)}{u-z} \, d u \right] = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{|f(u) - f(z)|^2}{|u-z|^2} |d u| .$$

Moreover, both the following limits simultaneously exist or not and in the first case

$$\lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} \left[\frac{z \overline{f(z)}}{\pi i} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{f'(u)}{u - z} \mathrm{d}u \right] = - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{\operatorname{Im}[f(u) \overline{f(z)}]}{|u - z|^2} | \mathrm{d}u | .$$

Given a continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ and $z \in \mathbb{T}$ set

$$V[f](z) := \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{|f(u) - f(z)|^2}{|u - z|^2} | \mathrm{d}u | , \quad (6)$$

$$V^*[f](z) := - \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\mathbb{T} \setminus \mathbb{T}(z, \varepsilon)} \frac{\operatorname{Im}[f(u) \overline{f(z)}]}{|u - z|^2} | \mathrm{d}u | , \quad (7)$$

provided the limits exist as well as $V[f](z) := +\infty$ and $V^*[f](z) := 0$ otherwise.

Theorem 4. ([16, Th. 1.2]) *If $f \in \operatorname{Hom}^+(\mathbb{T})$ is absolutely continuous on \mathbb{T} , then for a.e. $z \in \mathbb{T}$, the limit in (5) with f replaced by f' and the limits in (6) and (7) exist, and*

$$2 \mathcal{C}_{\mathbb{T}}[f'](z) = \bar{z} f(z) (V[f](z) + i V^*[f](z)) .$$

Lemma 2. ([16, Lemma 1.3]) *For every $K \geq 1$ the following inequalities hold:*

$$1 \leq M_K := \frac{4}{\pi} \int_0^{1/\sqrt{2}} \left(\frac{\Phi_K(r)}{r} \right)^{1+1/K} \frac{\mathrm{d}r}{\sqrt{1-r^2}} \leq K^2 2^{5(1-1/K^2)/2}$$

and

$$1 \geq L_K := \frac{4}{\pi} \int_0^{1/\sqrt{2}} \left(\frac{\Phi_{1/K}(r)}{r} \right)^{1+1/K} \frac{\mathrm{d}r}{\sqrt{1-r^2}} \geq \frac{K 2^{5(1-K^2)/(2K)}}{K^2 + K - 1} .$$

In particular, $L_K \rightarrow 1$ and $M_K \rightarrow 1$ as $K \rightarrow 1^+$.

Given a continuous function $f : \mathbb{T} \rightarrow \mathbb{C}$ and $z \in \mathbb{T}$ set

$$f^+(z) := \sup_{u \in \mathbb{T} \setminus \{z\}} \left| \frac{f(u) - f(z)}{u - z} \right| \in [0, +\infty] ,$$

$$f^-(z) := \inf_{u \in \mathbb{T} \setminus \{z\}} \left| \frac{f(u) - f(z)}{u - z} \right| \in [0, +\infty) .$$

For $K \geq 1$ write $\text{QC}(\mathbb{D}; K)$ for the class of all K -quasiconformal self-mappings of \mathbb{D} .

Theorem 5. ([16, Th. 1.4]) *Given $K \geq 1$ and $F \in \text{QC}(\mathbb{D}; K)$ let f be the boundary valued function of F . If $F(0) = 0$, then*

$$L_K (f^-(z))^{1-1/K} \leq V[f](z) \leq M_K (f^+(z))^{1-1/K} , \quad z \in \mathbb{T} .$$

Lemma 3. ([16, Lemma 1.5]) *Suppose that $f \in \text{Hom}^+(\mathbb{T})$ is absolutely continuous on \mathbb{T} . Then*

$$\sup_{z \in \mathbb{T}} f^+(z) = e_f := \text{ess sup}_{z \in \mathbb{T}} |f'(z)|$$

as well as

$$\inf_{z \in \mathbb{T}} f^-(z) = d_f$$

where d_f is defined by (1).

Corollary 1. ([16, Cor. 1.6]) *Given $K \geq 1$ and $F \in \text{QC}(\mathbb{D}; K)$ let f be the boundary valued function of F . If $F(0) = 0$ and f is absolutely continuous on \mathbb{T} , then*

$$L_K d_f^{1-1/K} \leq V[f](z) = 2 \text{Re} \left[z \overline{f(z)} C_{\mathbb{T}}[f'](z) \right] \leq M_K e_f^{1-1/K}$$

for a.e. $z \in \mathbb{T}$.

