Factorization algebras in algebraic geometry

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January 26, 2020

This is a summary of the talk I gave at the Algebra Symposium, held at Tohoku University in September 2019. It is based on the papers [6, 8], joint with B. Hennion and E. Vasserot. The common theme of these papers is application of the concept of factorization algebras, originally defined for the needs of Quantum Field Theory, to "purely algebro-geometric" problems, i.e., problems in Algebraic Geometry which a priori do not involve this concept.

1 Factorization algebras: algebro-geometric version

The concept of factorization algebras on an algebraic curve was introduced by Beilinson and Drinfeld in their 2004 book [1]. Their goal was to give a geometric axiomatization of the theory of vertex algebras or, in the physical language, of 2-dimensional Conformal Field Theory. This theory has been extended to varieties of arbitrary dimensions by Francis and Gaitsgory [3]. The fundamental tool here is the concept of the *Ran space*.

Let X be a smooth algebraic variety over a field $/\mathbf{k}$, $\mathrm{char}(\mathbf{k}) = 0$. The Ran space of X, is, informally, the "space"

$$\mathrm{Ran}(X) = \{\text{all finite, nonempty subsets } I \subset X\}.$$

In this definition a subset $I \subset X$ is considered without multiplicity. So, for example, a 2-element subset $\{x, y\}$ can, if x and y merge, degenerate into a 1-element subset $\{x\} = \{x, x\}$. In the opposite direction, one point can split into many, similarly to elementary particles in physics. Combining all finite subsets together can be seen as an instance of "second quantization of algebraic geometry" (Y.I. Manin).

There is no way of making $\mathrm{Ran}(X)$ into an algebraic variety in a rigorous sense of the word. However, we can do meaningful algebraic geometry on it: consider sheaves, $\mathcal{D}$-modules, etc. This is explained in [1, 3, 6].

In particular, a *factorization algebra* on X is a sheaf ($\mathcal{D}$-module) on $\mathrm{Ran}(X)$ with natural identifications

$$\mathcal{F}_{I \sqcup J} \simeq \mathcal{F}_I \otimes \mathcal{F}_J.$$

where $\mathcal{F}_I$ means the fiber of $\mathcal{F}$ at a point $I \in \mathrm{Ran}(X)$, i.e., a finite subset $I \subset X$. This corresponds to the principle of locality in Quantum Field Theory: points which are away from each other are “independent”. Tensor multiplication of vector spaces can be seen a a quantum analog of the usual multiplication of probabilities, something that happens for independent events.

There are various versions of the notion of a factorization algebra, corresponding to different meaning of the word “sheaf" that can be defined for $\mathrm{Ran}(X)$.

A factorization algebra has a global invariant, the *factorization cohomology*

$$\int_X \mathcal{F} = R\Gamma(\mathrm{Ran}(X), \mathrm{DR}(\mathcal{F})).$$

Here $R\Gamma$ is the derived functor of global sections, and DR is the de Rham complex of a $\mathcal{D}$-module.

Example 1.1. Let $X = \mathbb{A}^1$ be the affine line. A translation invariant factorization algebra $\mathcal{F}$ on X is the same as a vertex algebra V which is recovered as the fiber $\mathcal{F}_0$ of $\mathcal{F}$ at 0.

2 Smooth Manifold version

A different framework for factorization alebras, closer to the Quantum Field Theory intuition, was given by Lurie [9] and Costello-Gwilliam [2].

Let M be C^∞-manifold. It is not assumed to be algebraic but can be (in which one can compare with the previous setting). We consider M with the usual topology (so not Zariski if M is in fact algebraic).

A *factorization algebra* on M is a datum $\mathcal{A}$ associating:

(1) Any open $(U \subset M) \mapsto \mathcal{A}(U)$, a cochain complex $/\mathbf{k}$.

(2) Any $(U_1 \sqcup \cdots \sqcup U_m \subset U_0) \mapsto$ multiplication

$$\mu : \mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_m) \longrightarrow \mathcal{A}(U_0), \text{ so that:}$$

(3) For U_0 equalling $U_1 \sqcup \cdots \sqcup U_m$, μ is a quasi-isomorphism.

(4) The $m = 1$ part of data (pre-co-sheaf) is a cosheaf for a certain class of coverings, called *Weiss coverings*.

The condition (3) corresponds to ocality in Quantum Field Theory. A covering $V = \bigcup V_i$ is called a Weiss covering, if any finite subset $I \subset V$ is contained in one of the V_i. This may seem counterintuitive, but it simply means that $\mathrm{Ran}(V)$ is covered by the $\mathrm{Ran}(V_i)$. So the Weiss topology is simply a means to bring the Ran space into the consideration without mentioning it explicitly and working on M all along.

3 Locally constant factorization algebras: topological QFT

A factorization algebra $\mathcal{A}$ is called *locally constant*, if for any two *disks* $U_1 \subset U_0$ the map μ is a quasi-isomorphism. In the physical jargon this condition means that the de Rham differential d_{DR} acts "trivially" (in a Q-exact way). In other words, our quantum field theory is "topological".

In the flat case $M = \mathbb{R}^n$ it was proved by Lurie that we have a 1:1 correspondence

$$\text{Locally constant FA } \mathcal{A} \longleftrightarrow \; E_n - \text{algebras } A = \mathcal{A}(\mathbb{R}^n)$$

Here E_n is the operad of little n-disks. It describes commutativity "up to level n".

In the curved case (any M with G-structure in the tangent bundle, $G \subset O(n)$, e.g., $G = U(d)$ for d-dim $\mathbb{C}$-mflds) we have the following extension of the above, also due to Lurie:

Every E_n-algebra A equipped with a *homotopy G-action* gives a locally constant factorization algebra $\underline{A}_M$ on M. In particular, we can form the factortization homology $\int_M(A) := \underline{A}_M(M)$.

4 Nonabelian Poincaré duality (Salvatore-Lurie)

Suppose that we are given:
Y: a topological space with G-action;
M: an n-manifold with G-structure;
Let $Y_M \to M$ be the fibration with fiber Y associated to TM.
Let also $A = C^\bullet(Y, \mathbf{k})$ be the cochain algebra. It is:

- "Commutative": E_n for any n.
- Has a homotopy G-action.

One formulation of the Non-Abelian Poincaré duality is as follows: If Y is $(n-1)$-connected, then

$$\int_M (C^\bullet(Y)) \;=\; C^\bullet(\mathrm{Sect}(Y_M/M), \mathbf{k}).$$

Here Sect means the space of continuous sections. Statements of such kind goback to the work of Bott-Segal and Haefligher on cohomology of Lie algebras of vector fields (see bellow), well before factorization algebras.

Let us explain, roughly, the meaning of this statement. Suppose G acts trivially, so $\mathrm{Sect}(Y_M/M) = \mathrm{Map}(M, Y)$ is the space of maps $M \to Y$.

Approximate statement: if $A = H^\bullet(Y)$ has the form $A = \mathrm{Sym}^\bullet(V)$, then

$$\int_M (A) \;=\; \mathrm{Sym}^\bullet(V \otimes H_\bullet(M)).$$

How is this related? In such cases typically $V^\bullet = (\pi_\bullet(Y) \otimes \mathbf{k})^*$. Expecting the same for $\mathrm{Map}(M, Y)$ we reduce to the known fact (Haefligher)

$$\pi_\bullet \mathrm{Map}(M, Y) \otimes \mathbf{k} \; = \; H^\bullet(M, \mathbf{k}) \otimes \pi_\bullet(Y).$$

How is this related to the usual Poincaré duality? We have an alternative (Koszul dual) formulation (w.r.t. Koszul self-duality of the E_n-operad):

Suppose $y \in Y$ is G-fixed point. Then $Y_M \to M$ has a distinguised section $\underline{y}$ corresponding to y. Let also $B = \Omega^n(Y, y)$ be n-fold loop space, an E_n-algebra in spaces. Then (as spaces!):

$$\int_M (\Omega^n(Y, y)) \; = \; \mathrm{Sect}_c(Y_M/M).$$

Here Sect_c means the space of sections with "compact support", i.e., sections which coincide with $\underline{y}$ outside a compact set. If $Y = K(\pi, n)$, then $\Omega^n(Y) = \pi$. Assume G-action trivial. Then:

$\int_M(\pi)$ is the space with homotopy groups being the homology $H_\bullet(M, \pi)$, and we get

$$H_i(M, \pi) = \pi_i \mathrm{Map}_c(M, K(\pi, n)) = H_c^{n-i}(M, \pi).$$

5 Holomorphic FA: vertex algebras

On $M = \mathbb{C}$, we can speak about *holomorphic* factorization algebras $\mathcal{A}$ (in which the antiholomorphic Dolbeault differential $\overline{\partial}$, rather than the de Rham differential d, acts in a Q-exact way). As shown in [2], under appropriate assumptions,

$$\text{Such } \mathcal{A} \overset{1:1}{\longleftrightarrow} \text{Vertex Algebras } V = \mathcal{A}\{|z| < 1\}.$$

Remarks 5.1. (a) The Algebro-Geometric (AG) and and Smooth-Manifold versions are intuitively equivalent: the important U's are

$$\bigsqcup \text{small disks} \;\sim\; \text{finite sets of points}.$$

(b) Most "outside" applications of AG formalism have been in $\dim_{\mathbb{C}} = 1$ case: Geometric Langlands etc.

6 Application 1: Gelfand-Fuchs cohomology in AG

Let $X/\mathbf{k}$ be smooth algebraic variety. Let $\mathfrak{g} = T(X) = R\Gamma(X, T)$ be the (derived, dg-) Lie algebra of global vector fields. Usual global vector fields, if X affine. We want to find the Lie algebra cohomology (with coefficients in $\mathbf{k}$)

$$H^\bullet_{\mathrm{Lie}}(T(X)) = ?$$

This cohomology is important (e.g., $H^2_{\mathrm{Lie}} \sim$ central extensions) but usually hard to find. Already here:

Example 6.1. $X = \mathbb{A}^1 - \{0\}$ punctured affine line. In this case $T(X)$ has basis $L_i = z^{i+1}d/dz$, $i \in \mathbb{Z}$ with the well known commutation relations

$$[L_i, L_j] = (j - i)L_{i+j}.$$

It is known: $H^\bullet_{\mathrm{Lie}} = \mathbf{k}[\beta_2, \beta_3]$, with $\beta_2 =$ being the famous Virasoro cocycle. But no direct purely algebraic proof of this fact is known (!), the classical argument goes through the case of C^∞ vector fields on the circle (for $\mathbf{k} = \mathbb{R}$).

Example 6.2. An easier example is obtained for $X = \mathbb{A}^1$, so $\mathfrak{g} = \operatorname{Der} \mathbf{k}[z]$, with basis only $L_{-1}, L_0, L_1, \cdots$. Note: L_{-1}, L_0, L_1 span an $\mathfrak{sl}_2$.

Recall: any $\mathfrak{g}$ acts trivially on its own $H^\bullet_{\mathrm{Lie}}$. So

$$C^\bullet_{\mathrm{Lie}}(\mathfrak{g}) \sim C^\bullet_{\mathrm{Lie}}(\mathfrak{g})_{L_0} \quad \text{(degree 0 wedges)}.$$

$$\text{But this} = C^\bullet_{\mathrm{Lie}}(\mathfrak{sl}_2) \quad \text{(balanced wedges cannot involve } L_{\geqslant 2}).$$

$$H^\bullet_{\mathrm{Lie}}(\mathfrak{g}) \;=\; H^\bullet_{\mathrm{Lie}}(\mathfrak{sl}_2) \;=\; H^\bullet_{\mathrm{top}}(SU_2).$$

Now $SU(2) = S^3$ is the 3-sphere, so we get

$$H^\bullet_{\mathrm{Lie}}(\mathfrak{g}) = H^\bullet_{\mathrm{top}}(S^3) = \mathbf{k}[\beta_3].$$

Note: for $\operatorname{Der} \mathbf{k}[z, z^{-1}]$ the degree 0 complex is still very hard to analyze.

7 Classical Gelfand-Fuchs theory

The classical theory (explained in more detail in [4]) addresses the following question. Let M be C^∞ manifold, and consider the Lie algebra $\mathrm{Vect}(M)$ of C^∞ vector fields. What is its $H^\bullet_{\mathrm{Lie}}$? The theory then proceeds in two stages:

Stage 1: formal vector fields. We start with the Lie algebra

$$W_n = \left\{ \sum f_i \partial/\partial z_i, \;\; f_i \in \mathbb{R}[[z_1, \cdots, z_n]] \right\}.$$

For this Lie algebra, Gelfand and Fuchs found that:

$$H^\bullet_{\mathrm{Lie}}(W_n) \;=\; H^\bullet_{\mathrm{top}}(Y_n)$$

where Y_n is the fiber product

$$\begin{array}{ccc} Y_n & \longrightarrow & EGL_n(\mathbb{C}) = \text{Stiefel variety} \\ {\scriptstyle GL_n(\mathbb{C})}\downarrow & & \downarrow{\scriptstyle GL_n(\mathbb{C})} \\ \operatorname{sk}_{2n} BGL_n(\mathbb{C}) & \longrightarrow & BGL_n(\mathbb{C}) = \mathrm{Gr}(n, \mathbb{C}^\infty) \end{array}$$

and $\operatorname{sk}_{2n} BGL_n(\mathbb{C})$ is the $2n$-skeleton of the infinite Grassmannian in the cell decomposition by Schubert cells.

Stage 2: general M: We form the fibration $Y_M \xrightarrow{Y_n} M$, via T_M and $GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})$. Then the result is:

$$H^\bullet_{\mathrm{Lie}}(\mathrm{Vect}(M)) \;\simeq\; H^\bullet_{\mathrm{top}}(\mathrm{Sect}(Y_M/M)).$$

Example 7.1. Suppose $M = S^1$ is the circle. In this case:

$\mathrm{Vect}(S^1)$ is a completion of the $(L_i)_{i\in\mathbb{Z}}$ algebra above.

$$BGL_1(\mathbb{C}) = \mathbb{C}P^\infty; \quad \mathrm{sk}_2\, BGL_1(\mathbb{C}) = \mathbb{C}P^1 = S^2; \quad Y_1 \overset{\text{hom.eq.}}{\sim} S^3.$$

So we get the statement of Example 6.2. Further, T_{S^1} is trivial, so $Y_{S^1} \sim S^1 \times S^3$,

$$\mathrm{Sect}(Y_{S^1}/S^1) \;\sim\; \mathrm{Map}(S^1, S^3) \overset{S^3 \text{ is a group}}{=\!=} S^3 \times \Omega(S^3),$$

and the cohomology of this is $\mathbf{k}[\beta_3] \otimes \mathbf{k}[\beta_2]$.

8 Algebro-Geometric version

Let now $X/\mathbb{C}$ be a smooth algebraic variety, $\dim_{\mathbb{C}}(X) = n$. We then have $Y_X \xrightarrow{Y_n} X$, via T_X and $GL_n(\mathbb{C})$. Note that its fibers are identified with Y_n, not Y_{2n}, even though from the C^∞-point of view, X is of dimension $2n$.

Theorem 8.1 (B. Hennion-M.K.[6] [1])**.** *(a) We have a canonical map* $\lambda : H^\bullet_{\mathrm{Lie}}(T(X)) \longrightarrow H^\bullet_{\mathrm{top}}(\mathrm{Sect}(Y_X/X))$.

(b) If X is affine, λ is an isomorphism.

The proof is based on the theory of factorization algebras (both in the n-dimensional AG and $2n$-dimensional Smooth-Manifold versions). It is straightforward:

$\exists$ natural (algebro-geometric) FA $\check{\mathcal{C}}^\bullet$ on $\mathrm{Ran}(X)$ s.t.

$$H^\bullet_{\mathrm{Lie}}(T(X)) \;\simeq\; \int_X \check{\mathcal{C}}^\bullet.$$

The crucial tool for the next step is the Covariant Verdier Duality of Gaitsgory-Lurie [5] which can be explained as follows.

The "space" $\mathrm{Ran}(X)$ is ∞-dimensional, union of fin-dim skeleta

$$X = \mathrm{Ran}_1 \subset \mathrm{Ran}_2 \subset \cdots \subset \mathrm{Ran}(X).$$

Here $\mathrm{Ran}_p(X)$ is formed by finite subsets $I \subset X$ of cardinality $\leqslant p$. Given a FA $\mathcal{F}$ on $\mathrm{Ran}(X)$, there is a new FA $\psi(\mathcal{F})$ with

$$\psi(\mathcal{F})|_{\mathrm{Ran}_k} \;=\; \underline{R\Gamma}_{\mathrm{Ran}_k}(\mathcal{F}) \quad \text{(Cohomology with support)}$$

There is always a map

$$\lambda : \int_X \mathcal{F} \longrightarrow \int_X \psi(\mathcal{F})$$

[1] Conjectured by B. Feigin in the 80's.

Its image consists of classes "supported on a finite-dimensional skeleton".

In our case the Covariant Verdier Duality allows us to pass from algebro-geometric to C^∞ FA. Here are the main points.

Main Point 1: $\psi(\check{\mathcal{C}}^\bullet)$ is topological (locally constant) while $\check{\mathcal{C}}^\bullet$ itself is holomorphic, related to the vacuum module (with $c = 0$)

$$\mathcal{V} = \operatorname{Ind}_{W_n = T(\text{Formal } n\text{-disk})}^{T(\text{Punctured formal disk})} \mathbb{C},$$

which is a vertex algebra (derived (dg), for $n > 1$).

Main Point 2: Factorization homology of $\psi(\check{\mathcal{C}}^\bullet)$ are found topologically, via $\mathrm{Sect}(Y_X/X)$. This is done by associating to it a locally constant FA in the C^∞ sense and applying Non-Abelian Poincaré Duality.

9 Application 2: Cohomological Hall Algebras for surfaces

We recall the classical concept of the *Hall Algebra* in the following context. Let X be an algebraic variety *over a finite field* $\mathbb{F}_q$. Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X with proper support. Then for any two $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X))$ all the Ext-groups $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ are finite-dimensional $\mathbb{F}_q$-vector spaces, in particular, they are finite sets.

Let $H = \mathrm{Fun}(\mathrm{Coh}(X) \to \mathbb{C})$ be the space of isomorphism invariant functions on objects of $\mathrm{Coh}(X)$. This space carries the *Hall Multiplication* via the *Induction Diagram*

$$\begin{array}{ccc} & \{0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0\} & \\ {\scriptstyle (p',p'')} \swarrow & & \searrow {\scriptstyle p} \\ \mathrm{Coh} \times \mathrm{Coh} & & \mathrm{Coh} \end{array}$$

where on the top we have the category of short exact sequences. The multiplication is defined by the pullback and pushforward of functions:

$$H \otimes H = \mathrm{Fun}(\mathrm{Coh}(X) \times \mathrm{Coh}(X)) \xrightarrow{p_* \circ (p',p'')^*} \mathrm{Fun}(Coh(X)) = H.$$

This makes H into an associative algebra known as the *Hall algebra* of $\mathrm{Coh}(X)$.

Examples 9.1. (a)Suppose that X a smooth projective curve. In this case Coh splits into:

- Vector bundles. They give rise to an algebra H_{Bun} formed by unramified automorphic forms on all the GL_n (with r being the rank of the bundle). The multiplication is given by Eisenstein series, and the resulting algebra is related/similar to quantum affine algebras [7].

- Sheaves with 0-dimensional support. They give rise to an algebra H_0 formed by classical Hecke operators. It is commutative. Its multiplication with elements of H_{Bun} gives the classical action of Hecke operators on automorphic forms.

(b) In the case $\dim(X) > 1$, nothing interesting is known about such algebras. They have huge size, and it is not clear what are the good questions to ask.

In higher-dimensional case one can follow a different approach, forming the so-called *cohomological Hall algebra*, or COHA. For this, we consider X defined over $\mathbb{C}$, not over a finite field and consider $\mathrm{Coh}(X)$ as an algebraic stack. Instead of invariant functions on objects, we consider the stack-theoretic (co)homology of this stack. (roughly, equivariant (co)homology with respect to the stabilizer groups. Then we want to use the functoriality of (co)homology to produce a multiplication out of the Induction Diagram. However, the maps in the diagram are singular, so it is subtle, as we need both types of functorialities, covariant as well as contravariant, for a single theory (say, homology).

For $n = 1$ and 2 it was shown in [8] and [10] that one can define the multiplication using *virtual fundamental classes* to account for the singular nature of the maps. More precisely, we work with $H_\bullet^{\mathrm{BM}}(\mathrm{Coh}_0(X))$, the Borel-Moore homology of the stack of coherent sheaves with 0-dimensional support. So the resulting algebra is the analog of the classical algebra of Hecke operators for curves over a finite field.

Remark 9.2. This construction is related to the concept of COHA for 3-dimensional Calabi-Yau (CY^3) categories, as defined by Kontsevich and Soibelman. This COHA, is defined, intuitively, as the space of vanishing cycles for (holomorphic) Chern-Simons functional. Given a surface X, one can form the CY^3 manifolld $\mathrm{Tot}(K_X)$ (the total space of the canonical bundle), and the two constructions should match in this case. But there is so far, no geometric treatment (for CY^3 categories associated to manifolds), the examples considered being related to quivers.

10 Factorization of COHA

Observation (M.K. - E. Vasserot, [8]): $H(\mathrm{Coh}_0)$ is a factorization algebra on X, in the Smooth-Manifold sense:

$$\mathrm{Coh}_0(U_1 \sqcup U_2) = \mathrm{Coh}_0(U_1) \times \mathrm{Coh}_0(U_2).$$

Further, this factorization algebra is locally constant. Therefore, $H(\mathrm{Coh}_0(X))$ found from the case of the flat space $H_{\mathrm{flat}} = H(\mathrm{Coh}_0(\mathbb{A}^2))$ by Factorization Homology. Now, a coherent sheaf $\mathcal{F}$ on $\mathbb{A}^2$ with 0-dimensional support is the same as a finite-dimensional vector space $V = H^0(\mathcal{F})$ with action of $\mathbb{C}[x, y]$, i.e., with a pair of commuting operators $x, y : V \to V$. Therefore the stack $\mathrm{Coh}_0(\mathbb{A}^2)$ is identified with a disjoint union of quotient stacks:

$$\mathrm{Coh}_0(\mathbb{A}^2) = \bigsqcup_{n \geqslant 0} C_n /\!/ GL_n, \quad \text{where}$$

$$C_n = \{(A,B) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \big| [A,B]=0\} \quad \text{is the commuting variety.}$$

It is known (goes back to the Feit-Fine 1962 formula for $|C_n(\mathbb{F}_q)|$) that as a bigraded vector space (by n above and coh. degree),

$$H_{\text{flat}} \simeq \text{Sym}^\bullet(V),\ V = qt\mathbb{C}[q^{-1}, t],\ \deg(q) = (0,-2), \deg(t) = (1,0).$$

Theorem 10.1 ([8])**.** *(a)* $H_{\text{flat}} \simeq \text{Sym}^\bullet(V)$ *as an algebra, in particular, it is commutative.*

(b) (Poincaré-Birkhoff-Witt-type theorem for COHA) For any surface X, we have an identification as bigraded vector spaces

(10.2) $$H^{\text{BM}}_\bullet(\text{Coh}_0(X) \simeq \text{Sym}^\bullet(H^{\text{BM}}_\bullet(X) \otimes V').$$

Here $V' = V(0,4)$ (shifted grading), so that for $X = \mathbb{A}^2$ with $H^{\text{BM}}_4(\mathbb{A}^2) = \mathbb{C}$ we have

$$V = V' \otimes H^{\text{BM}}_\bullet(\mathbb{A}^2).$$

(Note that Eq. (10.2) is analogous to the Non-abelian Poincaré Duality.)

This means that we can find the size (graded dimension) of $H(\text{Coh}_0(X))$, and the answer looks like

$$H^\bullet_{\text{top}}(\text{Sect}(F_X/X))$$

for a fibraton $F_X \to X$ with fiber F s.t. $H^\bullet_{\text{top}}(F) = H_{\text{flat}}$. So the situation here is formally similar to Gelfand-Fuchs!

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