ATOM SPECTRA OF GROTHENDIECK CATEGORIES

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Abstract. This paper explains recent progress on the study of Grothendieck categories using the atom spectrum, which is a generalization of the prime spectrum of a commutative ring. As a part, we give a classification of localizing subcategories which can be applied to both locally noetherian schemes and noncommutative noetherian rings. It is shown that the atom spectrum of a Grothendieck category can have a rich poset structure compared with the prime spectrum of a commutative ring. We also show some properties on minimal elements of the atom spectrum for noncommutative noetherian rings.

1. Introduction

The aim of this paper is to explain recent progress on the study of Grothendieck categories. We investigate a Grothendieck category by using a kind of spectrum, which we call the *atom spectrum*. A typical example of a Grothendieck category is the category of modules over a ring. In the case where the ring is commutative, the atom spectrum of the module category coincides with the prime spectrum of the commutative ring. Therefore this theory can be regarded as an attempt to generalize the notion of the prime spectrum to noncommutative rings. It seems possible to understand and reformulate some classical noncommutative ring theory from the categorical viewpoint.

The theory of atom spectrum is not only for the study of noncommutative rings. Another example of a Grothendieck category is the category of quasi-coherent sheaves of a scheme. We can show that the atom spectrum of the category of quasi-coherent sheaves of a locally noetherian scheme coincides with the set of points of the scheme, and as a consequence, we can show a classification of localizing subcategories in a general setting including both the case of locally noetherian schemes and the case of noncommutative noetherian rings.

The reader may find the details of this paper in [Kan12a], [Kan12b], [Kan13], and [Kan14]. The reader who is unfamiliar with terms of abelian categories may be referred to [Pop73] or [Ste75].

We start with the definition of a Grothendieck category.

Definition 1.1. An abelian category $\mathcal{A}$ is called a *Grothendieck category* if it satisfies the following conditions.

(1) $\mathcal{A}$ admits arbitrary direct sums (and hence arbitrary direct limits), and for every direct system of short exact sequences in $\mathcal{A}$, its direct limit is also a short exact sequence.
(2) $\mathcal{A}$ has a generator G, that is, every object in $\mathcal{A}$ is isomorphic to a quotient object of the direct sum of some copies of G.

As we mentioned, the category $\operatorname{Mod} \Lambda$ of right modules over a ring Λ and the category $\operatorname{QCoh} X$ of quasi-coherent sheaves on a scheme X (see [Con00, Lemma 2.1.7]) are Grothendieck categories.

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One might think the notion of Grothendieck categories is quite an abstract setting given in order to include module categories. However, it is shown that every Grothendieck category is a part of some module category.

Theorem 1.2 (Gabriel and Popescu [PG64, Proposition])**.** *Let $\mathcal{A}$ be a Grothendieck category. Then there exist a ring Λ and a localizing subcategory $\mathcal{X}$ of* $\operatorname{Mod}\Lambda$ *such that $\mathcal{A}$ is equivalent to* $(\operatorname{Mod}\Lambda)/\mathcal{X}$.

In this paper, we adopt Grothendieck categories as main objects to study. Recall that for a commutative ring R, the prime spectrum $\operatorname{Spec} R$ plays an fundamental role. For a Grothendieck category $\mathcal{A}$, we will consider the *atom spectrum* $\operatorname{ASpec}\mathcal{A}$.

2. Atom spectrum

From now on, let $\mathcal{A}$ be a Grothendieck category. The atom spectrum of a Grothendieck category is defined by using the notion of monoform objects.

Definition 2.1. A nonzero object H in $\mathcal{A}$ is called *monoform* if for each nonzero subobject L of H, no nonzero subobject of H is isomorphic to a subobject of H/L.

In the case of a commutative ring, the following result shows how monoform objects are related to prime ideals.

Proposition 2.2 ([Sto72, Lemma 1.5])**.** *Let R be a commutative ring and $\mathfrak{a}$ an ideal of R. Then $R/\mathfrak{a}$ is a monoform object in* $\operatorname{Mod} R$ *if and only if $\mathfrak{a}$ is a prime ideal of R.*

We state basic properties of monoform objects.

Proposition 2.3. *Let H be a monoform object in $\mathcal{A}$.*

(1) ([Kan12a, Proposition 2.2]) *Every nonzero subobject of H is again monoform.*
(2) ([Kan12a, Proposition 2.6]) *H is uniform, that is, for every nonzero subobjects L_1 and L_2 of H, we have $L_1 \cap L_2 \neq 0$.*

Even in the case of a commutative ring R, the collection of monoform objects is quite different from the set of prime ideals. Indeed, it is known that the residue field $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a monoform object in $\operatorname{Mod} R$ for each prime ideal $\mathfrak{p}$ of R ([Sto72, p. 626]). Hence all its submodules are monoform. See [Kan12a, Example 8.3] for an example of a noncommutative ring. In order to obtain a generalization of the prime spectrum of a commutative ring, we introduce an equivalence relation between monoform objects.

Definition 2.4. We say that monoform objects H_1 and H_2 in $\mathcal{A}$ are *atom-equivalent* (denoted by $H_1 \sim H_2$) if there exists a nonzero subobject of H_1 isomorphic to a subobject of H_2.

Definition 2.5. The *atom spectrum* $\operatorname{ASpec}\mathcal{A}$ of $\mathcal{A}$ is defined by

$$\operatorname{ASpec}\mathcal{A} = \frac{\{\text{monoform objects in } \mathcal{A}\}}{\sim}.$$

Each element of $\operatorname{ASpec}\mathcal{A}$ is called an *atom* in $\mathcal{A}$. For each monoform object H in $\mathcal{A}$, the equivalence class of H is denoted by $\overline{H}$.

The notion of atoms was originally introduced by Storrer [Sto72], and the generalization to abelian categories was stated in [Kan12a].

The following result shows that the atom spectrum is a generalization of the prime spectrum of a commutative ring.

Theorem 2.6 (Storrer [Sto72, p. 631])**.** *Let R be a commutative ring. Then the map*

$$\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$$

given by

$$\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$$

is bijective.

For a locally noetherian scheme X, the atom spectrum of $\operatorname{QCoh} X$ coincides with the set of points of X.

Theorem 2.7 ([Kan14, Theorem 7.6]). *Let X be a locally noetherian scheme. Then the map*

$$|X| \to \operatorname{ASpec}(\operatorname{QCoh} X)$$

given by

$$x \mapsto j_{x*}k(x)$$

is bijective, where $k(x)$ is the residue field of x, and $j_x\colon \operatorname{Spec} \mathcal{O}_{X,x} \to X$ is the canonical morphism.

Matlis' correspondence between the prime ideals and the indecomposable injective modules can be generalized to a wide class of Grothendieck categories including the category $\operatorname{Mod} \Lambda$ for a right noetherian ring Λ.

For an object M in a Grothendieck category $\mathcal{A}$, the injective envelope $E(M)$ of M always exists and it is unique up to isomorphism (see [Pop73, Theorem 10.10]).

We recall the statement of Matlis' correspondence.

Theorem 2.8 (Matlis [Mat58, Proposition 3.1]). *Let R be a commutative noetherian ring. Then the map*

$$\operatorname{Spec} R \to \frac{\{\, \textit{indecomposable injective } R\textit{-modules} \,\}}{\cong}$$

given by

$$\mathfrak{p} \mapsto E(R/\mathfrak{p})$$

is bijective.

In order to generalize Matlis' correspondence, we need to consider some noetherianness of a Grothendieck category. The notion of the locally noetherianness is well-investigated one.

Definition 2.9. A Grothendieck category $\mathcal{A}$ is called *locally noetherian* if there exists a generating set $\mathcal{G}$ of $\mathcal{A}$ consisting of noetherian objects, that is, $\mathcal{A}$ admits a set $\mathcal{G}$ of noetherian objects such that $\bigoplus_{G \in \mathcal{G}} G$ is a generator of $\mathcal{A}$.

For a ring Λ, the Grothendieck category $\operatorname{Mod} \Lambda$ is locally noetherian if and only if Λ is right noetherian. Therefore the following generalization can be applied to right noetherian rings.

Theorem 2.10 ([Kan12a, Theorem 5.9]; see also [Sto72, Corollary 2.5]). *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then the map*

$$\operatorname{ASpec} \mathcal{A} \to \frac{\{\, \textit{indecomposable injective objects in } \mathcal{A} \,\}}{\cong}$$

given by

$$\overline{H} \mapsto E(H)$$

is bijective.

3. Classification of localizing subcategories

In this section, we state a classification of localizing subcategories.

Definition 3.1. A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called a *localizing subcategory* if the following conditions are satisfied.

(1) $\mathcal{X}$ is closed under subobjects, quotient objects, and extensions. In other words, for every exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathcal{A}$, we have $M \in \mathcal{X}$ if and only if $L, N \in \mathcal{X}$.

(2) $\mathcal{X}$ is closed under arbitrary direct sums, that is, for every set $\mathcal{S}$ of objects in $\mathcal{X}$, we have $\bigoplus_{M \in \mathcal{S}} M \in \mathcal{X}$.

We recall a classification of localizing subcategories for a commutative noetherian ring. This classification given by [Gab62] is regarded as an origin of many kinds of classification of subcategories.

For a commutative ring R, we say that a subset Φ of $\operatorname{Spec} R$ is *closed under specialization* if for every $\mathfrak{p} \subset \mathfrak{q}$ in $\operatorname{Spec} R$, the assertion $\mathfrak{p} \in \Phi$ implies $\mathfrak{q} \in \Phi$.

Theorem 3.2 (Gabriel [Gab62, Proposition VI.4]). *Let R be a commutative noetherian ring. Then the map*

$$\{\, \textit{localizing subcategories of } \operatorname{Mod} R \,\} \to \{\, \textit{specialization-closed subsets of } \operatorname{Spec} R \,\}$$

given by

$$\mathcal{X} \mapsto \bigcup_{M \in \mathcal{X}} \operatorname{Supp} M$$

is bijective. The inverse map is given by

$$\Phi \mapsto \{\, M \in \operatorname{Mod} R \mid \operatorname{Supp} M \subset \Phi \,\}.$$

The key notion to generalize Gabriel's classification is the "support" of an object in a Grothendieck category. It is defined in terms of atoms as follows.

Definition 3.3. For each object M in $\mathcal{A}$, define the subset $\operatorname{ASupp} M$ of $\operatorname{ASpec} \mathcal{A}$ by

$$\operatorname{ASupp} M = \{\, \overline{H} \in \operatorname{ASpec} \mathcal{A} \mid H \cong L'/L \text{ for some } L \subset L' \subset M \,\}.$$

This is called the *atom support* of M.

Proposition 3.4 ([Kan13, Proposition 3.2]). *The set*

$$\{\, \operatorname{ASupp} M \mid M \in \mathcal{A} \,\}$$

satisfies the axioms of open subsets of $\operatorname{ASpec} \mathcal{A}$.

This simple proposition is quite impressive from the viewpoint of ring theory. For a commutative ring R, the set of subsets of the form $\operatorname{Supp} M$ is exactly the set of specialization-closed subsets, and hence it is also closed under infinite intersection. However, this is not necessarily true for a Grothendieck category. Indeed, Example 4.3 gives a counter-example.

We call the topology on $\operatorname{ASpec} \mathcal{A}$ defined by Proposition 3.4 the *localizing topology*.

We define maps which will be used in the generalized classification of localizing subcategories.

Definition 3.5.

(1) For a full subcategory $\mathcal{X}$ of $\mathcal{A}$, define the subset $\operatorname{ASupp} \mathcal{X}$ of $\operatorname{ASpec} \mathcal{A}$ by

$$\operatorname{ASupp} \mathcal{X} = \bigcup_{M \in \mathcal{X}} \operatorname{ASupp} M.$$

(2) For a subset Φ of $\operatorname{ASpec} \mathcal{A}$, define the full subcategory $\operatorname{ASupp}^{-1} \Phi$ of $\mathcal{A}$ by

$$\operatorname{ASupp}^{-1} \Phi = \{\, M \in \mathcal{A} \mid \operatorname{ASupp} M \subset \Phi \,\}.$$

We introduce a class of Grothendieck categories, which includes all locally noetherian Grothendieck categories, in particular $\operatorname{Mod} \Lambda$ for a right noetherian ring Λ, and $\operatorname{QCoh} X$ for a locally noetherian scheme X (which is not necessarily a locally noetherian Grothendieck category. See [Har66, p. 135, Example]).

Definition 3.6. We say that a Grothendieck category $\mathcal{A}$ has *enough atoms* if $\mathcal{A}$ satisfies the following conditions.

(1) Every injective object in $\mathcal{A}$ has an indecomposable decomposition.
(2) Each indecomposable injective object in $\mathcal{A}$ is isomorphic to $E(H)$ for some monoform object H in $\mathcal{A}$.

See [Kan14] for more details on Grothendieck categories with enough atoms. It is shown in [Kan14, Theorem 7.6] that the Grothendieck category $\operatorname{QCoh} X$ has enough atoms for every locally noetherian scheme X.

Theorem 3.7 ([Kan14, Theorem 6.8]; see also [Her97, Theorem 3.8], [Kra97, Corollary 4.3], and [Kan12a, Theorem 5.5])**.** *Let $\mathcal{A}$ be a Grothendieck category with enough atoms. Then the map*

$$\{\, \textit{localizing subcategories of } \mathcal{A} \,\} \to \{\, \textit{specialization-closed subsets of } \operatorname{ASpec}\mathcal{A} \,\}$$

given by

$$\mathcal{X} \mapsto \bigcup_{M \in \mathcal{X}} \operatorname{ASupp} M$$

is bijective. The inverse map is given by

$$\Phi \mapsto \{\, M \in \mathcal{A} \mid \operatorname{ASupp} M \subset \Phi \,\}.$$

For a localizing subcategory $\mathcal{X}$ of $\mathcal{A}$, it is known that the categories $\mathcal{A}$ and $\mathcal{A}/\mathcal{X}$ are Grothendieck categories (see [Pop73, Corollary 4.6.2]). It is natural to ask how their atom spectra are related to each other.

Proposition 3.8 ([Kan13, Proposition 5.12 and Theorem 5.17]; see also [Kra97, Corollary 4.4] and [Her97, Proposition 3.6])**.** *Let $\mathcal{X}$ be a localizing subcategory.*

(1) $\operatorname{ASpec}\mathcal{X}$ *is homeomorphic to the open subset* $\operatorname{ASupp}\mathcal{X}$ *of* $\operatorname{ASpec}\mathcal{A}$.
(2) $\operatorname{ASpec}(\mathcal{A}/\mathcal{X})$ *is homeomorphic to the closed subset* $\operatorname{ASpec}\mathcal{A} \setminus \operatorname{ASupp}\mathcal{X}$ *of* $\operatorname{ASpec}\mathcal{A}$.

In particular, under the identifications by these homeomorphisms, we have

$$\operatorname{ASpec}\mathcal{A} = \operatorname{ASpec}\mathcal{X} \amalg \operatorname{ASpec}\frac{\mathcal{A}}{\mathcal{X}}$$

as a set.

4. Partial order

In this section, we introduce a partial order on the atom spectrum and investigate its structure.

Definition 4.1. Let $\alpha, \beta \in \operatorname{ASpec}\mathcal{A}$. We write $\alpha \leq \beta$ if α belongs to the topological closure $\overline{\{\beta\}}$ of β with respect to the localizing topology.

In fact, the relation $\leq$ is a partial order on $\operatorname{ASpec}\mathcal{A}$ (see [Kan13, Proposition 3.5]). The following result shows that this is a generalization of the inclusion relation between prime ideals of a commutative ring.

Proposition 4.2 ([Kan13, Proposition 4.3])**.** *For a commutative ring R, the bijection in Theorem 2.6 gives an isomorphism*

$$(\operatorname{Spec} R, \subset) \cong (\operatorname{ASpec}(\operatorname{Mod} R), \leq)$$

of posets.

For a commutative ring R, the open subsets of $\operatorname{Spec} R$ with respect to the localizing topology is exactly the specialization-closed subsets. Therefore the localizing topology on $\operatorname{Spec} R$ and the poset (partially ordered set) structure of $\operatorname{Spec} R$ can be recovered from each other. However, as the next example shows, the localizing topology cannot necessarily be recovered from the poset structure for a Grothendieck category.

Example 4.3 ([Pap02, Example 4.7])**.** Let k be a field. We consider the graded ring $k[x]$ with $\deg x = 1$. The category $\operatorname{GrMod} k[x]$ of $\mathbb{Z}$-graded $k[x]$-modules with degree-preserving homomorphisms is a locally noetherian Grothendieck category. For each object M in $\operatorname{GrMod} k[x]$ and $i \in \mathbb{Z}$, the object $M(i)$ in $\operatorname{GrMod} k[x]$ is defined by $M(i)_j = M_{i+j}$. Let $S := k[x]/(x)$. Then we have

$$\operatorname{ASpec}(\operatorname{GrMod} k[x]) = \{\overline{k[x]}\} \cup \{\, \overline{S(i)} \mid i \in \mathbb{Z} \,\}.$$

Note that $\overline{k[x]} = \overline{k[x](i)}$ for each $i \in \mathbb{Z}$ and that $\overline{S(i)} = \overline{S(j)}$ if and only if $i = j$.

A subset Φ of $\operatorname{ASpec}(\operatorname{GrMod} k[x])$ is open if and only if $\overline{k[x]} \notin \Phi$ or there exists $n \in \mathbb{Z}$ such that $\Phi_n \subset \Phi$, where

$$\Phi_n := \{\overline{k[x]}\} \cup \{\, \overline{S(i)} \mid i \leq n \,\}.$$

Although all Φ_n are open, their intersection

$$\bigcap_{n\in\mathbb{Z}} \Phi_n = \{\overline{k[x]}\}$$

is not open. Since every element of $\operatorname{ASpec}(\operatorname{GrMod} k[x])$ is a closed point, we have $\alpha \leq \beta$ in $\operatorname{ASpec}(\operatorname{GrMod} k[x])$ if only if $\alpha = \beta$.

Since we have the naturally defined partial order on $\operatorname{ASpec} \mathcal{A}$, it is expected to investigate its general property. Let us recall the case of commutative rings. For every commutative ring R, the poset $\operatorname{Spec} R$ has a maximal element and a minimal element. Some other properties of $\operatorname{Spec} R$ were also known (see for example, [Kap74, Theorem 11]). The next theorem, essentially shown by Hochster [Hoc69], states all general properties of the poset $\operatorname{Spec} R$. The precise statement was given by Speed [Spe72].

Theorem 4.4 (Hochster [Hoc69, Proposition 10] and Speed [Spe72, Corollary 1])**.** *Let P be a poset. Then the following assertions are equivalent.*

1. *There exists a commutative ring R such that $P \cong \operatorname{Spec} R$ as a poset.*
2. *P is an inverse limit of finite posets.*

We establish the same type of result for Grothendieck categories, but the conclusion is quite different from the case of commutative rings.

Theorem 4.5 ([Kan13, Theorem 7.27])**.** *For every poset P, there exists a Grothendieck category $\mathcal{A}$ such that $P \cong \operatorname{ASpec} \mathcal{A}$ as a poset.*

The construction uses colored quivers. See [Kan13] for the details.

Theorem 4.5 shows that there are quite various kinds of Grothendieck categories compared with commutative rings. By combining this theorem with Theorem 1.2 and Proposition 3.8, we also realize a diversity of noncommutative rings.

Corollary 4.6 ([Kan13, Corollary 5.19])**.** *For every poset P, there exists a ring Λ such that P is homeomorphic to a closed subset of* $\operatorname{ASpec}(\operatorname{Mod} \Lambda)$.

From now on, we state some results on the poset structure of the atom spectrum of a Grothendieck category with some noetherian property.

Proposition 4.7 ([Kan13, Proposition 4.6])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then* $\operatorname{ASpec} \mathcal{A}$ *satisfies the ascending chain condition.*

The previous proposition is expected from the analogous result on commutative noetherian rings. On the other hand, we will see a different phenomenon about minimal elements. Denote by $\operatorname{AMin} \mathcal{A}$ the set of minimal elements of $\operatorname{ASpec} \mathcal{A}$.

Proposition 4.8 ([Kan13, Proposition 8.2])**.** *There exists a locally noetherian Grothendieck category $\mathcal{A}$ such that* $\operatorname{AMin} \mathcal{A} = \emptyset$.

We regard this proposition as a consequence of the weakness of the condition of the locally noetherianness. Instead of this condition, we consider a Grothendieck category having a noetherian generator. Note that for every right noetherian ring Λ, the Grothendieck category $\operatorname{Mod} \Lambda$ has the noetherian generator Λ. We obtain the following result with an impressive proof.

Theorem 4.9. *Let $\mathcal{A}$ be a Grothendieck category with a noetherian generator.*

1. ([Kan13, Proposition 4.7]) *For each $\beta \in \operatorname{ASpec} \mathcal{A}$, there exists $\alpha \in \operatorname{AMin} \mathcal{A}$ such that $\alpha \leq \beta$.*
2. ([K]) $\operatorname{AMin} \mathcal{A}$ *is a finite set.*

Sketch of proof. (2) It can be shown that $\Phi := \operatorname{ASpec} \mathcal{A} \setminus \operatorname{AMin} \mathcal{A}$ is an open subset of $\operatorname{ASpec} \mathcal{A}$. Let $\mathcal{X} := \operatorname{ASupp}^{-1} \Phi$. Then we have $\operatorname{ASpec}(\mathcal{A}/\mathcal{X}) = \operatorname{AMin} \mathcal{A}$. Let G be a noetherian generator of $\mathcal{A}$ and G' its image in $\mathcal{A}/\mathcal{X}$. Then G' is a generator of $\mathcal{A}/\mathcal{X}$ which is of finite length. Therefore $\operatorname{ASpec}(\mathcal{A}/\mathcal{X}) = \operatorname{ASupp} G'$ is a finite set. □

Note the following result on Grothendieck categories.

Theorem 4.10 (Năstăsescu [Nas81, Theorem 3.3]). *Let $\mathcal{A}$ be a Grothendieck category with an artinian generator. Then there exists a right artinian ring Λ such that $\mathcal{A} \cong \operatorname{Mod} \Lambda$.*

For a given right noetherian ring Λ, the category $\operatorname{Mod} \Lambda$ is a Grothendieck category with the noetherian generator Λ. By the above argument, there exists a right artinian ring Λ' such that $\mathcal{A}/\mathcal{X} \cong \operatorname{Mod} \Lambda'$, where $\mathcal{X} = \operatorname{ASupp}^{-1}(\operatorname{ASpec} \mathcal{A} \setminus \operatorname{AMin} \mathcal{A})$. In particular, $\operatorname{AMin}(\operatorname{Mod} \Lambda) = \operatorname{ASpec}(\operatorname{Mod} \Lambda')$. Consequently, we obtain a right artinian ring (unique up to Morita equivalence) from a right noetherian ring in a categorical way.

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