日　本　数　学　会

２００３年度秋季総合分科会

# 函数論分科会
# 講演アブストラクト

２００３年9月

於　千　葉　大　学

# 函数論分科会委員会規則

1. 函数論分科会委員会（以下，委員会と略）の目的
   函数論研究者の研究活動を活発にし，研究討論及び研究連絡を円滑に行うことを目的とする．

2. 委員会の任務
   (a) 函数論分科会評議員候補者を選出する．
   (b) 数学会より依頼された分科会選出の各種委員 (たとえば，受賞候補推薦委員等) 候補者の推薦．
   (c) 科研費基盤研究（審査区分（１））の代表者の推薦および計画調書提出の依頼．
   (d) 科研費運用に関して代表者または分担者から相談された時はこれに応ずる．
   (e) 学会特別講演およびシンポジウム講演の講演候補者を順位を付して決定する．
   (f) 分科会の行事 (たとえば， シンポジウムの開催等) について決定する．
   (g) 次期委員会委員候補者の推薦．
   (h) その他評議員の要請する案件および分科会に関する一切の問題を協議決定する．

3. 委員会の構成及び委員の選出・任期
   (a) 委員の定数は特に定めないが10名程度をもって構成する．必要に応じ追加，削減できる．
   (b) 委員の任期は春季学会から２年間とする．再任は妨げないが，原則として再々任は認めない．
   (c) 委員の選出は秋季学会において投票によって行う．
      i. 委員会推薦の新任候補者について信任投票を行い，その結果，投票総数の過半数を得た候補者に委員を委嘱する．その際，函数論の研究分野のバランス，更に地区的にも偏しないよう候補者の推薦を配慮する．
      ii. 投票の結果，委員会推薦候補者以外に分科会会員の１０名以上から推薦された者があるときには得票数上位２名に委員を委嘱する．

4. 委員会の開催及び議決
   (a) 委員会は評議員が召集する．
   (b) 委員会は委員総数の過半数の出席で成立する．
   (c) 年３回 (春季，シンポジウム，秋季) 定期委員会を開催する．必要に応じ臨時委員会を開催する事ができる．
   (d) 案件の議決は投票によってはならない．決定できない時は懸案事項として次回に繰越す．緊急事項については評議員に処置を一任する．

5. 函数論分科会委員会における評議員の任務
   (a) 委員会の司会をする．
   (b) 数学会評議員会の決定事項を委員会に報告する．
   (c) 委員会で決定した事項 (シンポジウム，学会特別講演等) を施行する．
   (d) 委員会の了承を得て，決定事項を分科会会員に公表する．

付 則　この規則は，1974 年 10 月 12 日より施行する．
付 則　この規則の改正は，1996 年 8 月 1 日より施行する．

# 函 数 論 分 科 会

9 月 26 日 (金)　第 VIII 会場

**9:45 ～ 11:45**



**14:20 ～ 15:20**　特別講演

宮 本 育 子 (千 葉 大 理)* 正値 (優) 調和関数の挙動と除外集合

**15:45 ～ 16:45**　2003 年度 幾何学賞受賞特別講演 (第 VI 会場)

平 地 健 吾 (東 大 数 理)* 強擬凸領域の幾何とベルグマン核

9月27日(土)　第VIII会場

**9:20 ~ 12:00**



**14:20 ~ 15:45**



**16:00 ~ 17:00　特別講演**

田島慎一 (新潟大工)* Grothendieck 留数の代数解析と計算アルゴリズム
中村弥生 (お茶水女大人間文化)

# 1. On Some Double and Triple Infinite Sums
# ( A Serendipity in N- Fractional Calculus )

Katsuyuki Nishimoto  Descartes Press Co.

**Abstract**

In 1991, an infinite sum which is derived through the N- fractional calculus of products

$$(z^{\beta}\cdot\log az)\ (a\neq 0)$$

is reported as a serendipity in N- fractional calculus, by the author.

In this paper two kinds double infinite sums and a triple infinite sum which are obtained from the previous infinite sum are reported.

A triple infinite sum is shown as follows, for example.

**Theorem.** We have the triple infinite sum

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{\left[(m-1)!\cdot(n+m)!\left(\sum_{k=1}^{m}(n+k)^{-1}\right)\right]^{-1}}{k(k+m)(k+m+1)\cdots(k+m+n)}=1.59063\cdots ,$$

where $m\in Z^{+}$ and $n\in Z_{0}^{+}$.

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[5] K. Nishimoto ; On the infinite sum

$$Q_{m,n}=\sum_{k=1}^{\infty}\frac{1}{k(k+m)(k+m+1)\cdots(k+m+n)}\qquad (n\in Z^{+}\cup\{0\},\ m\in Z^{+})$$

( A serendipity in fractional calculus ). J. Coll. Engng. Nihon Univ. B-32 (1991),7 - 13.

[6] K. Nishimoto and S.T, Tu ; On the Infinite Sum

$$\sum_{k=1}^{\infty}\frac{(n-1)!\cdot 2^{n-1}}{\prod_{k=0}^{n}(2k+3)}\quad\text{and}\quad\sum_{k=1}^{\infty}\frac{(n-1)!\cdot(n+1)\cdot 2^{n-1}}{\prod_{k=0}^{n+1}(2k+3)}$$

( A serendipity in fractional calculus ). J. Coll. Engng. Nihon Univ. B-32 (1991),15 - 21.

[7] K. Nishimoto and S.T, Tu ; On the Infinite Sum

$$R_{m,\beta} = (-1)^n \sum_{k=1}^{\infty} \frac{(m+k-1)! \cdot (-1)^k}{(m-1)!\,k} \cdot \frac{\Gamma(-m-k-\beta)}{\Gamma(-\beta)}$$

( A serendipity in fractional calculus ). J. Coll. Engng. Nihon Univ. B-32 (1991),23- 30.

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# 2. Notes on Partial Sums of Analytic Functions

Shigeyoshi Owa (Kinki University)

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$. Let $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ which are starlike in $\mathbb{U}$. Also let $\mathcal{K}$ be the subclass of $\mathcal{S}$ consisting of functions $f(z)$ which are convex in $\mathbb{U}$. The partial sum $f_n(z)$ of $f(z) \in \mathcal{A}$ is given by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$

**Remark 1.** (1) If $f(z) \in \mathcal{S}$, then $f_n(z)$ is not univalent in $\mathbb{U}$ when $|a_n| \geqq \frac{1}{n}$.

(2) The function

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k$$

is the extremal function for the class $\mathcal{S}^*$. But $f_n(z)$ is not starlike in $\mathbb{U}$.

(3) The function

$$g(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k$$

is the extremal function for the class $\mathcal{K}$. But $g_n(z)$ is not convex in $\mathbb{U}$.

**Remark 2.** (1) $f_3(z) = z + 2z^2 + 3z^3$ satisfies

$$\mathrm{Re}\left(1 + \frac{z f_3''(z)}{f_3'(z)}\right) > \frac{3191}{15876} = 0.2009\cdots$$

for $0 \leqq r < \beta$, where $\beta$ is the positive root of the equation

$$81x^4 - 162x^3 + 74x^2 - 16x + 1 = 0 \qquad (0 < x < \frac{1}{3}).$$

(2) $g_3(z) = z + z^2 + z^3$ satisfies

$$\operatorname{Re}\left(\frac{zg_3'(z)}{g_3(z)}\right) > \frac{626}{961} = 0.6514\cdots$$

for $0 \leqq r < \beta$, where $\beta$ is the positive root of the equation

$$x^4 - 6x^3 + 9x^2 - 8x + 1 = 0 \qquad (0 < x < \frac{1}{\sqrt{3}}).$$

**Theorem 1.** $f_3(z) = z + 2z^2 + 3z^3$ satisfies

$$\operatorname{Re}\left(1 + \frac{zf_3''(z)}{f_3'(z)}\right) > 2\left(1 - \frac{\sqrt{10}}{5}\right) = 0.7350\cdots$$

for $0 \leqq r < \dfrac{7 - 2\sqrt{10}}{9} = 0.0750\cdots$.

**Theorem 2.** $g_3(z) = z + z^2 + z^3$ satisfies

$$\operatorname{Re}\left(\frac{zg_3'(z)}{g_3(z)}\right) > \frac{4 - \sqrt{5}}{2} = 0.9919\cdots$$

for $0 \leqq r < \dfrac{7 - 3\sqrt{5}}{2} = 0.1455\cdots$.

**Theorem 3.**

(1) $f_3(z) = z + 2z^2 + 3z^3$ satisfies

$$\operatorname{Re}\left(\frac{zf_3'(z)}{f_3(z)}\right) > \frac{3(89 - 16\sqrt{22})}{137} = 0.3055\cdots$$

for $0 \leqq r < \dfrac{5 - \sqrt{22}}{3} = 0.1031\cdots$.

(2) $g_3(z) = z + z^2 + z^3$ satisfies

$$\operatorname{Re}\left(1 + \frac{zg_3''(z)}{g_3'(z)}\right) > \frac{3(89 - 16\sqrt{22})}{137} = 0.3055\cdots$$

for $0 \leqq r < \dfrac{5 - \sqrt{22}}{3} = 0.1031\cdots$.

# 3. Gaussian convolutionの実逆変換について

群馬大工　斎藤三郎

関数の解析性をコンピュータで捉えられるかという問について、人はできないと思うかも知れない。　ラプラス変換やワイヤシュトラス変換（ガウス convolution) の像関数が解析関数になることから、それらの逆変換を実軸上の値あるいは実際にコンピュータで求めるには実軸上の有限個の点での値を用いて逆変換を確立する必要がある。　このため実逆変換には本質的な難しさがある。　実際、それらの問題は古い歴史を有し、いろいろ具体的な応用の観点からも要請されて現在でもこれらの難問への挑戦が行われている（[6]、[11])。

私たちはそれらの積分変換の像関数を完全に特徴づける方法を得たことから、新しい解析的な逆変換公式を確立し、　さらに像関数に条件をつけると誤差評価や逆変換の条件付き安定性が導けることを示した（[1 - 4,6,7,10])。　しかしながら、それらの公式は解析性などの像関数の特徴的な性質を正確に用いているため、コンピュータを用いて逆変換を実際に求められるようにはなっていないように思われる。そこで、コンピュータで実現できるような逆変換公式を求めて新しい公式を考えている。

一般に、　ヒルベルト空間の枠内での、フレッドフォルムの第１種積分方程式の解法について、　上記の観点から

１）　定義域の関数を再生核を持つヒルベルト空間（上記の変換では具体的には１階のソボレフ空間）と制限する、
２）　Tikhonov regularizations の考えを取り入れる、
３）　最良近似の考えを取り入れて、一般逆を考える、
４）　解析性を射影作用素で捉える、

などの考えで、新しい逆変換公式を得たのでこれらの一般的な考えと具体的な公式を紹介する。　それらの中のある公式は驚くべきものになっていると思われる。

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# 4. Bloch-type Spaces 上の Closed Range を持つ合成作用素について

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$D$を複素平面$C$上の開単位円板とする。 The Bloch space $B$を

$$\| f \|_{\beta} := |f(0)| + \| f \|_B < +\infty ,$$

where $\| f \|_B := \sup_{z \in D} (1 - |z|^2) \, |f'(z)|$を満たす$D$上の解析関数全体、 the space $B_{\log}$を

$$\| f \|_{\beta_{\log}} := |f(0)| + \| f \|_{B_{\log}} < +\infty ,$$

where $\| f \|_{B_{\log}} := \sup_{z \in D} (1 - |z|^2) \, |f'(z)| \left( \log \frac{2}{1 - |z|^2} \right)$を満たす$D$上の解析関数全体、the space $B_\alpha$を

$$\| f \|_{\beta_\alpha} := |f(0)| + \| f \|_{B_\alpha} < +\infty ,$$

where $\| f \|_{B_\alpha} := \sup_{z \in D} (1 - |z|^2)^\alpha \, |f'(z)|$を満たす$D$上の解析関数全体とする。

$\omega$ を positive continuous function on $[0,1]$ with $\omega(|z|) \to 0$ ( $|z| \to 1^-$ ) と仮定し、 the space $B_\omega$を

$$\| f \|_{B_\omega} := \sup_{z \in D} \omega(|z|) \, |f'(z)| < +\infty$$

を満たす$D$上の解析関数全体とする。　$\varphi$は a holomorphic function taking $D$ into $D$を表示し、$C_\varphi$は$\varphi$との合成作用素とする。そのとき、文献[3]で、P. Ghatage と J.Yan と D.Zheng は 次の結果を証明した：

**Theorem A.** *Let $\varphi$ be a holomorphic function taking $D$ into $D$. If $C_\varphi$ is bounded below on $B$, then there exist positive constants $\epsilon$,$r$ with $r < 1$ such that , for all $z \in D$, $\rho(\varphi(\Omega_\epsilon), z) \le r$ where $\Omega_\epsilon = \{z \in D : \left| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \varphi'(z) \right| > \epsilon\}$.*

**Theorem B.** *Let $\varphi$ be a holomorphic function taking $D$ into $D$. If for some constants $0 < r < \frac{1}{4}$, for each $w \in D$, there is a point $z_w \in D$ such that $\rho(\varphi(z_w), w) < r$ and $\left| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \varphi'(z) \right| > \epsilon$, then $C_\varphi$ is bounded below on $B$.*

我々は次の結果を証明した:

**Theorem 1.** *Let $\alpha > 1$. Let $\varphi$ be a holomorphic function taking $D$ into $D$ and $C_\varphi$ is bounded on $B_\alpha$. If $C_\varphi$ is bounded below on $B_\alpha$, then there exist positive constants $\epsilon$, $r$ with $r < 1$ such that , for all $w \in D$, $\rho(\varphi(\Omega_\epsilon), w) \le r$ where $\Omega_\epsilon = \{z \in D : \left| \left( \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^\alpha \varphi'(z) \right| > \epsilon\}$.*

**Theorem 2.** *Let $\alpha > 1$. Let $\varphi$ be a holomorphic function taking $D$ into $D$. If for some constants $0 < r < \gamma_2 := \min\{\frac{1}{2}, \frac{1}{\gamma_1}\}$, and $\epsilon > 0$, for each $w \in D$, there is a point $z_w \in D$ such that $\rho(\varphi(z_w), w) < r$* and $\left|\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{\alpha} \varphi'(z)\right| > \epsilon$, *then $C_\varphi$ is bounded below on $B_\alpha$.*

**Theorem 3.** *Let $\varphi$ be a holomorphic function taking $D$ into $D$ and $C_\varphi$ is bounded on $B_{\log}$. If $C_\varphi$ is bounded below on $B_{\log}$, then there exist positive constants $\epsilon$,$r$ with $r < 1$ such that , for all $w \in D$, $\rho(\varphi(\Omega_\epsilon), w) \leq r$ where* $\Omega_\epsilon = \{z \in D : \left|\frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|^2}\right)}{\left(1-|\varphi(z)|^2\right)\left(\log\frac{2}{1-|\varphi(z)|^2}\right)}\varphi'(z)\right| > \epsilon,\ \varphi(z) \in D(w,r)\ \}$.

**Theorem 4.** *Let $\varphi$ be a holomorphic function taking $D$ into $D$. If for some constants $0 < r < C_2 := \min\{\frac{1}{2}, \frac{1}{C_1}\}$, and $\epsilon > 0$, for each $w \in D$, there is a point $z_w \in D$ such that $\rho(\varphi(z_w), w) < r$ and* $\left|\frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|^2}\right)}{\left(1-|\varphi(z)|^2\right)\left(\log\frac{2}{1-|\varphi(z)|^2}\right)}\varphi'(z)\right| > \epsilon$, *then $C_\varphi$ is bounded below on $B_{\log}$.*

**Theorem 5.** *Let $\varphi$ be a holomorphic function taking $D$ into $D$. If for some constants $0 < r < K_2 := \min\{\frac{1}{2}, \frac{1}{K_1}\}$,and $\epsilon > 0$, for each $w \in D$, there is a point $z_w \in D$ such that $\rho(\varphi(z_w), w) < r$ and $\left|\frac{\omega(|z|)}{\omega(|\varphi(z)|)}\varphi'(z)\right| > \epsilon$, then $C_\varphi$ is bounded below on $B_\omega$.*

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# 5. 単位球上の重調和グリーンポテンシャルの $p$-乗球面積分平均の増大度

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Gardiner [1] や Mizuta [4] によって，$n$ 次元単位球 $\mathbf{B}$ や上半空間上の調和グリーンポテンシャルの $p$-乗球面積分平均の増大度の評価がなされている．本講演では，この問題を単位球上の重調和グリーンポテンシャルにおいて考察する．

$G_2(x,y)$ を単位球上の重調和グリーン関数とし，単位球上の非負測度 $\mu$ に対して，

$$G_2\mu(x) = \int_{\mathbf{B}} G_2(x,y)\, d\mu(y)$$

と定義する．重調和グリーン関数 $G_2(x,y)$ の評価により，$G_2\mu$ が重調和グリーンポテンシャル（すなわち、$G_2\mu \not\equiv \infty$）であるための必要十分条件は

$$\int_{\mathbf{B}} (1-|y|)^2\, d\mu(y) < \infty$$

である．

関数 $u$ の 原点が中心の半径 $r$ 球面 $S(r)$ 上の球面積分平均を

$$\mathcal{M}(u,r) = ⨍_{S(r)} u\, dS = \frac{1}{|S(r)|}\int_{S(r)} u\, dS$$

で表す．さらに，$p>0$ に対して，$\mathcal{M}_p(u,r) = \{\mathcal{M}(|u|^p, r)\}^{1/p}$ とする．

**定理 1.** $G_2\mu$ を $\mathbf{B}$ 上の重調和グリーンポテンシャルとする。

(1) $n \le 4$ のとき，$1 \le p < \infty$ ならば，

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p}\mathcal{M}_p(G_2\mu, r) = 0.$$

(2) $n \ge 5$ のとき，

(i) $1 \le p < (n-1)/(n-4)$ ならば，

$$\lim_{r\to 1}(1-r)^{n-2-(n-1)/p}\mathcal{M}_p(G_2\mu, r) = 0.$$

(ii) $(n-1)/(n-4) \le p < (n-1)/(n-5)$ ならば、

$$\liminf_{r\to 1}(1-r)^{n-2-(n-1)/p}\mathcal{M}_p(G_2\mu, r) = 0.$$

この定理の応用として，B 上の優重調和関数が重調和グリーンポテンシャルとなるための条件を球面積分平均の極限で特徴付ける．ここで，開集合 $\Omega$ 上の下半連続な局所可積分関数 $u$ が優重調和であるとは，

(i) $\displaystyle\int_\Omega u(x)\Delta^2\varphi(x)dx \geq 0 \qquad (\forall\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0)$

(ii) $\displaystyle u(x) = \lim_{r\to 0} \fint_{B(x,r)} u(y)\ dy \in (-\infty, \infty] \qquad (\forall x \in \Omega).$

**定理 2.** $u$ を B 上の優重調和関数とする．このとき，$u$ が

$$\liminf_{r\to 1}(1-r)^{-1}\mathcal{M}_1(u,r) = 0$$

を満たすならば，$u$ は重調和グリーンポテンシャルである．

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# 6. $\alpha$-parabolic Bergman spaces

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近年，Ramey や Yi および幾人かの研究者らにより，上半空間 $\boldsymbol{R}_+^{n+1}$ 上の harmonic Bergman space の研究がすすめられている ([7],[8],[9])．そこで，鍵となっていることは，調和関数 $u$ の Poisson 積分表示

$$u(x,t)=\int_{\boldsymbol{R}^n}P(x-y,t-s)u(y,s)dy \qquad (t>s>0)$$

である．ここで，$P$ は上半空間上の Poisson 核である．

これを見て，放物的に Bergman 空間を考えるのもおもしろいと考え，ここで，$\alpha$-parabolic Bergman space を導入し，基本的な性質を確かめることにした．

$0<\alpha\leq 1$ に対し，Euclid 空間 $\boldsymbol{R}^{n+1}$ 上の $\alpha$-次放物型作用素

$$L^{(\alpha)}:=\partial_t+(-\Delta)^\alpha$$

を考え，$W^{(\alpha)}$ を $L^{(\alpha)}$ の基本解とする：

$$W^{(\alpha)}(x,t)=\begin{cases}(2\pi)^{-n}\int_{\boldsymbol{R}^n}\exp(-t|\xi|^{2\alpha}+\sqrt{-1}\,x\cdot\xi)d\xi & \text{if} \quad t>0\\ 0 \quad \text{othewise.}\end{cases}$$

ここで注意することは，$\alpha=1/2$ のとき，$W^{(1/2)}(x,t)=P(x,t)$ と Poisson 核と一致するので，我々の理論は harmonic Bergman space を含む形ですすめられるという点である．

$1\leq p<\infty$ に対し，

$$\boldsymbol{b}_\alpha^p:=\{u:\boldsymbol{R}_+^{n+1}\to\boldsymbol{R}\in L^p|L^{(\alpha)}u=0 \quad \text{(distribution)}\}$$

とおき，$\alpha$-parabolic Bergman space とよぶ．

今回報告する結果は次のとおりである．

**定理 1.** $u\in\boldsymbol{b}_\alpha^p$ に対し，

$$u(x,t)=\int_{\boldsymbol{R}^n}W^{(\alpha)}(x-y,t-s)u(y,s)dy \qquad (t>s>0).$$

定理 1 は Huygens Property と呼ばれる ([11]) が，この結果と基本解の評価が，以下の結果を導くのに基本的に重要である．そして，定理 1 の証明には，$L^{(\alpha)}$ に関する調和測度と Riesz 核に関する掃散測度の関係が用いられる．

**定理 2.** $R_\alpha(x,t;y,s):=-2\partial_tW^{(\alpha)}(x-y,t+s)$ は，Bergman 核，すなわち，$L^2(\boldsymbol{R}_+^{n+1})$ から $\boldsymbol{b}_\alpha^2$ への直交射影を表現する再生核である．

**定理 3.** $(b^p_\alpha)^* \simeq b^q_\alpha \quad (1 < p < \infty, 1/q + 1/p = 1)$

**定理 4.** $(b^1_\alpha)^* \simeq \mathcal{B}_\alpha/\boldsymbol{R}$. ここで，$\mathcal{B}_\alpha$ は $\alpha$-parabolic Bloch space

$$\mathcal{B}_\alpha := \{u | L^{(\alpha)}u = 0, t\partial_t u \in L^\infty(\boldsymbol{R}^{n+1}_+)\}$$

であり，$\boldsymbol{R}$ は定数関数を表す．

**定理 5.** $(\mathcal{B}_{\alpha,0}/\boldsymbol{R})^* \simeq b^1_\alpha$. ここで，$\mathcal{B}_{\alpha,0}$ は $\alpha$-parabolic little Bloch space，すなわち，$\mathcal{A}$ を Alexiandroff point とし，

$$\mathcal{B}_{\alpha,0} := \{u \in \mathcal{B}_\alpha | \lim_{(x,t)\to\mathcal{A}} t\partial_t u(x,t) = 0\}$$

である．

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# 7. 容量関数を使った weak type のソボレフの不等式の評価

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$\Omega$ は $\mathbf{R}^n$ の有界な開集合で, $1 < p < n$ とする. ソボレフ空間 $W^{1,p}(\Omega)$ のソボレフの不等式を使って, O'Neil や Peetre の結果より次の不等式が導かれる.

$\|\nabla u\|_p \lneq 1$ であるようなどんな $u \in C_0^\infty(\Omega)$ に対しても

$$\int_0^\infty t^{p-1}|\{u > t\}|^{1-p/n}dt \lesssim c. \tag{1}$$

ここで, $c$ は $u$ に無関係な定数で, $|A|$ は集合 $A$ の $n$ 次元ルベーグ測度である.

$\Omega = \mathbf{R}^n$ のとき, この不等式が非整数次ソボレフ空間でどんな形で成り立つかかを考ええる.

$G_\alpha$ は $\alpha$ 次のベッセル核とする. 次の $\alpha$ 次ベッセルポテンシャルの集合

$$\mathcal{L}_\alpha^p = \{G_\alpha * f; f \in L^p(\mathbf{R}^n)\}$$

はノルムを $\|G_\alpha * f\|_{\alpha,p} = \|f\|_p$ で定義する。自然数 $m$ に対しては, ソボレフ空間 $W^{m,p}(\mathbf{R}^n)$ は $\mathcal{L}_m^p$ と一致し, それらのノルムは同値であることが, 知られている. したがって, $\mathcal{L}_\alpha^p$ は整数次とは限らないソボレフ空間と見なされる.

$B_{\alpha,p}(A)$ を集合 $A$ の $\alpha$-次ベッセル容量として, $\mathbf{R}^n$ 上の関数 $g$ に対し, ベッセル容量関数を

$$p(g) = \inf\{\|f\|_p;\ |g| \lneq G_\alpha * f\ B_{\alpha,p} - q.e.\ \text{on}\ \mathbf{R}^n,\ f \in L^p\}$$

と定義する.

このとき, 次の定理が得られる.

定理. $1 < p < n$, $\alpha p < n$, $u \in \mathcal{L}_\alpha^p$ とする. このとき

$$\int_0^\infty t^{p-1}|\{|u| > t\}|^{1-\alpha p/n}dt \lneq cp(u).$$

ここで, $c$ はに無関係な定数である.

この定理から $\alpha = 1$ の場合には, (1) が導かれる.

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# 特別講演

## Behaviour of Positive (Superharmonic) Harmonic Functions and Exceptinal Sets

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Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and all positive real numbers, respectively. We denote by $\mathbf{R}^n$ $(n \geq 2)$ the $n$-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, y), X = (x_1, x_2, , , x_{n-1})$. The unit sphere and the upper half unit sphere are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. When we introduce the system of the spherical coordinates

$$(r, \Theta), \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1}), \quad y = r\cos\theta_1$$

in $\mathbf{R}^n$, the half-space

$$\{(r, \Theta) \in \mathbf{R}^n : r \in \mathbf{R}_+,\ (1, \Theta) \in \mathbf{S}_+^{n-1}\} = \{(X, y) \in \mathbf{R}^n :\ y > 0\}$$

will be denoted by $\mathbf{T}_n$.

By $C_n(\Omega)$, we denote the set $\{(r, \Theta) \in \mathbf{R}^n : r \in \mathbf{R}_+,\ (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ with a domain $\Omega \subset \mathbf{S}^{n-1}(n \geq 2)$ having smooth boundary. We call it a *cone*. Then $\mathbf{T}_n$ is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$.

Consider the Dirichlet problem

$$\begin{aligned}(\Lambda_n + \lambda)f &= 0 \quad on\ \Omega \\ f &= 0 \quad on\ \partial\Omega,\end{aligned}$$

where $\Lambda_n$ is the spherical part of the Laplace operator $\Delta_n$

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + r^{-2}\Lambda_n.$$

We denote the least positive eigenvalue of this boundary value problem by $\tau_\Omega$ and the normalized positive eigenfunction corresponding to $\tau_\Omega$ by $f_\Omega(\Theta)$;

$$\int_\Omega f_\Omega^2(\Theta)d\sigma_\Theta = 1,$$

where $d\sigma_\Theta$ is the surface element on $\mathbf{S}^{n-1}$. We denote the solutions of the equation

$$t^2 + (n-2)t - \tau_\Omega = 0$$

by $\alpha_\Omega, -\beta_\Omega$ $(\alpha_\Omega, \beta_\Omega > 0)$. If $\Omega = \mathbf{S}_+^{n-1}$, then $\alpha_\Omega = 1,\ \beta_\Omega = n-1$, and

$$f_\Omega(\Theta) = (2ns_n^{-1})^{1/2}\cos\theta_1,$$

where $s_n$ is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

## 1. Behaviour of positive superharmonic functions and exceptional sets in a cone

### (1) Definitions of exceptional sets in a cone

It is known that the Martin boundary $\Delta$ of $C_n(\Omega)$ is the set $\partial C_n(\Omega) \cup \{\infty\}$ and each point of $\Delta$ is a minimal Martin boundary point. When we denote the Martin kernel by $K(P, Q)$ $(P \in C_n(\Omega), Q \in \partial C_n(\Omega) \cup \{\infty\})$ with respect to a reference point chosen suitably, we know

$$K(P, \infty) = r^{\alpha_\Omega} f_\Omega(\Theta) \quad (P \in C_n(\Omega)).$$

A subset $E$ of $C_n(\Omega)$ is said to be *minimally thin* at $\infty \in \Delta$ with respect to $C_n(\Omega)$, if there exists a point $P \in C_n(\Omega)$ such that

$$\hat{R}^E_{K(\cdot,\infty)}(P) \neq K(P, \infty),$$

where $\hat{R}^E_{K(\cdot,\infty)}(P)$ is the regularized reduced function of $K(\cdot, \infty)$ relative to $E$.

Let $E$ be a Borel subset of $C_n(\Omega)$ and

$$E_k = E \cap I_k(\Omega) \quad (k = 0, 1, 2, \ldots),$$

where

$$I_k(\Omega) = \{(r, \Theta) \in C_n(\Omega); 2^k \leq r < 2^{k+1}\}.$$

Since $E_k$ is a bounded subset of $C_n(\Omega)$, then $\hat{R}^{E_k}_{K(\cdot,\infty)}$ is bounded on $C_n(\Omega)$ and hence the greatest harmonic minorant of $\hat{R}^{E_k}_{K(\cdot,\infty)}$ is zero. When we denote the Green function of $C_n(\Omega)$ by $G(P, Q)$ $(P \in C_n(\Omega), Q \in C_n(\Omega))$, we see from the Riesz decomposition theorem that there exists a unique positive measure $\lambda_E$ on $C_n(\Omega)$ such that

$$\hat{R}^{E_k}_{K(\cdot,\infty)}(P) = G\lambda_{E_k}(P) \quad (P \in C_n(\Omega)).$$

We denote the total mass $\lambda_{E_k}(C_n(\Omega))$ of $\lambda_{E_k}$ by $\lambda_\Omega(E_k)$. The (Green) energy $\gamma_\Omega(E_k)$ of $\lambda_{E_k}$ is defined by

$$\gamma_\Omega(E_k) = \int_{C_n(\Omega)} (G\lambda_{E_k}) d\lambda_{E_k}$$

As in $\mathbf{T}_n$ (Essén and Jackson [9, DEFINITION 3.1. and REMARK 3.1]), we have

**Theorem 1.1** (Miyamoto and Yoshida [19, Theorem 1). *A subset $E$ of $C_n(\Omega)$ is minimally thin at $\infty$ with respect to $C_n(\Omega)$ if and only if*

$$\sum_{k=0}^{\infty} \gamma_\Omega(E_k) 2^{-k(\alpha_\Omega+\beta_\Omega)} < \infty.$$

As in $\mathbf{T}_n$ (Essén and Jackson [9, DEFINITION 3.2.]), A subset $E$ of $C_n(\Omega)$ is said to be *rarefied at $\infty$ with respect to $C_n(\Omega)$*, if

$$\sum_{k=0}^{\infty} 2^{-k\beta_\Omega} \lambda_\Omega(E_k) < +\infty.$$

## (2) Behaviour of positive superharmonic functions in a cone

We set

$$c(v) = \inf_{P \in C_n(\Omega)} \frac{v(P)}{K(P,\infty)}$$

for a positive superharmonic function $v(P)$ on $C_n(\Omega)$. We immediately see that $c(v) < +\infty$ (e.g. see Yoshida [24, Lemma 6.1]).

The following theorem 1.2 is immediately obtained by specializing the Fatou boundary limit theorem for the Martin space: for any positive superharmonic functin $v(P)$ in $C_n(\Omega)$

$$\text{mf} \lim_{P=(r,\Theta)\in C_n(\Omega), r\to+\infty} \frac{v(P)}{K(P,\infty)} = c(v),$$

where "mf limit" means minimal-fine limit (Naïm [21, THÉORÉME 10 and THÉORÉME 8'-17], also Doob [8, XII, 13. Theorem (a)]).

**Theorem 1.2.** *Let $v(P)$ be any positive superharmonic function in $C_n(\Omega)$. Then there exists a minimally thin set $E$ at $\infty$ with respect to $C_n(\Omega)$ such that $v(P)/K(P,\infty)$ uniformly converges to $c(v)$ on $C_n(\Omega)-E$ as $r \to +\infty$ ($P=(r,\Theta) \in C_n(\Omega)$). Conversely, for any minimally thin set $E$ at $\infty$ with respect to $C_n(\Omega)$ there exists a positive superharmonic function $v(P)$ such that*

$$\lim_{P=(r,\Theta)\in E, r\to+\infty} \frac{v(P)}{K(P,\infty)} = +\infty.$$

As in $\mathbf{T}_n$ (Essén and Jackson [9, THEOREM 4.6]), we have

**Theorem 1.3** (Miyamoto and Yoshida [19, Theoerem 3]). *Let $v(P)$ be a positive superharmonic function on $C_n(\Omega)$. Then there exists a rarefied set $E$ at $\infty$ with respect to $C_n(\Omega)$ such that $v(P)r^{-\alpha_\Omega}$ uniformly converges to $c(v)f_\Omega(\Theta)$ on $C_n(\Omega) - E$ as $r \to +\infty$ ($P=(r,\Theta) \in C_n(\Omega)$).*

**Remark 1.1.** This theorem is best possible in the following sense: Given any rarefied set $E$ at $\infty$ with respect to $C_n(\Omega)$, there exists a positive superharmonic function $v(P)$ on $C_n(\Omega)$ such that $v(P)r^{-\alpha_\Omega} \geq 1$ on $E$ and $c(v)=0$ (see Theorem 2.2). Hence $v(P)r^{-\alpha_\Omega}$ does not converge to $c(v)f_\Omega(\Theta) = 0$ on $E$ as $r \to +\infty$.

## (3) Relation between minimally thin sets and rarefied sets in a cone

A cone $C_n(\Omega')$ is called a *subcone* of $C_n(\Omega)$, if $\overline{\Omega'} \subset \Omega$ ($\overline{\Omega'}$ is the closure of $\Omega'$ on $\mathbf{S}^{n-1}$).

As in $\mathbf{T}_n$ ( Essén and Jackson [9, Remark 3.2]), we have

**Theorem 1.4** (Miyamoto and Yoshida [19, Theoerem 4]). *Let $E$ be a subset of $C_n(\Omega)$. If $E$ is rarefied at $\infty$ with respect to $C_n(\Omega)$, then $E$ is minimally thin at $\infty$ with respect to $C_n(\Omega)$. If $E$ is contained in a subcone of $C_n(\Omega)$ and $E$ is minimally thin at $\infty$ with respect to $C_n(\Omega)$, then $E$ is also rarefied at $\infty$ with respect to $C_n(\Omega)$.*

## 2. Characterizations of exceptional sets in a cone

### (1) Qualitative characterizations

### (A) Minimally thin sets

As for a subset of a Liapunov-Dini domain in $\mathbf{R}^n$ (Sjögren [22, THEOREME 2] and Dahlberg [7, THEOREM 1]), we have

**Theorem 2.1** (Miyamoto, Yanagishita and Yoshida [17, THEOREM 1]). *The following statements are equivalent.*
*(I) A subset $E$ of $C_n(\Omega)$ is minimally thin at $\infty$ with respect to $C_n(\Omega)$.*
*(II) (Sjögren type) There exists a positive superharmonic function $v(P)$ on $C_n(\Omega)$ such that*

$$\inf_{P\in C_n(\Omega)} \frac{v(P)}{K(P,\infty)} = 0 \tag{2.1}$$

*and*

$$E \subset \{P \in C_n(\Omega); v(P) \geq r^{\alpha_\Omega} f_\Omega(\Theta)\}.$$

*(III) (Dahlberg type) There exist a positive superharmonic function $v(P)$ on $C_n(\Omega)$ and a positive number $m$ such that even if $v(P) \geq mK(P,\infty)$ $(P \in E)$, there exists $P_0 \in C_n(\Omega)$ satisfying $v(P_0) < mK(P_0,\infty)$.*

### (B) Rarefied sets

As in $\mathbf{T}_n$ (Essén and Jackson [9, Remark 4.4], Aikawa and Essén [3, Definition 12.4, p.74]) and in $\mathbf{T}_2$ (Hayman [11, p.474]), we have

**Theorem 2.2** (Miyamoto and Yoshida [19, Theorem 2]). *A subset $E$ of $C_n(\Omega)$ is rarefied at $\infty$ with respect to $C_n(\Omega)$ if and only if there exists a positive superharmonic function $v(P)$ in $C_n(\Omega)$ satisfying (2.1) and*

$$E \subset \{P = (r,\Theta) \in C_n(\Omega);\ v(P) \geq r^{\alpha_\Omega}\}.$$

### (2) Quantitative characterizations

### (A) Minimally thin sets

As in $\mathbf{T}_n$ (Essén, Jackson and Rippon [10, COROLLARY 3]), we have

**Theorem 2.3** (Miyamoto and Yoshida [19]). *If a subset of $E$ of $C_n(\Omega)$ is minimally thin at $\infty$ with respect to $C_n(\Omega)$, then $E$ is covered by a sequence of balls $B_k$ satisfying*

$$\sum_k \left(\frac{r_k}{R_k}\right)^n < +\infty,$$

*where $r_k$ is the radius of $B_k$ and $R_k$ is the distance between the origin and the center of $B_k$.*

But we have a sharper result than in Theorem 2.3, because the energy $\gamma_\Omega(E)$ is countably sub-additive and quasi-additive. The following types of theorems were originally considered by Beurling [6] for a subset of a half-plane and then by Dahlberg [7] for a subset of a Liapunov-Dini domain in $\mathbf{R}^n$. As in $\mathbf{T}_n$ (Aikawa [1, Corollary 7 and Corollary 8]), we have

**Theorem 2.4** (Miyamoto, Yanagishita and Yoshida [17, Theorem 2]). *Let a subset $E$ of $C_n(\Omega)$ be minimally thin at $\infty$ with respect to $C_n(\Omega)$. Then we have*

$$\int_E \frac{dP}{(1+|P|)^n} < \infty. \tag{2.2}$$

For a subset $S$ in $\mathbf{R}^n$, the interior of $S$ and the diameter of $S$ are denoted by int $S$ and diam $S$, respectively. For two subsets $S_1$ and $S_2$ in $\mathbf{R}^n$, the distance between $S_1$ and $S_2$ is denoted by $\mathrm{dist}(S_1, S_2)$. A cube is of the form

$$[l_1 2^{-k}, (l_1+1)2^{-k}] \times \cdots \times [l_n 2^{-k}, (l_n+1)2^{-k}]$$

where $k, l_1, \ldots, l_n$ are integers. Let $\rho$ be a number satisfying $0 < \rho \leq \frac{1}{2}$. A family of the Whitney cubes of $C_n(\Omega)$ with $\rho$ is the set of cubes having the following properties;
(*i*) $\bigcup_i W_i = C_n(\Omega)$,
(*ii*) $\mathrm{int}\, W_i \cap \mathrm{int}\, W_j = \emptyset \quad (i \neq j)$,
(*iii*) $\left[\frac{8}{3\rho}\right] \mathrm{diam} W_i \leq \mathrm{dist}(W_i, \mathbf{R}^n \setminus C_n(\Omega)) \leq 2\left(\left[\frac{8}{3\rho}\right]+1\right) \mathrm{diam} W_i$,
where [a] denotes the integer satisfying $[a] \leq a < [a]+1$ (Stein [23, p.167, Theorem 1]).

The following types of theorems were stated in Aikawa [1, Corollary 8] for a subset of $\mathrm{T}_n$, Aikawa and Essén [2, Corollary 7.4.6 in p.158] for a subset of a Liapunov-Dini domain in $\mathbf{R}^n$.

**Theorem 2.5** (Miyamoto, Yanagishita and Yoshida [17, Theorem 3] ) *If $E$ is a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with $\rho$ $(0 < \rho \leq \frac{1}{2})$, then* (2.2) *is also sufficient for $E$ to be minimally thin at $\infty$ with respect to $C_n(\Omega)$.*

### (B) Rarefied sets

When we set

$$d(P) = \inf_{Q \notin C_n(\Omega)} |P - Q|$$

for any $P \in C_n(\Omega)$, we have the following result analogous to Theorem 2.5, because $\lambda(E)$ is countably sub-additivite.

**Theorem 2.6** (Miyamoto and Yoshida [20]). *Let $E$ be a union of cubes from a family of the Whitney cubes of $C_n(\Omega)$ with $\rho$ $(0 < \rho \leq \frac{1}{2})$ such that*

$$\int_E \frac{dP}{(1+|P|)^{n-1} d(P)} < \infty. \tag{2.3}$$

*Then $E$ is rarefied at $\infty$ with respect to $C_n(\Omega)$ .*

As in $\mathbf{T}_n$ (Essén, Jackson and Rippon [10, THEOREM 2 and p.397]), we have the following theorem, which gives Azarin [5, Theorem 2] with Theorem 1.3.

**Theorem 2.7** (Miyamoto and Yoshida [20]). *If a subset of $E$ of $C_n(\Omega)$ is rarefied at $\infty$ with respect to $C_n(\Omega)$, then $E$ is covered by a sequence of balls $B_k$ satisfying*

$$\sum_k \left(\frac{r_k}{R_k}\right)^{n-1} < +\infty,$$

*where $r_k$ is the radius of $B_k$ and $R_k$ is the distance between the origin and the center of $B_k$.*

**Remark 2.1.** Since $\lambda(E)$ is not countably quasi-additivite (see Aikawa and Essén [3, p.105]), we cannot always obtain (2.3) from the rarefiedness of $E$ as in Theorem 2.4.

## 3. Behaviour of positive harmonic functions and exceptional sets in a cone

### (1) Equivalence set

Following Beurling [6], we say that a subset $E$ of $C_n(\Omega)$ is an *equivalence set* for $\infty$, if every positive harmonic function in $C_n(\Omega)$ which majorizes $K(P,\infty)$ on $E$ also majorizes $K(P,\infty)$ everywhere on $C_n(\Omega)$, i.e.

$$\inf_{P\in C_n(\Omega)} \frac{h(P)}{K(P,\infty)} = \inf_{P\in E} \frac{h(P)}{K(P,\infty)}$$

for every positive harmonic function $h$ in $C_n(\Omega)$.

We set

$$B(P,r) = \{P' \in \mathbf{R}^n;\ |P'-P| < r\} \quad (r > 0)$$

for any $P \in C_n(\Omega)$. For a subset $E$ of $C_n(\Omega)$ and a number $\rho$ $(0 < \rho < 1)$ we put

$$E_\rho = \cup_{P\in E} B(P, \rho d(P)).$$

Aikawa [2, THEOREM 1] proved the following theorem for a subset of an $NTA$ domain in $\mathbf{R}^n$.

**Theorem 3.1** (Miyamoto, Yanagishita and Yoshida [18, Theorem 2] and Aikawa [2, THEOREM 1]). *Let $E$ be a subset of $C_n(\Omega)$. The following conditions on $E$ are equivalent:*
*(i) $E$ is an equivalence set for $\infty$;*
*(ii) for any $\rho$, $0 < \rho < 1$, $E_\rho$ is not minimally thin at $\infty$;*
*(iii) for some $\rho$, $0 < \rho < 1$, $E_\rho$ is not minimally thin at $\infty$ .*

The following types of theorems were obtained by Beurling [6, Lemma 1] for a subset of a half-plane and then by Dahlberg [7, Theorem 1] for a subset of a Liapunov-Dini domain in $\mathbf{R}^n$.

**Theorem 3.2** (Miyamoto, Yanagishita and Yoshida [18, Theorem 3]). *Let $E$ be a subset of $C_n(\Omega)$. Then the following conditions on $E$ are equivalent:*
(*i*) *$E$ is an equivalence set for $\infty$ ;*
(*ii*) *$\int_{E_\rho}(1+|P|)^{-n}dP = +\infty$ for every $\rho$ $(0<\rho<1)$;*
(*iii*) *$\int_{E_\rho}(1+|P|)^{-n}dP = +\infty$ for some $\rho$ $(0<\rho<1)$.*

### (2) Beurling's minimum Principle

Theorems 3.1 and 3.2 can be called "Beurling's minimum principle" by following Ancona [4] and Maz'ya [12]. We remark that the Martin function $K(P,\infty)$ is (up to a positive multiplicative constant) the only positive harmonic function in $C_n(\Omega)$ which vanishes on $\partial C_n(\Omega)$ and hence if $h(P)$ is a positive harmonic function of $C_n(\Omega)$, then

$$\liminf_{P\in C_n(\Omega),\ P\to P'} \{h(P) - K(P,\infty)\} \geq 0$$

for every $P' \in \partial C_n(\Omega)$. Thus if $h(P)$ satisfies an additional condition

$$\liminf_{P\in C_n(\Omega),\ P\to\infty} \{h(P) - K(P,\infty)\} \geq 0,$$

then it follows from the minimal principle of a harmonic function that $h(P)$ majorizes $K(P,\infty)$ everywhere on $C_n(\Omega)$.

In connection with the fact stated above, we shall say that a subset $E$ of $C_n(\Omega)$ is an *essential equivalence set* for $\infty$, if every positive harmonic function $h(P)$ on $C_n(\Omega)$ satisfying

$$\liminf_{P\in E,\ P\to\infty} \{h(P) - K(P,\infty)\} \geq 0,$$

majorizes $K(P,\infty)$ everywhere on $C_n(\Omega)$.

**Theorem 3.3** (Miyamoto and Yanagishita [16, Theorem 4]). *Let $E$ be a subset of $C_n(\Omega)$. It is necessary and sufficient condition for $E$ to be an essential equivalence set for $\infty$ that $E$ is an equivalence set for $\infty$.*

## 4. Behaviour of positive harmonic (superharmonic) functions and exceptional sets in a cylinder

Let $D$ be a bounded domain on $\mathbf{R}^{n-1} (n \geq 2)$ having smooth boundary. The set

$$\{(X,y) \in \mathbf{R}^n; X \in D, -\infty < y < +\infty\}$$

is usually called a *cylinder*. For a positive superharmonic (harmonic) functions in this cylinder, we can also obtain a result coresponding to each result in Sections 1, 2 and 3 (Miyamoto [13],[14], Miyamoto and Yanagishita[15],[16]).

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