Monstrous Moonshine over the integers

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1 What is Moonshine?

Moonshine is fundamentally about strange connections between finite groups and modular forms. These connections should be very special. In particular, if there are infinitely many cases of a phenomenon, then we typically do not call it moonshine. Instead, we should expect a general theory to explain the phenomenon.

2 Monstrous Moonshine

Monstrous moonshine began in 1978 with a numerical observation by McKay relating representations of the monster simple group with coefficients of the modular J function. This observation initially appeared to be a coincidence, but soon additional numerical evidence together with theoretical advances showed that there is a substantial connection.

The finite simple groups are now known to be naturally organized into 18 infinite families, together with 26 additional groups, called sporadic groups. The monster $\mathbb{M}$, constructed in [Griess 1982], is the largest of the 26 sporadic simple groups, with order about 8×10^{53}. This is roughly the number of protons in Jupiter, so it is not feasible to compute with all of the elements of the group at once. On the other hand, it is somewhat smaller than the number $52! \sim 8 \times 10^{67}$ of permutations in a standard deck of playing cards. We can compute by hand with this group of permutations, but computation with the monster is much more difficult despite its smaller size because $\mathbb{M}$ has no small representations. In particular, there are no faithful permutation representations of degree less than about 9×10^{19}, and no complex linear representations of dimension less than 196883.

The J-function is an analytic function on the complex upper half-plane that is invariant under the action of $SL_2(\mathbb{Z})$ by Möbius transformations. The quotient of the upper half-plane by this action is complex-analytically isomorphic to $\mathbb{C}$, i.e., it is a punctured genus zero curve, and J realizes such an isomorphism. In general, a function that generates the function field of an upper half-plane quotient is called a hauptmodul, or principal modulus. Thus, J is a hauptmodul for $SL_2(\mathbb{Z})$. J has the Fourier expansion

$$q^{-1} + 196884q + 21493760q^2 + \cdots \qquad (q = e^{2\pi i z})$$

and all of its coefficients are non-negative integers.

In 1978, McKay noted that $196884 = 196883 + 1$, where the left side is q-coefficient of the J function, and the numbers on the right side are the dimensions of the smallest irreducible representations of the monster. He suggested that there was a connection, and later computations by Thompson [Thompson 1979] strongly suggested this was not a coincidence, as the first few coefficients of J were easily written as very simple combinations of dimensions of irreducible representations of the monster:

$$196884 = 1 + 196883 \quad (1)$$
$$21493760 = 1 + 196883 + 21296876 \quad (2)$$
$$864299970 = 2 \times 1 + 2 \times 196883 + 21296876 + 842609326 \quad (3)$$

McKay's observation could be explained by the existence of a natural graded representation $V = \bigoplus_n V_n$ of the monster, such that $\sum(\dim V_n)q^{n-1} = J$. This existence problem is now known as the McKay-Thompson conjecture, and it was solved with the construction of the "Moonshine Module" $V^\natural$ by Frenkel, Lepowsky, and Meurman. They initially constructed it as a graded representation $V^\natural = \bigoplus V_n$ with graded dimension $\sum(\dim V_n)q^{n-1} = J$ in 1984, using vertex operators. However, Borcherds introduced a notion of vertex algebra in [Borcherds 1986], and claimed that their construction is an example. Frenkel, Lepowsky, and Meurman then showed that this was true, and furthermore, that the monster is the full automorphism group of $V^\natural$ as a vertex algebra [Frenkel-Lepowsky-Meurman 1988].

Conway and Norton, acting on a suggestion by Thompson, computed a conjectural list of graded traces of elements of the monster on the graded representation V [Conway-Norton 1979]. They noted that all of the trace functions appeared to be Hauptmoduls of genus zero discrete subgroups of $SL_2(\mathbb{R})$. From this evidence, they proposed the "Monstrous Moonshine" conjecture, a refinement of the McKay-Thompson conjecture that asserted the existence of a graded representation $V = \bigoplus_n V_n$ such that for each element g in the monster, the graded trace $\sum_n Tr(g|V_n)q^{n-1}$ matched the function they computed.

Atkin, Fong, and Smith showed in 1980 that a satisfactory virtual representation exists, but with no explicit construction. Borcherds showed in [Borcherds 1992] that $V^\natural$ satisfies Conway and Norton's conjecture, using the vertex algebra structure in an essential way. Specifically, he showed that for each $g \in \mathbb{M}$, the series $T_g(\tau) = \sum_n Tr(g|V_n^\natural)q^{n-1}$ is equal to the Hauptmodul for g proposed by Conway and Norton.

3 More monstrous moonshine

Conway and Norton suggested in 1979 that there may be similar behavior for other groups, and Queen computed several potential character functions in 1980. As an example of this behavior, the second largest sporadic group, called the Baby monster, has irreducible representations of dimension $1, 4371, 96255, \ldots$, and the Hauptmodul for $\Gamma_0(2)^+$ is $q^{-1} + 4372q + 96256q^2 + \cdots$.

One of the most interesting phenomena to appear in these computations was that if $g \in \mathbb{M}$ has prime order p, and lies in the conjugacy class named pA, then the graded trace function T_g has non-negative integer coefficients that look like dimensions of representations of the centralizer $C_{\mathbb{M}}(g)$. For $p = 2$, the centralizer of an element in class 2A is isomorphic to a central extension $2.\mathbb{B}$ of the Baby monster. Thus, our example is still connected to the Monster, but in a way that is different from the Conway-Norton conjecture.

This phenomenon now has two explanations, both involving the existence of representations of groups whose dimensions are given by the non-negative coefficients.

Conjecture 1: Generalized Moonshine (Norton 1987) For each $g \in \mathbb{M}$, there exists a $\frac{1}{N}\mathbb{Z}$-graded projective representation $V(g) = \bigoplus_n V(g)_n$ of $C_{\mathbb{M}}(g)$, such that the trace functions $Z(g, h; \tau) = \sum_n Tr(\tilde{h}|V(g)_n)q^{n-1}$ satisfy good modularity properties. Here, $\tilde{h}$ is some lift of h to a linear transformation on $V(g)$, and N is the level of $T_g(\tau)$. [Norton 1987]

Conjecture 2: Modular Moonshine (Ryba 1994) For each g in conjugacy class pA, there is a $\mathbb{Z}_{\geq 0}$-graded vertex algebra V_g over $\mathbb{F}_p$ with a $C_{\mathbb{M}}(g)$ action, such that the graded Brauer character of each p-regular $h \in C_{\mathbb{M}}(g)$ on V_g is equal to the Monstrous Moonshine function T_{gh}. [Ryba 1996]

Both conjectures were rather quickly given conjectural interpretations that placed the central objects $V(g)$ and V_g in a meaningful context. For Generalized Moonshine, the physicists

Dixon, Ginsparg, and Harvey proposed that the representations $V(g)$ are twisted sectors of a conformal field theory with $\mathbb{M}$ symmetry, and the functions $Z(g,h;\tau)$ are genus 1 partition functions with twisted boundary conditions [Dixon-Ginsparg-Harvey 1988]. For Modular Moonshine, Borcherds and Ryba proposed that V_g is the Tate cohomology group $\hat{H}^0(g, V^\natural_{\mathbb{Z}})$ with coefficients in a self-dual integral form $V^\natural_{\mathbb{Z}}$ of $V^\natural$ with $\mathbb{M}$-symmetry [Borcherds-Ryba 1996].

Both conjectures saw substantial progress in the mid 1990s. For Generalized Moonshine, the representations $V(g)$ were reinterpreted as irreducible g-twisted $V^\natural$-modules, and these were shown to exist in [Dong-Li-Mason 1997]. For Modular moonshine, the conjecture was proved in [Borcherds-Ryba 1996], [Borcherds 1998], and [Borcherds 1999] under the assumption that $V^\natural_{\mathbb{Z}}$ exists. The self-dual integral form was not known to exist at the time, but they got an unconditional result for odd primes by using a self-dual form over $\mathbf{Z}[1/2]$. However, the complete solutions to these conjectures required the following critical advance in the theory of vertex algebras:

Theorem [van Ekeren-Möller-Scheithauer 2015] If V is strongly regular and holomorphic, and $g \in Aut(V)$ is finite order, then there exists an abelian intertwining algebra structure on the direct sum of irreducible twisted modules

$$^{g}V := \bigoplus_{i=0}^{|g|-1} V(g^i)$$

Here, the notion of abelian intertwining algebra, introduced in [Dong-Lepowsky 1993], is a generalization of vertex operator algebra, where the multiplication operation is only commutative and associative after some adjustment with a braiding structure. The existence of abelian intertwining algebra structure was essential to the proof of the Hauptmodul property of $Z(g,h;\tau)$ in [Carnahan 2012], and the existence of $V^\natural_{\mathbb{Z}}$ in [Carnahan 2017b].

Another important corollary is the cyclic orbifold construction, which lets us build new holomorphic vertex algebras using finite order automorphisms of existing objects.

Corollary

Let V be a strongly regular and holomorphic vertex operator algebra, and $g \in Aut(V)$ finite order. Assume g is "anomaly-free" (i.e., eigenvalues of $L(0)$ on the twisted module $V(g)$ are in $\frac{1}{|g|}\mathbb{Z}$). Decompose $^{g}V := \bigoplus_{i=0}^{|g|-1} V(g^i)$ under the canonical g action to get a graded structure $\bigoplus_{i,j} V^{i,j}$, where $V = \bigoplus_j V^{0,j}$. Then $V/g := \bigoplus V^{i,0}$ is a strongly regular holomorphic vertex operator algebra, and there is a canonical automorphism g^* such that $V^{i,0}$ is the $e^{2\pi\sqrt{-1}i/|g|}$ eigenspace.

With this result, we get 51 constructions of $V^\natural$ from the Leech lattice vertex operator algebra V_Λ:

1. An order 2 orbifold [Frenkel-Lepowsky-Meurman 1988], the original construction.

2. An order 3 orbifold [Chen-Lam-Shimakura 2016]

3. Orders 5, 7, 13 [Abe-Lam-Yamada 2017]

4. 46 classes of composite order [Carnahan 2017a]

These constructions were conjectured in [Tuite 1993] as part of his orbifold correspondence between massless classes in the Conway group Co_0 and non-Fricke classes in $\mathbb{M}$.

For the construction of $V^\natural_{\mathbb{Z}}$, we need the prime order constructions: For $p \in \{2,3,5,7,13\}$, the pair $(V_\Lambda, p\mathrm{a})$ is orbifold dual to $(V^\natural, p\mathrm{B})$.

4 Cyclic orbifolds over small rings

The general idea behind the construction of $V^\natural_\mathbb{Z}$ is the construction of forms of $V^\natural$ over many cyclotomic S-integer rings, followed by a step where these forms are glued together. Thus, we need a theory of vertex algebras and cyclic orbifolds over these rings so we can construct suitable forms of $V^\natural$.

Definition: A vertex algebra over a commutative ring R is an R-module V, with an element $\mathbf{1} \in V$ and a multiplication map $V \otimes_R V \to V((z))$, written $u \otimes v \mapsto Y(u,z)v = \sum u_n v z^{-n-1}$, satisfying:

1. $Y(\mathbf{1}, z) = id_V z^0$ and $Y(a,z)\mathbf{1} \in a + zV[[z]]$.
2. For any $r, s, t \in \mathbb{Z}$, and any $u, v, w \in V$,

$$\sum_{i \geq 0} \binom{r}{i} (u_{t+i}v)_{r+s-i} w = \sum_{i \geq 0} (-1)^i \binom{t}{i} (u_{r+t-i}(v_{s+i}w) - (-1)^t v_{s+t-i}(u_{r+i}w))$$

When Borcherds introduced vertex algebras, he allowed coefficients in arbitrary commutative rings, and he gave the following example:

For any positive definite even unimodular lattice L there is a self-dual vertex algebra $(V_L)_\mathbb{Z}$ over $\mathbb{Z}$ (Borcherds 1986). It is a $\mathbb{Z}$-form of $Sym(t^{-1}(\mathbb{C} \otimes L)[t^{-1}]) \otimes \mathbb{C}[L]$ spanned by monomials of the form $s_{\alpha_1,n_1} \cdots s_{\alpha_k,n_k} e^\alpha$, where e^α is a basis element of $\mathbb{C}[L]$, α_i are chosen from a basis of L, and the operator $s_{\alpha,k}$ is the z^k-coefficient of $\exp(\sum_{n>0} \frac{\alpha(-n)}{n} z^n)$. Here, $Sym(t^{-1}(\mathbb{C} \otimes L)[t^{-1}])$ is a representation of the Heisenberg algebra, with generators $\alpha(n) = \alpha t^{-n} \in L[t, t^{-1}] \oplus \mathbb{C}K$.

We will need a more refined notion, where the underlying R-modules are direct sums of finite projective modules. The notion of vertex operator algebra over a field was introduced in [Frenkel-Lepowsky-Meurman 1988] as a variant of vertex algebras with similar finiteness properties, so we give one of perhaps many natural extensions of the definition to commutative rings:

Definition: A vertex operator algebra over R with half central charge c is a vertex algebra V over R equipped with a "conformal element" ω and a $\mathbb{Z}$-grading $V = \bigoplus V_n$, such that

1. If $u \in V_m$, $v \in V_n$, then $u_k v \in V_{m+n-k-1}$.
2. The coefficients of $Y(\omega, z) = \sum L_n z^{-n-2}$ satisfy Virasoro relations: $[L_m, L_n] = (m - n)L_{m+n} + c\binom{m+1}{3}\delta_{m+n,0} id_V$.
3. Each V_n is a finite rank projective R-module, and L_0 acts on V_n by $n \cdot id_{V_n}$.

We can also define abelian intertwining algebras over certain subrings of $\mathbb{C}$, as long as the subrings contain suitable denominators and enough roots of unity.

For the cyclic orbifold construction, we have the following key result

Lemma: Let $V = \bigoplus_{i,j \in \mathbb{Z}/N\mathbb{Z}} V^{i,j}$ be a self-dual abelian intertwining algebra over $\mathbb{C}$, where each $V^{i,j}$ is an irreducible $V^{0,0}$-module, and let $U = \bigoplus V^{0,j}$ and $W = \bigoplus V^{i,0}$. If R is a suitable subring of $\mathbb{C}$, and we are given self-dual R-forms U_R and W_R such that $U_R \cap V^{0,0} = W_R \cap V^{0,0}$, then they generate a self-dual R-form of V.

To apply this result to the construction of $V^\natural$, we use an intermediate orbifold method introduced in [Abe-Lam-Yamada 2017]:

Theorem: Let $P_0 = \{2, 3, 5, 7, 13\}$. If p, q are distinct in P_0, and $pq \notin \{65, 91\}$, then there is an automorphism $\bar{g}$ of the Leech lattice of order pq, such that no non-identity power of $\bar{g}$ has fixed points, and an order pq lift $g \in Aut(V_\Lambda)$. Then:

1. $V_\Lambda / g^p \cong V_\Lambda / g^q \cong V^\natural$
2. $V_\Lambda / g \cong V_\Lambda$.

In particular, there are 2 copies of $V^\natural$ inside the abelian intertwining algebra $\bigoplus_i V_\Lambda(g^i)$, which is generated by 2 copies of V_Λ.

Corollary: Let p, q be distinct elements of $P_0 = \{2, 3, 5, 7, 13\}$, such that $pq \notin \{65, 91\}$, and let $R_{pq} = \mathbb{Z}[1/pq, e^{\pi\sqrt{-1}/pq}]$. Then, there is a self-dual R_{pq}-form of the abelian intertwining algebra $\bigoplus_i V_\Lambda(g^i)$, and it contains 2 isomorphic self-dual R_{pq}-forms of $V^\natural$.

5 Monster symmetry

Before we can glue our forms of $V^\natural$, we need to show that they have full Monster symmetry. There are two reasons for this: First, the symmetry gives us more flexibility in finding isomorphisms between forms so that we can glue. Second, when we have control over symmetry, we can restrict how many isomorphism types can come from gluing.

Recall the Leech lattice Λ has $Co_0 = 2.Co_1$ symmetry. From this, the lattice vertex algebra has symmetry given by $AutV_\Lambda \cong (\mathbb{C}^\times)^{24}.Co_0$, a non-split extension.

Let $p \in P_0$, $\bar{g} \in Co_0$ fixed-point free, order p. Then any order p lift $g \in AutV_\Lambda$ has centralizer $(\mathbb{Z}/p\mathbb{Z})^{24/(p-1)}.C_{Co_0}(\bar{g})$. The same is true for suitably chosen automorphisms of the R_{pq}-form. We therefore obtain natural actions of large finite groups on abelian intertwining algebras, and on the forms of $V^\natural$. In particular, the self-dual R_{pq}-forms of $V^\natural$ naturally inherit an action of $G_p = p^{1+24/(p-1)}.(C_{Co_0}(\bar{g^q})/\bar{g^q})$ from an abelian intertwining algebra containing $V^\natural_{R_{pq}}$ and $(V_\Lambda)_{R_{pq}}$ (and similarly for G_q).

We therefore have R_{pq}-forms of $V^\natural$ with actions of the subgroups G_p and G_q of $\mathbb{M}$. By work of Wilson on maximal subgroups of $\mathbb{M}$ [Wilson 2017], we conclude that the only subgroup of $\mathbb{M}$ containing these groups is $\mathbb{M}$ itself. These forms therefore have monster symmetry.

6 Gluing forms over small rings

We now have a collection of forms of $V^\natural$ over rings R_{pq}, each with monster symmetry, and we wish to show that all of them arise from a form over $\mathbb{Z}$ by tensor product. In commutative algebra, this is known as a descent problem, and we may use the tools of Zariski and faithfully flat descent to prove this. For the data we have, it is easiest to phrase this as a gluing problem:

Definition: Given a diagram $R_1 \to R_3 \leftarrow R_2$ of commutative rings, a gluing datum for vertex operator algebras is a triple (V^1, V^2, f), where

1. V^1 is a vertex operator algebra over R_1,
2. V^2 is a vertex operator algebra over R_2, and
3. $f : V^1 \otimes_{R_1} R_3 \to V^2 \otimes_{R_2} R_3$ is an isomorphism of vertex operator algebras over R_3.

These form a category, where morphisms are pairs of maps satisfying a commutative square condition.

Proposition: Let $i_1 : R \to R_1$ and $i_2 : R \to R_2$ be maps of commutative rings, such that either

1. i_1 and i_2 form a Zariski open cover, or
2. i_1 and i_2 are faithfully flat.

Then, the category of gluing data for $R_1 \to R_1 \otimes_R R_2 \leftarrow R_2$ is equivalent to the category of vertex operator algebras over R.

The proof of this proposition easily reduces to a gluing problem for modules over commutative rings. I initially thought the gluing problem for modules would follow immediately from effectiveness of faithfully flat descent, but the proof turned out to be unexpectedly tricky.

To construct the gluing data, we first produce isomorphisms between certain fixed-point vertex operator subalgebras by comparing them to vertex operator subalgebras of V_Λ. We then extend them to isomorphisms of forms of $V^\natural$ using the action of elementary abelian subgroups of $\mathbb{M}$. This is the first place where monster symmetry is essential.

Proposition: Let $R_n = \mathbb{Z}[1/n, e^{\pi\sqrt{-1}/n}]$ and let $g \in p\mathrm{B}$. Recall $(V^\natural, p\mathrm{B})$ is orbifold dual to $(V_\Lambda, p\mathrm{a})$, and $V^g_{pq} \cong (V_\Lambda)^\sigma_{R_{pq}}$. Then $V^g_{pq} \otimes_{R_{pq}} R_{pqr} \cong (V_\Lambda)^\sigma_{R_{pqr}} \cong V^g_{pr} \otimes_{R_{pr}} R_{pqr}$.

By a theorem in [Wilson 1988], for each $p \in P_0$, there is an elementary subgroup $H_p \subset \mathbb{M}$ of order p^2, whose non-identity elements lie in conjugacy class pB. V_{pq} and V_{pr} are generated by g-fixed point subalgebras for g ranging over H_p, so $V_{pq} \otimes_{R_{pq}} R_{pqr} \cong V_{pr} \otimes_{R_{pr}} R_{pqr}$ by uniqueness of generated self-dual forms.

From our isomorphisms $V_{pq} \otimes_{R_{pq}} R_{pqr} \cong V_{pr} \otimes_{R_{pr}} R_{pqr}$, we may produce a self-dual $\mathbb{Z}$-form with $\mathbb{M}$-symmetry by repeated gluing. Uniqueness comes from the fact that the double coset space $\mathbb{M}\backslash\mathbb{M}/\mathbb{M}$ is a singleton. This is the second place where monster symmetry is essential. The main result is then:

Theorem: There is a unique self-dual $\mathbb{Z}$-form $V^\natural_\mathbb{Z}$ of $V^\natural$ such that $V^\natural_\mathbb{Z} \otimes R_{pq} \cong V_{pq}$. This form has $\mathbb{M}$-symmetry, and the natural inner product is positive definite.

Corollary (Modular moonshine conjecture): For any $g \in p$A, the vertex algebra $\hat{H}^0(g, V^\natural_\mathbb{Z})$ has an action of $C_\mathbb{M}(g)$ by automorphisms, and for any p-regular $h \in C_\mathbb{M}(g)$, the graded Brauer character is the q-expansion of the Monstrous Moonshine hauptmodul $T_{gh}(\tau)$.

Corollary: There exists a positive definite unimodular lattice of rank 196884 with a faithful monster action.

This lattice is just the weight 2 part of $V^\natural_\mathbb{Z}$, and in fact, it carries a commutative non-associative product structure, such that the automorphism group of the algebra is $\mathbb{M}$. This is essentially a self-dual $\mathbb{Z}$-form of the Griess algebra, with a slight modification at the identity element.

7 Further questions

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