TILTING THEORY OF PREPROJECTIVE ALGEBRAS

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Abstract. In this report, we explain recent developments about the representation theory of preprojective algebras, and its connection to path algebras and the Coxeter groups. In particular, we discuss our recent results based on joint work with Hugh Thomas [MT].

1. Introduction

Let Δ be a simply laced Dynkin diagram and Q a quiver whose underlying graph is Δ. In [G], Gabriel gave a close relationship between the representation theory of Q and the root system of Δ. More precisely, he proved a bijection between indecomposable representations of Q and positive roots of Δ. This result is one of the most fundamental connections between the quiver representation theory and the root system. Recently, it has turned out that preprojective algebras (Definition 2.3) allow us to give a stronger and more direct connection. Indeed, the preprojective algebra, which unifies all the path algebras of quivers whose underlying graph is Δ, gives a representation-theoretical interpretation of the Weyl group W of Δ (Theorem 2.7). This fact leads to the extensive study of connections between representation theory of algebras and combinatorics of W (for example [AM, AIRT, BIRS, GLS, IR1, IRRT, IRTT, L, ORT]). We also remark that the preprojective algebra naturally appears in many branch of mathematics such as simple singularities, quantum groups, quiver varieties and cluster algebras.

In this report, we explain some relationships between the representation theory of preprojective algebras, path algebras and the Coxeter groups. One of the key ingredients is the notion of *c-sortable elements* (Definition 2.16), which are some elements of W. c-sortable elements were originally defined in [R2] from the viewpoint of *Cambrian lattices* [R1]. In particular, an explicit map $\pi^c : W \to \{c\text{-sortable elements of } W\}$, where c denotes the Coxeter element, plays a quite important role to relate c-sortable elements with generalized associahedra and cluster algebras [FR, R3, RS1].

On the other hand, it is also shown that c-sortable elements are quite natural from the viewpoint of quiver representations [IT, AIRT] (Theorem 2.21, Corollary 2.23). One of the aims is to give a *categorical* interpretation of the above map π^c in terms of the representation theory of preprojective algebras and path algebras. Another aim is, using this map, to give answers to questions proposed by Oppermann–Reiten–Thomas [ORT] (Question 3.1). In [ORT], they gave a very fundamental bijection between the elements of W and quotient-closed subcategories of the path algebra of Q (Theorem 2.14). Then it is natural to ask a characterization of quotient-closed subcategories being extension-closed (that is, *torsion classes*) and indeed they proposed conjectures about the problems [ORT, Conjecture 11.1,11.2].

In this report, we will explain the background about these problems including necessary definitions, examples and results, and we discuss methods of our proofs of conjectures.

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2. Preliminary

Notation. Throughout the report, let K be an algebraically closed field. For a K-algebra Λ, we denote by $\mathsf{mod}\Lambda$ the category of finite dimensional right Λ-modules.

In this section, we collect some basic definitions and results.

2.1. **Path algebras and preprojective algebras.**

Definition 2.1. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets : Q_0 (whose elements are called vertices) and Q_1 (whose elements are called arrows), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $a \in Q_1$ its source $s(a) \in Q_0$ and its target $t(a) \in Q_0$, respectively. Thus, a quiver is nothing but an oriented graph without any restriction as to the number of arrows. A *path* of Q is a sequence

$$a_\ell \dots a_2 a_1,$$

where $a_k \in Q_1$ for all $1 \leq k \leq \ell$ and $t(a_k) = s(a_{k+1})$ for each $1 \leq k < \ell$. Moreover, we define e_i the path of length 0 which corresponds to each vertex $i \in Q_0$.

Let Q be a quiver. The *path algebra* KQ of Q is the K-algebra whose underlying K-vector space has as its basis the set of all paths of Q, and define the product for two paths $a_m \cdots a_1$ and $b_n \cdots b_1$ of KQ by

$$a_m \cdots a_1 \cdot b_n \cdots b_1 := \begin{cases} a_m \cdots a_1 b_n \cdots b_1 & (s(a_1) = t(b_n)) \\ 0 & (s(a_1) \neq t(b_n)). \end{cases}$$

In particular, we have $e_i^2 = e_i$ for any $i \in Q_0$, that is, it is an idempotent and we have $1 = \sum_{i \in Q_0} e_i$.

We give some examples of path algebras.

Example 2.2. (a) Let Q be the following quiver

$Q =$  .

Then we have $KQ \simeq K[x]$.

(b) Let Q be the following quiver

$$Q = (1 \xrightarrow{a} 2 \xrightarrow{b} 3).$$

Then we have

$$KQ \simeq \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ K & K & K \end{pmatrix}.$$

Moreover, we have

$$KQ/\langle ba \rangle \simeq \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ 0 & K & K \end{pmatrix},$$

where $\langle ba \rangle$ denotes the ideal of KQ generated by ba.

Next we give a definition of preprojective algebras.

Definition 2.3. Let Q be an acyclic quiver. The preprojective algebra of Q is the algebra

$$\Lambda = K\overline{Q}/\langle \sum_{a\in Q_1} (aa^* - a^*a)\rangle$$

where $\overline{Q}$ is the double quiver of Q, which is obtained from Q by adding for each arrow $a : i \to j$ in Q_1 an arrow $a^* : i \leftarrow j$ pointing in the opposite direction.

The following lemma is well-known (see [Ri] for example).

Lemma 2.4. *Let Q be an acyclic quiver and Λ the preprojective algebra of Q.*

(a) *Λ does not depend on the orientation of Q (namely it is determined by the underlying graph of Q).*
(b) *Λ is finite dimensional if and only if Q is a Dynkin (A,D,E) quiver.*

Notation. Fix a vertex $i \in Q_0$. Then we can define 1-dimensional simple module S_i by

$$S_i \cdot e_j = \begin{cases} K & (i = j) \\ 0 & (i \neq j). \end{cases}$$

Example 2.5. Let Q be the following quiver

$$Q = \ (1 \xrightarrow{a} 2 \xrightarrow{b} 3).$$

Then we have

$$\overline{Q} = \ (1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \underset{b^*}{\overset{b}{\rightleftarrows}} 3\ ).$$

In this case, $e_1\Lambda$ has K-basis $\{e_1, a^*, a^*b^*\}$ and, as a Λ-module, $e_1\Lambda$ has the following composition series

$$0 \subset M_3 \subset M_2 \subset M_1 = e_1\Lambda,$$

where $M_3 = S_3$, $M_2/M_3 \simeq S_2$ and $M_1/M_2 \simeq S_1$. Therefore we denote $e_1\Lambda$ by $\begin{smallmatrix} S_1 \\ S_2 \\ S_3 \end{smallmatrix}$

For simplicity, we denote S_i by i. Then we can write $e_1\Lambda$ as $\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}$. In this notation, we have

$$e_1\Lambda \oplus e_2\Lambda \oplus e_3\Lambda = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 31 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\ .$$

2.2. **A connection between preprojective algebras and the Coxeter groups.** Let Q be a finite connected acyclic quiver with vertices $Q_0 = \{1, \ldots, n\}$. We always assume for simplicity that we have an arrow $j \to i$, then $j < i$. Next we discuss an important relationship between preprojective algebras and the Coxeter groups.

Definition 2.6. The *Coxeter group* W associated to Q is defined by the generators $S := \{s_1, \ldots, s_n\}$ and relations

- $s_i^2 = 1$,
- $s_is_j = s_js_i$ if there is no arrow between i and j in Q,
- $s_is_js_i = s_js_is_j$ if there is precisely one arrow between i and j in Q.

We denote by $\mathbf{w}$ a word, that is, an expression in the free monoid generated by s_i for $i \in Q_0$ and w its equivalence class in the Coxeter group W. We regard W as a poset defined by the (*right*) *weak order*. An element $c = s_1 \ldots s_n$ is called a *Coxeter element* (Note that we require that the order of the product of simple generators of c is compatible with the orientation of the quiver).

Let Λ the preprojective algebra of Q. We denote by I_i the two-sided ideal of Λ generated by $1-e_i$, where e_i $(i \in Q_0)$ is the path of length 0 (= a primitive idempotent) of Λ. We denote by $\langle I_1, \ldots, I_n\rangle$ the set of ideals of Λ which can be written as $I_{u_1} \cdots I_{u_l}$ for some $l \geq 0$ and $u_1, \ldots, u_l \in Q_0$. Then we have the following result (see also [Mi, Theorem 2.14] in the case of Dynkin).

Theorem 2.7. [BIRS, Theorem III.1.9] *There exists a bijection* $W \to \langle I_1, \ldots, I_n\rangle$. *It is given by* $w \mapsto I_w = I_{u_l} \cdots I_{u_1}$ *for any reduced expression* $w = s_{u_1} \cdots s_{u_l}$.

Note that the product of ideals is taken in the opposite order to the product of expression of w. This is just because we follow the convention of [ORT, AIRT].

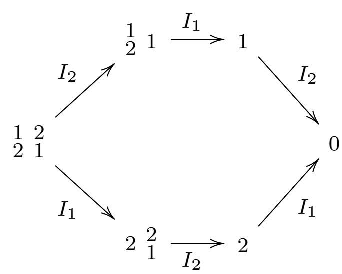
The following result shows that the object I_w is quite natural and important from the viewpoint of tilting theory (we refer to [IR2] and its literature for tilting and support τ-tilting theory).

Theorem 2.8. [BIRS, Theorem III.1.6][Mi, Theorem 2.2].

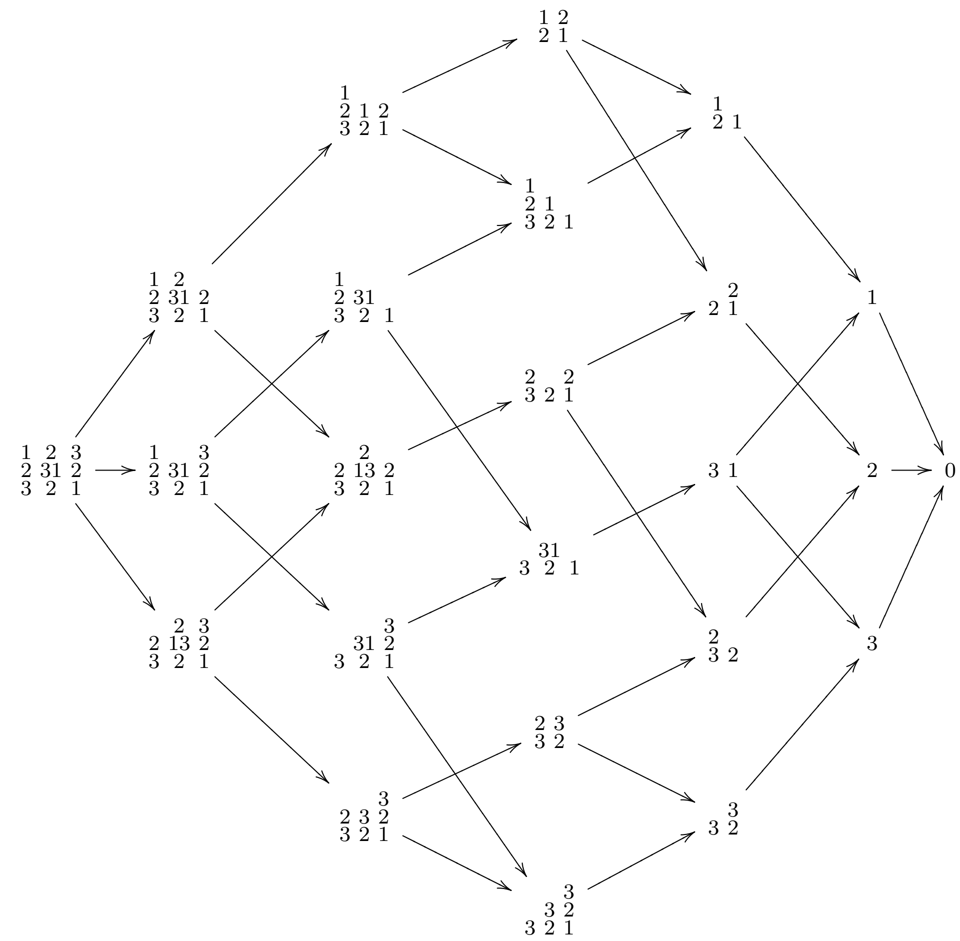
(a) *Let Λ be the preprojective algebra of a non-Dynkin quiver Q and* $\operatorname{tilt}\Lambda$ *the set of isoclasses of basic tilting Λ-modules. Then the map $w \mapsto I_w$ gives an order-reversing injection from W to* $\operatorname{tilt}\Lambda$.
(b) *Let Λ be the preprojective algebra of a Dynkin quiver Q and* $\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$ *the set of isoclasses of basic support τ-tilting Λ-modules. Then the map $w \mapsto I_w$ gives an order-reversing bijection from W to* $\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$.

Example 2.9. Take $i \in Q_0$. Then, for $X \in \mathsf{mod}\Lambda$, XI_i is the minimum amongst submodule Y of X such that any composition factor of X/Y has the form S_i (that is, the action of I_i delete i-top of X). Note that the Hasse quiver of $\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$ coincides with the mutation quiver of $\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$ [AIR, Corollary 2.34].

(a) Let Λ be the preprojective algebra of type A_2. In this case, the Hasse quiver of $\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$ is given as follows.



Here we write a direct sum $X \oplus Y$ by $X\ Y$. Note that I_i denotes a left multiplication (not right multiplication).
(b) Let Λ be the preprojective algebra of type A_3. In this case, the Hasse quiver of $\mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$ is given as follows



Remark 2.10. There is some generalization of Theorem 2.8 to non-simply laced Dynkin cases by Fu-Geng [FG] and Murakami [Mu].

Remark 2.11. A relationship with the construction of quiver varieties are explained in [ST].

2.3. **Results by Oppermann–Reiten–Thomas.** Next we briefly explain a main result of [ORT], which gives a fundamental connection between path algebras, preprojective algebras and the Coxeter groups.

Definition 2.12. Let Λ be the preprojective algebra of Q. For a Λ-module X, we denote by X_{KQ} the KQ-module by the restriction, that is, we forget the action of the arrows $a^* \in \overline{Q}$. We denote by $\mathsf{add}X_{KQ}$ the full subcategories of $\mathsf{mod}KQ$ whose objects are direct summands of finite direct sums of copies of X_{KQ}. Moreover we associate the subcategory

$$\mathsf{res}X = \mathsf{add}X_{KQ} \bigcap \mathsf{mod}KQ.$$

In the case of non-Dynkin quiver, we denote by $\overline{\mathsf{res}X}$ the additive category generated by $\mathsf{res}X$ together with all non-preprojective indecomposable KQ-modules (Note that in the case of Dynkin quiver, all modules are preprojective modules. We refer to [ASS, VIII] for the notion of *preprojective modules*).

Example 2.13. Let Q be the following quiver

$$1 \longrightarrow 2 \longrightarrow 3.$$

Then $KQ = {}_1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}$ (Note that our convention is different from [ASS] but the same as [ORT, AIRT]) and the preprojective algebra $\Lambda = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&1\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\\1\end{smallmatrix}$. Then $\mathsf{res}(\Lambda) = \mathsf{add}\{1, 2, 3, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\}$ (in this way, we always obtain all preprojective KQ-modules as $\mathsf{res}(\Lambda)$: this is why we call Λ *preprojective* algebras).

For example, $I_1I_3 = \begin{smallmatrix}2\\3\end{smallmatrix} \oplus \begin{smallmatrix}&2&\\3&&1\\&2&\end{smallmatrix} \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$ and $\mathsf{res}(I_1I_3) = \mathsf{add}\{2, 3, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\}$.

Consider the infinite word $\underline{c}^\infty = \underline{c}\ \underline{c}\ \underline{c} \ldots$, where $\underline{c} = s_1 \ldots s_n$. For w, we take the lexicographically first reduced expression for w in $\underline{c}^\infty$ (or equivalently, among all the reduced expressions $s_{u_1} \ldots s_{u_l}$ for w in $\underline{c}^\infty$, we choose the one such that s_{u_1} is as far to the left as possible in $\underline{c}^\infty$, and, among such expressions, s_{u_2} is as far to the left as possible, and so on for each s_{u_j}). It is uniquely determined and we denote it by $\underline{w}$. Then we can identify $\underline{c}^\infty$ with indecomposable preprojective KQ-modules $P_1, \ldots, P_n, \tau^- P_1, \ldots, \tau^- P_n, \tau^{-2} P_1, \ldots$, dropping any $\tau^i P_j$ if it is zero, where $P_i := e_i KQ$.

We call a subcategory $\mathcal{A}$ in $\mathsf{mod}KQ$ *cofinite* if there are only finitely many indecomposable KQ-modules which are not in $\mathcal{A}$ (in the case of Dynkin quiver, cofinite is automatic). Note that any cofinite quotient closed subcategory contains all the non-preprojective KQ-modules [ORT, Proposition 2.2]. Then we have the following result.

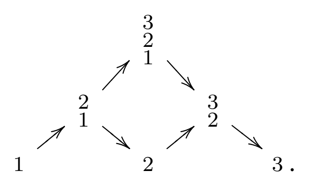
Theorem 2.14. [ORT] *Let Q be an acyclic quiver and W the Coxeter group of Q. There is a bijection*

$$W \longrightarrow \{\text{cofinite quotient-closed subcategories of } \mathsf{mod}KQ\}.$$

(a) *This bijection is obtained by removing from indecomposable preprojective KQ-modules corresponding to $\underline{w}$ (via the above identification)*
(b) *This bijection is also obtained by the map $w \mapsto \overline{\mathsf{res}I_w}$.*

Thus we have two maps : (a) uses combinatorics of the Auslander-Reiten quiver and (b) uses the representation theory of preprojective algebras.

Example 2.15. (a) Let $Q = (1 \to 2 \to 3)$. Then the AR quiver of $\mathsf{mod}KQ$ is given by



For example, take $w = s_1s_3 = s_3s_1$. Then $\underline{w} = s_1s_3$ and hence the corresponding indecomposable modules are $\{1, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\}$. Thus the subcategory $\mathsf{add}\{2, 3, \begin{smallmatrix}2\\1\end{smallmatrix}, \begin{smallmatrix}3\\2\end{smallmatrix}\}$, which was obtained by removing $\{1, \begin{smallmatrix}3\\2\\1\end{smallmatrix}\}$ from all KQ-modules, is quotient-closed (Theorem 2.14 (a)). On the other hand, one can obtain the same category as $\mathsf{res}I_w$ (Theorem 2.14 (b) and Example 2.13). For example, if $w = s_2s_3s_1s_2$, then $\mathsf{res}I_w = \mathsf{add}\{1, 3\}$ is quotient-closed.

(b) Let Q be the following quiver

$$\begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \longrightarrow & 3. \end{array}$$

We identify the infinite word $\underline{c}^{\infty} = s_1s_2s_3s_1s_2s_3s_1s_2s_3\ldots$ with infinitely many preprojective modules $P_1, P_2, P_3, \tau^{-1}P_1, \tau^{-1}P_2, \tau^{-1}P_3, \tau^{-2}P_1, \tau^{-2}P_2, \tau^{-2}P_3, \cdots$.

For example, take $w = s_1s_3s_2s_3s_1$. Then the left most word $\underline{w}$ is $s_1s_2s_3s_2s_1$. Hence the corresponding indecomposable modules are $\{P_1, P_2, P_3, \tau^{-1}P_2, \tau^{-2}P_1\}$ and hence the subcategory consists of modules of $\mathsf{mod}KQ$ by removing these modules is quotient-closed, which is also obtained as $\overline{\mathsf{res}I_w}$.

2.4. c**-sortable elements.**

Definition 2.16. Let c be a Coxeter element. Fix a reduced expression of c and regard c as a reduced word. For $w \in W$, we denote the support of W by $\mathrm{supp}(w)$, that is, the set of generators occurring in a reduced expression of w.

We call an element $w \in W$ *c-sortable* if there exists a reduced expression of w of the form $\mathbf{w} = c^{(0)}c^{(1)}\ldots c^{(m)}$, where all $c^{(t)}$ are subwords of c whose supports satisfy

$$\mathrm{supp}(c^{(m)}) \subseteq \mathrm{supp}(c^{(m-1)}) \subseteq \ldots \subseteq \mathrm{supp}(c^{(1)}) \subseteq \mathrm{supp}(c^{(0)}) \subseteq Q_0.$$

For the generators $S = \{s_1, \ldots, s_n\}$, we let $\langle s\rangle := S \setminus \{s\}$ and denote $W_{\langle s\rangle}$ by the subgroup of W generated by $\langle s\rangle$. For any $w \in W$, there is a unique factorization $w = w_{\langle s\rangle} \cdot w^{\langle s\rangle}$ maximizing $\ell(w_{\langle s\rangle})$ for $w_{\langle s\rangle} \in W_{\langle s\rangle}$ and $\ell(w_{\langle s\rangle}) + \ell(w^{\langle s\rangle}) = \ell(w)$.

Then we give the following map introduced by Reading [R2].

Definition 2.17. Let c be a Coxeter element and let s be initial in c. Then, define $\pi^c(\mathrm{id}) = \mathrm{id}$ and, for each $w \in W$, we define

$$\pi^c(w) := \begin{cases} s\pi^{scs}(sw) & \text{if } \ell(sw) < \ell(w) \\ \pi^{sc}(w_{\langle s\rangle}) & \text{if } \ell(sw) > \ell(w). \end{cases}$$

Then we have the following property.

Theorem 2.18. [R3, Proposition 3.2][RS2, Corollary 6.2] *For any $w \in W$, $\pi^c(w)$ is the unique maximal c-sortable element below w in the weak order.*

Example 2.19. Let Q be the following quiver

$$1 \longrightarrow 2 \longrightarrow 3.$$

Then $c = s_1s_2s_3$. For example $s_1s_2s_3s_2$ is a c-sortable element, and $s_1s_2s_3s_2s_1$ is not.

Let $w = s_1s_2s_3s_2s_1$. Then one can check that $\pi^c(w) = s_1s_2s_3s_2$ and it is a unique maximal c-sortable element below w.

2.5. c**-sortable elements and finite torsion-free classes.** Next we discuss a relationship between sortable elements and the notion of torsion(-free) classes.

Definition 2.20. (a) We call a subcategory of $\mathsf{mod}KQ$ *torsion class* (respectively, *torsion-free class*) if it is closed under factor modules (respectively, submodules) and extension-closed.

(b) We call a torsion class (or torsion-free class) *finite* if it has finitely many indecomposable modules.

For a given torsion class $\mathcal{T}$, we have the corresponding torsion-free class $\mathcal{F}$ as $\mathcal{T}^{\perp} := \{X \in \mathsf{mod}KQ \mid \mathrm{Hom}_{KQ}(\mathcal{T}, X) = 0\}$, and dually for a given torsion-free class $\mathcal{F}$, we have the corresponding torsion class ${}^{\perp}\mathcal{F}$. We call such a pair of torsion class and torsion-free class a *torsion pair*.

Next we recall layers following [AIRT]. For any reduced word $\mathbf{w} = s_{u_1} \ldots s_{u_l}$, we have the chain of ideals

$$\Lambda \supset I_{u_1} \supset I_{u_2} I_{u_1} \supset \ldots \supset I_{u_l} \ldots I_{u_2} I_{u_1} = I_{\mathbf{w}}.$$

For $j = 1, \ldots, l$, we define the *layer*

$$L_{\mathbf{w}}^{j} = e_{u_j} L_{\mathbf{w}}^{j} := \frac{I_{u_{j-1}} \ldots I_{u_1}}{I_{u_j} \ldots I_{u_1}}.$$

Note that the layer $L_{\mathbf{w}}^{j}$ is an indecomposable Λ-module for any $j = 1, \ldots, l$ [AIRT].

Then, for a c-sortable word, we can give a torsion-free class, which is explicitly described by layers, as follows.

Theorem 2.21. [AIRT, Theorems 3.3, 3.11 and Corollary 3.10] *Let c be a Coxeter element of Q and $\mathbf{w} = c^{(0)} c^{(1)} \ldots c^{(m)} = s_{u_1} \ldots s_{u_l}$ a c-sortable word.*

(a) *$L_{\mathbf{w}}^{j}$ is a non-zero indecomposable KQ-module for all $j = 1, \ldots, l$.*
(b) *We have $\mathsf{add}\{L_{\mathbf{w}}^{1}, \ldots, L_{\mathbf{w}}^{l}\} = \mathsf{res}(\Lambda/I_w)$ and it is finite torsion-free class.*

Example 2.22. Let Q be the following quiver

$$\begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & \longrightarrow & 3. \end{array}$$

Then $s_1 s_2 s_3$ is a Coxeter element of Q. Let $\mathbf{w} = s_1 s_2 s_3 s_1 s_2 s_1$. Then we have

$$L_{\mathbf{w}}^{1} = {\scriptstyle 1}, \ L_{\mathbf{w}}^{2} = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \ L_{\mathbf{w}}^{3} = \begin{smallmatrix} & 3 & & \\ 1 & & 2 & \\ & & & 1 \end{smallmatrix}, \ L_{\mathbf{w}}^{4} = \begin{smallmatrix} 2 & & 3 & & \\ & 1 & & 2 & \\ & & & & 1 \end{smallmatrix}, \ L_{\mathbf{w}}^{5} = \begin{smallmatrix} & 3 & & & & & \\ 1 & & 2 & & 3 & & \\ & & & 1 & & 2 & \\ & & & & & & 1 \end{smallmatrix}, \ L_{\mathbf{w}}^{6} = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}.$$

Hence we have

$$\mathsf{add}\{L_{\mathbf{w}}^{1}, \ldots, L_{\mathbf{w}}^{l}\} = \mathsf{res}(\Lambda/I_w).$$

Therefore, Theorem 2.21 implies that a c-sortable element gives a finite torsion-free class. Conversely, any finite torsion-free classes of $\mathsf{mod}KQ$ is given by a c-sortable element [AIRT, Theorem 3.16].

Thus, we provide the following correspondence, which is also shown in [T].

Corollary 2.23. *The map $w \mapsto \mathsf{res}(\Lambda/I_w)$ gives a bijection*

$$\{c\text{-sortable elements}\} \longrightarrow \{\text{finite torsion-free classes of } \mathsf{mod}KQ\}.$$

Remark 2.24. Many other interesting relationship with sortable elements are discussed in [IT, AIRT].

3. Our results

Let Q be a finite acyclic quiver, Λ the preprojective algebra of Q, W the Coxeter group of Q and c the Coxeter element of W (which depends on the orientation of Q).

In this section, we explain our main results of [MT]. Our investigation has one of its primary origins in the following natural questions and the related conjectures posed in [ORT, Conjecture 11.1,11.2].

Question 3.1. (a) *When is $\overline{\mathsf{res} I_w}$ a torsion class of $\mathsf{mod}KQ$ for $w \in W$?*

(b) *When* $\overline{\mathsf{res} I_w}$ *is a torsion class, how can we relate* w *to a* c*-sortable element* x *which provides the corresponding finite torsion-free class* $\mathsf{res}(\Lambda/I_x)$ *?*

With regard to this question, we note that, for a given finite torsion-free class, the corresponding torsion class is not necessary cofinite in the case of non-Dynkin quivers, and hence it is not necessary the form $\overline{\mathsf{res} I_w}$ in general. It is easy check that a finite torsion-free class consists of preprojective modules if and only if the corresponding torsion class is cofinite. Therefore, it is natural to give a characterization of a c-sortable element x when $\mathsf{res}(\Lambda/I_x)$ consists of preprojective modules. For this purpose, we give the following definition.

Definition 3.2. Let Q is a non-Dynkin quiver. A c-sortable element x is called *bounded* if there exists a positive integer N such that $x \leq c^N$. If Q is a Dynkin quiver, then we regard any c-sortable element as bounded. We denote by bc-sort$W$ the set of bounded c-sortable elements.

Example 3.3. (a) Let Q be the following quiver

$$1 \longrightarrow 2 \rightrightarrows 3.$$

Because

$$\begin{aligned} c^3 &= s_1s_2s_3s_1s_2s_3s_1s_2s_3 \\ &= s_1s_2s_3s_1s_2s_1s_3s_2s_3 \\ &= s_1s_2s_3s_2s_1s_2s_3s_2s_3, \end{aligned}$$

we have $s_1s_2s_3s_2 \leq c^3$ and hence $s_1s_2s_3s_2$ is bounded c-sortable.

(b) Let Q be the following quiver

$$\begin{array}{ccccc} & & 2 & & \\ & \nearrow & & \searrow & \\ 1 & & \longrightarrow & & 3. \end{array}$$

Then one can check that $s_1s_2s_3s_2$ is not bounded c-sortable.

Then we give the following lemma.

Lemma 3.4. *Let* x *be a* c*-sortable element. Then the following are equivalent.*

(a) x *is bounded* c*-sortable.*
(b) *Any module of* $\mathsf{res}(\Lambda/I_x)$ *is a preprojective module.*
(c) *The corresponding torsion class* $(\mathsf{res}(\Lambda/I_x))^{\perp}$ *is cofinite.*

Thus, bounded c-sortable elements are essential objects from the viewpoint of Question 3.1. To give a complete answer to the question, we also introduce the following terminology.

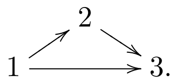
Definition 3.5. Let x be a c-sortable element. If there exists a maximum element amongst $\{w \in W \mid \pi^c(w) = x\}$, then we denote it by $\widehat{x}^c = \widehat{x}$ and call it *c-antisortable*, following the definition from [RS1]. We denote by c-antisort$W$ the set of c-antisortable elements of W.

Example 3.6. (a) Let Q be the following quiver

$$1 \longrightarrow 2 \rightrightarrows 3.$$

Take a c-sortable element $x = s_1s_2s_3s_2$. Then one can check that $\widehat{x} = s_1s_2s_3s_2s_1$.

(b) Let Q be the following quiver



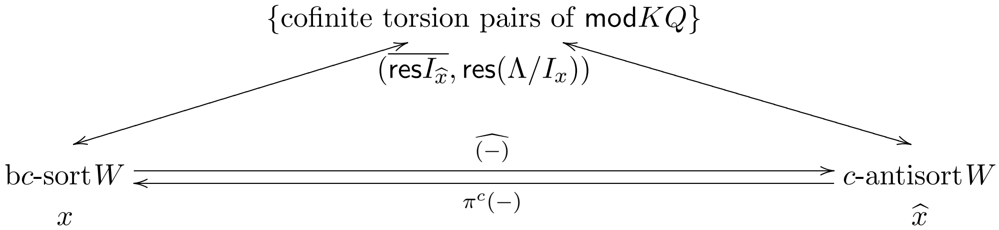
Take a c-sortable element $x = s_1s_2s_3s_2$. Consider the following infinite word

$$s_1s_2s_3s_2s_1s_3s_2s_1s_3s_2s_1s_3s_2\cdots.$$

Then from the word, we can take an arbitrary large element w such that $\pi^c(w) = x$. Thus, $\widehat{x}$ does not exist.

Using the above terminology, we obatin the following consequences and give an answer to Question 3.1.

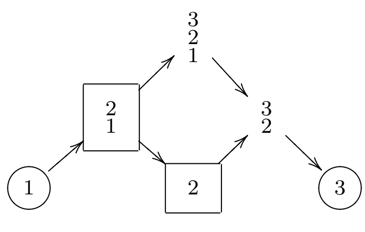
Theorem 3.7. *We have the following commutative diagram of bijections:*



In particular, a c-sortable element x is bouded if and only if it admits a maximum element in $\{w \in W \mid \pi^c(w) = x\}$.

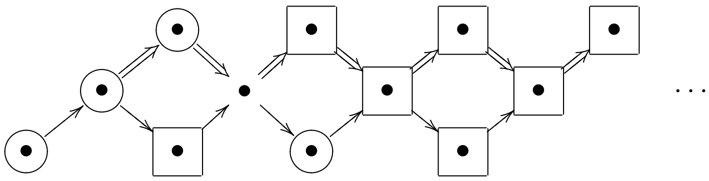
Thus, the answer of Question 3.1 is given by the notion of c-antisortable elements and the map $\pi^c(-)$ and $\widehat{(-)}$. Note that each category of $(\overline{\mathsf{res}I_{\widehat{x}}}, \mathsf{res}(\Lambda/I_x))$ can be explicitly described in terms of the Coxeter group (Theorem 2.14 and 2.21). As a consequence of the above results, we establish a proof of [ORT, Conjecture 11.1,11.2].

Example 3.8. (a) Let $Q = (1 \to 2 \to 3)$. Then the AR quiver of $\mathsf{mod}KQ$ is given by



For example, we take a c-sortable element $x = s_1s_3$. Then we have the torsion-free class $\mathsf{res}(\Lambda/I_x) = \mathsf{add}\{\,1\,,\,3\,\}$ whose modules are circled above. Then one can check that the corresponding torsion class is $\mathsf{add}\{\,{}^{2}_{1}\,,\,2\,\}$ whose modules are squared above. By Theorem 2.14, the torsion class is given as $\mathsf{res}I_w$ for $w = s_1s_3s_2s_1$ and the theorem implies that this element is $\widehat{x}$.

(b) Let $Q = (\,1 \to 2 \substack{\to\\ \to\\ \to} 3\,)$. Then the preprojective component of the AR quiver of $\mathsf{mod}KQ$ is given as the translation quiver. Thus it is given as the form



For example, we take a c-sortable element $x = s_1 s_2 s_3 s_2$, which is bounded c-sortable. Then $\mathsf{res}(\Lambda/I_x)$ consists of the modules which are circled above. The corresponding torsion class consists of the modules which are squared above and all the rest. It is given as $\overline{\mathsf{res} I_w}$ for $w = s_1 s_2 s_3 s_2 s_1$ and our theorem implies that we have $\widehat{x} = s_1 s_2 s_3 s_2 s_1$.

Finally we explain our strategy for a proof of the theorem. The first step is a parametrization of torsion pairs of $\mathsf{mod}\Lambda$.

Proposition 3.9. *For any $w \in W$,*

$$(\mathsf{Fac}(I_w), \mathsf{Sub}(\Lambda/I_w))$$

is a torsion pair of $\mathsf{mod}\Lambda$. *In particular,* $(\mathsf{Fac}(I_w) \cap \mathsf{mod}KQ, \mathsf{Sub}(\Lambda/I_w) \cap \mathsf{mod}KQ)$ *is a torsion pair of* $\mathsf{mod}KQ$.

We denote by $\mathrm{torf}\Lambda$ (respectively, $\mathrm{torf}KQ$) the set of torsion-free classes of $\mathsf{mod}\Lambda$ (respectively, $\mathsf{mod}KQ$). Then, there is a natural map from $\mathrm{torf}\Lambda$ to $\mathrm{torf}KQ$, by taking the intersection to $\mathsf{mod}KQ$. The following result recognizes a categorical map of $\pi^c : W \to c\text{-sort}W$, where $c\text{-sort}W$ denotes c-sortable elements.

Theorem 3.10. *We have the following commutative diagram* :

$$\begin{array}{ccc} W & \xrightarrow{\pi^c(-)} & c\text{-sort}W \\ {\scriptstyle \mathsf{Sub}(\Lambda/I_-)}\downarrow & & \downarrow{\scriptstyle \mathsf{res}(\Lambda/I_-)} \\ \mathrm{torf}\Lambda & \xrightarrow{-\cap \mathsf{mod}KQ} & \mathrm{torf}KQ \end{array}$$

Then Question 3.1 can be explained in this way: Assume that $\overline{\mathsf{res} I_w}$ is a torsion class. Then we can show that it is given as $\mathsf{Fac}(I_w) \cap \mathsf{mod}KQ$. Therefore the corresponding torsion-free class is $\mathsf{Sub}(\Lambda/I_w) \cap \mathsf{mod}KQ$ by Proposition 3.9. Thus Theorem 3.10 shows that it is $\mathsf{res}(\Lambda/I_{\pi^c(w)})$, that is, we can relate w by π^c to the c-sortable element which provides the corresponding torsion-free class. Moreover, assume that there exists $u \in W$ with $u > w$ and $\pi^c(u) = \pi^c(w)$. Then by Theorem 2.14 and the above argument, we have

$$\overline{\mathsf{res}(I_u)} \supset \mathsf{Fac} I_u \cap \mathsf{mod}KQ = {}^{\perp}(\mathsf{res}\Lambda_{\pi^c(w)}) = \overline{\mathsf{res} I_w}.$$

Because $\overline{\mathsf{res}(I_u)}$ (respectively, $\overline{\mathsf{res} I_w}$) consists of the category which is obtained by removing $\ell(u)$ (respectively, $\ell(w)$) indecomposable KQ-modules from the preprojective modules by Theorem 2.14, this is impossible. In this way, we can show that w is of maximal length in $\{w \in W \mid \pi^c(w) = x\}$, and indeed maximum element (but it is non-trivial). Thus we conclude $w = \widehat{\pi^c(w)}$, that is, w is a c-antisortable element.

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