VECTOR BUNDLES ON ALGEBRAIC SURFACES

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Contents

0. Introduction

In this survey, we shall explain some properties of moduli spaces of stable sheaves on smooth projective surfaces. We first explain classical results on the moduli spaces briefly. Then we explain recent development in detail. Thus we shall explain geometry associated to the derived category of coherent sheaves, which was started by Mukai and developed by Orlov, Bridgeland and other people. For other aspects of derived category of coherent sheaves, we recommend excellent articles [20], [38].

Due to lack of the author's ability and also enough time to write, this article is not written well. Moreover there will be many misunderstanding on the results in particular references. So if you are interested in these topic, it is better to check the detail as students do before seminar.

0.1. **Stability.**

Let X be a smooth projective surface over $\mathbb{C}$. For a coherent sheaf E on X, $v(E)$ denotes a topological invariant of E. Typical topological invariants are

(i) the Chern character $v(E) = \mathrm{ch}(E) \in H^{2*}(X, \mathbb{Q})$,

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2010 *Mathematics Subject Classification.* 14D20.

The author is supported by the Grant-in-aid for Scientific Research (No. 26287007), JSPS.

(ii) Mukai vector $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}_X} \in H^{2*}(X,\mathbb{Q})$, if K_X is numerically trivial.

(iii) the class in the numerical Grothendieck group

$$K(X)_{\mathrm{num}} = K(X)/\ker(\mathrm{ch}),$$

where $\mathrm{ch} : K(X) \to H^{2*}(X,\mathbb{Q})$ is the Chern character map or

(iv) $v(E) = (\mathrm{rk}\, E, c_1(E), \chi(E)) \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$.

Definition 0.1. Let H be an ample divisor on X and $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$. A coherent sheaf E is *β-twisted semi-stable* if E is torsion free and

$$\frac{\chi(F(-\beta+nH))}{\mathrm{rk}\, F} \le \frac{\chi(E(-\beta+nH))}{\mathrm{rk}\, E} \; (n \gg 0) \tag{0.1}$$

for all non-trivial subsheaf F of E. If $\beta = 0$, then β-semi-stability is nothing but the semi-stability of Gieseker.

Remark 0.2. Roughly speaking, torsion freeness means locally free except finitely many points of X, since X is smooth of dimension 2. Although we are mainly interested in locally free sheaves, in order to get a compact moduli space, we need to add torsion free sheaves in the boundary.

Theorem 0.3 (Gieseker [16], Matsuki-Wentworth [27])**.** *There is a coarse moduli scheme $M_H^\beta(v)$ of β-twisted semi-stable sheaves with topological invariant v. It is a projective scheme.*

If $\beta = 0$, then we denote the moduli space by $M_H(v)$. If H is general in $\mathrm{Amp}(X)$, then $M_H^\beta(v)$ is independent of the choice of β, and hence $M_H^\beta(v) = M_H(v)$. β-twisted semi-stability is a generalization of a more restrictive notion *slope stability*, which is defined by looking the coefficient of n in (0.1). The famous Kobayashi-Hitchin correspondence connects the algebro-geometric notion with the differential geometric notion:

(0.2) slope stable vector bundles $\Leftrightarrow$ irreducible Hermite-Einstein connections.

For smooth 4-manifolds, Donaldson constructed a very powerful invariant called *Donaldson invariant* by using the moduli of Hermite-Einstein connections. Then the Kobayashi-Hitchin correspondence says that the computations of Donaldson invariants of Kähler surfaces can be reduced to study $M_H(v)$. By this reason, the structure of $M_H(v)$ was extensively studied until a more useful invariant called *Seiberg-Witten invariant* appeared in 1995.

0.2. **Classical results.** We first pick up some general results on the structure of $M_H(v)$ which were obtained until 1995. There are many examples of smooth and irreducible moduli spaces if X is not of general type. However $M_H(v)$ is singular in general, and may be non-reduced and reducible. On the other hand, if $c_2 \gg 0$, then the singular locus of $M_H(v)$ is higher codimension, $M_H(v)$ is irreducible and locally complete intersection [34], [17]. In particular, $M_H(v)$ is a normal variety. If the geometric genus p_g is positive, then Jun Li [Li] and O'Grady proved that $M_H(v)$ is of general type under mild conditions. In order to prove the claim, a section of $H^0(X, K_X)$ is used to construct many canonical sections on a resolution of $M_H(v)$. Hence the proof does not work if $p_g = 0$. Indeed there is no general results of the moduli spaces for the case $p_g = 0$, as far as I know.

Problem 0.4. *Study $M_H(v)$ for a surface of general type with $p_g = 0$.*

For special surfaces such as $\mathbb{P}^2$, $K3$ surfaces, abelian surfaces, or elliptic surfaces, there are many works on the structure of $M_H(v)$. They were started in early 80's and continued after 1995. For these cases, there are many examples of smooth or normal moduli spaces, and we can expect beautiful properties of the moduli spaces. As examples, we pick (1) Drezet and Le Potier's result for $\mathbb{P}^2$ [15] and (2) Mukai's fundamental results for abelian and $K3$ surfaces [31],[32]. (1) In [15], they described the condition for the existence of stable sheaves. Although they only treated the case of a projective plane, their method[1] are powerful enough to be applied to other surfaces. We mention in section 2.6 recent developement on their work. For Mukai's works and related results, we shall explain them in next paragraph. For other results, I would like to recommend to see Huybrechts and Lehn's book and its references [21].

Assume that X is an abelian surface or a $K3$ surface. Then we usually use Mukai vector $v(E) \in H^{2*}(X,\mathbb{Z})_{\text{alg}} = \mathbb{Z} \oplus \text{NS}(X) \oplus \mathbb{Z}\varrho_X$, where ϱ_X is the fundamental class of X. We have an integral bilinear form $\langle \ , \ \rangle$ on $H^{2*}(X,\mathbb{Z})_{\text{alg}}$:

$$\langle (x_0, x_1, x_2), (x_0', x_1', x_2') \rangle = (x_1, x_1') - x_0 x_2 - x_2 x_0' \in \mathbb{Z}. \tag{0.3}$$

If $v = (r, \xi, a)$ is primitive, i.e., $\gcd(r, \xi, a) = 1$ and H is a general ample divisor, then Mukai showed $M_H(v)$ is a holomorphic symplectic manifold of dimension $\langle v^2 \rangle + 2$. Moreover the deformation class of $M_H(v)$ is determined by $\langle v^2 \rangle$ [42]. The main idea to determine the deformation class is to use the symmetry of X. For abelian surfaces and $K3$ surfaces, we have big symmetries to determine the deformation classes. More concretely, we use the following symmetry:

- (i) Deformation of the pair (X, H): For a polarized deformation of X, we have a monodromy representation on $H^2(X,\mathbb{Z})$. By this action, we can change ξ.
- (ii) Equivalences of the derived category (Fourier-Mukai transforms): By these actions, we can change r.

Let us explain (ii) more. A non-trivial example of equivalences $\Phi : \mathbf{D}(X) \to \mathbf{D}(X')$ of derived categories was first constructed by Mukai for abelian varieties [30]. Later similar equivalences were constructed for other varieties, e.g., a $K3$ surface [36], [10]. Here non-trivial implies the category of coherent sheaves are not preserved under Φ and the rank of objects change. Although these kind of equivalences are quite useful, technically it is not so easy to deal with: Indeed we need to find deformations (X', H') of (X, H) and Fourier-Mukai transforms $\Phi : \mathbf{D}(X') \to \mathbf{D}(X'')$ which induce isomorphisms $M_{H'}(v') \cong M_{H''}(v'')$. This can be achieved by solving some indeterminate equations on Mukai vectors. It was elementary but not interesting, since there was no systematic treatment, and which may make the topic of stable sheaves on surfaces unattractive.

0.3. **Recent development.**

A method to improve the situation is to appreciate the symmetries of the derived category and apply them to classical problems. For this purpose, we may need to consider complexes and their moduli spaces.

[1] The author imagines that they were influenced by the work of Atiyah and Bott [3], and Harder and Narasimhan to compute the Betti numbers of the moduli spaces on curves.

However derived categories of coherent sheaves themselves are too big to control, it is desirable to introduce good subcategories which is easier to control and capture rich geometric and algebraic structures. Bridgeland's theory of stability conditions is very suitable for this purpose. Before explaining Bridgeland's stability conditions and their applications, let us start with related results before Bridgeland's theory of stability conditions appeared.

First of all, we note that Beilinson constructed a derived equivalence

$$\mathbf{D}(\mathbb{P}^n) \cong \mathbf{D}(A), \tag{0.4}$$

where A is a non-commutative algebra and $\mathbf{D}(A)$ is the derived category of A-modules. In the theory of vector bundles, it is famous as Beilinson's spectral sequence, which was a very useful tool to study vector bundles on projective spaces. As we mention in section 2.6, instead of Beilinson's spectral sequence, the equivalence (0.4) itself plays important roles recently.

Similar equivalences are also found for some rational varieties by Russian mathematicians. Moreover many interesting results on the derived categories were proved. For the relation with this survey, Bondal and Orlov's contributions ([8],[9]) are noteworthy. They proved that the underlying manifold X can be recovered from $\mathbf{D}(X)$ if K_X or $-K_X$ is ample. They also studied the relation of 3-fold flops with the derived categories, and conjectured that 3 fold flops preserves the derived categories. Bridgeland [11] solved this conjecture by using the geometry of derived categories. Thus he constructed 3 fold flops as moduli spaces of some objects in $\mathbf{D}(X)$ and used the technique of Fourier-Mukai transforms which was developped mainly by Bridgeland. Since this work is the first well-known application of moduli problem of objects in $\mathbf{D}(X)$, we shall explain the construction in detail. Derived category of coherent sheaves also appears in Homological mirror symmetry, which also influenced recent development.

1. Construction of 3 fold flops

1.1. **Idea of the construcion.** Let us explain the idea of Bridgeland construction of 3 fold flops. Let $\varphi : X \to Z$ be a flopping contraction of a smooth 3-fold. Let X' be a moduli space of stable 0-dimensional sheaves E with $v(E) = v(\mathbb{C}_x)$ ($x \in X$). Then $E \cong \mathbb{C}_x$ for some $x \in X$, and we get $X' \cong X$. By perturbing the stability condition, we would like to get a different moduli space giving the flop of X. For simplicity, we first assume that φ is a contraction of $(-1,-1)$-curve, that is, a contraction of a smooth rational curve C with the normal bundle $\mathcal{O}_C(-1)^{\oplus 2}$. Since $\mathbb{C}_x$ is an irreducible object of $\mathrm{Coh}(X)$, there is no choice to modify the stability. For $x \in C$, we have an exact sequence in $\mathrm{Coh}(X)$.

$$0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C \to \mathbb{C}_x \to 0 \tag{1.1}$$

In the derived category, it can be understand as an exact triangle

$$0 \to \mathcal{O}_C \to \mathbb{C}_x \to \mathcal{O}_C(-1)[1] \to \mathcal{O}_C[1] \tag{1.2}$$

If there is an abelian category $\mathcal{C}$ such that $\mathcal{O}_C, \mathcal{O}_C(-1)[1] \in \mathcal{C}$, then we may have a stability condition such that $\mathbb{C}_x$ is not stable. Bridgeland introduced a modification of $\mathrm{Coh}(X)$ called a *category of perverse coherent sheaves*, and constructed the flop as a moduli space of stable perverse coherent sheaves. As the name indicates, this notion is influenced by Beilinson, Bernstein and Deligne's article [7]. In their paper, they introduced the notion of *t-structure*, which has been useful in representation theory and the theory of non-commutative algebra. By this work of Bridgeland, the importance of t-structure may be noticed by general algebraic geometer.

In the remaining of this section, let us explain a more detail of the construction of the flop. So it is possible to skip to the next section.

1.2. **Perverse coherent sheaves.** Let $\varphi : X \to Z$ be a flopping contraction of a smooth 3-fold X. Then we have

$$\mathbf{R}\varphi_*(\mathcal{O}_X) = \mathcal{O}_Z, \quad \dim \varphi^{-1}(z) \le 1 \text{ for any } z \in Z.$$

For simplicity, we assume that there is one singular point p of Z. Let $C_1, ..., C_n$ be the irreducible components of the exceptional locus $\varphi^{-1}(p)$. C_i are smooth rational curves. We set

$$\begin{aligned} T :=& \{E \in \operatorname{Coh}(X) \mid \operatorname{Hom}(E, \mathcal{O}_{C_i}(-1)) = 0\}, \\ F :=& \{E \in \operatorname{Coh}(X) \mid \ E \text{ is generated by } \mathcal{O}_{C_i}(a_i),\ a_i \le -1\}. \end{aligned} \tag{1.3}$$

Then (T, F) is a torsion pair of $\operatorname{Coh}(X)$, that is, there is a unique exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{1.4}$$

such that $E_1 \in T$ and $E_2 \in F$ for any $E \in \operatorname{Coh}(X)$. Let ${}^{-1}\operatorname{Per}(X/Z)$ be the tilting: For $E \in {}^{-1}\operatorname{Per}(X/Z)$,

$${}^{-1}\operatorname{Per}(X/Z) = \langle T, F[1] \rangle = \left\{ E \in \mathbf{D}(X) \,\middle|\, \begin{matrix} H^i(E) = 0, i \ne -1, 0, \\ H^{-1}(E) \in F, H^0(E) \in T \end{matrix} \right\}. \tag{1.5}$$

Thus we have an exact sequence in ${}^{-1}\operatorname{Per}(X/Z)$:

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0. \tag{1.6}$$

There is a locally free sheaf G such that $\mathbf{R}\varphi_*(G^\vee \otimes E) \in \operatorname{Coh}(Z)$ for $E \in {}^{-1}\operatorname{Per}(X/Z)$ and $\mathbf{R}\varphi_*(G^\vee \otimes E) = 0$ if and only if $E = 0$. G is called a *local projective generator* of ${}^{-1}\operatorname{Per}(X/Z)$ and $\mathbf{R}\varphi_*(G^\vee \otimes \bullet)$ induces a Morita equivalence [39]

$${}^{-1}\operatorname{Per}(X/Z) \to \operatorname{Coh}_{\mathcal{A}}(Z), \tag{1.7}$$

where $\mathcal{A} = \varphi_*(G^\vee \otimes G)$ is a sheaf of $\mathcal{O}_Z$-algebras. Then we define the dimension of $E \in {}^{-1}\operatorname{Per}(X/Z)$ by the dimension of $\mathbf{R}\varphi_*(G^\vee \otimes E)$. X' is constructed as a moduli space of $\mathcal{A}$-modules on Z.

We choose a divisor D on X. For 0-dimensional objects of ${}^{-1}\operatorname{Per}(X/Z)$, we have a notion of stability.

Definition 1.1. A 0-dimensional object E is D-semi-stable, if

$$\frac{\chi(G, E_1(-D))}{\chi(G, E_1)} \le \frac{\chi(G, E(-D))}{\chi(G, E)} \tag{1.8}$$

for all subobject E_1 of E.

Then there is a moduli space $M^D(v)$ of D-twisted semi-stable objects E of ${}^{-1}\operatorname{Per}(X/Z)$, where v is the Chern character of E [43, Prop. 1.5.4] (the assumption of $\dim X$ in [43, Prop. 1.5.4] is not needed).

We note that $\chi(G, E(-D)) = \chi(G, E) - (\mathrm{ch}_2(E), D)$. Hence (1.8) is equivalent to

$$\frac{-(\mathrm{ch}_2, D)}{\chi(G, E_1)} \leq \frac{-(\mathrm{ch}_2(E), D)}{\chi(G, E)}. \tag{1.9}$$

Let $\varrho_X \in H^3(X)$ be the fundamental class of X. Then $\varrho_X = v(\mathbb{C}_x)$. For $v(E) = \varrho_X$, (1.9) implies E is D-semi-stable iff $(\mathrm{ch}_2(F), D) \geq 0$ for all subobject F of E.

From now on, we assume that $-D$ is φ-ample. Then

Lemma 1.2. *$\mathbb{C}_x$ is not D-twisted semi-stable for $x \in \varphi^{-1}(p)$.*

Proof. We have an exact sequence in $\mathrm{Coh}(X)$

$$0 \to c \to \varphi^*(\varphi_*(\mathbb{C}_x)) \to \mathbb{C}_x \to 0, \tag{1.10}$$

where $\mathbf{R}\varphi_*(c) = 0$. We note that $\varphi^*(\varphi_*(\mathbb{C}_x)) = \mathcal{O}_{\varphi^{-1}(p)}$ and c is generated by $\mathcal{O}_{C_i}(-1)$. Since $\mathrm{ch}_2(\mathcal{O}_{\varphi^{-1}(p)}) = -\varphi^{-1}(p)$ and $\mathrm{ch}_2(\mathcal{O}_{C_i}) = -C_i$,

$$(\mathrm{ch}_2(\mathcal{O}_{\varphi^{-1}(p)}), D) > 0, \;\; (\mathrm{ch}_2(\mathcal{O}_{C_i}[1]), D) < 0 \tag{1.11}$$

for all i. Hence the claim holds. □

In the paper [11], Bridgeland constructed the moduli space of perverse coherent subsheaves I of $\mathcal{O}_X$ with $v(\mathcal{O}_X/I) = v(\mathbb{C}_x)$. As we shall see below, these two moduli spaces are the same.

Lemma 1.3. *Let E be a perverse coherent sheaf with $v(E) = v(\mathbb{C}_x)$. E is D-semi-stable if and only if there is a surjective morphism $f : \mathcal{O}_X \to E$.*

Proof. Assume that E is a quotient of $\mathcal{O}_X$. Let I be the kernel. Since

$$\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{C_i}(-1)[1]) = \mathrm{Hom}(I, \mathcal{O}_{C_i}(-1)) = 0,$$

E satisfies $\mathrm{Hom}(E, \mathcal{O}_{C_i}(-1)[1]) = 0$. Hence if $v(E) = v(\mathbb{C}_x)$, then it is D-twisted stable.

Conversely for a D-stable object E with $v(E) = v(\mathbb{C}_x)$, $\chi(E) = 1$ implies there is a morphism $f : \mathcal{O}_X \to E$. Since $\mathcal{O}_{C_i}[1]$ and $\mathcal{O}_{\varphi^{-1}(p)}$ are the irreducible objects supported on $\varphi^{-1}(p)$, we have

$$\begin{aligned} \operatorname{im} f &= \sum_i n_i \mathcal{O}_{C_i}[1] + n\mathcal{O}_{\varphi^{-1}(p)}, \;\; (n_i, n \geq 0) \\ \operatorname{coker} f &= \sum_i m_i \mathcal{O}_{C_i}[1] + m\mathcal{O}_{\varphi^{-1}(p)}, \;\; (m_i, m \geq 0). \end{aligned} \tag{1.12}$$

Since $1 = \chi(E) = \chi(\operatorname{im} f) + \chi(\operatorname{coker} f)$, we get $(n, m) = (1, 0)$ or $(0, 1)$. Since $\mathrm{Hom}(\operatorname{im} f, \mathcal{O}_{C_i}(-1)[1]) = 0$ for all i, $(n, m) = (1, 0)$. By the D-stability of E, $\operatorname{coker} f$ must be zero. Thus f is surjective. □

We set $X' := M^D(\varrho_X)$.

Remark 1.4. There is a universal family $\mathcal{E}$ on $X' \times X$. Since $L := p_{X'*}(\mathcal{E})$ is a line bundle on X', replacing $\mathcal{E}$ by $\mathcal{E} \otimes p_{X'}^*(L^\vee)$, we have a surjective morphism $\mathcal{O}_{X' \times X} \to \mathcal{E}$ in ${}^{-1}\mathrm{Per}(X' \times X/X' \times Z)$. Thus $\mathcal{E}$ is a quotient of $\mathcal{O}_{X' \times X}$.

Since $\pi_*(E)$ $(E \in X')$ is a structure sheaf of a point of Z, we have the following diagram:

$$\begin{array}{ccccc} X' & & & & X \\ & \searrow & & \swarrow & \\ & & Z & & \end{array} \tag{1.13}$$

Then Bridgeland proved that the universal family $\mathcal{E}$ induces an equivalence $\mathbf{D}(X) \cong \mathbf{D}(X')$ and also proved that $X' \to Z$ is the flop of $X \to Z$.

2. Bridgeland's stability condition

2.1. **Definition of stability condition.** Bridgeland introduced a notion of stability for objects of $\mathbf{D}(X)$. Moreover he showed that the space $\mathrm{Stab}(X)$ of stability conditions has a structure of a complex manifold. In this section, we shall briefly explain stability condition ([12], [13]). We start with the definition.

Definition 2.1. A *stability condition* $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$ on $\mathbf{D}(X)$ consists of a group homomorphism $Z_\sigma : K(X) \to \mathbb{C}$ and full additive subcategories $\mathcal{P}_\sigma(\phi) \subset \mathbf{D}(X)$ for all $\phi \in \mathbb{R}$ satisfying the following conditions:

(i) For $E \in \mathcal{P}_\sigma(\phi) \setminus \{0\}$, we have

$$Z_\sigma(E) = m(E) e^{\pi\sqrt{-1}\phi} \tag{2.1}$$

with some $m(E) \in \mathbb{R}_{>0}$.

(ii) $\mathcal{P}_\sigma(\phi + 1) = \mathcal{P}_\sigma(\phi)[1]$ for all $\phi \in \mathbb{R}$.

(iii) If $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}_\sigma(\phi_i)$ $(i = 1, 2)$, then $\mathrm{Hom}_{\mathbf{D}(X)}(E_1, E_2) = 0$.

(iv) For any $E \in \mathbf{D}(X) \setminus \{0\}$. we have the following collection of triangles

$$\begin{array}{ccccccccccc} 0 = E_0 & \longrightarrow & & E_1 & \longrightarrow & & E_2 & \longrightarrow \cdots & E_{n-1} & \longrightarrow & & E_n = E \\ & {\scriptstyle [1]}\nwarrow & \swarrow & & {\scriptstyle [1]}\nwarrow & \swarrow & & & & {\scriptstyle [1]}\nwarrow & \swarrow & \\ & & A_1 & & & A_2 & & & & & A_n & \end{array}$$

such that $A_i \in \mathcal{P}_\sigma(\phi_i)$ with $\phi_1 > \phi_2 > \cdots > \phi_n$.

A non-zero object $E \in \mathcal{P}_\sigma(\phi)$ is called *σ-semi-stable* of phase ϕ.

Definition 2.2. $\tilde{M}_\sigma(v)$ denotes the moduli space of σ-semi-stable objects E with $v(E) = v$ in a suitable sense (e.g. a scheme, an algebraic space or a stack). Since the phase is not determined by v, shift functor [2] acts on $\tilde{M}_\sigma(v)$. We denote the quotient by $M_\sigma(v)$.

We explain another formulation of the stability condition, which resemble Gieseker semi-stability: For a stability condition σ, let $\mathcal{A}_\sigma := \mathcal{P}_\sigma(0, 1]$ denotes the extension closed full subcategory of $\mathbf{D}(X)$ generated by $E \in \mathcal{P}_\sigma(\phi)$ with $\phi \in (0, 1]$. Then $\mathcal{A}_\sigma$ is an abelian category and the phase $\phi(E) \in (0, 1]$ is defined for a non-zero object $E \in \mathcal{A}_\sigma$ by (2.1). Moreover $E \in \mathcal{A}_\sigma$ is σ-semi-stable iff $\phi(F) \leq \phi(E)$ for any subobject F of E in $\mathcal{A}_\sigma$. Conversely for an abelian subcategory $\mathcal{A}$ (which is the heart of a t-structure) and a group homomorphism $Z : K(X) \to \mathbb{C}$ such that

$$Z(\mathcal{A} \setminus \{0\}) \subset \{\mathbb{R}_{>0} e^{\pi\sqrt{-1}\phi} \mid \phi \in (0, 1]\} = \mathbb{H} \cup \mathbb{R}_{<0}$$

and satisfying some conditions like the Harder-Narasimhan property, we have a stability condition $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$ with $Z_\sigma = Z$ and $\mathcal{A} = \mathcal{P}_\sigma(0,1]$. Therefore we have one to one correspondence:

$$\sigma = (Z_\sigma, \mathcal{P}_\sigma) \longleftrightarrow (Z_\sigma, \mathcal{A}_\sigma). \tag{2.2}$$

As in the β-twisted stability in Definition 0.1, the phase ϕ plays the role of a slope function on the abelian category $\mathcal{A}_\sigma$. Thus we have the correspondence

$$(\mathcal{A}_\sigma, \phi(\bullet)) \longleftrightarrow (\mathrm{Coh}(X), \tfrac{\chi(\bullet(-\beta+nH))}{\mathrm{rk}(\bullet)}).$$

Remark 2.3. Gieseker semi-stability is not a stability condition in the sense of Definition 0.1 if $\dim X \geq 2$.

The space $\mathrm{Stab}(X)$ of stability conditions has a structure of complex manifold. For a stability condition σ, we have $\Pi(\sigma) \in H^*(X,\mathbb{C})_{\mathrm{alg}}$ such that $Z_\sigma(\bullet) = \langle \Pi(\sigma), \bullet \rangle$. Then $\Pi : \mathrm{Stab}(X) \to H^*(X,\mathbb{C})_{\mathrm{alg}}$ is a covering map over the image. We require that $\mathrm{im}\,\Pi$ is an open subset from now on.[2]

Proposition 2.4. *We fix a Mukai vector v. There is wall/chamber structure on* $\mathrm{Stab}(X)$ *and $M_\sigma(v)$ is constant on each chamber.*

2.2. **Group actions on** $\mathrm{Stab}(X)$**.** On the space of stability conditions, there is a natural right action of the universal cover $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ of $\mathrm{GL}^+(2,\mathbb{R})$. By an identification $\mathbb{C} = \mathbb{R} + \mathbb{R}\sqrt{-1}$, $g \in \mathrm{GL}^+(2,\mathbb{R})$ acts on $Z_\sigma : K(X) \to \mathbb{C}$ by the composition $g^{-1} \circ Z_\sigma$. In order to get an action on the phase function ϕ, we need to take the universal cover of $\mathrm{GL}^+(2,\mathbb{R})$. Here we only explain the action of a subgroup $\mathbb{C} \subset \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$:

$$\begin{array}{ccc} \mathbb{C} & \subset & \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathbb{C}^* & \subset & \mathrm{GL}^+(2,\mathbb{R}). \end{array} \tag{2.3}$$

For $\lambda \in \mathbb{C}$,

$$\lambda(\sigma) = (e^{-\pi\sqrt{-1}\lambda} Z_\sigma, \mathcal{P}'), \;\; \mathcal{P}'(\phi) = \mathcal{P}_\sigma(\phi + \mathrm{Re}(\lambda)). \tag{2.4}$$

Remark 2.5. The second formulation of the stability condition may be familiar to whom working on vector bundles. However the description of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is not easier for this formulation. Indeed it seems there is no simple relation of abelian categories $\mathcal{P}'(0,1]$ and $\mathcal{P}_\sigma(0,1]$ unless $\mathrm{Re}(\lambda) \in \mathbb{Z}$, where $\lambda \in \mathbb{C}$.

There is also an action of $\mathrm{Aut}(\mathbf{D}(X))$. For an equivalence $\Phi : \mathbf{D}(X) \to \mathbf{D}(X')$ and a stability condition $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$ on X, we define a stability condition $\Phi(\sigma)$ on X' by

$$\Phi(\sigma) := (Z_\sigma \circ \Phi^{-1}, \mathcal{P}_{\Phi(\sigma)}), \;\; \mathcal{P}_{\Phi(\sigma)}(\phi) = \Phi(\mathcal{P}_\sigma(\phi)). \tag{2.5}$$

By this property (and the first formulation of stability condition), Φ induces an isomorphism

$$\Phi : M_\sigma(v) \to M_{\Phi(\sigma)}(\Phi(v)). \tag{2.6}$$

[2]Thus we require the support property.

Assume that X is a $K3$ surface or an abelian surface, so that there are enough Fourier-Mukai transforms. The stability conditions of Matsuki-Wentworth (classical stability) are parameterized by

$$(\beta, H) \in \mathrm{NS}(X)_{\mathbb{Q}} \times \mathrm{Amp}(X)_{\mathbb{Q}}. \tag{2.7}$$

We would like to extend the notion of stability which behaves well in the derived category. However we don't want to generalize the notion of stability blindly, since it makes hard to analyse. In order to minimize our consideration, we would like to impose the following conditions for our parameter space $\mathcal{S}$ of stability conditions:

(a) For each Mukai vector v, there is a wall/chamber structure on $\mathcal{S}$ and the stability is constant on each chamber.
(b) For a general (β, H), there is $\sigma \in \mathrm{Stab}(X)$ such that $M_H^\beta(v) = M_\sigma(v)$.
(c) Stability is preserved under any Fourier-Mukai transform

$$\Phi : \mathbf{D}(X) \to \mathbf{D}(X').$$

Thus for $\sigma \in \mathcal{S}$, there is $\Phi(\sigma) \in \mathcal{S}$ such that $E \in \mathbf{D}(X)$ is σ-semi-stable if and only if $\Phi(E)$ is $\Phi(\sigma)$-semi-stable.
(d) For each chamber $\mathcal{C}$, there is a Fourier-Mukai transform Φ which induces an isomorphism $M_\sigma(v) \to M_{H'}^{\beta'}(w)$, where $\sigma \in \mathcal{C}$ and $M_{H'}^{\beta'}(w)$ is a moduli space of β'-twisted semi-stable sheaves.

Theorem 2.6. *Bridgeland stability conditions satisfy (a)–(d).*

For (a), (b), see [13]. (c) is (2.6). (d) is in [5] and [26].

Remark 2.7. For other surfaces, we cannot expect property (d) in general.

2.3. **Examples of stability conditions.** Let us explain examples of stability conditions in [13]. Let X be a $K3$ surface or an abelian surface. Assume that $(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$. For $E \in K(X)$, we set

$$Z_{(\beta,\omega)}(E) := \langle e^{\beta + \sqrt{-1}\omega}, v(E) \rangle.$$

Let $\mathcal{A}_{(\beta,\omega)}$ be the tilt of $\mathrm{Coh}(X)$ with respect to the torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ defined by

(i) $\mathcal{T}_{(\beta,\omega)}$ is generated by β-twisted stable sheaves with $Z_{(\beta,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$.
(ii) $\mathcal{F}_{(\beta,\omega)}$ is generated by β-twisted stable sheaves with $-Z_{(\beta,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$.

$\mathcal{A}_{(\beta,\omega)}$ is the abelian category in [13] and it depends only on β and the ray $\mathbb{Q}_{>0}\omega$.

Then the pair $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$ satisfies the requirement of stability conditions on $\mathbf{D}(X)$[3] [13]. By the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $\mathrm{Aut}(\mathbf{D}(X))$, a general stability condition is represented by $\sigma_{(\beta,\omega)}$. Moreover if X is an abelian surface, then

$$\mathrm{Stab}(X)/\widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \cong \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}. \tag{2.8}$$

Every object $E \in \mathcal{A}_{(\beta,\omega)}$ fits in an exact sequence in $\mathcal{A}_{(\beta,\omega)}$:

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0 \tag{2.9}$$

[3] We need one more condition if X is a $K3$ surface.

where $H^{-1}(E) \in \mathcal{F}_{(\beta,\omega)}$ and $H^0(E) \in \mathcal{T}_{(\beta,\omega)}$. Thus E is a two-term complex. If X is a $K3$ surface, more complicated complexes appear by applying autoequivalences. On the other hand for an abelian surface, (2.8) implies only 2-terms complexes appear.

Definition 2.8. For a Mukai vector v, $\mathcal{M}_{(\beta,\omega)}(v)$ denotes the moduli stack of $\sigma_{(\beta,\omega)}$-semi-stable objects E of $\mathcal{A}_{(\beta,\omega)}$ with $v(E) = v$. $M_{(\beta,\omega)}(v)$ denotes the moduli scheme of the S-equivalence classes of $\sigma_{(\beta,\omega)}$-semi-stable objects E of $\mathcal{A}_{(\beta,\omega)}$ with $v(E) = v$, if it exists.

As we claimed, there is a region called large volume limit, where Bridgeland stability coincides with the classical stability.

Proposition 2.9 (Large volume limit)**.** *For a Mukai vector $v = (r, \xi, a)$ with $r \geq 0$, assume that $(\omega^2) \gg 0$ (depending on v). If $(\xi - r\beta, \omega) > 0$, then $M_{(\beta,\omega)}(v) = M_\omega^\beta(v)$.*

2.4. **Birational geometry of $M_H(v)$.** In this subsection, we shall explain the birational geometry of $M_H(v)$ in [6], [26], [44]. These results are the significant application of Bridgeland stability conditions to the structure of the moduli spaces. We start with some definitions.

Definition 2.10. Let $f : M \to S$ be a projective morphism.

(1) $D \in \operatorname{Pic}(M)$ is f-movable, if

$$\operatorname{codim} \operatorname{coker}(f^* f_*(\mathcal{O}_M(D)) \to \mathcal{O}_M(D)) \geq 2.$$

(2) A relative movable cone $\operatorname{Mov}(M/S)$ is a cone generated by f-movable divisors.

We shall consider relative cones over the albanese map.

Definition 2.11. (1) Two albanese maps $a' : M' \to \operatorname{Alb}(M')$ and $a : M \to \operatorname{Alb}(M)$ are equivalent if there is a birational map $f : M' \cdots \to M$ and an isomorphism $g : \operatorname{Alb}(M') \to \operatorname{Alb}(M)$ with a commutative diagram

$$\begin{array}{ccc} M' & \xrightarrow{f} & M \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ \operatorname{Alb}(M') & \xrightarrow{g} & \operatorname{Alb}(M) \end{array} \tag{2.10}$$

(2) For a normal variety M, $\operatorname{Amp}_{\mathrm{rel}}(M)$ denotes the relative ample cone of the albanese map $a : M \to \operatorname{Alb}(M)$. We set $\operatorname{Mov}_{\mathrm{rel}}(M) := \operatorname{Mov}(M/\operatorname{Alb}(M))$. $\operatorname{Mov}_{\mathrm{rel}}(M)_0$ denotes the interior of $\operatorname{Mov}_{\mathrm{rel}}(M)$.

For the moduli space $M_H(v)$, all the (relative) minimal models are the moduli spaces of Bridgeland's stable objects. Thus by extending our consideration from sheaves to complexes, we get a desirable description of birational properties.

We take $\sigma \in \mathcal{C}$. Then we have a map $\xi_\sigma : \operatorname{Stab}(X) \to P^+(M_\sigma(v))$. Up to the action of $\mathbb{R}_{>0}$, ξ_σ is surjective. Then $\xi_\sigma(\mathcal{C}) = \operatorname{Amp}_{\mathrm{rel}}(M_\sigma(v))$. So we set $\mathcal{C}_{\mathrm{Amp}} := \mathcal{C}$. There is another chamber decomposition of $\operatorname{Stab}(X)$ such that each chamber $\mathcal{C}_{\mathrm{Mov}}$ containing σ corresponds to the movable cone via ξ_σ.

$$
\begin{array}{ccc}
\mathrm{Stab}(X) & \overset{\xi_\sigma}{\longrightarrow} & P^+(M_\sigma(v)) \\
\cup & & \cup \\
\mathcal{C}_{\mathrm{Mov}} & \longrightarrow & \mathrm{Mov}_{\mathrm{rel}}(M_\sigma(v))_0 \\
\cup & & \cup \\
\mathcal{C}_{\mathrm{Amp}} & \longrightarrow & \mathrm{Amp}_{\mathrm{rel}}(M_\sigma(v))
\end{array} \tag{2.11}
$$

We also have a decomposition.

$$
\mathrm{Mov}_{\mathrm{rel}}(M_\sigma(v)) = \bigcup_{M' \cdots \to M_\sigma(v)} \overline{\mathrm{Amp}_{\mathrm{rel}}(M')}. \tag{2.12}
$$

By the characterization of walls for nef cones of each birational models, we get the following result.

Corollary 2.12 ([24]). *Kawamata and Morrison's Nef cone conjecture holds: There is a rational finite polyhedral fundamental domain for the action of* $\mathrm{Aut}(K_\sigma(v))$ *on* $\mathrm{Nef}(K_\sigma(v))$*, where* $K_\sigma(v)$ *is a fiber of the albanese map.*

2.5. Abelian surface with $\mathrm{NS}(X) = \mathbb{Z}H$ and $(H^2) = 2$.

2.5.1. *Stability condition.* In order to give an easy example of $\mathrm{Stab}(X)$, we assume that X is a principally polarized abelian surface with $\mathrm{NS}(X) = \mathbb{Z}H$, i.e., $(H^2) = 2$. In this case, we have an identification

$$
\begin{array}{ccc}
\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}} & \cong & \mathbb{H} \\
(s,t) & \leftrightarrow & s + t\sqrt{-1}
\end{array}
$$

Then the action of the autoequivalence group $\mathrm{Aut}(\mathbf{D}(X))$ on $\mathbb{H}$ factors through the natural action of $\mathrm{SL}(2,\mathbb{Z})$:

$$
\begin{array}{ccc}
\mathrm{Aut}(\mathbf{D}(X)) & \longrightarrow & \mathrm{Aut}(\mathbb{H}) \\
& \searrow \quad \nearrow & \\
& \mathrm{SL}(2,\mathbb{Z}) &
\end{array}
$$

So we have

$$
\mathrm{SL}(2,\mathbb{Z}) \backslash \mathrm{Stab}(X) / \widetilde{\mathrm{GL}}^+(2,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z}) \backslash \mathbb{H} \cong \mathbb{C}.
$$

By adding a cusp[4] ∞, we get a compactification.

$$
\mathrm{SL}(2,\mathbb{Z}) \backslash (\mathbb{H} \cup \mathbb{P}^1_{\mathbb{Q}}) \cong \mathbb{P}^1.
$$

Remark 2.13. If $n = (H^2)/2 > 1$, we still have $\mathrm{Stab}(X)^*/\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \cong \mathbb{H}$, and $\mathrm{Aut}(\mathbf{D}(X))$ acts as the action of $\Gamma_0(n) \subset \mathrm{SL}(2,\mathbb{Z})$.

The large volume limit (Proposition 2.9) corresponds to a neighborhood of $\infty \in \mathbb{P}^1_{\mathbb{C}}$. $M_z(r, dH, a) = M_H(r, dH, a)$ for $|z| \gg 0$ and $z < d/r$ ($M_z(r, dH, a) = \{E^\vee \mid E \in M_H(r, -dH, a)\}$ for $|z| \gg 0$ and $z > d/r$). Each wall is a semi-circle C in $\mathbb{H} \cup \{\infty\}$. If $\infty \in C$, then C is a half line ($rz - d = 0$). Walls do not intersect each other.

[4] Cusps are related to Foourier-Mukai transforms [23]

We illustrate walls for $v = (1, 0, -3)$ (see Figure 1). C_0 is a wall passing ∞. By the action of linear fractional transformation[5]

$$f(z) = \frac{2z - 3}{-z + 2},$$

C_0 is transformed to C_1 and C_1 is transformed a semi-circle C_2 which is inside of C_1. Applying $f^n(z)$, we have infinitely many walls $C_n = f^n(C_0)$ ($n \in \mathbb{Z}$). We also have infinitely many walls $W_n = f^n(W_0)$. C_n ($n > 0$) and W_n ($n \geq 0$) are semi-circles in the left hand side of C_0. Let $\mathcal{C}_0^-$ be the chamber bounded by C_0 and C_1, and $\mathcal{C}_0^+$ the chamber bounded by C_0 and C_{-1}. We set $\mathcal{C}_n^\pm := f^n(\mathcal{C}_0^\pm)$. These are the walls and chambers for $v = (1, 0, -3)$. The walls and chambers are symmetric with respect to t-axis. Each bounded chamber $\mathcal{C}_n^\pm$ ($n \neq 0$) is annulus. There are two unbounded chambers $\mathcal{C}_0^\pm$. The chamber $\mathcal{C}_0^-$ parameterizes $M_H(1, 0, -3)$ and the chamber $\mathcal{C}_0^+$ parameterizes the dual. For $\frac{q}{p} \in \overline{\mathcal{C}_n^\pm}$, by a Fourier-Mukai transform Φ inducing an isomorphism $f(z)$ of $\mathbb{P}^1$ with $f(q/p) = \infty$, $\mathcal{C}_n^\pm$ is transformed to an unbounded chamber, which gives the claim (d). In this example, the movable chamber is a single chamber. In particular, $\operatorname{Mov}_{\mathrm{rel}}(M_\sigma(v))_0 = \operatorname{Amp}_{\mathrm{rel}}(M_\sigma(v)) = \xi_\sigma(\mathcal{C}_n^\pm)$, where $\sigma \in \mathcal{C}_n^\pm$.

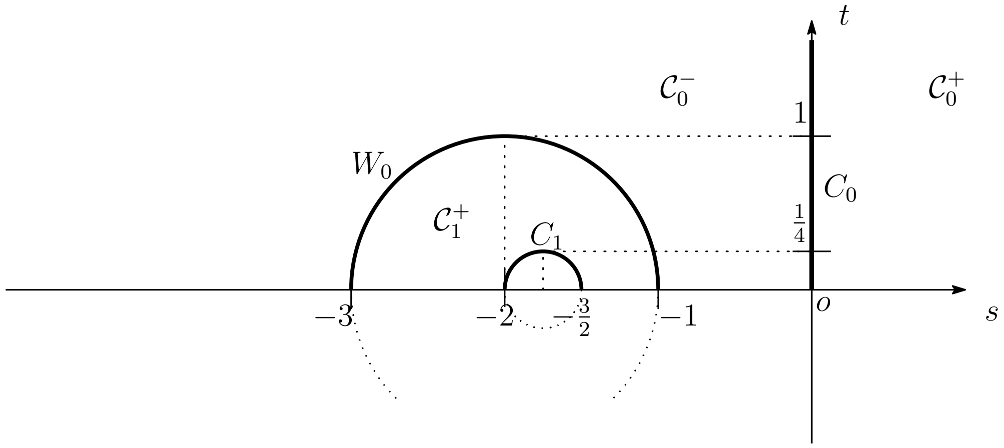

FIGURE 1. Walls for $v = (1, 0, -3)$ between C_0 and C_1.

2.5.2. *Relation with binary quadratic form.* Let Q be the space of binary quadratic forms:

$$Q := \{rx^2 + 2dxy + ay^2 \mid r, d, a \in \mathbb{Z}\}. \tag{2.13}$$

If X is a principally polarized abelian surface with $\operatorname{NS}(X) = \mathbb{Z}H$, then the Mukai lattice $H^*(X, \mathbb{Z})_{\mathrm{alg}}$ is isomorphic to Q via

$$\begin{array}{ccc} H^*(X, \mathbb{Z})_{\mathrm{alg}} & \to & Q \\ (r, dH, a) & \mapsto & rx^2 + 2dxy + ay^2. \end{array} \tag{2.14}$$

Then for $v = (r, dH, a)$, $\langle v^2 \rangle/2 = d^2 - ra$ is the discriminant of $rx^2 + 2dxy + ay^2$. Under this identification, the cohomological action of the Fourier-Mukai transforms factors through the natural action of $\mathrm{SL}(2, \mathbb{Z})$ on Q.

[5] $f(z)$ is the generator of the action of $\mathbf{D}(X)$ on $\mathbb{H}$ preserving $\pm(1, 0, -3)$. It is related to the fundamental unit of $\mathbb{Z}[\sqrt{3}]$.

Theorem 2.14 ([40],[25],[44]). *Let (X, H) be a principally polarized abelian surface with* $\mathrm{NS}(X) = \mathbb{Z}H$.

(1) *The birational equivalence class of $M_H(v)$ depends on the equivalence class of the associated quadratic form. In particular,*

$$\#\{M_H(v) \mid \gcd(r,d,a) = 1,\ \langle v^2 \rangle/2 = D\}/(\textit{birational equiv.}) \leq h_D, \tag{2.15}$$

where h_D is the class number of quadratic forms of discriminant D.

(2) *There is a birational map*

$$M_H(v) \cdots \to X \times \mathrm{Hilb}_X^{\langle v^2 \rangle/2}$$

if and only if

$$rx^2 + 2dxy + ay^2 = \pm 1$$

has a solution.

Remark 2.15. Finally we would like to remark that almost all results on $M_H(v)$ in this subsection are found or conjectured by Mukai around 1980 ([28],[29]). It is really surprizing to the author that these discoveries were done without Bridgeland stability condition.

2.6. **Related results and some problems.** The example of stability condition $\sigma_{(\beta,\omega)}$ in section 2.3 is generalized to arbitrary surface by Arcara and Bertram [1]. If $X = \mathbb{P}^2$, then Arcara, Bertram, Coskun and Huizenga [2] studied moduli spaces $M_{(\beta,\omega)}(v)$ of stable objects. By the equivalence (0.4), we can use the theory of quiver in order to study Bridgeland stability condition. In particular, we can use King's result to construct the moduli space $M_{(\beta,\omega)}(v)$ as a projective scheme. Moreover they showed that $M_{(\beta,\omega)}(v)$ is a log Fano variety, which implies $M_{(\beta,\omega)}(v)$ is a Mori dream space. In particular, the movable cone is rational finite polyhedral cone. Moreover for a non-commutative $\mathbb{P}^2$, Li and Zhao [22] studied moduli spaces of stable objects, and showed that they are Fano varieties. See also [14] for recent development.

For the stability condition on Enriques surface, Nuer [33] studied moduli of stable objects.

Finally we list the following problems.

Problem 2.16. (1) *Construction of moduli space.*

(2) *Smoothness of the moduli spaces. In birational geometry, smooth varieties do not form a good category. So it may not be so important.*

As we explained, Bridgeland stability condition is related to birational geometry of moduli spaces. So it is desirable to have projective moduli spaces. Note that Inaba [18] constructed moduli of simple complexes $\mathcal{M}(v)^{\mathrm{spl}}$. Let $\mathcal{M}_\sigma(v)$ be the subset consisting of σ-stable objects. On the other hand, Bayer and Macri [5] constructed nef line bundle $\mathcal{L}_\sigma$ on $\mathcal{M}_\sigma(v)$. For all known cases, $\mathcal{L}_\sigma$ is ample. For example, if X is a $K3$ surface or an abelian surface, $\mathcal{L}_\sigma$ is nothing but ξ_σ in (2.11). It is interesting to show the ampleness of $\mathcal{L}_\sigma$ for general cases.

References

[1] Arcara, D., Bertram, A., *Bridgeland-stable moduli spaces for K-trivial surfaces*, J. Eur. Math. Soc. **15** (2013), 1–38.

[2] Arcara, D., Bertram, A., Coskun, I., Huizenga, J., *The Minimal Model Program for the Hilbert Scheme of Points on P^2 and Bridgeland Stability,* arXiv:1203.0316.

[3] Atiyah, M. F., Bott, R., *The Yang-Mills equations over Riemann surfaces,* Philos. Trans. Roy. Soc. London Ser. A **308** (1982), pp. 523–615

[4] Bayer, A., Hassett, B., Tschinkel, Y., *Mori cones of holomorphic symplectic varieties of K3 type,* arXiv:1307.2291

[5] Bayer, A., Macri, E., *Projectivity and Birational Geometry of Bridgeland moduli spaces,* arXiv:1203.4613

[6] Bayer, A., Macri, E., *MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations,* arXiv:1301.6968

[7] Beilinson, A. A., Bernstein, J., Deligne, P., *Faisceaux pervers,* Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982

[8] Bondal, A., Orlov, D., *Semiorthogonal decomposition for algebraic varieties,* arXiv:alg-geom/9506012

[9] Bondal, A., Orlov, D., *Reconstruction of a variety from the derived category and groups of autoequivalences,* Compositio Math. **125** (2001), no. 3, 327–344.

[10] Bridgeland, T., *Equivalences of triangulated categories and Fourier-Mukai transforms,* Bull. London Math. Soc. **31** (1999), 25–34, math.AG/9809114

[11] Bridgeland, T., *Flops and derived categories,* Invent. Math. **147** (2002), 613-632.

[12] Bridgeland, T., *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345.

[13] Bridgeland, T., *Stability conditions on K3 surfaces,* math.AG/0307164, Duke Math. J. **141** (2008), 241–291

[14] Coskun, I., Huizenga, J., Woolf M., *The effective cone of the moduli space of sheaves on the plane*, arXiv:1401.1613.

[15] Drezet, J.-M., Le-Potier, J., *Fibrés stables et fibrés exceptionnels sur $\mathbb{P}^2$,* Ann. scient. Éc. Norm. Sup., 4^e série, t. **18** (1985), pp. 193–244

[16] Gieseker, D. *On the moduli of vector bundles on an algebraic surface,* Ann. of Math. **106** (1977), 45–60

[17] Gieseker, D., Li, J., *Moduli of high rank vector bundles over surfaces,* J. Amer. Math. Soc. **9** (1996), 107–151

[18] Inaba, M., *Toward a definition of moduli of complexes of coherent sheaves on a projective scheme*, J. Math. Kyoto Univ. **42** (2002), no. 2, 317–329.

[19] Inaba, M., *Moduli of stable objects in a triangulated category,* arXiv:math/0612078,J. Math. Soc. Japan **62** (2010), 395–429

[20] Inaba, M., *Sankakukenn jyou niokeru stability condition to moduli,* Sugaku **65** (2013), 160–

[21] Huybrechts, D., Lehn, M., *The geometry of moduli spaces of sheaves,* Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997

[Li] Li, J., *Kodaira dimension of moduli space of vector bundles on surfaces,* Invent. Math. **115** (1994), 1–40

[22] Li, C., Zhao, X., *The MMP for deformations of Hilbert schemes of points on the projective plane*, arXiv:1312.1748.

[23] Ma, S., *Fourier-Mukai partners of a K3 surface and the cusps of its Kahler moduli,* arXiv:0804.4047

[24] Markman, E., Yoshioka, K., *A proof of the Kawamata-Morrison Cone Conjecture for holomorphic symplectic varieties of $K3^{[n]}$ or generalized Kummer deformation type,* arXiv:1402.2049.

[25] Minamide, H., Yanagida, S., Yoshioka, K., *Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions*, arXiv:1106.5217.

[26] Minamide, H., Yanagida, S., Yoshioka, K., *Some moduli spaces of Bridgeland's stability conditions*, arXiv:1111.6187, Int. Math. Res. Not. IMRN 2014, No.19, 5264–5327, doi:10.1093/imrn/rnt126.

[27] Matsuki, K. and Wentworth, R. *Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface,* Internat. J. Math. **8** (1997), 97–148

[28] Mukai, S., *On Fourier functors and their applications to vector bundles on abelian surfaces* (in Japanese), in the proceeding of *Daisū-kikagaku Symposium* (Tohoku University, June 1979), 76–93.

[29] Mukai, S., *On classification of vector bundles on abelian surfaces* (in Japanese), RIMS Kôkyûroku **409** (1980), 103–127.

[30] Mukai, S., *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves,* Nagoya Math. J., **81** (1981), 153–175

[31] Mukai, S., *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface,* Invent. math. **77** (1984), 101–116

[32] Mukai, S., *On the moduli space of bundles on K3 surfaces I,* Vector bundles on Algebraic Varieties, Oxford, 1987, 341–413

[33] Nuer, H., *Projectivity and Birational Geometry of Bridgeland Moduli spaces on an Enriques Surface,* arXiv:1406.0908

[34] O'Grady, K., *Moduli of vector bundles on projective surfaces: some basic results,* Invent. Math. **123** (1996), 141–207

[35] O'Grady, K., *Moduli of vector-bundles on surfaces,* Algebraic geometry—Santa Cruz 1995, 101–126, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997

[36] Orlov, D., *Equivalences of derived categories and $K3$ surfaces,* alg-geom/9606006, Algebraic geometry, 7. J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381.

[37] Simpson, C., *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47–129

[38] Toda, Y., *3jigen Calabi-Yau tayoutai jyou no kyokusenn no kazoeageriron.* Sugaku **66** (2014) 337–365

[39] Van den Bergh, M., *Three-dimensional flops and noncommutative rings*, Duke Math. J. 122 (2004), no. 3, 423–455.

[40] Yanagida, S., Yoshioka, K., *Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on abelian surfaces*, J. Reine Angew. Math. **684** (2013), 31–86.

[41] Yanagida, S., Yoshioka, K., *Bridgeland's stabilities on abelian surfaces,* Math. Z. **276** (2014), 571–610. DOI:10.1007/s00209-013-1214-1.

[42] Yoshioka, K., *Moduli spaces of stable sheaves on abelian surfaces,* Math. Ann. **321** (2001), 817–884, math.AG/0009001

[43] Yoshioka, K., *Perverse coherent sheaves and Fourier-Mukai transforms on surfaces I*, Kyoto J. Math. **53** (2013), 261–344.

[44] Yoshioka, K., *Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface,* arXiv:1206.4838 v2, Adv. Stud. Pure Math. to appear.

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