Deformation quantization and localization of noncommutative algebras and vertex algebras

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The present paper is a report for the proceeings of 59th Algebra Symposium held at the University of Tokyo on 8–11th September 2014.

0. Introduction

We discuss certain infinite-dimensional noncommutative associative algebras. One well-known class of infinite-dimensional associative algebras is universal enveloping algebras of Lie algebras. For a semi-simple Lie algebra $\mathfrak{g}$, the central quotient of its universal enveloping algebra $U(\mathfrak{g})/Z(\mathfrak{g})$ is naturally isomorphic to the algebra of differential operators $\mathcal{D}(G/B)$ on the corresponding flag manifold G/B. Such an isomorphism connects the representation theory of the Lie algebra $\mathfrak{g}$ and geometrical properties of the flag manifold G/B and/or its cotangent bundle $T^*(G/B)$. Namely, the isomorphism leads an equivalence of categories between the category of $(U(\mathfrak{g})/Z(\mathfrak{g}))$-modules and the category of $\mathcal{D}_{G/B}$-modules where $\mathcal{D}_{G/B}$ is the sheaf of differential operators. The equivalence is known as the Beilinson-Bernstein correspondence and it is one of the most fundamental facts of the representation theory of semi-simple Lie algebras.

The above situation can be generalized, and it achieve the notion of quantizations. Note that, in the above, the universal enveloping algebra $U(\mathfrak{g})$ is equipped with a natural filtration, and the associated graded algebra $\operatorname{gr} U(\mathfrak{g})/Z(\mathfrak{g})$ with respect to the filtration is naturally isomorphic to the coordinate ring $\mathbb{C}[T^*(G/B)]$. The commutator $[a, b] = ab-ba$ on $U(\mathfrak{g})$ induces a Poisson bracket on $\operatorname{gr} U(\mathfrak{g})/Z(\mathfrak{g})$, while $\mathbb{C}[T^*(G/B)]$ is also equipped with a Poisson bracket induced from the symplectic structure of the cotangent bundle $T^*(G/B)$. The isomorphism between $\operatorname{gr} U(\mathfrak{g})/Z(\mathfrak{g})$ and $\mathbb{C}[T^*(G/B)]$ is not just an isomorphism of commutative algebras, but it is an isomorphism of Poisson algebras. We say that $U(\mathfrak{g})/Z(\mathfrak{g})$ is a quantization of $\mathbb{C}[T^*(G/B)]$ (or $T^*(G/B)$). Namely, in general cases, for a symplectic manifold X, a filtered associative algebra is called a quantization of $\mathbb{C}[X]$ (or of X), if the associated graded algebra is isomorphic to $\mathbb{C}[X]$ as a Poisson algebra.

In the past decades, several noncommutative algebras which give quantizations of certain symplectic manifolds were introduced. These algebras are constructed by using so called the quantum Hamiltonian reduction, which is noncommutative analogue of Hamiltonian reduction in symplectic geometry. These algebras, constructed by the quantum Hamiltonian reduction, include finite $\mathcal{W}$-algebras and rational Cherednik algebras, which play important and fundamental roles in the representation theory of semi-simple and/or affine Lie algebras and Ariki-Koike algebras.

When an associative algebra A is a quantization of a certain symplectic manifold X, a natural question is if we can localize A as a sheaf of noncommutative algebras on X: Namely, does there exist a sheaf of noncommutative algebras on X whose algebra of global sections is isomorphic to A? And if one exists, is there an equivalence of categories between their module categories like the Beilinson-Bernstein correspondence? If the algebra A is constructed by the quantum Hamiltonian reduction, such problems were well studied and now we know when such a method of localization works well.

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The present paper is a brief and introductory survey which is intended to show an outline of the constructions of quantizations and their localization, and also known results and applications. We also discuss quantizations of certain infinite-dimensional manifolds, called arc spaces, and their localization. Such a quantization is equipped with a certain infinite-dimensional algebraic structure, called a vertex algebra. On the other hand, while we will discuss the construction by quantum Hamiltonian reduction in a general setting, the detail of constructions of certain concrete algebras, e.g. the rational Cherednik algebras, will not be included in the present survey. In [K3], we have a catalog of several algebras obtained by quantum Hamiltonian reduction such as the rational Cherednik algebras and quantized toric algebras. Also, for the finite $\mathcal{W}$-algebras, we have a nice survey [Lo1]. Refer them for these examples.

1. Quantization of Poisson structure

Consider a complex symplectic manifold X and let $\mathcal{O}_X$ be its structure sheaf. Its symplectic form makes $\mathcal{O}_X$ a sheaf of Poisson algebras on X. Namely, $\mathcal{O}_X$ is equipped with a $\mathbb{C}$-bilinear form $\{—,—\}$, called a Poisson bracket, such that $\{—,—\}$ is a Lie bracket on $\mathcal{O}_X$ and $\{f,—\}$ is a derivation on $\mathcal{O}_X$ for any $f \in \mathcal{O}_X$. In particular, the coordinate ring $\mathbb{C}[X] = \Gamma(X, \mathcal{O}_X)$ is a Poisson algebra. In this survey, we consider noncommutative algebras connected with such a Poisson algebra structure.

Let A be a noncommutative associative algebra over $\mathbb{C}$, and we assume that there exists a filtration $\{F_i A\}_{i \geq 0}$ of algebras, satisfying the condition:

$$(1) \qquad [a,b] \underset{\text{def}}{=} ab - ba \in F_{i+j-1}A, \qquad \text{for any } a \in F_i A,\ b \in F_j A.$$

In this situation, the associated graded algebra $\operatorname{gr}_F A$ turns out to be a Poisson algebra whose the Poisson bracket is given by

$$\{\bar{a}, \bar{b}\} \underset{\text{def}}{=} [a,b] \mod F_{i+j-2}A, \quad \text{for } a \in F_i A,\ b \in F_j A$$

where $\bar{a}$ (resp. $\bar{b}$) is a class in $\operatorname{gr}_F A$ in which $a \in A$ (resp. b) belongs. Then, we say that the noncommutative algebra A is a quantization, i.e. noncommutative analogue, of the Poisson algebra $\mathbb{C}[X]$ if there exists an isomorphism $\operatorname{gr}_F A$ and $\mathbb{C}[X]$ as Poisson algebras with the above Poisson bracket.

1.1. **Examples.** Since Lie algebras are an algebraic structure which appears as an algebra consisting of vector fields on a certain manifold, such a situation appears frequently in Lie algebra theory. For example, here we see two classical and typical examples of quantization:

1. The algebra of differential operators with polynomial coefficient (the Weyl algebra) $\mathcal{D}(\mathbb{C}^d)$ on $\mathbb{C}^d$. The algebra $\mathcal{D}(\mathbb{C}^d)$ is equipped with a filtration given by order of differential operators. Namely, the filtration given by $\deg x_i = 0$ and $\deg \partial/\partial x_i = 1$ where $x_1, \ldots, x_d$ is the standard coordinates of $\mathbb{C}^d$. The associated graded algebra is naturally isomorphic to the coordinate ring $\mathbb{C}[T^*\mathbb{C}^d]$ of the cotangent bundle $T^*\mathbb{C}^d$, and moreover the isomorphism is an isomorphism of Poisson algebras. Thus we regard the noncommutative algebra $\mathcal{D}(\mathbb{C}^d)$ as a quantization of the Poisson algebra $\mathbb{C}[T^*\mathbb{C}^d]$. Note that we can choose a different filtration given by $\deg x_i = 1/2$ and $\deg \partial/\partial x_i = 1/2$, and this filtration also induces the same associated graded Poisson algebra $\mathbb{C}[T^*\mathbb{C}^d]$.

2. The universal enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional simple algebra $\mathfrak{g}$. Let $\underline{X} = G/B$ be the flag manifold associated with a Lie group G with the Lie algebra $\mathfrak{g}$ (where B is its Borel subgroup). We have an action of G on $\underline{X} = G/B$ by left-multiplication. This action induces a homomorphism of Lie algebras $\mu_D : \mathfrak{g} \longrightarrow \operatorname{Vect}(\underline{X})$ where $\operatorname{Vect}(\underline{X})$ is a Lie algebra consisting

of vector fields on $\underline{X}$. This homomorphism induces an isomorphism of algebras $\mu_D : U(\mathfrak{g})/Z(\mathfrak{g}) \xrightarrow{\sim} \mathcal{D}(\underline{X})$, where $Z(\mathfrak{g})$ is the center of $U(\mathfrak{g})$. That is, the quotient algebra $U(\mathfrak{g})/Z(\mathfrak{g})$ is a quantization of the coordinate ring $\mathbb{C}[T^*\underline{X}] \simeq \mathbb{C}[\mathcal{N}]$ where $\mathcal{N} \subset \mathfrak{g}$ is the nilpotent cone of the Lie algebra $\mathfrak{g}$. Note that there exists a resolution of singularities $T^*\underline{X} \longrightarrow \mathcal{N}$ called the Springer resolution, and this resolution induces the isomorphism between the coordinate rings $\mathbb{C}[T^*\underline{X}]$ and $\mathbb{C}[\mathcal{N}]$ as Poisson algebras. Namely, $U(\mathfrak{g})/Z(\mathfrak{g})$ is a quantization of $\mathbb{C}[T^*\underline{X}]$.

2. Localization of quantized algebras

Let X be a symplectic manifold with structure sheaf $\mathcal{O}_X$. Assume that we have a noncommutative algebra A which gives a quantization of the coordinate ring $\mathbb{C}[X] = \Gamma(X, \mathcal{O}_X)$. Theory of algebraic geometry allows us to consider localization of the commutative algebra $\mathbb{C}[X]$, that is the structure sheaf $\mathcal{O}_X$. Namely, localization of $\mathbb{C}[X]$ is a sheaf of commutative algebras whose algebra of global sections turns out to be $\mathbb{C}[X]$. Since our quantization A is a natural noncommutative analogue of the coordinate ring $\mathbb{C}[X]$, it is a natural question if there exists a sheaf of noncommutative algebras on X whose algebra of global sections turns out to be our algebra A. We call it localization of the noncommutative algebra A. We will require that localization $\mathcal{A}$ of A is a sheaf of noncommutative algebra on X, equipped with a filtration satisfying the condition (1), and the associated graded algebra is isomorphic to the structure sheaf $\mathcal{O}_X$ as a sheaf of Poisson algebras. We also require that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{gr}} & \mathbb{C}[X] \\ {\scriptstyle \Gamma(X,—)}\uparrow & & \uparrow{\scriptstyle \Gamma(X,—)} \\ \mathcal{A} & \xrightarrow{\mathrm{gr}} & \mathcal{O}_X \end{array}$$

In such a situation, we also call $\mathcal{A}$ a quantization of the structure sheaf $\mathcal{O}_X$.

To be precise, we need to consider a sheaf of $\mathbb{C}[[\hbar]]$-algebras on X equipped with a certain $\mathbb{C}^*$-action, and consider the $\mathbb{C}^*$-finite part of the algebra of its global sections to extract a $\mathbb{C}$-algebra from it. In the following example, we see why we need to consider a sheaf of $\mathbb{C}[[\hbar]]$-algebras, not a sheaf of $\mathbb{C}$-algebras.

2.1. **Microlocalization of sheaves of differential operators.** Consider the second example of the quantizations. If we consider the flag manifold $\underline{X} = G/B$ associated with the simple Lie algebra $\mathfrak{g} = Lie(G)$, the algebra of differential operators $\mathcal{D}(\underline{X})$ on $\underline{X}$ is isomorphic to the quotient algebra of the universal enveloping algebra $U(\mathfrak{g})/Z(\mathfrak{g})$. Thus, if we consider the sheaf of differential operators $\mathcal{D}_{\underline{X}}$ on $\underline{X}$, we have a natural isomorphism $\Gamma(\underline{X}, \mathcal{D}_{\underline{X}}) \simeq U(\mathfrak{g})/Z(\mathfrak{g})$. Nevertheless, the sheaf $\mathcal{D}_{\underline{X}}$ is not localization of $U(\mathfrak{g})/Z(\mathfrak{g})$ in the above sense, because the sheaf $\mathcal{D}_{\underline{X}}$ is a sheaf on $\underline{X}$, not on its cotangent bundle $X \underset{\mathrm{def}}{=} T^*\underline{X}$. Thus, to construct localization of $U(\mathfrak{g})/Z(\mathfrak{g})$ in the above sense, we need to construct a sheaf on X, i.e. we need to localize $\mathcal{D}_{\underline{X}}$ in the cotangent direction. Now we consider how such a localization $\mathcal{A}$ of $\mathcal{D}(\underline{X})$ on the cotangent bundle $X = T^*\underline{X}$ looks like.

Assume that we have local coordinates $(x_1, \ldots, x_d)$ of $\underline{X}$. Let $\partial_1, \ldots, \partial_d \in \mathcal{D}_{\underline{X}}$ be the partial differential operators $\partial_i = \partial/\partial x_i$ associated with this coordinates. We denote the equivalent class of ∂_i in $\mathrm{gr}\,\mathcal{D}_{\underline{X}} \simeq \mathcal{O}_{T^*\underline{X}}$ by y_i and the equivalent class of x_i by the same symbol x_i. Then $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ gives local symplectic coordinates of $X = T^*\underline{X}$. On the open subset $\{y_i \neq 0\} \subset X$, the localization sheaf $\mathcal{A}$ has a local section ∂_i^{-1}, which is the inverse of ∂_i. Since $\mathcal{A}$ is localization of $\mathcal{D}(\underline{X})$, it is natural that the multiplication $\cdot$ in $\mathcal{A}$ satisfies the Leibniz rule. Then, by the Leibniz rule, we have

$$\partial_i^{-1} \cdot x_i^{-1} = x_i^{-1}\partial_i^{-1} + x_i^{-2}\partial_i^{-2} + 2x_i^{-3}\partial_i^{-3} + \dots,$$

and RHS turns an infinite sum. Here each term of RHS means symbols of differential operators. Thus, to construct localization of $\mathcal{D}(\underline{X})$ on $X = T^*\underline{X}$, we need to allow such infinite sums. Algebraically, it can be obtained by introducing a Rees algebra with respect to the filtration of $\mathcal{D}_{\underline{X}}$ and taking completion of it.

Now we introduce the precise definition of deformation-quantization of the structure sheaf $\mathcal{O}_X$. Let $\hbar$ be an indeterminate and let A be an associative $\mathbb{C}[[\hbar]]$-algebra such that the quotient algebra $A/\hbar A$ is a commutative $\mathbb{C}$-algebra. Then, for any a, $b \in A$, $[a, b] = ab - ba$ lies in $\hbar A$, and hence $\{\bar{a}, \bar{b}\} = \frac{1}{\hbar}[a, b]$ is well-defined on $A/\hbar A$, where $\bar{a}$ (resp. $\bar{b}$) be the class to which a (resp. b) belongs. It is easy to see that $\{—, —\}$ is a Poisson bracket on the quotient algebra $A/\hbar A$.

Definition 2.1. *For a symplectic manifold X, a deformation-quantization of $\mathcal{O}_X$ (or of X) is a sheaf $\mathcal{A}$ of associative $\mathbb{C}[[\hbar]]$-algebras, which is flat over $\mathbb{C}[[\hbar]]$ and complete in $\hbar$-adic topology, such that*

$$(\mathcal{A}/\hbar\mathcal{A}, h^{-1}[—, —]) \simeq (\mathcal{O}_X, \{—, —\})$$

as a sheaf of Poisson algebras.

A basic example of deformation-quantization is the following $\hbar$-deformed Weyl algebra $A_{T^*\mathbb{C}^d}$. As a vector space, we define

$$A_{T^*\mathbb{C}^d} = \mathbb{C}[[\hbar]][x_1, \dots, x_d, y_1, \dots, y_d],$$

and its defining relations are given by $[y_i, x_j] = y_i x_j - x_j y_i = \hbar\delta_{ij}$, $[x_i, x_j] = [y_i, y_j] = 0$ where δ_{ij} is Kronecker's delta. With localizing it in a straight-forward way, we have a sheaf $\mathcal{A}_{T^*\mathbb{C}^d} \simeq_{\mathbb{C}} \mathcal{O}_{T^*\mathbb{C}^d} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$, and this sheaf gives a deformation-quantization of the structure sheaf $\mathcal{O}_{T^*\mathbb{C}^d}$. Note that in $\mathcal{A}_{T^*\mathbb{C}^d}$, we have a local section x_i^{-1} and y_i^{-1}, and their product is given by the infinite sum

$$y_i^{-1} \cdot x_i^{-1} = x_i^{-1}y_i^{-1} + \hbar x_i^{-2}y_i^{-2} + 2\hbar^2 x_i^{-3}y_i^{-3} + \dots,$$

which is well-defined in $\mathcal{A}_{T^*\mathbb{C}^d}$.

To construct the localization of $\mathcal{D}(\underline{X})$ for the manifold $\underline{X}$ on its cotangent bundle $X = T^*\underline{X}$, we patch up $\mathcal{A}_{T^*\mathbb{C}^d}$ with local symplectic coordinates by gluing. Let $\{U_\alpha\}_\alpha$ be an affine open covering of $\underline{X}$. Then we have the open covering $\{T^*U_\alpha\}_\alpha$ of $X = T^*\underline{X}$, and on each T^*U_α, we have local symplectic coordinates $(x_1^\alpha, \dots, x_d^\alpha, y_1^\alpha, \dots, y_d^\alpha)$. Now we can introduce a sheaf $\mathcal{A}_{T^*U_\alpha}$ on T^*U_α, which is a restriction of $\mathcal{A}_{T^*\mathbb{C}^d}$ with respect to the symplectic coordinates. These sheaves can be glued together into a sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{A}_{T^*\underline{X}} \simeq_{\mathbb{C}} \mathcal{O}_{T^*\underline{X}} \otimes_{\mathbb{C}} \mathbb{C}[[\hbar]]$. This sheaf $\mathcal{A}_{T^*\underline{X}}$ gives a deformation-quantization of $\mathcal{O}_{T^*\underline{X}}$.

One natural and fundamental problem of deformation-quantizations is if there exists a deformation-quantization of $\mathcal{O}_X$ and how many deformation-quantizations exist for given complex symplectic manifold X. In this survey, we do not care this problem and consider deformation-quantizations which can be constructed explicitly. For the existence and classification problem of deformation-quantization, refer [BeKa], [Lo2] and references therein.

2.2. **Quantum Hamiltonian reduction.** As the above example, for a symplectic manifold X which is the cotangent bundle $X = T^*\underline{X}$ of a complex manifold $\underline{X}$, we have a deformation-quantization of $\mathcal{O}_X$. But an important point of deformation-quantization is that we can also construct a deformation-quantization on a symplectic manifold which is not a cotangent bundle of a certain manifold. Quantum Hamiltonian reduction is a method to construct a noncommutative algebra which is a quantization of a certain Poisson algebra.

First we review the notion of Hamiltonian reduction. It is a method to construct a symplectic variety. The quantum Hamiltonian reduction is noncommutative analogue of the Hamiltonian reduction.

Let $\underline{X}$ be a complex manifold. We assume that an algebraic/Lie group M acts on $\underline{X}$. Then this action naturally induces an action of M on $T^*\underline{X}$ and on its structure sheaf $\mathcal{O}_{T^*\underline{X}}$, and the action preserves the Poisson bracket on $\mathcal{O}_{T^*\underline{X}}$. Moreover, by differentiating the induced action of M on the structure sheaf $\mathcal{O}_{\underline{X}}$ of $\underline{X}$ (not $T^*\underline{X}$), we have a homomorphism of Lie algebras

$$\mu_D : \mathfrak{m} = Lie M \longrightarrow \mathrm{Vect}(\underline{X}) \subset \mathcal{D}(\underline{X}),$$

and a homomorphism of $\mathbb{C}$-algebras,

$$\mu_D : U(\mathfrak{m}) \longrightarrow \mathcal{D}(\underline{X}).$$

The homomorphism μ_D induces a homomorphism of commutative algebras between their associated graded algebras

$$\mu^* : S(\mathfrak{m}) = \mathbb{C}[\mathfrak{m}^*] \longrightarrow \mathbb{C}[T^*\underline{X}]$$

where $S(\mathfrak{m})$ is the symmetric algebra over the vector space $\mathfrak{m}$, and $\mathfrak{m}^*$ is the dual vector space of $\mathfrak{m}$. By considering associated morphism with μ^* between algebraic varieties, we have a morphism $\mu : T^*\underline{X} \longrightarrow \mathfrak{m}^*$, called a moment map associated with the M-action on $T^*\underline{X}$. By the construction, this morphism μ is compatible with the M-action on $T^*\underline{X}$. Namely, the subset $\mu^{-1}(\chi) \subset T^*\underline{X}$ is closed under the action of M for $\chi \in \mathfrak{m}^*$. Then, we set

$$X^0_\chi = \mu^{-1}(\chi)/\!/M \underset{\mathrm{def}}{=} \mathrm{Spec}\, \mathbb{C}[\mu^{-1}(\chi)]^M.$$

The affine scheme X^0_χ may have singularities, but it is a Poisson variety, whose Poisson bracket is naturally induced from the Poisson bracket on $\mathbb{C}[T^*\underline{X}]$. We may also construct nonsingular symmetric manifold associated with the M-action by using geometric invariant theory. Here we do not care details of geometric invariant theory, and we assume that we can take a Zariski open subset of $\mu^{-1}(\chi)$ denoted by $\mu^{-1}(\chi)^{ss}$ such that the group M acts free on $\mu^{-1}(\chi)^{ss}$. The subset $\mu^{-1}(\chi)^{ss}$ is called semistable locus with respect to the action of M. Since the M-action on $\mu^{-1}(\chi)^{ss}$ is free, the quotient with respect to the M-action

$$X_\chi = \mu^{-1}(\chi)^{ss}/M$$

turns out to be a nonsingular scheme. Moreover, the symplectic form on $T^*\underline{X}$ induces a symplectic form on X_χ and thus X_χ is a symplectic manifold. The symplectic manifold X_χ is called Hamiltonian reduction of $T^*\underline{X}$ with respect to the M-action. Note that X_χ is a quotient of subset of the original symplectic manifold $T^*\underline{X}$ with respect to the M-action.

Important known algebras are obtained by quantum Hamiltonian reduction for the case where $\underline{X}$ is a linear representation of the group M (thus is a $\mathbb{C}$-vector space), and $\chi = 0$. For known examples of such algebras, refer [K3, Section 5]. Below we consider such a case and denote $X = X_0$ simply.

The structure sheaf of the new symplectic manifold X can be constructed explicitly from the structure sheaf of $T^*\underline{X}$. While X is a quotient of subset of $T^*\underline{X}$, its structure sheaf $\mathcal{O}_X$ can be written in the form of the M-invariant subalgebra of the quotient of $\mathcal{O}_{T^*\underline{X}}$ as follows:

$$\mathcal{O}_X \simeq \left(p_*\left(\mathcal{O}_{(T^*\underline{X})^{ss}} \;/\; \mathcal{O}_{(T^*\underline{X})^{ss}}\mu^*(\mathfrak{m})\right)\right)^M$$

where $p : \mu^{-1}(0)^{ss} \longrightarrow X$ is the projection. Namely, the structure sheaf $\mathcal{O}_X$ is the algebra of M-invariants of M-coinvariants of the structure sheaf $\mathcal{O}_{T^*\underline{X}}$.

The quantum Hamiltonian reduction is quantization of the Hamiltonian reduction. Namely, we consider noncommutative analogue of the above construction by replacing the structure sheaf $\mathcal{O}_{T^*\underline{X}}$ by its deformation-quantization $\mathcal{A}_{T^*\underline{X}}$. Let $c : \mathfrak{m} \longrightarrow \mathbb{C}$ be a linear function on $\mathfrak{m}$ which is invariant under the adjoint action of $\mathfrak{m}$. This function c is a parameter of the quantization. Consider the following sheaf of $\mathbb{C}[[\hbar]]$-algebras on the symplectic manifold X, which is a subquotient of the deformation-quantization $\mathcal{A}_{(T^*\underline{X})^{ss}}$:

$$\mathcal{A}_{X,c} = \left(p_*\left(\mathcal{A}_{(T^*\underline{X})^{ss}} \;/\; \mathcal{A}_{(T^*\underline{X})^{ss}}(\mu_D - \hbar c)(\mathfrak{m})\right)\right)^M .$$

The construction naturally implies that the sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{A}_{X,c}$ is a deformation-quantization of $\mathcal{O}_X$. Indeed, under the following geometric conditions, we can show that $\mathcal{A}_{X,c}$ is a deformation-quantization of $\mathcal{O}_X$:

(1) $\mu^{-1}(0)^{ss} \neq \emptyset$,
(2) the moment map μ is a flat morphism,
(3) the action of M on $\mu^{-1}(0)^{ss}$ is free,
(4) the morphism $X \longrightarrow X_0$ is birational and X_0 has only normal singularities.

2.3. **Algebra of global sections.** Now we make precise the connection between the quantization of the coordinate ring $\mathbb{C}[X]$ and the deformation-quantization of the structure sheaf $\mathcal{O}_X$.

Let X be a symplectic manifold obtained by Hamiltonian reduction, and let $\mathcal{A}_{X,c}$ be a deformation-quantization of $\mathcal{O}_X$ as in Section 2.2. Note that $\mathcal{A}_{X,c}$ is a sheaf of $\mathbb{C}[[\hbar]]$-algebras and thus its algebra of global sections $\Gamma(X, \mathcal{A}_{X,c})$ is also a $\mathbb{C}[[\hbar]]$-algebra. By construction, we have $\Gamma(X, \mathcal{A}_{X,c})/\hbar\Gamma(X, \mathcal{A}_{X,c}) \simeq \mathbb{C}[X]$. In Section 1, we defined a quantization of the Poisson algebra $\mathbb{C}[X]$ as a filtered $\mathbb{C}$-algebra. Thus the algebra of global sections $\Gamma(X, \mathcal{A}_{X,c})$ is not a quantization in that sense. But we can obtain a quantization of $\mathbb{C}[X]$ by modifying the notion of "the algebra of global sections" a little.

Recall that we assumed that $\underline{X}$ is a linear representation of M and thus $\underline{X}$ is a $\mathbb{C}$-vector space. Consider a $\mathbb{C}^*$-action on $T^*\underline{X}$ such that the induced equivariant $\mathbb{C}^*$-action on $\mathcal{O}_{T^*\underline{X}}$ makes the coordinate functions x_i, y_i of $T^*\underline{X}$ semi-invariant elements of weight one. By letting $\hbar$ be also a semi-invariant element of weight two, the $\mathbb{C}^*$-action lift to an equivariant action on the deformation-quantization $\mathcal{A}_{T^*\underline{X}}$. These $\mathbb{C}^*$-actions induce a $\mathbb{C}^*$-action on the symplectic manifold X, and also an equivariant $\mathbb{C}^*$-action on the sheaf $\mathcal{A}_{X,c}$ on X. Then the algebra of global sections $\widehat{A}_c \underset{\text{def}}{=} \Gamma(X, \mathcal{A}_{X,c})$ is a $\mathbb{C}[[\hbar]]$-algebra with a $\mathbb{C}^*$-action.

The algebra $\widehat{A}_c$ is decomposed into a direct product of weight spaces with respect to the $\mathbb{C}^*$-action: $\widehat{A}_c = \prod_m (\widehat{A}_c)_m$ where $(\widehat{A}_c)_m$ is the weight space of weight m. Consider the direct sum $\bigoplus_m (\widehat{A}_c)_m$. Then, it is a $\mathbb{C}[\hbar]$-subalgebra of $\widehat{A}_c$. We call it the finite part of $\widehat{A}_c$ with respect to the $\mathbb{C}^*$-action, and denote $(\widehat{A}_c)_{\mathbb{C}^*\text{-fin}}$. Now consider the specialization $\hbar = 1$, and we obtain a $\mathbb{C}$-algebra

$$A_c = \Gamma_F(X, \mathcal{A}_{X,c}) \underset{\text{def}}{=} (\widehat{A}_c)_{\mathbb{C}^*\text{-fin}}/(\hbar - 1)(\widehat{A}_c)_{\mathbb{C}^*\text{-fin}}.$$

The $\mathbb{C}[[\hbar]]$-algebra $\widehat{A}_c$ is graded by the power of $\hbar$, and the grading induces a filtration of the $\mathbb{C}$-algebra A_c.

Proposition 2.2. *Under the conditions (1)–(4) in Section 2.2, the algebra $A_c = \Gamma_F(X, \mathcal{A}_{X,c})$ is a quantization of the Poisson algebra $\mathbb{C}[X]$.*

We also have the following commutative diagram:

$$\begin{array}{ccc} A_c & \xrightarrow{\mathrm{gr}} & \mathbb{C}[X] \\ {\scriptstyle \Gamma_F(X,\text{—})}\uparrow & & \uparrow{\scriptstyle \Gamma(X,\text{—})} \\ \mathcal{A}_{X,c} & \xrightarrow{\hbar=0} & \mathcal{O}_X \end{array}$$

Similarly, we can construct "the algebra of global sections" also for the deformation-quantization $\mathcal{A}_{T^*\underline{X}}$ of the cotangent bundle $T^*\underline{X}$ for a manifold $\underline{X}$.

3. Deformation-quantization of arc spaces

3.1. **Arc spaces.** The notion of deformation-quantization and quantum Hamiltonian reduction works also for certain infinite-dimensional manifolds, not only finite-dimensional ones. For a finite-dimensional manifold X, consider an infinite-dimensional manifold $J_\infty X$, called an arc space or a ∞-Jet scheme on X. We regard X as a scheme of finite type over $\mathbb{C}$. Then the arc space $J_\infty X$ is a scheme whose R-valued points for a $\mathbb{C}$-algebra R is given by

$$J_\infty X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, J_\infty X) \underset{\mathrm{def}}{=} \mathrm{Hom}(\mathrm{Spec}\, R[[t]], X) = X(R[[t]]).$$

Namely, the arc space $J_\infty X$ is a scheme, whose $\mathbb{C}$-valued points are infinitesimal arcs on X. Then, for an R-valued point $x(t)$ of $J_\infty X$, we can regard $x(t)$ as an infinite-dimensional arc on X and its head $x(0)$ is an R-valued point of X. Thus, we have a canonical projection $J_\infty X \longrightarrow X$, $x(t) \mapsto x(0)$.

We consider a finite-dimensional symplectic manifold X. Then its structure sheaf $\mathcal{O}_X$ is equipped with a Poisson bracket $\{\text{—},\text{—}\}$. Along with it, the structure sheaf $\mathcal{O}_{J_\infty X}$ has an additional structure, called a vertex Poisson algebra. Vertex Poisson algebras are vertex algebra analogue of Poisson algebras, which are naturally obtained as an associated graded algebra of a certain vertex algebra. First, we review briefly the definition of vertex algebras.

A vertex algebra is a quadruple of data $(V, \mathbf{1}, T, Y(\text{—}, z))$, where

- V — a vector space,
- $\mathbf{1} \in V$ — a vector in V, called the vacuum vector
- $T : V \longrightarrow V$ — a linear operator, called the translation operator
- $Y(\text{—}, z) : V \longrightarrow \mathrm{End}\, V[[z, z^{-1}]]$ — a linear operator, where $Y(A, z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1}$ is a formal series of linear operators $A_{(n)}$ on V for each vector $A \in V$. The operator $Y(A, z)$ is called the vertex operator associated with A.

These data are subject to the following axioms:

- (vacuum axiom) The vacuum vector satisfies $Y(\mathbf{1}, z) = \mathrm{id}_V$, and for any $A \in V$, we have $Y(A, z)\mathbf{1} \in V[[z]]$, so that $Y(A, z)\mathbf{1}$ can be specialized at $z = 0$. Then, we have $Y(A, z)\mathbf{1}|_{z=0} = A$. In other words, we have $A_{(n)}\mathbf{1} = 0$ for any $n \geq 0$ and $A_{(-1)}\mathbf{1} = A$ in V.
- (translation axiom) For any $A \in V$, we have
$$[T, Y(A, z)] = \partial_z Y(A, z),$$
and $T\mathbf{1} = 0$.
- (locality axiom) For any A, $B \in V$, the vertex operators $Y(A, z)$ and $Y(B, w)$ are mutually local: Namely, there exists $N \in \mathbb{Z}_{\geq 0}$
$$(z - w)^N [Y(A, z), Y(B, w)] = 0.$$

From the above definition of vertex algebras, we obtain the following identity between coefficients of vertex operators $Y(A,z)$ and $Y(B,z)$ for A, $B \in V$, so called Borcherds' identity:

$$\sum_{j\geq 0}\binom{m}{j}(A_{(n+j)}B)_{(m+l-j)} = \sum_{j\geq 0}(-1)^j\binom{n}{j}\{A_{(m+n-j)}B_{(l+j)} - (-1)^n B_{(n+l-j)}A_{(m+j)}\}$$

for any l, m and $n \in \mathbb{Z}$.

A vertex algebra V is called a commutative vertex algebra if $A_{(n)} = 0$ for any $A \in V$ and $n \in \mathbb{Z}_{\geq 0}$. In this situation, operators $A_{(-m)}$ and $B_{(-n)}$ commute for any A, $B \in V$ and m, $n \in \mathbb{Z}_{\geq 1}$, and we can identify the commutative vertex algebra V as a commutative $\mathbb{C}$-algebra in infinitely many variables with the derivation T. For any manifold X, the structure sheaf $\mathcal{O}_{J_\infty X}$ of the arc space $J_\infty X$ has a structure of a sheaf of commutative vertex algebras with variables $f_{(-n)} = (1/n!)T^n f$ for $f \in \mathcal{O}_X$ and $n \in \mathbb{Z}_{\geq 1}$. The canonical projection $J_\infty X \longrightarrow X$ induces an embedding $\mathcal{O}_X \hookrightarrow \mathcal{O}_{J_\infty X}$ given by $f \mapsto f_{(-1)}$ for $f \in \mathcal{O}_X$.

A vertex Poisson algebra is a tuple $(V, \mathbf{1}, T, Y_+(—,z), Y_-(—,z))$, such that the quadruple $(V, \mathbf{1}, T, Y_+(—,z))$ is a commutative vertex algebra with the vertex operator $Y_+(A,z) = \sum_{n\leq -1} A_{(n)}z^{-n-1}$, and coefficients of $Y_-(A,z) = \sum_{n\geq 0} A_{(n)}z^{-n-1}$ satisfies a truncation of the Borcherds' identity. It is vertex-algebraic analogue of Poisson algebras in the following sense; if $(V, \mathbf{1}, T, Y(—))$ is a vertex algebra with filtration and the associated graded vertex algebra $\operatorname{gr} V$ with respect to the filtration is commutative, we can define the structure of a vertex Poisson algebra on the associated graded vertex algebra $\operatorname{gr} V$.

Assume that X is a symplectic manifold. Consider the arc space $J_\infty X$, and then the structure sheaf $\mathcal{O}_{J_\infty X}$ is a sheaf of commutative vertex algebras as above. Moreover, the Poisson bracket $\{—,—\}$ on $\mathcal{O}_X$ induces a structure of a vertex Poisson algebra on $\mathcal{O}_{J_\infty X}$ which satisfies

$$f_{(n)}g = \begin{cases}\{f,g\} & \text{if } n = 0,\\ 0 & \text{if } n \geq 1,\end{cases}$$

for f, $g \in \mathcal{O}_X$ and also satisfies the (truncated) Borcherds' identity.

3.2. **Deformation-quantization of $J_\infty X$.** As deformation-quantization of the vertex Poisson algebras, we obtain the notion of $\hbar$-adic vertex algebras, which are introduced by H. Li in [Li]. An $\hbar$-adic vertex algebra is a quadruple $(V, \mathbf{1}, T, Y(—,z))$ where V is a flat $\mathbb{C}[[\hbar]]$-module which is complete in $\hbar$-adic topology, such that $(V/\hbar^N V, \mathbf{1}, T, Y(—,z))$ is a vertex algebra over $\mathbb{C}$ for any $N \geq 0$. Note that a $\hbar$-adic vertex algebra need not a vertex algebra over $\mathbb{C}[[\hbar]]$.

Assume that, for an $\hbar$-adic vertex algebra $(V, \mathbf{1}, T, Y(—,z))$, the quotient vertex algebra $(V/\hbar V, \mathbf{1}, T, Y(—,z))$ is a commutative vertex algebra. Then, $V/\hbar V$ has naturally the structure of vertex Poisson algebra defined by

$$\bar{a}_{(n)}\bar{b} = \hbar^{-1}a_{(n)}b \mod \hbar$$

for a, $b \in V$ and $n \in \mathbb{Z}_{\geq 0}$, and where $\bar{a} = a \mod \hbar \in V/\hbar V$ is the equivalent class of a.

Now we introduce the notion of deformation-quantization for vertex algebras. Let X be a symplectic manifold, and consider the structure sheaf $\mathcal{O}_{J_\infty X}$ of the arc space $J_\infty X$ as a sheaf of vertex Poisson algebras. We say that the sheaf of $\hbar$-adic vertex algebras $\mathcal{A}_X^{ch}$ on $J_\infty X$ is a deformation-quantization of $\mathcal{O}_{J_\infty X}$ (or of $J_\infty X$) if its quotient $\mathcal{A}_X^{ch}/\hbar\mathcal{A}_X^{ch}$ is isomorphic to $\mathcal{O}_{J_\infty X}$ as a sheaf of vertex Poisson algebras on $J_\infty X$.

As the usual deformation-quantization, we can construct a deformation-quantization of $\mathcal{O}_{J_\infty X}$ as a sheaf of $\hbar$-adic vertex algebras in the following two cases:

(1) When X is the cotangent bundle of a certain manifold $\underline{X}$, i.e. $X = T^*\underline{X}$, we may have a sheaf of vertex algebras called an algebra of chiral differential operators (CDO) on X, denoted by $\mathcal{D}_{\underline{X}}^{ch}$, which was introduced in [BD] and [GMS] independently. Note that there exists an obstruction for existence of such a sheaf, but it is known that if the second Chern class $ch_2(\mathcal{T}_{\underline{X}})$ vanishes, such a sheaf $\mathcal{D}_{\underline{X}}^{ch}$ exists. See [GMS] for details.

By a similar method for localizing the sheaf of differential operators $\mathcal{D}_{\underline{X}}$ on $T^*\underline{X}$ as a deformation-quantization of $X = T^*\underline{X}$, we can construct localization of the sheaf of vertex algebras $\mathcal{D}_{\underline{X}}^{ch}$ as a sheaf of $\hbar$-adic vertex algebras on $X = T^*\underline{X}$, and moreover, as a sheaf on its arc space $J_\infty X$. This construction gives a deformation-quantization of the structure sheaf $\mathcal{O}_{J_\infty X}$ in the above sense. We denote it $\mathcal{A}_{X,\hbar}^{ch}$.

(2) When a symplectic manifold X is constructed by Hamiltonian reduction $X = \mu^{-1}(\chi)^{ss}/M$ as above for an action of a certain unipotent Lie group M on a manifold $\underline{X}$ and the CDO $\mathcal{D}_{\underline{X}}^{ch}$ exists, we can construct a deformation-quantization of $\mathcal{O}_{J_\infty X}$ as quantum Hamiltonian reduction of $\mathcal{A}_{X,\hbar}^{ch}$. In [AKM], with using a certain cohomological and quantum Hamiltonian reduction, called a BRST cohomology, a deformation-quantization of $\mathcal{O}_{J_\infty X}$ is constructed for a Slodowy variety X, a symplectic manifold which is obtained by Hamiltonian reduction of a flag variety.

As "a vertex algebra of global sections" with respect to a certain $\mathbb{C}^*$-action, we obtain (1) a vertex algebra at critical level associated with the affine Lie algebra corresponding to the flag variety, or (2) an affine $\mathcal{W}$-algebra at critical level associated with the Slodowy variety X, respectively. Namely, these deformation-quantizations are regarded as localization of such vertex algebras.

4. Applications of localization to representation theory

When an associative algebra A is localized by a deformation-quantization of a certain symplectic manifold X, the representation theory of A may be connected with the geometrical structure of the underlying manifold X. And indeed many applications of such a localization to the representation theory are known. In this section, we summarize some of such results which are recently studied for algebras obtained by quantum Hamiltonian reduction.

Throughout this section, we use the following notation. Let X be a symplectic manifold, and let $\mathcal{A}_{X,c}$ be a deformation-quantization of the structure sheaf $\mathcal{O}_X$ with a parameter of quantization c. Set $A_c = \Gamma(X, \mathcal{A}_{X,c})_{fin}|_{\hbar=1}$ be an associative $\mathbb{C}$-algebra which is obtained as "the algebra of global sections". In this section, we may assume that the symplectic manifold X is obtained by Hamiltonian reduction, and the Hamiltonian reduction satisfies the assumptions (1)–(4).

4.1. **Beilinson-Bernstein type correspondence.** The most fundamental results are certain equivalences of categories between the category of A_c-modules and the category of $\mathcal{A}_{X,c}$-modules with an equivariant torus action, which the functor of taking "global sections" gives. In the classical case of the universal enveloping algebra of simple Lie algebra $U(\mathfrak{g})$, such an equivalence of categories essentially coincides with an equivalence of abelian categories known as the Beilinson-Bernstein correspondence. In the case of quantum Hamiltonian reduction, we have (1) equivalence of triangulated categories between derived categories of module categories and (2) equivalence of abelian categories which is direct analogue of the Beilinson-Bernstein correspondence.

(1) Let A_c-mod be the abelian category of finitely-generated A_c-modules, and let $\mathcal{A}_{X,c}$-$\mathrm{mod}_{\mathbb{C}^*\text{-equiv}}$ be the abelian category of coherent $\mathcal{A}_{X,c}$-modules with the $\mathbb{C}^*$-equivariant action, which are torsion-free over $\mathbb{C}[[\hbar]]$. Then an object of $\mathcal{A}_{X,c}$-$\mathrm{mod}_{\mathbb{C}^*\text{-equiv}}$ are a sheaf on X and thus we can consider the module of its global sections. For an object $\mathcal{M} \in \mathcal{A}_{X,c}$-$\mathrm{mod}_{\mathbb{C}^*\text{-equiv}}$, the module of global sections $\Gamma(X,\mathcal{M})$ is a $\mathbb{C}[[\hbar]]$-module with $\mathbb{C}^*$-action, since $\mathcal{M}$ is equipped with the equivariant $\mathbb{C}^*$-action. Then, taking finite part with respect to $\mathbb{C}^*$-action, denoted $\Gamma(X,\mathcal{M})_{\mathbb{C}^*\text{-fin}}$, we have a $\mathbb{C}[\hbar]$-module. Finally, substituting $\hbar = 1$, we obtain a $\mathbb{C}$-vector space

$$\Gamma_F(X,\mathcal{M}) \underset{\text{def}}{=} \Gamma(X,\mathcal{M})_{\mathbb{C}^*\text{-fin}}|_{\hbar=1} = \Gamma(X,\mathcal{M})_{\mathbb{C}^*\text{-fin}} \otimes_{\mathbb{C}[\hbar]} \mathbb{C}_1,$$

where $\mathbb{C}_1$ is the $\mathbb{C}[\hbar]$-module by augmentation at $\hbar = 1$. Since $\Gamma_F(X,\mathcal{A}_{X,c}) \simeq A_c$, $\Gamma_F(X,\mathcal{M})$ is an A_c-module. This construction gives a functor of abelian categories

$$\Gamma_F(X, —) : \mathcal{A}_{X,c}\text{-mod}_{\mathbb{C}^*\text{-equiv}} \longrightarrow A_c\text{-mod}.$$

Consider the derived functor, and we have

$$R\Gamma_F(X, —) : D^b(\mathcal{A}_{X,c}\text{-mod}_{\mathbb{C}^*\text{-equiv}}) \longrightarrow D^b(A_c\text{-mod})$$

where $D^b(—)$ is a bounded derived category.

The following result is due to I. Gordon and I. Losev [GL]. Independently, K. McGerty and T. Nevins also studied essentially the same result independently in a little different manner in [MN1].

Theorem 4.1. *If the algebra A_c has finite global dimension, we have an equivalence of triangulated categories*

$$R\Gamma_F(X, —) : D^b(\mathcal{A}_{X,c}\text{-mod}_{\mathbb{C}^*\text{-equiv}}) \xrightarrow{\sim} D^b(A_c\text{-mod})$$

with the quasi-inverse functor $\mathcal{A}_{X,c} \otimes^L_{A_c} (—)$.

(2) Under a certain condition, the above functor $\Gamma_F(X, —)$ also gives an equivalence of abelian categories, not only the derived equivalence. Note that the Hamiltonian reduction X is defined as a projective variety over X^0:

$$X = \mathrm{Proj} \bigoplus_{m\geq 0} \mathbb{C}[\mu^{-1}(0)]^M_{\theta^m} \longrightarrow X^0 = \mathrm{Spec}\, \mathbb{C}[\mu^{-1}(0)]^M$$

where θ is a certain character of the group M. From the definition, we have a line bundle $\mathcal{O}(1)$ which is associated with the $\bigoplus_m \mathbb{C}[\mu^{-1}(0)]^M_{\theta^m}$-module $\bigoplus_m \mathbb{C}[\mu^{-1}(0)]^M_{\theta^{(m+1)}}$. Moreover, we can construct a sheaf $\mathcal{A}^\theta_{X,c}$ which gives a quantization of this line bundle $\mathcal{O}(1)$ in the sense that $\mathcal{A}^\theta_{X,c} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_0 \simeq \mathcal{O}(1)$ where $\mathbb{C}_0$ is a $\mathbb{C}[[\hbar]]$-module on $\mathbb{C}$ by augmentation at $\hbar = 0$. This sheaf is an $(\mathcal{A}_{X,c+d\theta}, \mathcal{A}_{X,c})$-bimodule where $d\theta$ is a character of the Lie algebra $\mathfrak{m}$ obtained by differentiating θ. By considering the tensor product with $\mathcal{A}^{\theta^m}_{X,c}$ over $\mathcal{A}_{X,c}$, we have a functor

$$\mathcal{A}^{\theta^m}_{X,c} \otimes_{\mathcal{A}_{X,c}} (—) : \mathcal{A}_{X,c}\text{-mod}_{\mathbb{C}^*\text{-equiv}} \longrightarrow \mathcal{A}_{X,c+md\theta}\text{-mod}_{\mathbb{C}^*\text{-equiv}}$$

for each $m \in \mathbb{Z}$. We have $\mathcal{A}^{\theta^m}_{X,c} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}_0 \simeq \mathcal{O}(m) \simeq \mathcal{O}(1)^{\otimes m}$, and the functor is an equivalence of categories whose quasi-inverse functor is given by $\mathcal{A}^{\theta^{-m}}_{X,c+md\theta}$. By applying "the functor of taking global sections" $\Gamma_F(X, —)$, we have functors

$$\Gamma_F(X, \mathcal{A}^{\theta^m}_{X,c}) \otimes_{A_c} (—) : A_c\text{-mod} \longrightarrow A_{c+md\theta}\text{-mod}$$

for each $m \in \mathbb{Z}$.

Theorem 4.2. *Assume that the functor* $\Gamma_F(X, \mathcal{A}_{X,c}^{\theta^m}) \otimes_{A_c} (\text{—})$ *is an equivalence of abelian categories (with the quasi-inverse given by* $\Gamma_F(X, \mathcal{A}_{X,c+md\theta}^{\theta^{-m}}) \otimes_{A_{c+md\theta}} (\text{—})$*) for all* $m \in \mathbb{Z}_{\geq 0}$*. Then,*

$$\Gamma_F(X, \text{—}) : \mathcal{A}_{X,c}\text{-mod}_{\mathbb{C}^*\text{-equiv}} \xrightarrow{\sim} A_c\text{-mod}$$

is an equivalence of abelian categories with the quasi-inverse $\mathcal{A}_{X,c} \otimes_{A_c} (\text{—})$*.*

The theorem is analogue of well-known theorem in the representation theory of simple Lie algebras, so called the Beilinson-Bernstein correspondence, studied in [BB1, BB2], and also in [BrKa]. For algebras obtained by quantum Hamiltonian reduction, it was first established for the rational Cherednik algebra of the symmetric group $\mathfrak{S}_n$ by M. Kashiwara and R. Rouquier in [KR], and later for some other cases in [BeKu], [DK] separately. Recently, K. McGerty and T. Nevins in [MN2] gave a general criterion when such an abelian equivalence holds by using Kashiwara's equivalence and Kirwan-Ness stratification.

In known cases of quantum Hamiltonian reduction, the sheaf of $\mathbb{C}[[\hbar]]$-algebras $\mathcal{A}_{X,c}$ is locally isomorphic to the $\hbar$-deformed Weyl algebra

$$\mathcal{D}_{\mathbb{C}^d,\hbar} = \mathbb{C}[[\hbar]][x_1, \ldots, x_d, y_1, \ldots, y_d]$$

with defining relation given by $[y_i, x_j] = \delta_{ij}\hbar$, $[x_i, x_j] = [y_i, y_j] = 0$. Thus the above equivalences connect the representation theory of A_c with microlocal analysis on the symplectic manifold X through the sheaf $\mathcal{A}_{X,c}$, and hence we have many applications in the representation theory of A_c.

First, for an A_c-module M, the support of the corresponding sheaf of modules $\mathcal{A}_{X,c} \otimes_{A_c} M$ in the symplectic manifold X is an invariant of modules. It is analogue of characteristic varieties in $\mathcal{D}$-module theory. We also have analogue of characteristic cycles, cycles on X (with multiplicities) associated with the module. For certain quantum Hamiltonian reduction A_c, characteristic cycles of some important A_c-modules were studied in [GS] (for the rational Cherednik algebra for $\mathfrak{S}_n$) and in [K1] (for the rational Cherednik algebra for $\mathbb{Z}/l\mathbb{Z}$).

Moreover, for the rational Cherednik algebra for $\mathbb{Z}/l\mathbb{Z}$, we can construct explicitly sheaves of modules corresponding to irreducible modules and standard modules in the category $\mathcal{O}$ of A_c, a highest weight category analogous to the Bernstein-Gelfand-Gelfand category for a simple Lie algebra. ([K2]) As a consequence, it follows that sheaves of modules corresponding to modules in the category $\mathcal{O}$ are regular holonomic in the sense of microlocal analysis. Conjecturally the same fact holds for the category $\mathcal{O}$ of other type of rational Cherednik algebra, but it is still an open problem.

4.2. **BRST cohomologies.** The construction of the algebra A_c as quantum Hamiltonian reduction associated with the M-action on $\mathcal{A}_{T^*\underline{X}}$ induces a certain cohomology, called a BRST cohomology. The BRST cohomology (or so called BRST reduction) is first introduced by theoretical physicist in the area of quantum field theory. Mathematically, it is known that the BRST cohomology gives a cohomological interpretation of the (quantum) Hamiltonian reduction (cf. [KS], [F]). In [K3], the explicit description of the BRST cohomology associated with the quantum Hamiltonian reduction is given in the case where $\underline{X}$ is a linear representation of the group M. We denote the BRST cohomology associated with the action of M on the ring of differential operators $\mathcal{D}(\underline{X})$ by $H^\bullet_{BRST,c}(\mathfrak{m}, \mathcal{D}(\underline{X}))$, where $\mathfrak{m}$ is the Lie algebra of M and c is the parameter of the quantum Hamiltonian reduction A_c. We can also define the sheaf version of the BRST cohomology, and it is denoted $\mathcal{H}^\bullet_{BRST,c}(\mathfrak{m}, \mathcal{A}_{(T^*\underline{X})^{ss}})$. Then, we have the following two isomorphisms of graded algebras (with respect to the degree of cohomologies) under certain conditions:

$$\mathcal{H}^{\bullet}_{BRST,c}(\mathfrak{m}, \mathcal{A}_{(T^*\underline{X})^{ss}}) \simeq \mathcal{A}_{X,c} \otimes_{\mathbb{C}} H^{\bullet}_{DR}(M), \tag{2}$$

$$H^{\bullet}_{BRST,c}(\mathfrak{m}, \mathcal{D}(\underline{X})) \simeq A_c \otimes_{\mathbb{C}} H^{\bullet}_{DR}(M), \tag{3}$$

where $H^{\bullet}_{DR}(M)$ is the de Rham cohomology of M. The former isomorphism follows from the geometrical conditions in Section 2.2, and indeed it is essentially a geometrical fact. On the other hand, to prove the latter isomorphism we need to make advantage of the representation theory of A_c. Indeed, by using the abelian and derived equivalences of categories in Section 4.1, we obtain (3) from (2).

4.3. **Isomorphism of quantizations.** Some noncommutative algebras give quantizations of the same Poisson algebra. In such a case, it is a natural problem if these quantizations are isomorphic with each other. But usually it is not easy to compare their different constructions directly. On the other hand, the deformation theory for deformation-quantization of a symplectic manifold is studied by R. Bezrukavnikov and D. Kaledin in [BeKa]. As an application of their result, I. Losev proved isomorphisms between certain noncommutative algebras, which are constructed by quantum Hamiltonian reduction in [Lo2]. For example, deformed preprojective algebras introduced in [CBH] and finite $\mathcal{W}$-algebras associated with a subregular nilpotent orbit in simple Lie algebra of type ADE are isomorphic with each other.

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