TOWARD CRITERIA FOR K-STABILITY OF LOG FANO PAIRS

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Abstract. This is my proceedings of "64th Algebra Symposium" at Tohoku university. In the proceedings, we give a simplification for the proof of "a valuative criterion" for the uniform K-stability of log Fano pairs.

Contents

1. K-stability of log Fano pairs

We work over an arbitrary algebraically closed field $\Bbbk$ of characteristic zero. Throughout this proceedings, we always assume that (X, Δ) is an n-dimensional *log Fano pair*, that is, X is a normal projective variety over $\Bbbk$ and Δ is an effective $\mathbb{Q}$-Weil divisor with (X, Δ) a klt pair and $L := -(K_X+\Delta)$ an ample $\mathbb{Q}$-divisor. (For the theory of minimal model program, we refer the readers to [KM98].) We recall the *K-semistability* and the *uniform K-stability* of (X, Δ).

Definition 1.1 (see, e.g., [Tia97, Don02]). (1) The following data

- a normal projective variety $\mathcal{X}$ and a surjective morphism $p\colon \mathcal{X} \to \mathbb{P}^1$,
- a p-semiample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$,
- a $\mathbb{G}_m$-action $\mathbb{G}_m \curvearrowright (\mathcal{X}, \mathcal{L})$ commuting with the action $\mathbb{G}_m \curvearrowright \mathbb{P}^1_t$ with $(a, t) \mapsto at$,

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- a $\mathbb{G}_m$-equivariant isomorphism
$$\left(\mathcal{X}\setminus\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}\setminus\mathcal{X}_0}\right)\simeq\left(X\times(\mathbb{P}^1\setminus\{0\}), p_1^*L\right),$$
where $\mathcal{X}_0$ is the fiber of p at $0\in\mathbb{P}^1$,

is said to be a *test configuration* of (X, L). We simply say that "$(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is a test configuration of (X, L)". For a test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L), let $\Delta_{\mathcal{X}}$ be the $\mathbb{Q}$-Weil divisor on $\mathcal{X}$ defined by the closure of $\Delta\times(\mathbb{P}^1\setminus\{0\})$.

(2) A test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L) is said to be *trivial* if the ample model of $\mathcal{L}$ over $\mathbb{P}^1$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X\times\mathbb{P}^1, p_1^*L)$.

(3) (see [LX14]) A test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L) is said to be *special* if $\mathcal{L}$ is ample over $\mathbb{P}^1$ and the pair $(\mathcal{X}, \Delta_{\mathcal{X}}+\mathcal{X}_0)$ is a plt pair.

(4) (see [Wan12, Oda13]) For a test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L), the *Donaldson-Futaki invariant* $\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L})$ of $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is defined as follows:
$$\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L}):=\frac{n}{n+1}\cdot\frac{(\mathcal{L}^{\cdot n+1})}{(L^{\cdot n})}+\frac{(\mathcal{L}^{\cdot n}\cdot(K_{\mathcal{X}/\mathbb{P}^1}+\Delta_{\mathcal{X}}))}{(L^{\cdot n})},$$
where $K_{\mathcal{X}/\mathbb{P}^1}:=K_{\mathcal{X}}-p^*K_{\mathbb{P}^1}$.

In the paper [Fuj19a], the *Ding invariant*, introduced by [Ber16] (see also [Fuj18]), plays an important role.

Definition 1.2 (see [Ber16, Fuj18])**.** For a test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L), the *Ding invariant* $\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L})$ of $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is defined as follows:
$$\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L}):=-\frac{(\mathcal{L}^{\cdot n+1})}{(n+1)(L^{\cdot n})}-1+\mathrm{lct}\left(\mathcal{X}, \Delta_{\mathcal{X}}+D_{((\mathcal{X},\Delta_{\mathcal{X}}),\mathcal{L})}; \mathcal{X}_0\right),$$
where $D_{((\mathcal{X},\Delta_{\mathcal{X}}),\mathcal{L})}$ is the $\mathbb{Q}$-Weil divisor on $\mathcal{X}$ supported on $\mathcal{X}_0$ with
$$D_{((\mathcal{X},\Delta_{\mathcal{X}}),\mathcal{L})}\sim_{\mathbb{Q}}-(K_{\mathcal{X}/\mathbb{P}^1}+\Delta_{\mathcal{X}})-\mathcal{L},$$
and lct is the *log canonical threshold*, that is,
$$\begin{aligned}&\mathrm{lct}\left(\mathcal{X}, \Delta_{\mathcal{X}}+D_{((\mathcal{X},\Delta_{\mathcal{X}}),\mathcal{L})}; \mathcal{X}_0\right)\\ :=\ &\max\{c\in\mathbb{R}\mid\left(\mathcal{X}, \Delta_{\mathcal{X}}+D_{((\mathcal{X},\Delta_{\mathcal{X}}),\mathcal{L})}+c\mathcal{X}_0\right)\text{ is a sub-lc pair}\}.\end{aligned}$$

Definition 1.3. A log Fano pair (X, Δ) is said to be *K-semistable* (resp., *Ding semistable*) if $\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L})\geq 0$ (resp., $\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L})\geq 0$) holds for any test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L).

In the papers [Der16, BHJ17], they systematically treat the *norm* of test configurations.

Definition 1.4 (see [Der16, BHJ17])**.** For a test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L), let us consider the normalization of the graph

$$\begin{array}{ccc} & \mathcal{Y} & \\ {}^{\Pi}\swarrow & & \searrow^{\Theta} \\ X \times \mathbb{P}^1 & & \mathcal{X} \end{array}$$

of the rational map $X \times \mathbb{P}^1 \dashrightarrow \mathcal{X}$. The *minimal norm* (resp., the *non-archimedean J-norm*) $\|(\mathcal{X}, \mathcal{L})\|_m$ (resp., $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$) of $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is defined as follows:

$$\begin{aligned} \|(\mathcal{X}, \mathcal{L})\|_m &:= \frac{(\Pi^* p_1^* L \cdot \Theta^* \mathcal{L}^{\cdot n})}{(L^{\cdot n})} - \frac{n(\mathcal{L}^{\cdot n+1})}{(n+1)(L^{\cdot n})}, \\ J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &:= \frac{(\Pi^* p_1^* L^{\cdot n} \cdot \Theta^* \mathcal{L})}{(L^{\cdot n})} - \frac{(\mathcal{L}^{\cdot n+1})}{(n+1)(L^{\cdot n})}. \end{aligned}$$

Remark 1.5.
1. The definition of test configurations in [Fuj19a] and the above definition differs. In [Fuj19a], we consider $(\mathcal{X}, \mathcal{L})$ over $\mathbb{A}^1$. If we canonically compactify $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ over $\mathbb{P}^1$, then we get the same notion.
2. In [Fuj19a] and [BBJ15], we focused on $J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$. Recently, I recognized that considering $\|(\mathcal{X}, \mathcal{L})\|_m$ is more natural in order to prove a "valuative criterion" for K-stability of log Fano pairs. It is the purpose of the proceedings explaining this observation.

Definition 1.6. A log Fano pair (X, Δ) is said to be *uniformly K-stable* (resp., *unifomrly Ding stable*) if there exists $\delta \in (0, 1)$ such that $\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L}) \geq \delta \cdot J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$ (resp., $\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L}) \geq \delta \cdot J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$) holds for any test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of (X, L).

We recall basic results:

Proposition 1.7. *Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a test configuration of (X, L) with $\mathcal{L}$ ample over $\mathbb{P}^1$.*
1. *We have the inequalities*
$$\frac{1}{n} \cdot J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \|(\mathcal{X}, \mathcal{L})\|_m \leq n \cdot J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$
2. *We have $\|(\mathcal{X}, \mathcal{L})\|_m \geq 0$, and equality holds if and only if $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is trivial.*
3. *We have $\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L}) \geq \mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L})$, and equality holds if and only if $\mathcal{L} \sim_{\mathbb{Q}, \mathbb{P}^1} -(K_{\mathcal{X}/\mathbb{P}^1} + \Delta_{\mathcal{X}})$ and the pair $(\mathcal{X}, \Delta_{\mathcal{X}} + \mathcal{X}_0)$ is an lc pair. (In particular, we have $\mathrm{DF}_\Delta(\mathcal{X}, \mathcal{L}) = \mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L})$ for any special test configuration.)*

Proof. (1) See [BHJ17, Proposition 7.8 and Remark 7.12].
(2) See [BHJ17] or [Der16].
(3) See [Ber16] or [Fuj18]. □

2. A valuative criterion and the purpose of this proceedings

We recall "a valuative criterion" for the K-stability of log Fano pairs introduced in [Li17] and [Fuj19a] independently.

Definition 2.1. Let F be a prime divisor over X, that is, there exists a log resolution $\sigma\colon \tilde{X} \to X$ of (X, Δ) such that F is a prime divisor on $\tilde{X}$. (The following definitions does not depend on the choice of σ.)

(1) Let $A(F)$ be the *log discrepancy* of (X, Δ) along F, that is, $A(F) := 1 + \mathrm{ord}_F(K_{\tilde{X}} - \sigma^*(K_X + \Delta))$.
(2) The divisor F is said to be *dreamy* if the graded $\Bbbk$-algebra
$$\bigoplus_{k,j\geq 0} H^0(\tilde{X}, \sigma^*(krL) - jF)$$
is finitely generated, where r is some (hence, any) positive integer with rL Cartier.
(3) For any $x \in \mathbb{R}_{\geq 0}$, let us set
$$\mathrm{vol}(L - xF) := \mathrm{vol}_{\tilde{X}}(\sigma^* L - xF)$$
(see [Laz04a, Laz04b]). We define
$$\tau(F) := \min\{\tau \in \mathbb{R}_{>0} \mid \mathrm{vol}(L - \tau F) = 0\}.$$
(4) (see [BJ17]) We set
$$S(F) := \frac{1}{(L^{\cdot n})} \int_0^\infty \mathrm{vol}(L - xF)dx.$$
(5) (see [Fuj19a, Li17]) We set
$$\begin{aligned} \beta(F) &:= (L^{\cdot n})(A(F) - S(F)), \\ j(F) &:= (L^{\cdot n})(\tau(F) - S(F)). \end{aligned}$$
(6) (see [Fuj19b]) We set
$$\hat{\beta}(F) := \frac{\beta(F)}{A(F)(L^{\cdot n})} = 1 - \left(\frac{A(F)}{S(F)}\right)^{-1}.$$

More generally, for a divisorial valuation $v = c \cdot \mathrm{ord}_F$ with $c \in \mathbb{Q}_{>0}$, we naturally define $A(v) := c \cdot A(F)$, $\tau(v) := c \cdot \tau(F)$ $S(v) := c \cdot S(F)$, etc. See [Fuj19a] for detail.

In [Fuj19a], I proved the following "valuative criterion" for the uniform K-stability of (X, Δ). For the K-semistability, the result was proved by [Li17] and [Fuj19a] independently.

Theorem 2.2 (see [Fuj19a])**.** *The following are equivalent:*

(1) (X, Δ) *is uniformly K-stable.*
(2) *There exists* $\delta \in (0, 1)$ *such that* $\beta(F) \geq \delta \cdot j(F)$ *holds for any prime divisor* F *over* X*.*
(3) *There exists* $\delta \in (0, 1)$ *such that* $\beta(F) \geq \delta \cdot j(F)$ *holds for any prime divisor* F *over* X *which is dreamy.*

Nowadays, it has been known that the invariant $\hat{\beta}(F)$, more precisely, the invariant

$$\frac{A(F)}{S(F)},$$

is more important than $\beta(F)$ and $j(F)$. See, for example, [FO18, BJ17]. Actually, I proved the following result in [Fuj19b]:

Theorem 2.3 (see [Fuj19b])**.** *The following are equivalent:*

(1) (X, Δ) *is uniformly K-stable.*
(2) *There exists* $\varepsilon \in (0, 1)$ *such that* $\hat{\beta}(F) \geq \varepsilon$ *holds for any prime divisor* F *over* X*.*
(3) *There exists* $\varepsilon \in (0, 1)$ *such that* $\hat{\beta}(F) \geq \varepsilon$ *holds for any prime divisor* F *over* X *which is dreamy.*

The purpose of this proceedings is to prove Theorem 2.3 directly, by changing the original proof of Theorem 2.2 a bit.

Remark 2.4. It is more convenient in many situations that considering not only divisorial valuations but also all valuations in order to consider K-stability of (X, Δ). Actually, in [BJ17], they showed that the uniform K-stability of (X, Δ) is equivalent to

$$\inf_{v} \frac{A(v)}{S(v)} > 1,$$

where v runs through all valuations on X with $A(v) < +\infty$. See [BJ17] in detail.

3. From test configurations to $\hat{\beta}$

Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a test configuration of (X, L) with $\mathcal{L}$ ample over $\mathbb{P}^1$ and $\mathcal{X}_0$ integral. As we have seen in [Fuj19a, Proposition 2.10], we can naturally get the divisorial valuation $v_{\mathcal{X}_0}$ on X obtained by the restriction of the valuation $\mathrm{ord}_{\mathcal{X}_0}$. The following theorem is important in [Fuj19a].

Theorem 3.1 (see [Fuj19a, Theorem 5.1])**.** *We have*

$$\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L}) = A(v_{\mathcal{X}_0})\cdot\hat{\beta}(v_{\mathcal{X}_0}) = A(v_{\mathcal{X}_0}) - S(v_{\mathcal{X}_0}).$$

The following theorem is important in this proceedings.

Theorem 3.2. *We have*

$$\|(\mathcal{X},\mathcal{L})\|_m = A(v_{\mathcal{X}_0})\cdot\left(1-\hat{\beta}(v_{\mathcal{X}_0})\right) = S(v_{\mathcal{X}_0}).$$

Proof. The proof is similar to the proof of [Fuj19a, Theorem 5.1]. Since $\mathcal{X}_0$ is integral, we may assume that $\mathcal{L} = -(K_{\mathcal{X}/\mathbb{P}^1}+\Delta_{\mathcal{X}})$. Let

$$\begin{array}{ccc} & \mathcal{Y} & \\ {}^{\Pi}\swarrow & & \searrow^{\Theta} \\ X\times\mathbb{P}^1 & & \mathcal{X} \end{array}$$

be the normalization of the graph. Set

$$\begin{aligned} B &:= K_{\mathcal{Y}/\mathbb{P}^1}+\Delta_{\mathcal{Y}} - \Pi^*(K_{X\times\mathbb{P}^1/\mathbb{P}^1}+\Delta_{X\times\mathbb{P}^1}) \\ &- \left(K_{\mathcal{Y}/\mathbb{P}^1}+\Delta_{\mathcal{Y}} - \Theta^*(K_{\mathcal{X}/\mathbb{P}^1}+\Delta_{\mathcal{X}})\right), \end{aligned}$$

where $\Delta_{\mathcal{Y}}$ and $\Delta_{X\times\mathbb{P}^1}$ are the strict transforms of $\Delta_{\mathcal{X}}$. By [BHJ17, Proposition 4.11], we get

$$\begin{aligned} \mathrm{ord}_{\mathcal{X}_0} B &= \mathrm{ord}_{\mathcal{X}_0}\left(K_{\mathcal{Y}/\mathbb{P}^1}+\Delta_{\mathcal{Y}} - \Pi^*(K_{X\times\mathbb{P}^1/\mathbb{P}^1}+\Delta_{X\times\mathbb{P}^1})\right) \\ &= A(v_{\mathcal{X}_0}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|(\mathcal{X},\mathcal{L})\|_m &= = \frac{1}{(L^{\cdot n})}\left(\frac{1}{n+1}(\mathcal{L}^{\cdot n+1}) + (\Theta^*\mathcal{L}^{\cdot n}\cdot(\Pi^* p_1^* L - \Theta^*\mathcal{L}))\right) \\ &= -\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L}) + \frac{1}{(L^{\cdot n})}\left(\mathcal{L}^{\cdot n}\cdot\Theta_* B\right) \\ &= -A(v_{\mathcal{X}_0})\cdot\hat{\beta}(v_{\mathcal{X}_0}) + \frac{1}{(L^{\cdot n})}\left(\mathcal{L}^{\cdot n}\cdot A(v_{\mathcal{X}_0})\mathcal{X}_0\right) \\ &= A(v_{\mathcal{X}_0})\cdot\left(1-\hat{\beta}(v_{\mathcal{X}_0})\right) \end{aligned}$$

by Theorem 3.1. □

Remark 3.3. In [Fuj19a], I showed the equality

$$J^{\mathrm{NA}}(\mathcal{X},\mathcal{L}) = \frac{1}{(L^{\cdot n})}\cdot j(v_{\mathcal{X}_0}).$$

Thanks to Theorem 3.2, it is more natural to focus on $\|(\mathcal{X},\mathcal{L})\|_m$ than to focus on $J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ in order to evaluate $\hat{\beta}(F)$.

4. The uniform K-stability and the uniform Ding stability

Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a test configuration of (X, L) with $\mathcal{L}$ ample over $\mathbb{P}^1$. In [Fuj19a, Section 3], I considered the behavior of the invariant

$$\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L}) - \delta \cdot J^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}),$$

under the processes of certain minimal model programs (MMP, in short) achieved in the important paper [LX14]. In the proceedings, we consider the behavior of the invariant

$$\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L}) - \delta \cdot \|(\mathcal{X}, \mathcal{L})\|_m.$$

Actually, the invariant also non-increases under the processes of certain MMP in [LX14]. We briefly see the proof. The proof is more or less same as the argument in [Fuj19a, Section 3].

Theorem 4.1 (cf., [Fuj19a, Theorem 3.1])**.** *Let $\pi\colon \mathcal{X}^{\mathrm{lc}} \to \mathcal{X}$ be the log canonical modification of $(\mathcal{X}, \Delta_{\mathcal{X}}+\mathcal{X}_0)$, that is, the pair $(\mathcal{X}^{\mathrm{lc}}, \Delta_{\mathcal{X}^{\mathrm{lc}}}+\mathcal{X}_0^{\mathrm{lc}})$ is lc and $K_{\mathcal{X}^{\mathrm{lc}}/\mathbb{P}^1} + \Delta_{\mathcal{X}^{\mathrm{lc}}} + \mathcal{X}_0^{\mathrm{lc}}$ is π-ample. Let E be the $\mathbb{Q}$-divisor supported on $\mathcal{X}_0^{\mathrm{lc}}$ with*

$$E \sim_{\mathbb{Q}} K_{\mathcal{X}^{\mathrm{lc}}/\mathbb{P}^1} + \Delta_{\mathcal{X}^{\mathrm{lc}}} + \pi^*\mathcal{L}.$$

(Of course, E is π-ample.) For any $0 < t \ll 1$ with $t \in \mathbb{Q}$, let us set the ample $\mathbb{Q}$-line bundle

$$\mathcal{L}_t := \pi^*\mathcal{L} + tE.$$

Then, for any $\delta \in [0,\, 1/n]$, we have

$$\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L}) - \delta \cdot \|(\mathcal{X}, \mathcal{L})\|_m \geq \mathrm{Ding}_\Delta(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}_t^{\mathrm{lc}}) - \delta \cdot \|(\mathcal{X}^{\mathrm{lc}}, \mathcal{L}_t^{\mathrm{lc}})\|_m.$$

Proof. From the definition of E, we have

$$D_t := D_{((\mathcal{X}^{\mathrm{lc}}, \Delta_{\mathcal{X}^{\mathrm{lc}}}), \mathcal{L}_t^{\mathrm{lc}})} = -(1+t)E.$$

Let $\mathcal{X}_0^{\mathrm{lc}} = \sum_{i=1}^p E_i$ be the irreducible decomposition and let us set $E = \sum_{i=1}^p e_i E_i$. We may assume that $e_1 \leq \cdots \leq e_p$. Under the setting, we have

$$\mathrm{lct}\left(\mathcal{X}^{\mathrm{lc}}, \Delta_{\mathcal{X}^{\mathrm{lc}}} + D_t; \mathcal{X}_0^{\mathrm{lc}}\right) = 1 + (1+t)e_1.$$

Let

$$X \times \mathbb{P}^1 \xleftarrow{\ \Pi\ } \mathcal{Y} \xrightarrow{\ \Theta\ } \mathcal{X}^{\mathrm{lc}}$$

be the normalization of the graph. Moreover, let us set $\phi_t := \Theta^*\mathcal{L}_t^{\mathrm{lc}}$ and $\phi_{\mathrm{triv}} := \Pi^* p_1^* L$. Then we have

$$\begin{aligned}\mathrm{Ding}_\Delta(\phi_t) &= -\frac{(\phi_t^{\cdot n+1})}{(n+1)(L^{\cdot n})} + (1+t)e_1, \\ \|\phi_t\|_m &= \frac{1}{(L^{\cdot n})}\left((\phi_{\mathrm{triv}}\cdot\phi_t^{\cdot n}) - \frac{n}{n+1}(\phi_t^{\cdot n+1})\right).\end{aligned}$$

Thus we get

$$\begin{aligned}& (n+1)(L^{\cdot n})\left((\mathrm{Ding}_\Delta(\phi_0) - \delta\cdot\|\phi_0\|_m) - (\mathrm{Ding}_\Delta(\phi_t) - \delta\cdot\|\phi_t\|_m)\right) \\ = \ & -(\phi_0^{\cdot n+1}) + (n+1)e_1(L^{\cdot n}) - \delta(n+1)(\phi_{\mathrm{triv}}\cdot\phi_0^{\cdot n}) + \delta n(\phi_0^{\cdot n+1}) \\ - \ & \left(-(\phi_t^{\cdot n+1}) + (n+1)(1+t)e_1(L^{\cdot n}) - \delta(n+1)(\phi_{\mathrm{triv}}\cdot\phi_t^{\cdot n}) + \delta n(\phi_t^{\cdot n+1})\right) \\ = \ & (1-\delta n)\left((\phi_t^{\cdot n+1}) - (\phi_0^{\cdot n+1})\right) - t(n+1)e_1(L^{\cdot n}) \\ + \ & \delta(n+1)(\phi_{\mathrm{triv}}\cdot(\phi_t^{\cdot n} - \phi_0^{\cdot n})) \\ = \ & (1-\delta n)\sum_{j=0}^{n}\left((\phi_t^{\cdot j+1}\cdot\phi_0^{\cdot n-j}) - (\phi_t^{\cdot j}\cdot\phi_0^{\cdot n+1-j}) - te_1(L^{\cdot n})\right) \\ + \ & \delta(n+1)\sum_{j=0}^{n-1}\left((\phi_{\mathrm{triv}}\cdot\phi_t^{\cdot j+1}\cdot\phi_0^{\cdot n-1-j}) - (\phi_{\mathrm{triv}}\cdot\phi_t^{\cdot j}\cdot\phi_0^{\cdot n-j}) - te_1(L^{\cdot n})\right) \\ = \ & (1-\delta n)t\sum_{j=0}^{n}\left(\phi_t^{\cdot j}\cdot\phi_0^{\cdot n-j}\cdot\Theta^*(E - e_1\mathcal{X}_0^{\mathrm{lc}})\right) \\ + \ & \delta(n+1)t\sum_{j=0}^{n-1}\left(\phi_{\mathrm{triv}}\cdot\phi_t^{\cdot j}\cdot\phi_0^{\cdot n-1-j}\cdot\Theta^*(E - e_1\mathcal{X}_0^{\mathrm{lc}})\right) \\ \geq \ & 0.\end{aligned}$$

This completes the proof. □

Theorem 4.2 (cf. [Fuj19a, Theorem 3.2])**.** *Assume that $(\mathcal{X}, \Delta_{\mathcal{X}}+\mathcal{X}_0)$ is lc. Let $\sigma\colon \mathcal{X}^0 \to \mathcal{X}$ be a $\mathbb{G}_m$-equivariant small $\mathbb{Q}$-factorial modification. Fix $l \gg 0$ such that*

$$\mathcal{H}^0 := \frac{1}{l+1}\left(l\mathcal{L}^0 - (K_{\mathcal{X}^0/\mathbb{P}^1} + \Delta_{\mathcal{X}^0})\right)$$

is semiample and big over $\mathbb{P}^1$, where $\mathcal{L}^0 := \sigma^*\mathcal{L}$. As in [LX14, Theorem 3]*, let us consider a $(K_{\mathcal{X}^0/\mathbb{P}^1} + \Delta_{\mathcal{X}^0})$-MMP*

$$\mathcal{X}^0 \dashrightarrow \mathcal{X}^1 \dashrightarrow \cdots \dashrightarrow \mathcal{X}^k$$

over $\mathbb{P}^1$ with scaling $\mathcal{H}^0$. Set $\lambda_0 := l+1$ and

$$\lambda_{j+1} := \min\{\lambda \mid K_{\mathcal{X}^j/\mathbb{P}^1} + \Delta_{\mathcal{X}^j} + \lambda\mathcal{H}^j \text{ is nef over } \mathbb{P}^1\},$$

where $\mathcal{H}^j$ *is the strict transform of* $\mathcal{H}^0$ *on* $\mathcal{X}^j$*. Then we get*

$$l+1=\lambda_0>\lambda_1\geq\cdots\geq\lambda_k>\lambda_{k+1}=1$$

(*see* [LX14, Theorem 3]). *For any* $0\leq j\leq k-1$ *and* $\lambda\in[\lambda_{j+1},\lambda_j]\cap\mathbb{Q}$, *let us set*

$$\mathcal{L}^j_\lambda:=\frac{1}{\lambda-1}\left(K_{\mathcal{X}^j/\mathbb{P}^1}+\Delta_{\mathcal{X}^j}+\lambda\mathcal{H}^j\right).$$

Moreover, let $(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}})/\mathbb{P}^1$ *be the ample model of* $(\mathcal{X}^k,\mathcal{L}^k_{\lambda_k})$ *over* $\mathbb{P}^1$*. Then, for any* $\delta\in[0,\,1/n]$*, we have*

$$\mathrm{Ding}_\Delta(\mathcal{X},\mathcal{L})-\delta\cdot\|(\mathcal{X},\mathcal{L})\|_m\geq\mathrm{Ding}_\Delta(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}})-\delta\cdot\|(\mathcal{X}^{\mathrm{ac}},\mathcal{L}^{\mathrm{ac}})\|_m.$$

Proof. Let E be the $\mathbb{Q}$-divisor on $\mathcal{X}^0$ supported on $\mathcal{X}^0_0$ such that $E\sim_{\mathbb{Q}} K_{\mathcal{X}^0/\mathbb{P}^1}+\Delta_{\mathcal{X}^0}+\mathcal{H}^0$. Then we have

$$D_\lambda:=D_{((\mathcal{X}^j,\Delta_{\mathcal{X}^j}),\mathcal{L}^j_\lambda)}=-\frac{\lambda}{\lambda-1}E^j,$$

where E^j is the strict transform of E on $\mathcal{X}^j$. In order to prove Theorem 4.2, it is enough to show the inequality

$$\mathrm{Ding}_\Delta(\mathcal{X}^j,\mathcal{L}^j_{\lambda_j})-\delta\cdot\|(\mathcal{X}^j,\mathcal{L}^j_{\lambda_j})\|_m\geq\mathrm{Ding}_\Delta(\mathcal{X}^j,\mathcal{L}^j_{\lambda_{j+1}})-\delta\cdot\|(\mathcal{X}^j,\mathcal{L}^j_{\lambda_{j+1}})\|_m$$

for any $0\leq j\leq k-1$. Let

$$\begin{array}{ccc} & \mathcal{Y} & \\ {\scriptstyle\Pi}\swarrow & & \searrow{\scriptstyle\Theta} \\ X\times\mathbb{P}^1 & & \mathcal{X}^j \end{array}$$

be the normalization of the graph. Let $E^j=\sum_{i=1}^p E_i$ be the irreducible decomposition and let us set $E^j=\sum_{i=1}^p e_iE_i$. We may assume that $e_1\leq\cdots\leq e_p$. Moreover, let us set $\phi_\lambda:=\Theta^*\mathcal{L}^j_\lambda$ and $\phi_{\mathrm{triv}}:=\Pi^*p_1^*L$. Then we have

$$\mathrm{Ding}_\Delta(\phi_\lambda)=-\frac{(\phi_\lambda^{\cdot n+1})}{(n+1)(L^{\cdot n})}+\frac{\lambda}{\lambda-1}e_1.$$

Note that

$$\phi_{\lambda_{j+1}}-\phi_{\lambda_j}=-D_{\lambda_{j+1}}+D_{\lambda_j}=\left(\frac{\lambda_{j+1}}{\lambda_{j+1}-1}-\frac{\lambda_j}{\lambda_j-1}\right)E^j.$$

Thus we have

$$
\begin{aligned}
& (n+1)(L^{\cdot n})\left(\left(\mathrm{Ding}_\Delta(\phi_{\lambda_j}) - \delta\cdot\|\phi_{\lambda_j}\|_m\right) - \left(\mathrm{Ding}_\Delta(\phi_{\lambda_{j+1}}) - \delta\cdot\|\phi_{\lambda_{j+1}}\|_m\right)\right)\\
= \; & -(\phi_{\lambda_j}^{\cdot n+1}) + \frac{\lambda_j}{\lambda_j-1}(n+1)e_1(L^{\cdot n}) - \delta(n+1)(\phi_{\mathrm{triv}}\cdot\phi_{\lambda_j}^{\cdot n}) + \delta n(\phi_{\lambda_j}^{\cdot n+1})\\
- \; & \left(-(\phi_{\lambda_{j+1}}^{\cdot n+1}) + \frac{\lambda_{j+1}}{\lambda_{j+1}-1}(n+1)e_1(L^{\cdot n}) - \delta(n+1)(\phi_{\mathrm{triv}}\cdot\phi_{\lambda_{j+1}}^{\cdot n}) + \delta n(\phi_{\lambda_{j+1}}^{\cdot n+1})\right)\\
= \; & (1-\delta n)\left((\phi_{\lambda_{j+1}}^{\cdot n+1}) - (\phi_{\lambda_j}^{\cdot n+1})\right) - \left(\frac{\lambda_{j+1}}{\lambda_{j+1}-1} - \frac{\lambda_j}{\lambda_j-1}\right)(n+1)e_1(L^{\cdot n})\\
+ \; & \delta(n+1)(\phi_{\mathrm{triv}}\cdot(\phi_{\lambda_{j+1}}^{\cdot n} - \phi_{\lambda_j}^{\cdot n}))\\
= \; & \left(\frac{\lambda_{j+1}}{\lambda_{j+1}-1} - \frac{\lambda_j}{\lambda_j-1}\right)\left((1-\delta n)\sum_{j=0}^{n}\left(\phi_{\lambda_{j+1}}^{\cdot j}\cdot\phi_{\lambda_j}^{\cdot n-j}\cdot\Theta^*(E-e_1\mathcal{X}_0^j)\right)\right.\\
+ \; & \left.\delta(n+1)\sum_{j=0}^{n-1}\left(\phi_{\mathrm{triv}}\cdot\phi_{\lambda_{j+1}}^{\cdot j}\cdot\phi_{\lambda_j}^{\cdot n-1-j}\cdot\Theta^*(E-e_1\mathcal{X}_0^j)\right)\right)\\
\geq \; & 0.
\end{aligned}
$$

This completes the proof. □

Theorem 4.3 (cf. [Fuj19a, Theorem 3.3])**.** *Assume that* $(\mathcal{X}, \Delta_{\mathcal{X}} + \mathcal{X}_0)$ *is lc and* $\mathcal{L} = -(K_{\mathcal{X}/\mathbb{P}^1} + \Delta_{\mathcal{X}})$*. Assume also that there exists a birational map*

$$\mathcal{X} \dashrightarrow \mathcal{X}^{\mathrm{s}}$$

over $\mathbb{P}^1$ *such that* $(\mathcal{X}^s, -(K_{\mathcal{X}^{\mathrm{s}}/\mathbb{P}^1} + \Delta_{\mathcal{X}^{\mathrm{s}}}))/\mathbb{P}^1$ *is a special test configuration and the discrepancy of* $(\mathcal{X}, \Delta_{\mathcal{X}})$ *along* $\mathcal{X}_0^{\mathrm{s}}$ *is equal to zero. (By* [LX14, Theorem 4]*, such birational map always exists after a base change of* $\mathcal{X}$ *over* $\mathbb{P}^1$*.) Then, for any* $\delta \in [0, 1/n]$*, we have*

$$\mathrm{Ding}_\Delta(\mathcal{X}, \mathcal{L}) - \delta\cdot\|(\mathcal{X}, \mathcal{L})\|_m \geq \mathrm{Ding}_\Delta(\mathcal{X}^{\mathrm{s}}, \mathcal{L}^{\mathrm{s}}) - \delta\cdot\|(\mathcal{X}^{\mathrm{s}}, \mathcal{L}^{\mathrm{s}})\|_m.$$

Proof. There exists the extraction $\pi\colon \mathcal{X}' \to \mathcal{X}$ of $\mathcal{X}_0^{\mathrm{s}}$ and we have $K_{\mathcal{X}'/\mathbb{P}^1} + \Delta_{\mathcal{X}'} = \pi^*(K_{\mathcal{X}/\mathbb{P}^1} + \Delta_{\mathcal{X}})$. Let

$$
\begin{array}{ccccc}
X\times\mathbb{P}^1 & \xleftarrow{\ \Pi\ } & \mathcal{Y} & \xrightarrow{\ \Xi\ } & \mathcal{X}^{\mathrm{s}}\\
 & & \big\downarrow{\scriptstyle\Theta} & & \\
 & & \mathcal{X}' & &
\end{array}
$$

be a common resolution of the base locus of birational maps. Set $\phi_0 := \Theta^*(-(K_{\mathcal{X}'/\mathbb{P}^1} + \Delta_{\mathcal{X}'}))$, $\phi_1 := \Xi^*(-(K_{\mathcal{X}^{\mathrm{s}}/\mathbb{P}^1} + \Delta_{\mathcal{X}^{\mathrm{s}}}))$ and $\phi_{\mathrm{triv}} := \Pi^* p_1^* L$. Let E be the $\mathbb{Q}$-divisor on $\mathcal{Y}$ supported on $\mathcal{Y}_0$ with $E \sim_{\mathbb{Q}} \phi_1 - \phi_0$. Since

$-E$ is Ξ-nef and Ξ-exceptional, E is effective by negativity lemma. Therefore, we have

$$\begin{aligned}
& (n+1)(L^{\cdot n})\left((\mathrm{Ding}_\Delta(\phi_0)-\delta\cdot\|\phi_0\|_m)-(\mathrm{Ding}_\Delta(\phi_1)-\delta\cdot\|\phi_1\|_m)\right)\\
= & -(\phi_0^{\cdot n+1})-\delta(n+1)(\phi_{\mathrm{triv}}\cdot\phi_0^{\cdot n})+\delta n(\phi_0^{\cdot n+1})\\
- & \left(-(\phi_1^{\cdot n+1})-\delta(n+1)(\phi_{\mathrm{triv}}\cdot\phi_1^{\cdot n})+\delta n(\phi_1^{\cdot n+1})\right)\\
= & (1-\delta n)\left((\phi_1^{\cdot n+1})-(\phi_0^{\cdot n+1})\right)+\delta(n+1)(\phi_{\mathrm{triv}}\cdot(\phi_1^{\cdot n}-\phi_0^{\cdot n}))\\
= & (1-\delta n)\sum_{j=0}^{n}\left(\phi_1^{\cdot j}\cdot\phi_0^{\cdot n-j}\cdot E\right)+\delta(n+1)\sum_{j=0}^{n-1}\left(\phi_{\mathrm{triv}}\cdot\phi_1^{\cdot j}\cdot\phi_0^{\cdot n-1-j}\cdot E\right)\\
\geq & \ 0.
\end{aligned}$$

This completes the proof. □

Corollary 4.4 (cf. [Fuj19a, Corollary 3.4])**.** *Take any $\delta\in[0,\,1/n]$. The following are equivalent:*

1. *For any test configuration $(\mathcal{X},\mathcal{L})/\mathbb{P}^1$, we have the inequality* $\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})\geq\delta\cdot\|(\mathcal{X},\mathcal{L})\|_m$.
2. *For any test configuration $(\mathcal{X},\mathcal{L})/\mathbb{P}^1$, we have the inequality* $\mathrm{Ding}_\Delta(\mathcal{X},\mathcal{L})\geq\delta\cdot\|(\mathcal{X},\mathcal{L})\|_m$.
3. *For any special test configuration $(\mathcal{X},\mathcal{L})/\mathbb{P}^1$, we have the inequality* $\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L})\geq\delta\cdot\|(\mathcal{X},\mathcal{L})\|_m$.

Proof. Follows immediately from Theorems 4.1, 4.2 and 4.3. □

5. A simplified proof

We recall the following theorem:

Theorem 5.1 ([Fuj19a, Theorem 4.1])**.** *Assume that there exists $\delta\in[0,\,1)$ such that, for any test configuration $(\mathcal{X},\mathcal{L})/\mathbb{P}^1$ of (X,L), the inequality* $\mathrm{Ding}_\Delta(\mathcal{X},\mathcal{L})\geq\delta\cdot J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ *holds. Then, for any prime divisor F over X, we have $\beta(F)\geq\delta\cdot j(F)$.*

Remark 5.2. One of the idea of the proof of Theorem 5.1, which was already appeared in [Fuj18], is to consider a sequence of test configurations and taking a kind of limit. The proof is technical and complicated. It is an interesting problem to simplify the proof of the Theorem 5.1. In particular, I want to rephrase Theorem 5.1 without using the language of $J^{\mathrm{NA}}(\mathcal{X},\mathcal{L})$ and $j(F)$. More precisely, I want to get a direct proof of Corollary 5.4.

We also recall the following easy lemma, proven by the log-concavity of the volume function (and the restricted volume function).

Lemma 5.3 (see [FO18, Lemma 1.2] and [Fuj19b, Theorem 2.3])**.** *For any prime divisor F over X, we have*

$$\frac{1}{n+1}\tau(F) \leq \frac{j(F)}{(L^{\cdot n})} \leq \frac{n}{n+1}\tau(F).$$

As a consequence, we have the following corollary.

Corollary 5.4. *If there exists $\delta \in [0,1)$ such that, for any test configuration $(\mathcal{X},\mathcal{L})/\mathbb{P}^1$ of (X,L), the inequality $\mathrm{Ding}_\Delta(\mathcal{X},\mathcal{L}) \geq \delta \cdot \|(\mathcal{X},\mathcal{L})\|_m$ holds. Then, for any prime divisor F over X, we have*

$$\hat{\beta}(F) \geq \frac{\delta}{n(n+1)}.$$

Proof. By Proposition 1.7 (1) and Theorem 5.1, we have $\beta(F) \geq (\delta/n)j(F)$. If $\tau(F) \leq A(F)$, then we have

$$\hat{\beta}(F) \geq 1 - \left(\frac{A(F)}{\frac{n}{n+1}\tau(F)}\right)^{-1} \geq \frac{1}{n+1}$$

by Lemma 5.3. Thus we may assume that $\tau(F) > A(F)$. In this case, by Lemma 5.3, we have

$$\hat{\beta}(F) \geq \frac{\delta}{n} \cdot \frac{j(F)}{A(F)(L^{\cdot n})} \geq \frac{\delta}{n(n+1)} \cdot \frac{\tau(F)}{A(F)} > \frac{\delta}{n(n+1)}.$$

Thus the assertion follows. □

As a consequence, although there is an unsatisfactory point (the proof of Corollary 5.4), we can get a simple proof of Theorem 2.3.

Simplified proof of Theorem 2.3. By Corollary 4.4, we have already seen that the following three conditions are equivalent:

(i) (X,Δ) is uniformly K-stable.
(ii) (X,Δ) is uniformly Ding stable.
(iii) There exists $\delta \in (0,1)$ such that, for any special test configuration $(\mathcal{X},\mathcal{L})/\mathbb{P}^1$ of (X,L), we have $\mathrm{DF}_\Delta(\mathcal{X},\mathcal{L}) \geq \delta \cdot \|(\mathcal{X},\mathcal{L})\|_m$.

By Corollary 5.4, the condition (ii) implies that the condition (2) in Theorem 2.3. Obviously, the condition (2) in Theorem 2.3 implies the condition (3) in Theorem 2.3. By Theorems 3.1 and 3.2, the condition (3) in Theorem 2.3 implies that the condition (iii). □

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