DERIVED EQUIVALENCES AND GORENSTEIN PROJECTIVE DIMENSION

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ABSTRACT. In this note, we introduce the notion of complexes of finite Gorenstein projective dimension and show that a derived equivalence induces an equivalence between the full triangulated subcategories consisting of complexes of finite Gorenstein projective dimension provided that the equivalence satisfies a certain condition.

This work is based on a joint work with M. Hoshino.

Derived equivalences appear in various fields of current research in mathematics. For instance, in [3] Beilinson showed that there exists an algebra A such that the derived category of A is triangle equivalent to the derived category of coherent sheaves on $\mathbb{P}^n$, in [5] Broué conjectured abelian defect group conjecture and in [12] Kontsevich formulated mirror symmetry in terms of derived equivalences. So it is more and more important to study derived equivalences. It is natural to ask when two abelian categories are derived equivalent. A way to answer this question is to compare invariants under derived equivalences. It is well-known that for derived equivalent rings finiteness of selfinjective dimension is an invariant (see e.g. [11]). Finiteness of selfinjective dimension is closely related to Gorenstein projective dimension (see [9, 10]). So one can expect that there are some invariants associated with Gorenstein projective dimension.

In this note, we introduce the notion of complexes of finite Gorenstein projective dimension and show that a derived equivalence induces an equivalence between the full triangulated subcategories consisting of complexes of finite Gorenstein projective dimension provided that the equivalence satisfies a certain condition. Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough projectives. Denote by $\mathcal{P}_\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of projective objects and by $\mathcal{GP}_\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects. A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is said to have finite Gorenstein projective dimension if it is isomorphic to a bounded complex of Gorenstein projective objects in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ (see Definition 11). We denote by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ consisting of complexes of finite Gorenstein projective dimension. Let $F : \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ be a triangle equivalence. Assume that there exists an integer $a > 0$ such that

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(FP, Q[i]) = 0 = \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Q, FP[i])$$

for all $P \in \mathcal{P}_\mathcal{A}$ and $Q \in \mathcal{P}_\mathcal{B}$ unless $-a \leq i \leq a$. Then our main result states that F induces an equivalence between $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fGpd}}$ (see Theorem 18). Note that in case $\mathcal{A}$ and $\mathcal{B}$ are module categories then such an integer a always exists for any derived

The detailed version of this paper will be submitted for publication elsewhere.

equivalence F. As corollaries we have the following: the equivalence F induces a triangle equivalence between $\mathcal{GP}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}}$ and $\mathcal{GP}_{\mathcal{B}}/\mathcal{P}_{\mathcal{B}}$ (see Corollary 19); $\mathcal{GP}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$ if and only if $\mathcal{GP}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}$, $\widehat{\mathcal{GP}}_{\mathcal{A}} = \mathcal{GP}_{\mathcal{A}}$ if and only if $\widehat{\mathcal{GP}}_{\mathcal{B}} = \mathcal{GP}_{\mathcal{B}}$, and $\widehat{\mathcal{GP}}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$ if and only if $\widehat{\mathcal{GP}}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}$ where $\widehat{\mathcal{GP}}_{\mathcal{A}}$ stands for the full subcategory of $\mathcal{A}$ consisting of objects $X \in \mathcal{A}$ with $\mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{P}_{\mathcal{A}}) = 0$ for $i > 0$ (see Corollary 20); and letting A, B be rings, A and B are derived equivalent if and only if $\mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\text{Mod-}B)_{\mathrm{fGpd}}$ are equivalent as triangulated categories (see Corollary 22).

This work is supported by JSPS KAKENHI Grant Number 26887034.

1. Preliminaries

In this note, complexes are cochain complexes and objects are considered as complexes concentrated in degree zero. Let $\mathcal{A}$ be an abelian category with enough projectives. We denote by $\mathcal{P}_{\mathcal{A}}$ the full subcategory of $\mathcal{A}$ consisting of all projective objects in $\mathcal{A}$. We denote by $\mathcal{D}(\mathcal{A})$ the derived category of complexes over $\mathcal{A}$ and by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ consisting of complexes with bounded cohomology. Also, we denote by $\mathrm{Hom}^{\bullet}_{\mathcal{A}}(-,-)$ the associated single complex of the double hom complex.

For an additive category $\mathcal{X}$ we denote by $\mathcal{K}(\mathcal{X})$ the homotopy category of cochain complexes over $\mathcal{X}$ and by $\mathcal{K}^{+}(\mathcal{X})$ and $\mathcal{K}^{\mathrm{b}}(\mathcal{X})$ the full triangulated subcategories of $\mathcal{K}(\mathcal{X})$ consisting of bounded below and bounded complexes, respectively.

For a ring A we denote by Mod-A the category of right A-modules.

We refer to [4], [8] and [14] for basic results in the theory of derived categories.

Definition 1. *For a complex $X^{\bullet}$, we denote by $\mathrm{Z}^i(X^{\bullet})$ and $\mathrm{H}^i(X^{\bullet})$ the ith cycle and the ith cohomology of $X^{\bullet}$, respectively.*

Definition 2 ([8])**.** *A complex $X^{\bullet} \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is said to have finite projective dimension if $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^{\bullet}[-i], -)$ vanishes on $\mathcal{A}$ for $i \gg 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$ the épaisse subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ consisting of complexes of finite projective dimension.*

Note that the canonical functor $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ gives rise to equivalences of triangulated categories

$$\mathcal{K}^{\mathrm{b}}(\mathcal{P}_{\mathcal{A}}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}.$$

Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$.

Definition 3. *A complex $X^{\bullet} \in \mathcal{K}(\mathcal{A})$ is said to be $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})$-exact if $\mathrm{Hom}_{\mathcal{A}}(X^{\bullet}, C)$ is exact for all $C \in \mathcal{C}$.*

Definition 4. *An exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to \cdots$ in $\mathcal{A}$ is said to be a $\mathcal{C}$-coresolution of $M \in \mathcal{A}$ if $C^i \in \mathcal{C}$ for all i and the exact sequence is $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{C})$-exact.*

Definition 5 ([1, 7])**.** *An object $M \in \mathcal{A}$ is said to be Gorenstein projective if M admits a $\mathcal{P}$-coresolution. We denote by $\mathcal{GP}_{\mathcal{A}}$ the full subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects $M \in \mathcal{A}$.*

We refer to [6] for basic facts on Gorenstein projective dimension.

2. Gorenstein projective dimension

In this section, we study Gorenstein projective objects and introduce the notion of complexes of finite Gorenstein projective dimension.

Definition 6. *We denote by $\widehat{\mathcal{GP}}_{\mathcal{A}}$ the full subcategory of $\mathcal{A}$ consisting of objects $X \in \mathcal{A}$ with $\mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{P}_{\mathcal{A}}) = 0$ for $i > 0$.*

Definition 7. *Let $\widehat{\mathcal{GP}}_0 := \widehat{\mathcal{GP}}_{\mathcal{A}}$. For $n \geq 1$ we denote by $\widehat{\mathcal{GP}}_n$ the full subcategory of $\widehat{\mathcal{GP}}_{\mathcal{A}}$ consisting of objects X admitting right resolutions in $\mathcal{A}$ $0 \to X \to P^1 \to \cdots \to P^n \to Y \to 0$ with $Y \in \widehat{\mathcal{GP}}_{\mathcal{A}}$ and $P^i \in \mathcal{P}$ for $1 \leq i \leq n$.*

Proposition 8. *We have $\mathcal{GP}_{\mathcal{A}} = \bigcap_{n \geq 0} \widehat{\mathcal{GP}}_n$.*

Theorem 9. *Let $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{GP}_{\mathcal{A}})$ with $X^i = 0$ unless $0 \leq i \leq l$. Then there exists a quasi-isomorphism $X^\bullet \to P^\bullet$ with $P^\bullet \in \mathcal{K}^{+}(\mathcal{P}_{\mathcal{A}})$ such that $\mathrm{Z}^{l+1}(P^\bullet) \in \mathcal{GP}_{\mathcal{A}}$ and $\mathrm{H}^{-i}(\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, \mathcal{P}_{\mathcal{A}})) = 0$ for $i > l$.*

Proposition 10. *Let $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$. The followings are equivalent:*

1. *$X^\bullet \cong Y^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ for some $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{GP}_{\mathcal{A}})$.*
2. *There exists a distinguished triangle $X^\bullet \to Y^\bullet \to Z[l] \to$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ with $Y^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$, $Z \in \mathcal{GP}_{\mathcal{A}}$ and $l \in \mathbb{Z}$.*
3. *$X^\bullet \cong Z[l]$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$ with $Z \in \mathcal{GP}_{\mathcal{A}}$ and $l \in \mathbb{Z}$.*

Definition 11. *A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is said to have finite Gorenstein projective dimension if $X^\bullet$ satisfies the equivalent condition in Proposition 10. We denotes by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ the full subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ consisting of all complexes in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ having finite Gorenstein projective dimension.*

Theorem 12 (cf. [2] and [9, Proposition 3.5])**.** *The followings hold:*

1. *The embedding $\widehat{\mathcal{GP}}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ induces a fully faithful functor*
$$\widehat{\mathcal{GP}}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$$
2. *The embedding $\mathcal{GP}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ induces an equivalence*
$$\mathcal{GP}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$$

At the end of this section, using the quotient category, we characterize Gorenstein projective objects.

Theorem 13. *Let $X \in \widehat{\mathcal{GP}}_{\mathcal{A}}$. Then $X \in \mathcal{GP}_{\mathcal{A}}$ if and only if for each $i > 0$ there exists $Y_i \in \widehat{\mathcal{GP}}_{\mathcal{A}}$ such that $X \cong Y_i[-i]$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$.*

3. Derived equivalences

In this section, we deal with derived equivalences of abelian categories with enough projectives. Let $\mathcal{B}$ be an abelian category with enough projectives. Throughout this section we assume that there exists an equivalence of triangulated categories $F : \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ with an integer $a > 0$ such that

$$(*) \qquad \mathrm{Hom}_{D(\mathcal{B})}(FP, Q[i]) = 0 = \mathrm{Hom}_{D(\mathcal{B})}(Q, FP[i])$$

for all $P \in \mathcal{P}_{\mathcal{A}}$ and $Q \in \mathcal{P}_{\mathcal{B}}$ unless $-a \leq i \leq a$ and G stands for a quasi-inverse of F.

Lemma 14. $\mathrm{H}^i(FP) = 0$ *for all* $P \in \mathcal{P}_{\mathcal{A}}$ *unless* $-a \leq i \leq a$.

Remark 15. $\mathrm{H}^i(GQ) = 0$ *for all* $Q \in \mathcal{P}_{\mathcal{B}}$ *unless* $-a \leq i \leq a$.

Proposition 16. *The equivalence F induces an equivalence of triangulated categories between $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fpd}}$.*

Proof. See [13, Proposition 8.2]. □

Lemma 17. *For each $X \in \widehat{\mathcal{GP}}_{\mathcal{A}}$ there exists $X' \in \widehat{\mathcal{GP}}_{\mathcal{B}}$ such that $FX \cong X'[a]$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{B})/\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fpd}}$.*

Theorem 18. *Let $F : \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ be an equivalence of triangulated categories. If there exists $a > 0$ such that*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(FP, Q[i]) = 0 = \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Q, FP[i])$$

for all $P \in \mathcal{P}_{\mathcal{A}}$ and $Q \in \mathcal{P}_{\mathcal{B}}$ unless $-a \leq i \leq a$ then F induces an equivalence of triangulated categories between $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fGpd}}$.

Corollary 19. *The equivalence F induces an equivalence between $\mathcal{GP}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}}$ and $\mathcal{GP}_{\mathcal{B}}/\mathcal{P}_{\mathcal{B}}$.*

Corollary 20. *The following hold.*

(1) $\mathcal{GP}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$ *if and only if* $\mathcal{GP}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}$.
(2) $\widehat{\mathcal{GP}}_{\mathcal{A}} = \mathcal{GP}_{\mathcal{A}}$ *if and only if* $\widehat{\mathcal{GP}}_{\mathcal{B}} = \mathcal{GP}_{\mathcal{B}}$.
(3) $\widehat{\mathcal{GP}}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$ *if and only if* $\widehat{\mathcal{GP}}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}$.

Proposition 21. *Let $F' : \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fGpd}}$ be an equivalence of triangulated categories. Then F' induces an equivalence of triangulated categories $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fpd}}$ if both $\mathcal{A}$ and $\mathcal{B}$ satisfy the condition Ab4.*

Corollary 22. *Let A, B be rings. Then A and B are derived equivalent, i.e., $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}A)$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}B)$ are equivalent as triangulated categories if and only if $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}A)_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}B)_{\mathrm{fGpd}}$ are equivalent as triangulated categories.*

References

[1] M. Auslander and M. Bridger, *Stable module theory*, Mem. Amer. Math. Soc., **94**, Amer. Math. Soc., Providence, R.I., 1969.

[2] L. L. Avramov, Gorenstein dimension and complete resolutions. Conference on commutative algebra to the memory of professor Hideyuki Matsumura, Nagoya, Aug 26-28, 1996, Nagoya University: Nagoya, Japan, 1996, 28–31.

[3] A. A. Beilinson, Coherent sheaves on $\mathbf{P}^n$ and problems in linear algebra. (Russian) Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 68–69.

[4] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, Compositio Math. **86** (1993), no. 2, 209–234.

[5] M. Broué, Isométries parfaites, types de blocs, catégories dérivées (French), Astérisque No. 181–182 (1990), 61–92.

[6] L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Math, **1747**, Springer-Verlag, Berlin, 2000.

[7] E. E. Enochs and O. M. G. Jenda, Gorenstein injective and projective modules, Math. Z. **220** (1995), no. 4, 611–633.

[8] R. Hartshorne, *Residues and duality*, in: *Lecture Notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64*, Lecture Notes in Math., **20**, Springer, Berlin, 1966.
[9] M. Hoshino and H. Koga, Zaks' lemma for coherent rings, Algebr. Represent. Theory **16** (2013), no. 6, 1647–1660.
[10] M. Hoshino and H. Koga, Finiteness of selfinjective dimension for noetherian algebras, Comm. Algebra **41** (2013), no. 9, 3414–3428.
[11] Y. Kato, On derived equivalent coherent rings, Comm. Algebra **30** (2002), no. 9, 4437–4454.
[12] M. Kontsevich, Homological algebra of mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zurich, 1994), 120–139, Birkhauser, Basel, 1995.
[13] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.
[14] J. L. Verdier, Catégories dérivées, état 0, in: *Cohomologie étale*, 262–311, Lecture Notes in Math., **569**, Springer, Berlin, 1977.

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