Boundary of Cohen-Macaulay cone and asymptotic behavior of system of ideals

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1 Introduction

On a smooth projective variety, we can define the intersection number for a given divisor and a given curve. By this pairing, we can define the numerical equivalence on divisors and curves. We get a (finitely generated) lattice if we divide the set of Weil divisors or curves by the numerical equivalence. In order to study the intersection pairing, we have some concepts of "positive" elements, e.g., ample, base point-free, nef, etc.. Consider the cone spanned by positive elements in the lattice tensored with the field of real numbers. This cone gives us many informations on the given algebraic variety.

In this note, we are interested in the intersection pairing around a fixed singular point of a scheme, or the vertex of the affine cone of a smooth projective variety. Let R be a Noetherian (Cohen-Macaulay) local ring corresponding to the given point. We first define a pairing between a finitely generated module, and a module of finite length and finite projective dimension. Consider the Grothendieck group of finitely generated R-modules, and divide it by the numerical equivalence. Then, we get a finitely generated lattice. It is natural to think that Cohen-Macaulay modules are positive elements under the pairing. So, we study the cone spanned by Cohen-Macaulay modules in the numerical Grothendieck group tensored with $\mathbb{R}$.

We always assume that R is a d-dimensional Noetherian Cohen-Macaulay local domain such that one of the following conditions are satisfied[1]:

[1] If either (a) or (b) is satisfied, there exists a regular alteration of $\operatorname{Spec} R$ by de Jong's

(a) R is a homomorphic image of an excellent regular local ring containing $\mathbb{Q}$.

(b) R is essentially of finite type over a field, $\mathbb{Z}$ or a complete DVR.

In this note, modules are always assumed to be finitely generated.

2 Intersection pairing on $\operatorname{Spec} R$ and the Cohen-Macaulay cone

Let $G_0(R)$ be the Grothendieck group of finitely generated R-modules. The symbol $[M]$ means the element in $G_0(R)$ corresponding to an R-module M. Let C_R be the category of modules of finite length and finite projective dimension. Here, note that $R/(x_1, \dots, x_d) \in C_R$ for a system of parameters $x_1, \dots, x_d$. In particular, C_R is not empty.[2] For $L \in C_R$, we define

$$\chi_L : G_0(R) \longrightarrow \mathbb{Z} \quad \text{by} \quad \chi_L([M]) = \textstyle\sum_i (-1)^i \ell_R(\mathrm{Tor}_i^R(L, M)).$$

Consider the map

$$C_R \times G_0(R) \to \mathbb{Z} \quad \text{defined by} \quad (L, [M]) \mapsto \chi_L([M]). \tag{1}$$

Here, we define *numerical equivalence* as follows. For $\alpha, \beta \in G_0(R)$,

$$\alpha \equiv \beta \stackrel{\text{def}}{\Longleftrightarrow} \chi_L(\alpha) = \chi_L(\beta) \ \text{ for any } \ L \in C_R.$$

Here, we put

$$\overline{G_0(R)} = G_0(R)/\{\alpha \in G_0(R) \mid \alpha \equiv 0\}.$$

By Theorem 3.1 and Remark 3.5 in [9], we have the following result.

Theorem 1 *$\overline{G_0(R)}$ is a finitely generated torsion-free abelian group.*

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theorem [6].

[2] By the new intersection theorem due to Roberts, we know that, for a Noetherian local ring R, C_R is not empty if and only if R is Cohen-Macaulay.

Remark 2 MCM (Maximal Cohen-Macaulay) modules behave as "positive elements" under the pairing (1) by the following reason.

Let L be an object in C_R. Then, by Auslander-Buchsbaum formula, we have

$$\operatorname{depth} L + \operatorname{pd}_R L = \operatorname{depth} R = d.$$

Then, we have $\operatorname{pd}_R L = d$. Let $\mathbb{F}.$ be the minimal free resolution of L. Then, it is very easy to check that the complex $\mathbb{F}.$ has a depth sensitive property, i.e., for any module N, we have

$$\operatorname{depth} N = d - \max\{i \mid H_i(\mathbb{F}. \otimes_R N) \neq 0\}.$$

We say that M is a MCM module if $\operatorname{depth} M = d$. By the depth sensitivity, if M is MCM, then $\operatorname{Tor}_i^R(L, M) = 0$ for any $i > 0$. Therefore, we have

$$\chi_L([M]) = \ell_R(L \otimes_R M) > 0.$$

By Auslander-Buchsbaum formula, any MCM module over a regular local ring is free. We say that a ring R is of finite (Cohen-Macaulay) representation type if there are only finitely many isomorphism classes of indecomposable MCM's. If R is of finite representation type, then R has only isolated singularity. It was proved that a Gorenstein local ring of finite representation type has a simple singularity. Simple singularities are of finite representation type. We refere the reader to Yoshino [17] for the representation theory of MCM's.

Bad Cohen-Macaulay rings have many MCM's in general. But, if we do not assume that R is Cohen-Macaulay, it is not known whether there exists an MCM module. This open problem is called the small Mac conjecture [5].

Example 3

1. If $L = R/(x_1, \ldots, x_d)$ for a system of parameters $x_1, \ldots, x_d$, then $\chi_L([R]) \neq 0$. Hence, $\overline{G_0(R)} \neq 0$.
2. If $d \leq 2$, then $\operatorname{rank} \overline{G_0(R)} = 1$. See Proposition 3.7 in [9].
3. Let X be a smooth projective variety with embedding $X \hookrightarrow \mathbb{P}^n$. Let R (resp. D) be the affine cone (resp. the very ample divisor) of this

embedding. Then, we have the following commutative diagram:

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \stackrel{\sim}{\longrightarrow} & A_*(R)_{\mathbb{Q}} & \stackrel{\sim}{\longleftarrow} & CH^{\cdot}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}(X)_{\mathbb{Q}} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{G_0(R)}_{\mathbb{Q}} & \stackrel{\sim}{\longrightarrow} & \overline{A_*(R)}_{\mathbb{Q}} & \stackrel{\phi}{\longleftarrow} & CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}} \end{array}$$

(a) By the commutativity of this diagram, ϕ is a surjection. Therefore, we have

$$\operatorname{rank} \overline{G_0(R)} \leq \dim_{\mathbb{Q}} CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}}. \tag{2}$$

(b) If $CH^{\cdot}(X)_{\mathbb{Q}} \simeq CH^{\cdot}_{num}(X)_{\mathbb{Q}}$, then ϕ is an isomorphism ([9], [15]). In this case, the equality holds in (2).

(c) There exists an example such that ϕ is not an isomorphism [15].

Further, Roberts and Srinivas [15] proved the following: Assume that the standard conjecture and Bloch-Beilinson conjecture are true. Then ϕ is an isomorphism if the defining ideal of R is generated by polynomials with coefficients in the algebraic closure of the prime field.

4. It is conjectured that $\overline{G_0(R)}_{\mathbb{Q}} \simeq \mathbb{Q}$ if R is complete intersection isolated singularity with d even.

It is true if R is the affine cone of a smooth projective variety X over $\mathbb{C}$ ([2]). In fact, since we have an injection

$$CH^i_{hom}(X)_{\mathbb{Q}} \longrightarrow H^{2i}(X, \mathbb{Q}) = \mathbb{Q}$$

and the natural surjection

$$CH^i_{hom}(X)_{\mathbb{Q}} \longrightarrow CH^i_{num}(X)_{\mathbb{Q}} \neq 0,$$

we know $CH^i_{num}(X)_{\mathbb{Q}} = \mathbb{Q}$ for each $i = 0, 1, \ldots, \dim X$. Here, remark that $H^{2i}(X, \mathbb{Q}) = \mathbb{Q}$ since the dimension of X is odd. Then, we have

$$CH^{\cdot}_{num}(X)_{\mathbb{Q}}/D\cdot CH^{\cdot}_{num}(X)_{\mathbb{Q}} = \mathbb{Q}.$$

Therefore, the rank of $\overline{G_0(R)}$ is one by 3 (a) as above.

Definition 4 *We define the Cohen-Macaulay cone as follows:*

$$C_{CM}(R) = \sum_{M:MCM} \mathbb{R}_{\geq 0}[M] \subset \overline{G_0(R)}_{\mathbb{R}}.$$

Here $\overline{G_0(R)}_{\mathbb{R}} = \overline{G_0(R)} \otimes_{\mathbb{Z}} \mathbb{R}$.

We refer the reader to [1] for basic properties on Cohen-Macaulay cones. It is easy to see that the dimension of the cone is equal to the rank of $\overline{G_0(R)}$. Further, we have

$$\overline{G_0(R)}_{\mathbb{R}} \supset C_{CM}(R)^- \supset C_{CM}(R) \supset Int(C_{CM}(R)^-) = Int(C_{CM}(R)) \ni [R],$$

where $C_{CM}(R)^-$ is the closure of $C_{CM}(R)$ with respect to the classical topology on $\overline{G_0(R)}_{\mathbb{R}}$, and $Int(-)$ is the interior.

If R is of finite representation type, then $C_{CM}(R)$ is a strongly convex polyhedral cone, in particular $C_{CM}(R)^- = C_{CM}(R)$.

We have no example that $C_{CM}(R)^-$ is not equal to $C_{CM}(R)$, or $C_{CM}(R)$ is not a polyhedral cone.

Remark that, for any $L \in C_R$, χ_L induces $\overline{\chi_L}$ which makes the following diagram commutative:

$$\begin{array}{ccc} G_0(R) & \xrightarrow{\chi_L} & \mathbb{Z} \\ \downarrow & \nearrow \overline{\chi_L} & \\ \overline{G_0(R)} & & \end{array}$$

The map $\overline{\chi_L}$ induces

$$(\overline{\chi_L})_{\mathbb{R}} : \overline{G_0(R)}_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

Let $x_1, \ldots, x_d$ be a system of parameters. Consider the map

$$\overline{\chi_{R/(\underline{x})}} : \overline{G_0(R)} \longrightarrow \mathbb{Z}.$$

Let $\mathbb{K}.$ be the Koszul complex with respect to $\underline{x}$. This map satisfies

$$\overline{\chi_{R/(\underline{x})}}([M]) = \operatorname{rank} M \cdot \overline{\chi_{R/(\underline{x})}}([R]),$$

since $\mathbb{K}.$ is the minimal free resolution of $R/(\underline{x})$ and $\mathbb{K}.$ admits this property. Therefore, we have a map

$$\mathrm{rk} : \overline{G_0(R)} \longrightarrow \mathbb{Z}$$

and

$$\mathrm{rk}_{\mathbb{R}} : \overline{G_0(R)}_{\mathbb{R}} \longrightarrow \mathbb{R}$$

defined by $\mathrm{rk}([M]) = \mathrm{rank}\, M$. (Here, $\mathrm{rk} = \frac{1}{\overline{\chi_{R/(\underline{x})}}([R])}\overline{\chi_{R/(\underline{x})}}$.)

Let F be the kernel of the map rk. Then, F is generated by cycles $[M]$ with $\dim M < d$. Thus, we have

$$\overline{G_0(R)} = \mathbb{Z}[R] \oplus F \quad \text{and} \quad \overline{G_0(R)}_{\mathbb{R}} = \mathbb{R}[R] \oplus F_{\mathbb{R}}.$$

Example 5

1. Put $R = k[x,y,z,w]_{(x,y,z,w)}/(xy - zw)$, where k is a field. Then, $F = \mathbb{Z}[R/(x,z)] \simeq \mathbb{Z}$. This ring has only three indecomposable MCM modules, R, (x,z) and (x,w).

   Then, the Cohen-Macaulay cone is spanned by

   $$[(x,z)] = ([R], -[R/(x,z)]) \quad \text{and} \quad [(x,w)] = ([R],[R/(x,z)])$$

   in $\overline{G_0(R)}_{\mathbb{R}} = \mathbb{R}[R] \oplus F_{\mathbb{R}}$.

2. Put $R = k[x_1, x_2, \ldots, x_6]_{(x_1,x_2,\ldots,x_6)}/(x_1x_2 + x_3x_4 + x_5x_6)$, where k is a field. Then, $F = \mathbb{Z}[R/(x_1,x_3,x_5)] \simeq \mathbb{Z}$. This ring has only three indecomposable MCM modules, R, M_1 and M_2, where M_1 and M_2 are MCM modules of rank 2.

   Then, the Cohen-Macaulay cone is spanned by

   $$[M_1] = (2[R],[R/(x_1,x_3,x_5)]) \quad \text{and} \quad [M_2] = (2[R], -[R/(x_1,x_3,x_5)])$$

   in $\overline{G_0(R)}_{\mathbb{R}} = \mathbb{R}[R] \oplus F_{\mathbb{R}}$.

   The Cohen-Macaulay cone of this ring is not spanned by classes of MCM modules of rank one.

3. Put $R = k[x,y,z,w]_{(x,y,z,w)}/(xy - f_1f_2\cdots f_t)$, where k is an algebraically closed field of characteristic zero. Here, we assume that $f_1, f_2, \ldots, f_t$ are pairwise coprime linear forms in $k[z,w]$. In this case, we have

   $$F = (\oplus_i \mathbb{Z}[R/(x,f_i)]) / \mathbb{Z}([R/(x,f_1)] + \cdots + [R/(x,f_t)]) \simeq \mathbb{Z}^{t-1}.$$

   We can prove that the Cohen-Macaulay cone is minimally spanned by the following $2^t - 2$ MCM's of rank one.

   $$\{\{(x, f_{i_1}f_{i_2}\cdots f_{i_s}) \mid 1 \le s < t, \quad 1 \le i_1 < i_2 < \cdots < i_s \le t\}$$

   This ring is of finite representation type if and only if $t \le 3$.

3 Maximal Cohen-Macaulay modules of rank one

Let R be a Noetherian standard graded normal domain such that R_0 is a field of characteristic zero. Assume that R has at most isolated singularity, and $[H^2_{R+}(R)]_0 = 0$. Then, it is known that R has at most finitely many MCM modules of rank one. (Prof. Flenner kindly taught me this result.)

If R is a Noetherian local ring with at most isolated singularity. Assume that R is a complete intersection and $\dim R \geq 4$. Then, R is factorial. In particular, R is only one MCM modules of rank one.

Assume that R is a Noetherian local ring of dimension 2. Even if R is a hypersurface with at most isolated singularity, R may have infinitely many MCM modules of rank one. (For example, the affine cone of an elliptic curve actually has infinitely many MCM modules of rank one.)

So, it is important to consider the case of $\dim R = 3$. In this section (Theorem 9), we show that R has only finitely many MCM's of rank one if R is a 3-dimensional isolated hypersurface singularity with desingularization.

By the following result, we know that $C_{CM}(R)^-$ is a strongly convex cone, that is, $C_{CM}(R)^-$ does not contain a line through the origin.

Lemma 6 *Let R be a d-dimensional Cohen-Macaulay local domain which satisfies (a) or (b) in the introduction. Then, $(\mathrm{rk}_{\mathbb{R}})^{-1} \cap C_{CM}(R)^-$ is a compact set.*

Corollary 7 *Assume that R is a Cohen-Macaulay local domain. Then, for any positive integer r,*

$$\{[M] \in \overline{G_0(R)} \mid M \textit{ is a MCM module of rank } r\ \}$$

is a finite subset of $\overline{G_0(R)}$.

Further, assume that R is a normal domain. Then, we have the *determinant map* (or the *first Chern class map*) $c_1 : G_0(R) \to A_{d-1}(R)$.

We can also define numerical equivalence on $A_{d-1}(R)$. Then, we define the class group modulo numerical equivalence to be

$$\overline{A_{d-1}(R)} = A_{d-1}(R)/\equiv .$$

By Proposition 3.7 and Example 4.1 in [9], we know that it is also a finitely generated torsion-free abelian group.

Here we can prove that there exists the map $\overline{c_1}$ which makes the following diagram commutative:

$$\begin{array}{ccc} G_0(R) & \xrightarrow{c_1} & A_{d-1}(R) \\ \downarrow & & \downarrow \\ \overline{G_0(R)} & \xrightarrow{\overline{c_1}} & \overline{A_{d-1}(R)} \end{array}$$

By the commutativity of the above diagram, we have the following:

Corollary 8 *Let R be a d-dimensional Cohen-Macaulay local normal domain. Assume that*

(*) the kernel of the natural map $A_{d-1}(R) \longrightarrow \overline{A_{d-1}(R)}$ is a finite group.

Then, for any positive integer r,

$$\{c_1([M]) \in A_{d-1}(R) \mid M \text{ is a MCM module of rank } r\ \}$$

is a finite subset of $A_{d-1}(R)$.

In particular, R has only finitely many MCM modules of rank one up to isomorphism.

Theorem 9 (Dao-Kurano, [2]) *Let R be a 3-dimensional isolated hypersurface singularity with desingularization. Then, the natural map*

$$A_2(R) \longrightarrow \overline{A_2(R)}$$

is an isomorphism. In particular (*) in Corollary 8 is satisfied. Therefore R has only finitely many MCM's of rank one.

Remark 10 *Put $B = \oplus_{n \geq 0} B_n = \mathbb{C}[B_1] = \mathbb{C}[y_0, y_1, \ldots, y_n]/I$, $R = B_{B_+}$, and $X = \mathrm{Proj}(B)$. Assume that X is smooth over $\mathbb{C}$. (Since $\dim R = d$, $\dim X = d-1$.)*

$$\begin{array}{ccccc} \mathrm{CH}^1(X) & \longrightarrow & \mathrm{CH}^1(X)/c_1(\mathcal{O}_X(1))\mathrm{CH}^0(X) & = & A_{d-1}(R) \\ \downarrow & & \downarrow f & & \\ \mathrm{CH}^1_{\mathrm{num}}(X) & \longrightarrow & \mathrm{CH}^1_{\mathrm{num}}(X)/c_1(\mathcal{O}_X(1))\mathrm{CH}^0_{\mathrm{num}}(X) & \xrightarrow{g} & \overline{A_{d-1}(R)} \end{array}$$

1. *Assume that R is a Cohen-Macaulay local normal ring with $d \geq 3$. Then,* $\mathrm{CH}^1(X)$ *is finitely generated and $f \otimes \mathbb{Q}$ is an isomorphism.*

2. *Assume that the ideal I is generated by some elements in $\overline{\mathbb{Q}}[y_0, y_1, \ldots, y_n]$. If some famous conjectures (the standard conjecture and Bloch-Beilinson conjecture) are true, then $g \otimes \mathbb{Q}$ is an isomorphism. (Roberts-Srinivas [15])*

Therefore, if R is a Cohen-Macaulay local normal ring with $d \geq 3$ such that X is defined over $\overline{\mathbb{Q}}$, and if some conjectures are true, then (*) is satisfied.

It is also proved in the case of positive characteristic.

If we remove the assumption that X is defined over $\overline{\mathbb{Q}}$, then there exists an example that $g \otimes \mathbb{Q}$ is not an isomorphism (Roberts-Srinivas [15]).

However, remark that if R is a standard graded Cohen-Macaulay normal graded domain over $\mathbb{C}$ with $\dim R \geq 3$. *Then, there exist only finitely many MCM modules of rank one.*

4 The fundamental class of a Noetherian local ring

We define the strictly nef cone $SN(R)$, and the fundamental class $\overline{\mu_R}$ for a Noetherian local domain R. They satisfy the following:

$$\begin{array}{ccccc} \overline{G_0(R)_{\mathbb{R}}} & \supset & SN(R) & \supset & C_{CM}(R) - \{0\} \\ \cup & & & & \\ \overline{G_0(R)_{\mathbb{Q}}} & \ni & \overline{\mu_R} & & \end{array}$$

The fundamental class is deeply related to the homological conjectures as in Fact 15.

We are mainly interested in the problem whether $\overline{\mu_R}$ is in such cones or not. Theorem 18 is the main result in this section, which states that if R is FFRT or F-rational, then $\overline{\mu_R}$ is in $C_{CM}(R)$. We shall give a corollary (Corollary 21).

Definition 11 We define the *strictly nef cone* by

$$SN(R) = \{\alpha \mid \ \chi_L(\alpha) > 0 \text{ for any } L \in C_R\}.$$

By the depth sensitivity, $\chi_L([M]) = \ell_R(H_0(L \otimes M)) > 0$ for any MCM module M $(\neq 0)$ and $L \in C_R$. Therefore,

$$SN(R) \supset C_{CM}(R) - \{0\}.$$

We can also define $SN(R)$ for non-Cohen-Macaulay local ring R using some perfect complexes instead of C_R.

Definition 12 We put

$$\mu_R = {\tau_R}^{-1}([\mathrm{Spec}\, R]) \in G_0(R)_{\mathbb{Q}},$$

where $\tau_R : G_0(R)_{\mathbb{Q}} \xrightarrow{\sim} A_*(R)_{\mathbb{Q}}$ is the singular Riemann-Roch map.

$$\begin{array}{ccc} G_0(R)_{\mathbb{Q}} & \longrightarrow & \overline{G_0(R)_{\mathbb{Q}}} \\ \mu_R & \mapsto & \overline{\mu_R} \end{array}$$

We call the image $\overline{\mu_R}$ in $\overline{G_0(R)_{\mathbb{Q}}}$ the *fundamental class* of R.

Remark that $\overline{\mu_R} \neq 0$ since $\mathrm{rank}_R\, \mu_R = 1$.

Put $R = T/I$, where T is a regular local ring. The map τ_R is defined using not only R but also T. Therefore, $\mu_R \in G_0(R)_{\mathbb{Q}}$ may depend on the choice of T.[3] However, we can prove that $\overline{\mu_R} \in \overline{G_0(R)_{\mathbb{Q}}}$ is independent of T (Theorem 5.1 in [9]).

We shall explain why we call $\overline{\mu_R}$ the fundamental class of R.

Remark 13 1. If $X = \mathrm{Spec}\, R$ is a d-dimensional affine variety over $\mathbb{C}$, we have the cycle map cl

$$\begin{array}{ccccc} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} & \xrightarrow{cl} & H_*(X, \mathbb{Q}) \\ \mu_R & \mapsto & [\mathrm{Spec}\, R] & \mapsto & \mu_X \end{array}$$

such that $cl([\mathrm{Spec}\, R])$ is the fundamental class μ_X in $H_{2d}(X, \mathbb{Q})$ in the usual sense, where $H_*(X, \mathbb{Q})$ is the Borel-Moore homology. Here μ_X is the generator of $H_{2d}(X, \mathbb{Q}) \simeq \mathbb{Z}$.

Hence, we call $\overline{\mu_R}$ the fundamental class of R.

[3]There is no example that the map τ_R actually depend on the choice of T. For some excellent rings, it had been proved that τ_R is independent of the choice of T (Proposition 1.2 in [8]).

2. Let R have a subring S such that S is a regular local ring and R is a localization of a finite extension of S. Let L be a finite-dimensional normal extension of $Q(S)$ containing $Q(R)$. Let B be the integral closure of R in L. Then, we have

$$\mu_R = \frac{1}{\operatorname{rank}_R B}[B] \text{ in } G_0(R)_{\mathbb{Q}}.$$

In particular, $\overline{\mu_R} = \frac{[B]}{\operatorname{rank}_R B}$ in $\overline{G_0(R)}_{\mathbb{Q}}$.

3. Assume that R is of characteristic $p > 0$ and F-finite. Assume that the residue class field is algebraically closed. By the singular Riemann-Roch theorem, we have

$$\overline{\mu_R} = \lim_{e \to \infty} \frac{[{}^e R]}{p^{de}} \text{ in } \overline{G_0(R)}_{\mathbb{R}},$$

where ${}^e R$ is the eth Frobenius direct image.

Example 14 1. If R is a complete intersection, then μ_R is equal to $[R]$ in $G_0(R)_{\mathbb{Q}}$, therefore $\overline{\mu_R} = [R]$ in $\overline{G_0(R)}_{\mathbb{Q}}$. There exists a Gorenstein ring such that $\overline{\mu_R} \neq [R]$. However there exist many examples of rings satisfying $\overline{\mu_R} = [R]$. Roberts ([12], [13]) proved the vanishing property of intersection multiplicity for rings satisfying $\overline{\mu_R} = [R]$.

2. Let R be a normal domain. Then, we have

$$\begin{array}{ccl} G_0(R)_{\mathbb{Q}} & \xrightarrow{\tau_R} & A_*(R)_{\mathbb{Q}} = A_d(R)_{\mathbb{Q}} \oplus A_{d-1}(R)_{\mathbb{Q}} \oplus \cdots \\ [R] & \mapsto & [\operatorname{Spec} R] - \frac{K_R}{2} + \cdots \\ [\omega_R] & \mapsto & [\operatorname{Spec} R] + \frac{K_R}{2} + \cdots \end{array}$$

If $\tau_R^{-1}(K_R) \neq 0$ in $\overline{G_0(R)}_{\mathbb{Q}}$, then $[R] \neq \overline{\mu_R}$.

Sometimes $\overline{\mu_R} = \frac{1}{2}([R] + [\omega_R])$ is satisfied. But it is not true in general.

3. Let $R = k[x_{ij}]/I_2(x_{ij})$, where (x_{ij}) is the generic $(m+1) \times (n+1)$-matrix, and k is a field. Suppose $0 < m \leq n$.

Then, we have

$$
\begin{array}{rcl}
G_0(R)_{\mathbb{Q}} \simeq \overline{G_0(R)_{\mathbb{Q}}} & \simeq & \mathbb{Q}[a]/(a^{m+1}) \\
{[R]} & \mapsto & \left(\frac{a}{1-e^{-a}}\right)^m \left(\frac{-a}{1-e^{a}}\right)^n \\
 & & = 1 + \frac{1}{2}(m-n)a + \frac{1}{24}(\cdots)a^2 + \cdots \\
{[\omega_R]} & \mapsto & \left(\frac{-a}{1-e^{a}}\right)^m \left(\frac{a}{1-e^{-a}}\right)^n \\
\overline{\mu_R} & \mapsto & 1 \\
\tau_R^{-1}(K_R) & \mapsto & (n-m)a
\end{array}
$$

Here, we shall explain the relationship between the fundamental class $\overline{\mu_R}$ and homological conjectures.

Fact 15 1. The small Mac conjecture is true if and only if $\overline{\mu_R} \in C_{CM}(R)$ for any R.

Even if R is an equi-characteristic Gorenstein ring, it is not known whether $\overline{\mu_R}$ is in $C_{CM}(R)$ or not. If R is a complete intersection, then $\overline{\mu_R} = [R] \in C_{CM}(R)$ as in 1) in Example 14.

2. If $\overline{\mu_R} = [R]$ in $\overline{G_0(R)_{\mathbb{Q}}}$, then the vanishing property of intersection multiplicity holds (Roberts [12], [13]).

3. Roberts [14] proved $\overline{\mu_R} \in SN(R)$ if $ch(R) = p > 0$. Using it, he proved the new intersection theorem in the mixed characteristic case.

4. $\overline{\mu_R} \in SN(R)$ if R contains a field (Kurano-Roberts [11]). *Even if R is a Gorenstein ring (of mixed characteristic), we do not know whether $\overline{\mu_R} \in SN(R)$ or not.*

5. If $\overline{\mu_R} \in SN(R)$ for any R, then Serre's positivity conjecture is true in the case where one of two modules is (not necessary maximal) Cohen-Macaulay.

If $\overline{\mu_R} \in C_{CM}(R)$ for any R, then small Mac conjecture is true, and so Serre conjecture is true in the case.

Remark 16 1. If R is Cohen-Macaulay of characteristic $p > 0$, then ${}^eR$ is a MCM module. Since $\overline{\mu_R}$ is the limit of $[{}^eR]/p^{de}$ in $\overline{G_0(R)_{\mathbb{R}}}$, $\overline{\mu_R}$ is contained in $C_{CM}(R)^-$. *In the case where R is not of characteristic*

$p > 0$, we do not know whether $\overline{\mu_R}$ is contained in $C_{CM}(R)^-$ even if R is Gorenstein.

2. As we have already seen, if R is Cohen-Macaulay, then $[R] \in Int(C_{CM}(R)) \subset C_{CM}(R)$.

   There is an example of non-Cohen-Macaulay ring R such that $[R] \notin SN(R)$.[4] On the other hand, it is expected that $\overline{\mu_R} \in SN(R)$ for any R. Therefore, for the non-Cohen-Macaulay local ring R, $\overline{\mu_R}$ behaves better than $[R]$ in a sence.

The fundamental class $\overline{\mu_R}$ is deeply related to homological conjectures. Therefore, we propose the following question.

Question 17 Assume that R is a "good" Cohen-Macaulay local domain (for example, equi-characteristic, Gorenstein, etc). Is $\overline{\mu_R}$ in $C_{CM}(R)$?

We can prove the following:

Theorem 18 (Kurano-Ohta [10]) *Assume that R is an F-finite Cohen-Macaulay local domain of characteristic $p > 0$ with residue class field algebraically closed.*

1. *If R is FFRT, then $\overline{\mu_R}$ is contained in $C_{CM}(R)$.*
2. *If R is F-rational, then $\overline{\mu_R}$ is contained in $Int(C_{CM}(R))$.*

The most important point in this proof is to use the dual F-signature defined by Sannai [16].

Remark 19 If the rank of $\overline{G_0(R)}$ is one for a Cohen-Macaulay local domain R, then $\overline{\mu_R} \in C_{CM}(R)$.

If R is a toric ring (a normal semi-group ring over a field k), then we can prove $\overline{\mu_R} \in C_{CM}(R)$ as in the case of FFRT without assuming that $ch(k)$ is positive.

[4] It was conjectured that $[R] \in SN(R)$.

Problem 20 1. As in the above proof, if there exists a MCM module in $Int(C_{CM}(R))$ such that its generalized F-signature or its dual F-signature is positive, then $\overline{\mu_R}$ is in $Int(C_{CM}(R)^-)$.

Without assuming that R is F-rational, do there exist such a MCM module?

2. How do we make mod p reduction? (the case of rational singularity)

3. If R is Cohen-Macaulay, is $\overline{\mu_R}$ in $C_{CM}(R)^-$? If R is a Cohen-Macaulay ring containing a field of positive characteristic, then $\overline{\mu_R}$ in $C_{CM}(R)^-$ as in 1) in Remark 16.

4. If R is of finite representation type, is $\overline{\mu_R}$ in $C_{CM}(R)$?

5. Find more examples of $C_{CM}(R)$ and $SN(R)$.

In order to prove the following corollary, we use the fact $\overline{\mu_R} \in Int(C_{CM}(R))$ for some F-rational ring R.

Corollary 21 (Chan-Kurano [1]) *Let d be a positive integer and p a prime number. Let ϵ_0, ϵ_1, ..., ϵ_d be integers such that*

$$\epsilon_i = \begin{cases} 1 & i = d, \\ -1,\ 0\ or\ 1 & d/2 < i < d, \\ 0 & i \leq d/2. \end{cases}$$

Then, there exists a d-dimensional Cohen-Macaulay local ring R of characteristic p, a maximal primary ideal I of R of finite projective dimension, and positive rational numbers α, β_{d-1}, β_{d-2},..., β_0 such that

$$\ell_R(R/I^{[p^n]}) = \epsilon_d \alpha p^{dn} + \sum_{i=0}^{d-1} \epsilon_i \beta_i p^{in}$$

for any $n > 0$.

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