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Mabuchi solitons and Mabuchi constants on Fano admissible manifolds

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Abstract. In this paper, we study the existence of Mabuchi solitons on admissible manifolds as defined by Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman. We prove that a Fano admissible manifold admits a Mabuchi soliton if and only if the Mabuchi constant is less than 1. We also provide an explicit formula for the Mabuchi constant on Fano admissible manifolds, which generalizes that of Mabuchi. Using this formula, we completely determine the existence and non-existence of Mabuchi solitons on Fano admissible manifolds over the complex projective space $\mathbf{P}^n$.

1. Introduction

1.1. Background

A *Mabuchi soliton*, introduced by Mabuchi in [**8**], is a generalization of a Kähler–Einstein metric. Let X be an n-dimensional Fano manifold, and let G be a maximal compact subgroup of $\mathrm{Aut}^0(X)$. Let ω be a G-invariant Kähler metric on X representing $2\pi c_1(X)$. By the dd^c-lemma, there exists a unique function $h_\omega \in C^\infty(X,\mathbf{R})^G$ satisfying

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c h_\omega, \quad \int_X (1 - e^{h_\omega}) \frac{\omega^n}{n!} = 0.$$

This function h_ω is called the *Ricci potential* of ω.

A Kähler metric $\omega \in 2\pi c_1(X)^G$ is called a *Mabuchi soliton* if the gradient

$$\mathrm{grad}_\omega (1 - e^{h_\omega})$$

with respect to ω is a real holomorphic vector field on X. It is immediate that any G-invariant Kähler–Einstein metric is a Mabuchi soliton. Moreover, a Mabuchi soliton is a Kähler–Einstein metric if and only if the Futaki invariant of X vanishes (see [**8**, Section 1]).

One motivation for studying Mabuchi solitons comes from the moment map interpretation of the Ding functional. Let $\mathcal{H}(X)^G$ be the space of all G-invariant Kähler metrics representing $2\pi c_1(X)$. Then, Donaldson showed in [**4**] that the map

$$\mathcal{H}(X)^G \ni \omega \mapsto 1 - e^{h_\omega} \in C^\infty(X,\mathbf{R})$$

may be viewed as a moment map, whose zeros are precisely Kähler–Einstein metrics. In analogy with the Calabi energy in the scalar curvature moment map picture, one can consider the Ding energy

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$$\mathrm{Ding}(\omega) \coloneqq \frac{1}{V}\int_X (1-e^{h_\omega})^2 \frac{\omega^n}{n!}, \quad V \coloneqq \int_X \frac{\omega^n}{n!}.$$

Yao showed that the critical points of this functional are precisely Mabuchi solitons ([**14**, Theorem 2.6]). Thus, in the Ding-theoretic setting, Mabuchi solitons play a role analogous to that of Calabi's extremal Kähler metrics ([**3**]) in the scalar curvature setting. Moreover, the existence of Mabuchi solitons is related to relative Ding stability through a Yau–Tian–Donaldson type correspondence ([**6**]). These features make Mabuchi solitons natural canonical metrics on Fano manifolds for which the Futaki invariant does not necessarily vanish.

An obstruction to the existence of Mabuchi solitons is given by the *Mabuchi constant*. Let P_ω be the space of Killing potentials with respect to ω, and let $\Pi^G_\omega \colon C^\infty(X, \mathbf{R}) \to P_\omega$ be the L^2-projection. Then Mabuchi showed that the constant

$$M_X \coloneqq \max_X \Pi^G_\omega (1-e^{h_\omega})$$

is independent of the choice of ω ([**8**, Section 2]). This M_X is called the *Mabuchi constant* of X. The Mabuchi constant satisfies $M_X < 1$ if X admits a Mabuchi soliton. In this paper, we study a special class of manifolds, called admissible manifolds ([**1**, Section 1]), and prove that the converse holds for Fano admissible manifolds. (We also point out that a similar equivalence holds for toric Fano manifolds; see [**14**] for details) Moreover, we give an explicit formula for the Mabuchi constant of Fano admissible manifolds, generalizing [**8**, Theorem 6.5]. Using this, we completely determine the existence and non-existence of Mabuchi solitons on admissible manifolds over a complex projective space $\mathbf{P}^n$.

1.2. Main results

An *admissible manifold* is a projective bundle $\mathbf{P}_Y(E_0 \oplus E_\infty) \to Y$ satisfying the following conditions:

- Y is covered by a product $\tilde{Y} \coloneqq \prod_{a\in\mathcal{A}} Y_a$ ($\mathcal{A} \subset \mathbf{Z}_{>0}, \#\mathcal{A} < \infty$) of simply-connected compact Kähler manifolds (Y_a, g_a, ω_a) of real dimension $2d_a$, with (g_a, ω_a) being pullbacks of tensors on Y.
- E_0 and E_∞ are holomorphic projectively-flat Hermitian vector bundles over Y of ranks d_0+1 and $d_\infty+1$, respectively, satisfying

$$\frac{c_1(E_\infty)}{d_\infty+1} - \frac{c_1(E_0)}{d_0+1} = \left[\frac{\omega_Y}{2\pi}\right], \quad \omega_Y \coloneqq \sum_{a\in\mathcal{A}} \varepsilon_a \omega_a,$$

  where $\varepsilon_a \in \{-1, 1\}$ ($a \in \mathcal{A}$).

Assume further that each ω_a is a Ricci positive Kähler–Einstein metric with $\mathrm{Ric}(\omega_a) = \varepsilon_a s_a \omega_a$ for $s_a \in \mathbf{R}$. Then $X = \mathbf{P}_Y(E_0 \oplus E_\infty)$ is a Fano manifold if and only if

$$\begin{cases} s_a > d_0 + 1 & (\text{if } \varepsilon_a = 1), \\ s_a < -(d_\infty + 1) & (\text{if } \varepsilon_a = -1) \end{cases}$$

for any $a \in \mathcal{A}$ ([**2**, Theorem 3.1]). Such an X is called a *Fano admissible manifold*.

Our first result shows that, in Fano admissible manifolds, the existence of a Mabuchi soliton can be completely determined by the Mabuchi constant.

THEOREM 1.1 (CF. COROLLARY 5.11). A Fano admissible manifold X admits a Mabuchi soliton if and only if $M_X < 1$.

The proof of Theorem 1.1 is obtained as a consequence of a more general existence theory for *v-solitons*, and will be given in Section 5.2. In general, the existence of a v-soliton is obstructed by the v-Futaki invariant (cf. Proposition 5.6). For Fano admissible manifolds, however, we show that the existence of an admissible v-soliton is in fact equivalent to the vanishing of the v-Futaki invariant (see Theorem 5.8). On the other hand, the v-Futaki invariant corresponding to Mabuchi solitons always vanishes. Therefore, 1.1 follows immediately from Theorem 5.8.

By Theorem 1.1, we can determine the existence of Mabuchi solitons on Fano admissible manifolds in terms of the Mabuchi constant. We next compute the Mabuchi constant for Fano admissible manifolds. Let

$$\Omega \coloneqq 2\pi c_1(X), \quad x_a \coloneqq \frac{d_0 + d_\infty + 2}{2s_a + d_\infty - d_0} \quad (a \in \mathcal{A}),$$

and define the *characteristic polynomial* p_Ω of Ω by

$$p_\Omega(x) \coloneqq (1+x)^{d_0}(1-x)^{d_\infty} \prod_{a \in \mathcal{A}} \left(\frac{\varepsilon_a}{x_a} + \varepsilon_a x \right)^{d_a}$$

(cf. Section 3.3). For $i \in \{0, 1, 2\}$, set

$$b_i \coloneqq \int_{-1}^{1} x^i p_\Omega(x)\, dx.$$

In the case $d_0 = d_\infty = 0$, Mabuchi obtained an explicit formula for M_X in terms of b_0, b_1, b_2 ([**8**, Theorem 6.5]). We extend this to general Fano admissible manifolds as follows.

THEOREM 1.2 (CF. THEOREM 6.1). Let $w \coloneqq \dfrac{d_0 - d_\infty}{d_0 + d_\infty + 2}$. Then

$$M_X = \frac{b_0|b_1 - wb_0| - b_1(b_1 - wb_0)}{b_0 b_2 - b_1^2} = 1 + \frac{b_0(|b_1 - wb_0| - (b_2 - wb_1))}{b_0 b_2 - b_1^2}.$$

Finally, we consider admissible manifolds over the complex projective space $\mathbf{P}^n$ and discuss the existence of Mabuchi solitons on them. Any Fano admissible manifold X over $\mathbf{P}^n$ can be written as

$$X = \mathbf{P}_{\mathbf{P}^n}(\mathcal{O}^{\oplus(d_0+1)} \oplus \mathcal{O}(k)^{\oplus(d_\infty+1)})$$

for some $k \in \mathbf{Z}$ satisfying $k \geq 1$ and $n + 1 > k(d_0 + 1)$. Moreover, the existence of a Mabuchi soliton on X is equivalent to the condition $M_X < 1$ (see Corollary 5.11). Using Theorem 1.2, we prove the following theorem.

Theorem 1.3 (cf. Theorem 7.1). The Fano admissible manifold X admits a Mabuchi soliton if and only if

$$(k, d_\infty) = (1, 0) \quad \text{or} \quad (n, k, d_0, d_\infty) = (1, 1, 0, 1).$$

In all other cases, $M_X > 1$.

Theorem 1.3 can also be interpreted from the viewpoint of stability. By the Yau–Tian–Donaldson type correspondence for Mabuchi solitons, established by Han and Li [**6**], the existence of a Mabuchi soliton on a Fano manifold is equivalent to *uniform relative Ding stability* (see also [**14, 12, 13, 7**]). Moreover, by [**13**, Theorem 1], the Mabuchi constant of any relatively Ding semistable Fano manifold is bounded above by 1. Combining these results with Theorem 1.3, we obtain the following.

Corollary 1.4. X is uniformly relatively Ding stable if and only if

$$(k, d_\infty) = (1, 0) \quad \text{or} \quad (n, k, d_0, d_\infty) = (1, 1, 0, 1).$$

Otherwise, X is relatively Ding unstable.

1.3. Organization of the paper

We start by reviewing basic notions on Mabuchi solitons in Section 2. The Mabuchi constant is also introduced. In Section 3, following [**1**], we introduce the notion of admissible manifolds. After introducing admissible manifolds, we discuss admissible Kähler classes and admissible Kähler metrics, which are the main tools of this paper, as well as the characteristic polynomial associated with an admissible Kähler class. In Section 4, we focus on those admissible manifolds that are Fano. We give an explicit formula for the Ricci potential of admissible Kähler metrics on Fano admissible manifolds in Section 4.2 (cf. Proposition 4.4). This is a refinement of the result of Maschler and Tønnesen-Friedman [**10**, Corollary 3.3]. Using Proposition 4.4, we obtain Proposition 4.7 for the normalization of the moment map, which plays an important role in the proof of Theorem 1.2. In Section 5, we introduce the notion of v-solitons on general Fano manifolds and explain that Mabuchi solitons form a special class of v-solitons. We also describe the v-Futaki invariant, which is an obstruction to the existence of v-solitons, and show that, for Fano admissible manifolds, the existence of an admissible v-soliton is equivalent to the vanishing of the v-Futaki invariant (cf. Theorem 5.8). From Theorem 5.8 and the fact that the v-Futaki invariant corresponding to Mabuchi solitons always vanishes (see Remark 5.7), we obtain Theorem 1.1. In Section 6, we study the Mabuchi constant for Fano admissible manifolds and give its explicit formula (Theorem 1.2). Finally, in Section 7, we prove Theorem 1.3. The proof proceeds by computing the Mabuchi constant for Fano admissible manifolds over $\mathbf{P}^n$ using Theorem 1.2, and applying Theorem 1.1 to determine the existence of Mabuchi solitons.

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2. Mabuchi solitons and the Mabuchi constant

We first recall the definitions of Mabuchi solitons and the Mabuchi constant. Let X be an n-dimensional Fano manifold, and let G be a maximal compact subgroup of $\mathrm{Aut}^0(X)$. Let ω be a G-invariant Kähler metric on X representing $2\pi c_1(X)$, and denote by h_ω the Ricci potential of ω.

Definition 2.1. Let $\omega \in 2\pi c_1(X)^G$ be a G-invariant Kähler metric on X. We say that ω is a *Mabuchi soliton* if the gradient

$$\mathrm{grad}_\omega(1 - e^{h_\omega})$$

with respect to ω is a real holomorphic vector field on X.

Example 2.2. Let $\omega \in 2\pi c_1(X)^G$ be a G-invariant Kähler–Einstein metric on X: $\mathrm{Ric}(\omega) = \omega$. Then $h_\omega = 0$ and hence

$$\mathrm{grad}_\omega(1 - e^{h_\omega}) = \mathrm{grad}_\omega 0 = 0.$$

Therefore, ω is a Mabuchi soliton.

Example 2.3 ([**8**, Example 5.8]). Let $X \coloneqq \mathbf{P}_{\mathbf{P}^n}(\mathcal{O}_{\mathbf{P}^n} \oplus \mathcal{O}_{\mathbf{P}^n}(1))$ be the Fano manifold obtained by blowing up $\mathbf{P}^{n+1}$ at a point. Then X admits a Mabuchi soliton but does not admit a Kähler–Einstein metric.

Next, we explain the Mabuchi constant, which is known as an obstruction to the existence of Mabuchi solitons. Let P_ω denote the space of Killing potentials with respect to ω, i.e., the space of real-valued smooth functions $f \in C^\infty(X, \mathbf{R})$ such that $\mathrm{grad}_\omega f$ is a real holomorphic vector field on X. Let $\Pi^G_\omega \colon C^\infty(X, \mathbf{R})^G \to P_\omega$ be the L^2-projection with respect to ω. Then a G-invariant Kähler metric $\omega \in 2\pi c_1(X)^G$ is a Mabuchi soliton if and only if $\Pi^G_\omega(1 - e^{h_\omega}) = 1 - e^{h_\omega}$. Moreover, Mabuchi showed in [**8**] that the constant

$$\max_X \Pi^G_\omega(1 - e^{h_\omega})$$

is independent of the choice of $\omega \in 2\pi c_1(X)^G$.

Definition 2.4. We define $M_X \coloneqq \max_X \Pi^G_\omega(1 - e^{h_\omega})$ and call it the *Mabuchi constant* of the Fano manifold X.

The following examples show that the Mabuchi constant serves as an obstruction to the existence of Kähler–Einstein metrics and Mabuchi solitons.

Example 2.5. Assume that $2\pi c_1(X)^G$ contains a Kähler–Einstein metric ω. Then $h_\omega = 0$, and hence

$$\Pi^G_\omega(1 - e^{h_\omega}) = \Pi^G_\omega 0 = 0.$$

Therefore, we have $M_X = \max_X \Pi^G_\omega(1 - e^{h_\omega}) = 0$.

Example 2.6 ([**8**, Theorem 3.1]). Assume that $2\pi c_1(X)^G$ contains a Mabuchi soliton ω. Then

$$\Pi_\omega^G(1-e^{h_\omega}) = 1 - e^{h_\omega} < 1$$

holds on X. Hence, $M_X = \max_X \Pi_\omega^G(1-e^{h_\omega}) < 1$.

3. Admissible manifolds

3.1. Definition of admissible manifolds

In this section, following the description in [**1**], we explain the definition of admissible manifolds. For details, we refer the reader to [**1**].

Definition 3.1. An *admissible manifold* is a projective bundle $\mathbf{P}_Y(E_0 \oplus E_\infty) \to Y$ satisfying the following conditions:

- Y is covered by a product $\tilde{Y} \coloneqq \prod_{a\in\mathcal{A}} Y_a$ ($\mathcal{A} \subset \mathbf{Z}_{>0}, \#\mathcal{A} < \infty$) of simply-connected compact Kähler manifolds (Y_a, g_a, ω_a) of real dimension $2d_a$, with (g_a, ω_a) being pullbacks of tensors on Y.
- E_0 and E_∞ are holomorphic projectively-flat Hermitian vector bundles over Y of ranks d_0+1 and $d_\infty+1$, respectively, satisfying

$$\frac{c_1(E_\infty)}{d_\infty+1} - \frac{c_1(E_0)}{d_0+1} = \left[\frac{\omega_Y}{2\pi}\right], \quad \omega_Y \coloneqq \sum_{a\in\mathcal{A}} \varepsilon_a \omega_a, \tag{3.1}$$

  where $\varepsilon_a \in \{-1, 1\}$ ($a \in \mathcal{A}$).

Let $X = \mathbf{P}_Y(E_0 \oplus E_\infty)$ be an admissible manifold. By the condition (3.1) above, there exist Hermitian metrics on E_0 and E_∞ such that the curvatures of their Chern connections are given by $\Omega_0 \otimes \mathrm{id}_{E_0}$, $\Omega_\infty \otimes \mathrm{id}_{E_\infty}$, and satisfy $\Omega_\infty - \Omega_0 = \omega_Y$. Let (g_0, ω_0) and $(g_\infty, \omega_\infty)$ denote the fiberwise Fubini–Study metrics of $\mathbf{P}(E_0)$ and $\mathbf{P}(E_\infty)$. We normalize them so that their scalar curvatures are $2d_0(d_0+1)$ and $2d_\infty(d_\infty+1)$, respectively. We also set $\varepsilon_0 \coloneqq 1$ and $\varepsilon_\infty \coloneqq -1$.

We summarize below the notation for admissible manifolds used in this paper:

(1) We set $\hat{\mathcal{A}} \coloneqq \{a \in \mathcal{A} \cup \{0, \infty\} \mid d_a > 0\}$.

(2) Let $\tilde{E}_0$ and $\tilde{E}_\infty$ be the pullbacks of E_0 and E_∞ to $\tilde{Y}$, respectively. Since $\tilde{Y}$ is simply-connected, they can be written as

$$\tilde{E}_0 = \mathcal{E}_0 \otimes \mathcal{O}^{\oplus(d_0+1)}, \quad \tilde{E}_\infty = \mathcal{E}_\infty \otimes \mathcal{O}^{\oplus(d_\infty+1)}$$

for some holomorphic line bundles $\mathcal{E}_0$ and $\mathcal{E}_\infty$ over $\tilde{Y}$. Let $\mathcal{L} \coloneqq \mathcal{E}_0^{-1} \otimes \mathcal{E}_\infty$. Then $\mathcal{L}$ is a holomorphic line bundle on $\tilde{Y}$, and can be written as

$$\mathcal{L} = \bigotimes_{a\in\mathcal{A}} \mathcal{L}_a,$$

where each $\mathcal{L}_a$ is a holomorphic line bundle over Y_a satisfying

$$c_1(\mathcal{L}_a) = \left[\frac{\varepsilon_a \omega_a}{2\pi}\right].$$

(3) Let $e_0 \coloneqq \mathbf{P}_Y(E_0 \oplus 0)$, $e_\infty \coloneqq \mathbf{P}_Y(0 \oplus E_\infty)$. Their universal covers are, respectively, $Y_0 \times \tilde{Y} \coloneqq \mathbf{P}^{d_0} \times \tilde{Y}$ and $\tilde{Y} \times Y_\infty \coloneqq \tilde{Y} \times \mathbf{P}^{d_\infty}$.

(4) The blow-up of X along $e_0 \cup e_\infty$ is given by $\hat{X} \coloneqq \mathbf{P}_{\hat{Y}}(\mathcal{O} \oplus \hat{\mathcal{L}}) \to \hat{Y}$, where $\hat{Y} \coloneqq \mathbf{P}_Y(E_0) \times_Y \mathbf{P}_Y(E_\infty) \to Y$ and $\hat{\mathcal{L}} \coloneqq \mathcal{O}_{E_0}(1) \otimes \mathcal{O}_{E_\infty}(-1)$. In this case, for $\omega_{\hat{Y}} \coloneqq \sum_{a \in \hat{\mathcal{A}}} \varepsilon_a \omega_a$ we have $c_1(\hat{\mathcal{L}}) = [\omega_{\hat{Y}}/2\pi]$. We say *a blow-down occurs* if $d_0 > 0$ or $d_\infty > 0$.

(5) Let $\hat{e}_0$ and $\hat{e}_\infty$ denote the zero and infinity sections of $\hat{X}$, respectively.

3.2. Admissible Kähler classes and metrics

In this section, we explain a special class of Kähler metrics on admissible manifolds, called *admissible Kähler metrics*. As in the previous section, we refer the reader to [**1**] for further details.

We begin by describing admissible Kähler classes, which serve as the natural setting for admissible Kähler metrics. Let $X = \mathbf{P}_Y(E_0 \oplus E_\infty)$ be an admissible manifold.

Definition 3.2. A Kähler class Ω on X is called an **admissible Kähler class** if its pullback to $\hat{X}$ can be written as

$$c\left(\sum_{a \in \hat{\mathcal{A}}} \frac{[\varepsilon_a \omega_a]}{x_a} + \hat{\Xi}\right) \tag{3.2}$$

for some $c \in \mathbf{R}$, $x_a \in \mathbf{R} \setminus \{0\}$ with $x_0 = 1$, $x_\infty = -1$, where $\hat{\Xi}$ is the Poincaré dual of $2\pi[\hat{e}_0 + \hat{e}_\infty]$. If $c = 1$, the admissible Kähler class is said to be *normalized*, or is called a *normalized admissible Kähler class*.

Let Ξ be a 2-form on X whose pullback to $\hat{X}$ is $[\omega_0] - [\omega_\infty] + \hat{\Xi}$. Then a Kähler class Ω is admissible if and only if

$$\Omega = c\left(\sum_{a \in \mathcal{A}} \frac{[\varepsilon_a \omega_a]}{x_a} + \Xi\right)$$

for some $c \in \mathbf{R}$ and $x_a \in \mathbf{R} \setminus \{0\}$.

Proposition 3.3 ([**1**, p.557] by the discussion therein). Let $c \in \mathbf{R}$ and $x_a \in \mathbf{R} \setminus \{0\}$. Let a cohomology class $\Omega \in H^{1,1}(X, \mathbf{R})$ be given by (3.2). Then the following two conditions are equivalent:

(1) Ω is an admissible Kähler class of X.

(2) $c \in \mathbf{R}_{>0}$ and $0 < \varepsilon_a x_a < 1$ for any $a \in \mathcal{A}$.

Let Ω be an admissible Kähler class of X given by (3.2). By Proposition 3.3, $c \in \mathbf{R}_{>0}$ and $0 < \varepsilon_a x_a < 1$ for any $a \in \mathcal{A}$. The scalar multiplication on the fiber of E_0 induces a $\mathbf{C}^*$-action on X. By identifying $X_0 \coloneqq X \setminus (e_0 \cup e_\infty) \cong \hat{X} \setminus (\hat{e}_0 \cup \hat{e}_\infty)$ with an open subset of $\hat{X}$, X_0 is a principal $\mathbf{C}^*$-bundle over $\hat{Y}$ associated to the holomorphic line bundle $\hat{\mathcal{L}}$.

We now consider the S^1-action on X obtained by restricting the $\mathbf{C}^*$-action. The Hermitian metrics on E_0 and E_∞ induce a fiberwise moment map $z\colon X \to [-1,1]$. The critical submanifolds of z are given by

$$z^{-1}(\{1\}) = e_0, \quad z^{-1}(\{-1\}) = e_\infty.$$

(A concrete construction of z can be found in [**1**, Section 2.1]) Let T denote the infinitesimal generator of the S^1-action on X. Let θ be a connection form on X_0 satisfying

$$d\theta = \sum_{a \in \hat{\mathcal{A}}} \varepsilon_a \omega_a. \tag{3.3}$$

DEFINITION 3.4. A Kähler metric $\omega \in \Omega$ on X is called an *admissible Kähler metric* if

$$\begin{aligned} g &= c\left(\sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} \varepsilon_a g_a + \frac{dz^2}{\Theta(z)} + \Theta(z)\theta^2\right), \\ \omega &= c\left(\sum_{a \in \hat{\mathcal{A}}} \frac{1 + x_a z}{x_a} \varepsilon_a \omega_a + dz \wedge \theta\right) \end{aligned} \tag{3.4}$$

on X_0, where g is the Riemannian metric associated to ω, and Θ is a smooth function on $[-1,1]$ satisfying

(1) $\Theta(x) > 0$ $(x \in (-1,1))$,

(2) $\Theta(\pm 1) = 0$, $\Theta'(\pm 1) = \mp 2$.

If $c = 1$, the admissible Kähler metric ω is said to be *normalized*.

Let $\omega \in \Omega$ be an admissible Kähler metric in the class Ω defined by (3.4). It is immediate that z is a moment map for $\frac{1}{c}\omega$:

$$-dz = \iota_T \frac{1}{c}\omega.$$

Moreover, the complex structure on X corresponding to this Kähler structure is obtained by taking the pullback of the complex structures on each Y_a $(a \in \mathcal{A})$, together with the condition $d^c z = \Theta(z)\theta$.

REMARK 3.5. Let ω be a Kähler metric on X, and suppose that it can be written in the form (3.4) on X_0. Then, using (3.3), one can show that ω belongs to Ω. Furthermore, the 2-form $d(z\theta)$, regarded as a differential form on $X_0 \subset \hat{X}$, extends to the whole of $\hat{X}$, and its de Rham cohomology class coincides with $\hat{\Xi}$. For details, see [**1**, pp. 556–557].

3.3. Characteristic polynomials

In this section, we explain the characteristic polynomial associated with an admissible Kähler class. As will be stated in Proposition 3.7 below, this polynomial coincides, up to a constant multiple, with the *Duistermaat–Heckman polynomial* associated with the S^1-action on X.

Let $X = \mathbf{P}_Y(E_0 \oplus E_\infty)$ be a d-dimensional admissible manifold, and let Ω be an admissible Kähler class on X given by (3.2).

DEFINITION 3.6. Let $\lambda_a \coloneqq \varepsilon_a/x_a$ for each $a \in \hat{\mathcal{A}}$ (note that $\lambda_0 = \lambda_\infty = 1$), and define

$$p_\Omega(x) \coloneqq \prod_{a\in\hat{\mathcal{A}}} (\lambda_a + \varepsilon_a x)^{d_a} = (1+x)^{d_0}(1-x)^{d_\infty} \prod_{a\in\mathcal{A}} (\lambda_a + \varepsilon_a x)^{d_a}.$$

We call p_Ω the *characteristic polynomial* of the admissible Kähler class Ω.

The following proposition can be deduced from a careful reading of the proof of [**1**, Proposition 6], but we include a short proof for the reader's convenience.

PROPOSITION 3.7. Let $\omega \in \Omega$ be an admissible Kähler metric on X given by (3.4). Then

$$\int_X f(z)\frac{\omega^d}{d!} = 2\pi c^d \left(\prod_{a\in\hat{\mathcal{A}}} \mathrm{vol}(Y_a, \omega_a)\right) \int_{-1}^{1} f(x) p_\Omega(x)\, dx$$

for any integrable function f on the closed interval $[-1, 1]$.

PROOF. By the well-known multinomial theorem, we have

$$\begin{aligned}
\frac{\omega^d}{d!} &= \frac{c^d}{d!}\left(\sum_{a\in\hat{\mathcal{A}}} \frac{1+x_a z}{x_a}\varepsilon_a\omega_a + dz\wedge\theta\right)^d \\
&= \frac{c^d}{(d-1)!}\left(\sum_{a\in\hat{\mathcal{A}}} \frac{1+x_a z}{x_a}\varepsilon_a\omega_a\right)^{d-1} \wedge dz\wedge\theta \\
&= c^d p_\Omega(z)\left(\bigwedge_{a\in\hat{\mathcal{A}}} \frac{\omega_a^{d_a}}{d_a!}\right) \wedge dz\wedge\theta.
\end{aligned}$$

Then, integration along the fibers yields

$$\begin{aligned}
\int_X f(z)\frac{\omega^d}{d!} &= c^d \int_X f(z) p_\Omega(z)\left(\bigwedge_{a\in\hat{\mathcal{A}}} \frac{\omega_a^{d_a}}{d_a!}\right) \wedge dz\wedge\theta \\
&= 2\pi c^d \left(\prod_{a\in\hat{\mathcal{A}}} \mathrm{vol}(Y_a, \omega_a)\right) \int_{-1}^{1} f(x) p_\Omega(x)\, dx,
\end{aligned}$$

as required. □

4. Fano admissible manifolds

4.1. Definition of Fano admissible manifolds

In this section we explain a necessary and sufficient condition for an admissible manifold to be Fano, following [**2**]. We here assume that $Y = \tilde{Y} = \prod_{a\in\mathcal{A}} Y_a$ and each ω_a is a Ricci positive Kähler–Einstein metric on Y_a: $\mathrm{Ric}(\omega_a) = \varepsilon_a s_a \omega_a$ $(a \in \mathcal{A})$. Let X be an admissible manifold over Y. Since Y is simply-connected, X can be written as

$$X = \mathbf{P}_Y(\mathcal{O}^{\oplus(d_0+1)} \oplus (\mathcal{L} \otimes \mathcal{O}^{\oplus(d_\infty+1)})),$$

where $\mathcal{L}$ is a holomorphic line bundle over Y satisfying $\mathcal{L} = \otimes_{a\in\mathcal{A}}\mathcal{L}_a$ with each $\mathcal{L}_a$ a holomorphic line bundles over Y_a satisfying $c_1(\mathcal{L}_a) = [\varepsilon_a\omega_a/2\pi]$.

Theorem 4.1 ([**2**, Theorem 3.1]). For X, the following conditions are equivalent:

(1) X is a Fano manifold.

(2) For each $a \in \mathcal{A}$,

$$\begin{cases} s_a > d_0 + 1 & (\text{if } \varepsilon_a = 1), \\ s_a < -(d_\infty + 1) & (\text{if } \varepsilon_a = -1). \end{cases}$$

Moreover, if X satisfies these conditions, then $2\pi c_1(X)$ is an admissible Kähler class, and satisfies

$$2\pi c_1(X) = c\left[\left(\sum_{a\in\mathcal{A}} \frac{[\varepsilon_a\omega_a]}{x_a}\right) + \Xi\right] \tag{4.1}$$

for

$$c = \frac{d_0 + d_\infty + 2}{2}, \quad x_a = \frac{d_0 + d_\infty + 2}{2s_a + d_\infty - d_0} \quad (a \in \mathcal{A}). \tag{4.2}$$

Definition 4.2. If X satisfies the conditions of Theorem 4.1, X is called a *Fano admissible manifold*.

Here we note that it is shown in [**2**] that any such X admits a Kähler–Ricci soliton.

4.2. Ricci potentials

Let $X = \mathbf{P}_Y(\mathcal{O}^{\oplus(d_0+1)} \oplus (\mathcal{L} \otimes \mathcal{O}^{\oplus(d_\infty+1)}))$ be a Fano admissible manifold. Let $\Omega := 2\pi c_1(X)$, and let $\omega \in \Omega$ be an admissible Kähler metric given by (3.4). The goal of this section is to compute explicitly the Ricci potential of ω. First, by [**2**, (3.2)] we have

$$\mathrm{Ric}(\omega) = \mathrm{Ric}\left(\frac{\omega}{c}\right) = \sum_{a\in\hat{\mathcal{A}}} \mathrm{Ric}(\omega_a) - \frac{1}{2}dd^c \log(p_\Omega(z)\Theta(z))$$

on $X \setminus (e_0 \cup e_\infty)$. On the other hand,

$$\begin{aligned}\omega &= c\left(\sum_{a\in\hat{\mathcal{A}}}\frac{1+x_a z}{x_a}\varepsilon_a\omega_a + dz\wedge\theta\right)\\ &= \sum_{a\in\hat{\mathcal{A}}}\frac{c}{x_a}\varepsilon_a\omega_a + cz\sum_{a\in\hat{\mathcal{A}}}\varepsilon_a\omega_a + c(dz\wedge\theta),\end{aligned}$$

which can be written as

$$\omega = \sum_{a\in\hat{\mathcal{A}}} s_a\varepsilon_a\omega_a + \left(\frac{d_\infty - d_0}{2} + cz\right)\sum_{a\in\hat{\mathcal{A}}}\varepsilon_a\omega_a + c(dz\wedge\theta)$$

since for any $a\in\hat{\mathcal{A}}$

$$\frac{c}{x_a} = s_a + \frac{d_\infty - d_0}{2}.$$

Hence we have

$$\begin{aligned}\mathrm{Ric}(\omega) - \omega = &\sum_{a\in\hat{\mathcal{A}}}\mathrm{Ric}(\omega_a) - \frac{1}{2}dd^c\log(p_\Omega(z)\Theta(z))\\ &- \sum_{a\in\hat{\mathcal{A}}} s_a\varepsilon_a\omega_a - \left(\frac{d_\infty - d_0}{2} + cz\right)\sum_{a\in\hat{\mathcal{A}}}\varepsilon_a\omega_a - c(dz\wedge\theta).\end{aligned}$$

Using $\mathrm{Ric}(\omega_a) = \varepsilon_a s_a\omega_a$, this simplifies to

$$\mathrm{Ric}(\omega) - \omega = -\frac{1}{2}dd^c\log(p_\Omega(z)\Theta(z)) - \left(\frac{d_\infty - d_0}{2} + cz\right)\sum_{a\in\hat{\mathcal{A}}}\varepsilon_a\omega_a - c(dz\wedge\theta).$$

Since $d^c z = \Theta(z)\theta$, we have

$$dz\wedge\theta = d(z\theta) - zd\theta = d\left(\frac{z}{\Theta(z)}d^c z\right) - zd\theta,$$

and using $d\theta = \sum_{a\in\hat{\mathcal{A}}}\varepsilon_a\omega_a$, we obtain

$$\mathrm{Ric}(\omega) - \omega = -\frac{1}{2}dd^c\log(p_\Omega(z)\Theta(z)) - \left(\frac{d_\infty - d_0}{2}\right)d\theta - cd\left(\frac{z}{\Theta(z)}d^c z\right).$$

Define functions on $(-1,1)$ by

$$a_\Theta(x) \coloneqq \frac{1}{\Theta(x)}, \quad b_\Theta(x) \coloneqq \frac{x}{\Theta(x)},$$

and let A_Θ, B_Θ be their primitives. Then

$$d\theta = d\left(a_\Theta(z)d^c z\right) = d\left(A'_\Theta(z)d^c z\right) = dd^c A_\Theta(z),$$

$$cd\left(\frac{z}{\Theta(z)}d^cz\right) = cd\left(b_\Theta(z)d^cz\right) = cd\left(B'_\Theta(z)d^cz\right) = dd^c(cB_\Theta(z)),$$

and hence

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2}dd^c\left(-(d_\infty - d_0)A_\Theta(z) - 2cB_\Theta(z) - \log(p_\Omega(z)\Theta(z))\right).$$

Lemma 4.3. Define a smooth function $h\colon (-1,1) \to \mathbf{R}$ by

$$h(x) \coloneqq -(d_\infty - d_0)A_\Theta(x) - 2cB_\Theta(x) - \log(p_\Omega(x)\Theta(x)).$$

Then h extends uniquely to a smooth function on the closed interval $[-1,1]$.

Proof. It suffices to show that the derivative of h

$$\begin{aligned} h'(x) &= \left(-(d_\infty - d_0)A_\Theta(x) - 2cB_\Theta(x) - \log(p_\Omega(x)\Theta(x))\right)' \\ &= \frac{-(d_\infty - d_0) - 2cx - \Theta'(x)}{\Theta(x)} - \frac{p'_\Omega(x)}{p_\Omega(x)} \end{aligned}$$

extends smoothly to the closed interval $[-1,1]$. We first consider a neighborhood of $x = 1$. Since $\Theta(1) = 0$ and $\Theta'(1) = -2$, we can write

$$\Theta(x) = -2(x-1) + \Theta_1(x)(x-1)^2$$

for some smooth function Θ_1 defined near $x = 1$. Therefore, we have

$$\Theta'(x) = -2 + 2\Theta_1(x)(x-1) + \Theta'_1(x)(x-1)^2,$$

and hence

$$\begin{aligned} -(d_\infty - d_0) - 2cx - \Theta'(x) = -2d_\infty - (d_0 + d_\infty + 2)(x-1) \\ - 2\Theta_1(x)(x-1) - \Theta'_1(x)(x-1)^2 \end{aligned}$$

and

$$\begin{aligned} \frac{-(d_\infty - d_0) - 2cx - \Theta'(x)}{\Theta(x)} &= \frac{-2d_\infty}{(-2 + \Theta_1(x)(x-1))(x-1)} \\ &\quad + \frac{-(d_0 + d_\infty + 2) - 2\Theta_1(x) - \Theta'_1(x)(x-1)}{-2 + \Theta_1(x)(x-1)} \\ &= \frac{d_\infty}{x-1} + \Theta_2(x), \end{aligned}$$

where Θ_2 is a smooth function defined near $x = 1$. On the other hand, writing $p_\Omega(x)$ near $x = 1$ as

$$p_\Omega(x) = (1-x)^{d_\infty}p_1(x),$$

where p_1 is a polynomial function satisfying $p_1(1) \neq 0$, we have

$$p'_\Omega(x) = -d_\infty(1-x)^{d_\infty-1}p_1(x) + (1-x)^{d_\infty}p'_1(x).$$

Thus,

$$\begin{aligned}-\frac{p'_\Omega(x)}{p_\Omega(x)} &= \frac{d_\infty(1-x)^{d_\infty-1}p_1(x)}{(1-x)^{d_\infty}p_1(x)} - \frac{(1-x)^{d_\infty}p'_1(x)}{(1-x)^{d_\infty}p_1(x)} \\ &= \frac{d_\infty}{(1-x)} + p_2(x),\end{aligned}$$

where p_2 is a smooth function defined near $x = 1$. Combining the above expressions, we obtain

$$\begin{aligned}h'(x) &= \frac{-(d_\infty - d_0) - 2cx - \Theta'(x)}{\Theta(x)} - \frac{p'_\Omega(x)}{p_\Omega(x)} \\ &= \frac{d_\infty}{x-1} + \Theta_2(x) + \frac{d_\infty}{(1-x)} + p_2(x) \\ &= \Theta_2(x) + p_2(x)\end{aligned}$$

which shows that h' extends smoothly near $x = 1$. The argument near $x = -1$ is similar. Therefore, h' extends smoothly to the closed interval $[-1, 1]$, and so does h. □

We still denote this extension by h, and define $h_\omega \in C^\infty(X, \mathbf{R})$ by

$$h_\omega \coloneqq h(z).$$

Noting that A_Θ and B_Θ are defined only up to additive constants, we obtain the following proposition.

PROPOSITION 4.4. By choosing A_Θ and B_Θ so that

$$\int_X (1 - e^{h_\omega})\frac{\omega^d}{d!} = 0,$$

the function $h_\omega = h(z)$ is the Ricci potential of ω.

REMARK 4.5. It is shown that the Ricci potential is a function of z in [**10**, Corollary 3.3]. Proposition 4.4 provides an explicit expression of this function.

By Proposition 4.4, we see that $1 - e^{h_\omega}$ is a function of z. In fact the following stronger statement is also known. This fact plays a very important role in Sections 5.2 and 6.

PROPOSITION 4.6 ([**1**, Proposition 6], [**9**, Theorem 9.1]). There exist constants $\alpha, \beta \in \mathbf{R}$ such that

$$\Pi^G_\omega(1 - e^{h_\omega}) = \alpha z + \beta.$$

Finally, we show the following proposition, which is crucial for proving Theorem 1.2 in the introduction.

Proposition 4.7. Let $w \coloneqq \dfrac{d_0 - d_\infty}{d_0 + d_\infty + 2}$. Then

$$\int_X (z - w) e^{h_\omega} \frac{\omega^d}{d!} = 0.$$

Proof. We first apply Proposition 3.7 to obtain

$$\int_X z e^{h_\omega} \frac{\omega^d}{d!} = 2\pi c^d \left(\prod_{a \in \hat{\mathcal{A}}} \mathrm{vol}(Y_a, \omega_a) \right) \int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{x}{\Theta(x)}\, dx.$$

We next compute the integral on the right-hand side. Observe that

$$\int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{x}{\Theta(x)}\, dx = \int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x)} \left(-\frac{1}{2c} e^{-2cB_\Theta} \right)'(x)\, dx.$$

Integrating by parts, we obtain

$$\begin{aligned} &\int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{x}{\Theta(x)}\, dx \\ &= \left[-\frac{1}{2c} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \right]_{-1}^{1} + w \int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{1}{\Theta(x)}\, dx. \end{aligned}$$

By Lemma 4.3, we have

$$\begin{aligned} \lim_{x \to 1} (-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)) &= h(1) + \lim_{x \to 1} \log(p_\Omega(x)\Theta(x)) = -\infty, \\ \lim_{x \to -1} (-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)) &= h(-1) + \lim_{x \to -1} \log(p_\Omega(x)\Theta(x)) = -\infty. \end{aligned}$$

Hence the boundary term vanishes:

$$\left[-\frac{1}{2c} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \right]_{-1}^{1} = 0.$$

Therefore, we obtain

$$\int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{x}{\Theta(x)}\, dx = w \int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{1}{\Theta(x)}\, dx.$$

Substituting this into the expression for the integral over X, we have

$$\begin{aligned} \int_X z e^{h_\omega} \frac{\omega^d}{d!} &= 2\pi c^d w \left(\prod_{a \in \hat{\mathcal{A}}} \mathrm{vol}(Y_a, \omega_a) \right) \int_{-1}^{1} e^{-(d_\infty - d_0) A_\Theta(x) - 2cB_\Theta(x)} \frac{1}{\Theta(x)}\, dx \\ &= w \int_X e^{h_\omega} \frac{\omega^d}{d!}. \end{aligned}$$

□

5. v-solitons on Fano admissible manifolds

5.1. v-solitons

We now temporarily leave the setting of admissible manifolds and consider a general Fano manifold. Let X be a d-dimensional Fano manifold, and let $T \subset \mathrm{Aut}^0(X)$ be a compact torus. Let $\Omega \coloneqq 2\pi c_1(X)$, and let $\omega \in \Omega$ be a T-invariant Kähler metric. Denote by $\mu_\omega \colon X \to \mathfrak{t}^*$ the moment map associated with the T-action. We normalize μ_ω so that for every $\xi \in \mathfrak{t}$,

$$\int_X \langle \mu_\omega, \xi \rangle e^{h_\omega} \frac{\omega^d}{d!} = 0,$$

where $\langle \cdot, \cdot \rangle \colon \mathfrak{t}^* \times \mathfrak{t} \to \mathbf{R}$ denotes the natural pairing. With this normalization, the image $P \coloneqq \mu_\omega(X)$ is independent of the choice of T-invariant Kähler metric $\omega \in \Omega$.

Definition 5.1. Let $v \in C^\infty(P, \mathbf{R}_{>0})$. A T-invariant Kähler metric $\omega \in \Omega$ is called a *v-soliton* if it satisfies

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c \log(v(\mu_\omega)). \tag{5.1}$$

Example 5.2. If $v(x) = 1$ for all $x \in P$, then the equation (5.1) becomes

$$\mathrm{Ric}(\omega) - \omega = 0,$$

so a v-soliton reduces to a Kähler–Einstein metric.

Example 5.3. Let $\xi \in \mathfrak{t}$, and set $v(x) = e^{\langle x, \xi \rangle}$ for each $x \in P$. Then the equation (5.1) becomes

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c \langle \mu_\omega, \xi \rangle,$$

and such a metric is known to be a Kähler–Ricci soliton.

Example 5.4. Mabuchi showed in [**9**] that Mabuchi solitons are a special case of v-solitons as follows. Let X be a Fano manifold and assume $M_X < 1$. Fix a maximal compact subgroup $G \subset \mathrm{Aut}^0(X)$. For a Kähler metric $\omega \in \Omega$, the vector field $\mathrm{grad}_\omega \Pi^G_\omega(1 - e^{h_\omega})$ is real holomorphic on X, and in fact induces an S^1-action on X (see [**5**] and [**9**]). For a suitable constant $a \in \mathbf{R}$, define $\mu_\omega \coloneqq \Pi^G_\omega(1 - e^{h_\omega}) - a \colon X \to \mathbf{R}$. Then μ_ω is a normalized moment map for this S^1-action. Define a function $v \colon P \to \mathbf{R}$ by $v(x) \coloneqq 1 - a - x$. Since $M_X < 1$, we have $v > 0$ on P. A direct computation shows that ω is a Mabuchi soliton if and only if

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c \log(v(\mu_\omega)).$$

Thus, Mabuchi solitons are v-solitons for this choice of v.

Next, we introduce an obstruction to the existence of v-solitons, analogous to the

Futaki invariant for Kähler–Einstein metrics. Let $\omega \in \Omega$ be a T-invariant Kähler metric, and let $\mu_\omega \colon X \to \mathfrak{t}^*$ be the normalized moment map. Let μ_{DH} denote the Borel measure on P obtained as the pushforward of the Liouville measure $\omega^d/d!$ under μ_ω. It is known that μ_{DH} is independent of the choice of $\omega \in \Omega$. This measure is known as the *Duistermaat–Heckman measure* on P associated with the T-action on X.

DEFINITION 5.5. Let $v \in C^\infty(P, \mathbf{R}_{>0})$ be a weight function and define a linear map $\mathrm{Fut}_v \colon \mathfrak{t} \to \mathbf{R}$ by

$$\mathrm{Fut}_v(\xi) \coloneqq \int_P \langle x, \xi \rangle v(x)\, d\mu_{\mathrm{DH}}.$$

This is called the *v-Futaki invariant* of X.

PROPOSITION 5.6. If Ω contains a v-soliton, then $\mathrm{Fut}_v = 0$.

PROOF. Let $\omega \in \Omega$ be a v-soliton on X. Then by (5.1) there exists a constant $a \in \mathbf{R}$ such that

$$\log(v(\mu_\omega)) = a + h_\omega,$$

hence $v(\mu_\omega) = e^{a+h_\omega}$. Therefore, for any $\xi \in \mathfrak{t}$ we have

$$\begin{aligned} \mathrm{Fut}_v(\xi) &= \int_X \langle \mu_\omega, \xi \rangle v(\mu_\omega) \frac{\omega^d}{d!} = \int_X \langle \mu_\omega, \xi \rangle e^{a+h_\omega} \frac{\omega^d}{d!} \\ &= e^a \int_X \langle \mu_\omega, \xi \rangle e^{h_\omega} \frac{\omega^d}{d!} = 0 \end{aligned}$$

by the normalization of μ_ω. □

REMARK 5.7. It is known that the v-Futaki invariant associated with Mabuchi solitons always vanishes. In fact, using the notation of Example 5.4, this follows from

$$\begin{aligned} \mathrm{Fut}_v(\xi) &= \int_X \langle \mu_\omega, \xi \rangle v(\mu_\omega) \frac{\omega^d}{d!} \\ &= \int_X \langle \mu_\omega, \xi \rangle \frac{\omega^d}{d!} - \int_X \langle \mu_\omega, \xi \rangle \Pi^G_\omega (1 - e^{h_\omega}) \frac{\omega^d}{d!} \\ &= \int_X \langle \mu_\omega, \xi \rangle \frac{\omega^d}{d!} - \int_X \langle \mu_\omega, \xi \rangle (1 - e^{h_\omega}) \frac{\omega^d}{d!} \\ &= \int_X \langle \mu_\omega, \xi \rangle \frac{\omega^d}{d!} - \int_X \langle \mu_\omega, \xi \rangle \frac{\omega^d}{d!} \\ &= 0. \end{aligned}$$

5.2. Existence result for Fano admissible manifolds

In this section, we study the existence problem of v-solitons on Fano admissible manifolds. Let $X = \mathbf{P}(\mathcal{O}^{\oplus(d_0+1)} \oplus (\mathcal{L} \otimes \mathcal{O}^{\oplus(d_\infty+1)}))$ be a Fano admissible manifold, and let $\Omega \coloneqq 2\pi c_1(X)$ be the admissible Kähler class given by (3.2). Consider the S^1-action

on X induced by scalar multiplication on $\mathcal{O}^{\oplus(d_0+1)}$, and let T, z and θ be as defined in Section 3.2. If $\omega \in \Omega$ is an admissible Kähler metric, then by Proposition 4.7, the normalized moment map is given by

$$\mu_\omega \coloneqq c(z - w),$$

where $w = (d_0 - d_\infty)/(d_0 + d_\infty + 2)$. Let $P \coloneqq \mu_\omega(X)$, and let $v \in C^\infty(P, \mathbf{R}_{>0})$. Define a function $u \in C^\infty([-1,1], \mathbf{R}_{>0})$ by $u(x) \coloneqq v(c(x-w))$. Then ω is a v-soliton if and only if it satisfies

$$\mathrm{Ric}(\omega) - \omega = \frac{1}{2} dd^c \log(u(z)).$$

The goal of this section is to prove the following theorem.

THEOREM 5.8. For a Fano admissible manifold X, the following three conditions are equivalent:

(1) X admits an admissible v-soliton.

(2) The v-Futaki invariant of X vanishes.

(3) $\displaystyle\int_{-1}^{1} (x-w)u(x)p_\Omega(x)\,dx = 0.$

REMARK 5.9. The equivalence in Theorem 5.8 is likely known to experts (see [**11**, Theorem 1.2] for a related result). However, since no explicit proof appears in the literature, we include one here for the reader's convenience.

Before proving Theorem 5.8, we present some corollaries. From Example 5.2, we immediately obtain the following, which was obtained in [**2**].

COROLLARY 5.10 ([**2**, Corollary 3.1]). For a Fano admissible manifold X, the following three conditions are equivalent:

(1) X admits a G-invariant Kähler–Einstein metric.

(2) The Futaki invariant of X vanishes.

(3) $\displaystyle\int_{-1}^{1} (x-w)p_\Omega(x)\,dx = 0.$

Noting that the v-Futaki invariant corresponding to Mabuchi solitons always vanishes (cf. Proposition 4.6 and Remark 5.7), Theorem 5.8 yields the following corollary.

COROLLARY 5.11. A Fano admissible manifold X admits a Mabuchi soliton if and only if $M_X < 1$.

Now we shall prove Theorem 5.8. The following proof proceeds largely along the lines of [**2**, Theorem 3.1]. The implication (1) $\Rightarrow$ (2) follows from Proposition 5.6. The equivalence of (2) and (3) is clear. Thus, it remains to prove (3) $\Rightarrow$ (1). Let $\omega \in \Omega$ be an

admissible Kähler metric given by (3.4), and define $F \coloneqq p_\Omega \Theta \in C^\infty([-1,1], \mathbf{R})$. Then F satisfies

$$F(x) > 0 \quad (x \in (-1,1)), \tag{5.2}$$
$$F(\pm 1) = 0, \quad F'(\pm 1) = \mp 2 p_\Omega(\pm 1). \tag{5.3}$$

Proposition 5.12. The following conditions are equivalent:

(1) ω is a v-soliton.

(2) F satisfies

$$\frac{(uF)'(z)}{(up_\Omega)(z)} = -(d_0 + d_\infty + 2)(z - w). \tag{5.4}$$

Proof. From the computation in Section 4.2, on $X \setminus (e_0 \cup e_\infty)$,

$$\mathrm{Ric}(\omega) - \omega = \sum_{a \in \hat{\mathcal{A}}} \mathrm{Ric}(\omega_a) - \sum_{a \in \hat{\mathcal{A}}} c \left(\frac{1}{x_a} + z \right) \varepsilon_a \omega_a - \frac{1}{2} dd^c \log(F(z)) - c(dz \wedge \theta).$$

Thus, ω is a v-soliton if and only if

$$\sum_{a \in \hat{\mathcal{A}}} \mathrm{Ric}(\omega_a) - \sum_{a \in \hat{\mathcal{A}}} c \left(\frac{1}{x_a} + z \right) \varepsilon_a \omega_a - \frac{1}{2} dd^c \log(u(z)F(z)) - c(dz \wedge \theta) = 0. \tag{5.5}$$

Computing $dd^c \log(u(z)F(z))$, we obtain

$$\begin{aligned} dd^c \log(u(z)F(z)) &= d \left(\frac{(uF)'(z)}{(uF)(z)} d^c z \right) = d \left(\frac{(uF)'(z)}{(up_\Omega)(z)} \theta \right) \\ &= \left(\frac{(uF)'}{up_\Omega} \right)' (z) dz \wedge \theta + \frac{(uF)'(z)}{(up_\Omega)(z)} d\theta \\ &= \left(\frac{(uF)'}{up_\Omega} \right)' (z) dz \wedge \theta + \frac{(uF)'(z)}{(up_\Omega)(z)} \sum_{a \in \hat{\mathcal{A}}} \varepsilon_a \omega_a. \end{aligned}$$

Thus, the equation (5.5) is equivalent to

$$\begin{aligned} &\sum_{a \in \hat{\mathcal{A}}} \left[\mathrm{Ric}(\omega_a) - \left\{ \frac{1}{2} \frac{(uF)'(z)}{(up_\Omega)(z)} + c \left(\frac{1}{x_a} + z \right) \right\} \varepsilon_a \omega_a \right] \\ &\quad - \frac{1}{2} \left\{ \left(\frac{(uF)'}{up_\Omega} \right)' (z) + 2c \right\} dz \wedge \theta = 0, \end{aligned}$$

in other words,

$$\mathrm{Ric}(\omega_a) - \left\{ \frac{1}{2} \frac{(uF)'(z)}{(up_\Omega)(z)} + c \left(\frac{1}{x_a} + z \right) \right\} \varepsilon_a \omega_a = 0 \quad (a \in \hat{\mathcal{A}}), \tag{5.6}$$

$$\left(\frac{(uF)'}{up_\Omega}\right)'(z) + 2c = 0. \tag{5.7}$$

Furthermore, since each ω_a is a Kähler–Einstein metric the equation (5.6) reduces to

$$\frac{1}{2}\frac{(uF)'(z)}{(up_\Omega)(z)} + c\left(\frac{1}{x_a} + z\right) = s_a \quad (a \in \hat{\mathcal{A}}). \tag{5.8}$$

The equation (5.7) is obtained by the derivative of (5.8). Thus, ω is a v-soliton if and only if F satisfies (5.8). As in (4.2), we have

$$\begin{aligned} s_a - c\left(\frac{1}{x_a} + z\right) &= \frac{1}{2}[(d_0+1)(1-z) - (d_\infty+1)(1+z)] \\ &= -\frac{1}{2}(d_0 + d_\infty + 2)(z - w), \end{aligned}$$

and hence (5.8) reduces to a single equation (5.4). □

We are now ready to prove the implication (3) ⇒ (1). Assume X satisfies (3), and consider the boundary value problem

$$\frac{(uF)'(x)}{(up_\Omega)(x)} = -(d_0 + d_\infty + 2)(x - w), \tag{5.9}$$

$$F(\pm 1) = 0. \tag{5.10}$$

The solution of (5.9) satisfying $F(-1) = 0$ is given by

$$F(x) = -\frac{d_0 + d_\infty + 2}{u(x)} \int_{-1}^{x} (t - w)u(t)p_\Omega(t)\, dt.$$

This function F is smooth on the closed interval $[-1, 1]$. Moreover, by assumption (3), we also have $F(1) = 0$.

Next, we compute the derivatives at the endpoints. We obtain

$$\begin{aligned} F'(\pm 1) &= -(d_0 + d_\infty + 2)(\pm 1 - w)p_\Omega(\pm 1) \\ &= \begin{cases} 2(d_0+1)p_\Omega(-1) & \text{if } x = -1, \\ -2(d_\infty+1)p_\Omega(1) & \text{if } x = 1. \end{cases} \end{aligned}$$

In particular, if $d_0 > 0$, then $p_\Omega(-1) = 0$ and hence $F'(-1) = 2p_\Omega(-1) = 0$. If $d_0 = 0$, then clearly $F'(-1) = 2p_\Omega(-1)$. Thus, in all cases, we have $F'(-1) = 2p_\Omega(-1)$. A similar argument shows that $F'(1) = -2p_\Omega(1)$. Therefore, F satisfies the condition (5.3).

We now verify that F satisfies the condition (5.2). It suffices to show that $uF > 0$ on the open interval $(-1, 1)$. First note that $w \in (-1, 1)$ because

$$w = \frac{d_0 - d_\infty}{d_0 + d_\infty + 2} \geq -\frac{d_\infty}{d_0 + d_\infty + 2} > -1,$$

and

$$w = \frac{d_0 - d_\infty}{d_0 + d_\infty + 2} \leq \frac{d_0}{d_0 + d_\infty + 2} < 1.$$

Let $x \in (-1, 1)$ arbitrarily. If $x \leq w$, then $t - w < 0$ on $[-1, x)$, and hence

$$(uF)(x) = -(d_0 + d_\infty + 2) \int_{-1}^{x} (t - w)u(t)p_\Omega(t)\, dt > 0.$$

If $x > w$, then $t - w > 0$ on $[x, 1)$, and we obtain

$$\begin{aligned}(uF)(x) &= -(d_0 + d_\infty + 2) \left[\int_{-1}^{1} (t - w)u(t)p_\Omega(t)\, dt - \int_{x}^{1} (t - w)u(t)p_\Omega(t)\, dt \right] \\ &= (d_0 + d_\infty + 2) \int_{x}^{1} (t - w)u(t)p_\Omega(t)\, dt \\ &> 0.\end{aligned}$$

Thus, $uF > 0$ holds for all $x \in (-1, 1)$.

Finally, define $\Theta\colon (-1, 1) \to \mathbf{R}$ by $\Theta \coloneqq F/p_\Omega$. It is clear that Θ is a positive smooth function on the open interval $(-1, 1)$, and from the expression

$$\Theta(x) = -\frac{d_0 + d_\infty + 2}{u(x)p_\Omega(x)} \int_{-1}^{x} (t - w)u(t)p_\Omega(t)\, dt,$$

we analyze its behavior near $x = -1$. Since $u(x) \neq 0$ and $x - w \neq 0$ near $x = -1$, we can write

$$u(x)p_\Omega(x) = (1 + x)^{d_0} g_-(x), \quad \int_{-1}^{x} (t - w)u(t)p_\Omega(t)\, dt = (1 + x)^{d_0+1} h_-(x),$$

where g_-, h_- are smooth around $x = -1$ and nonvanishing at $x = -1$. Then, we have

$$\begin{aligned}(1 + x)^{d_0}(x - w)g_-(x) &= (x - w)u(x)p_\Omega(x) \\ &= \left(\int_{-1}^{x} (t - w)u(t)p_\Omega(t)\, dt \right)' \\ &= (d_0 + 1)(1 + x)^{d_0} h_-(x) + (1 + x)^{d_0+1} h'_-(x) \\ &= (1 + x)^{d_0} \left[(d_0 + 1)h_-(x) + (1 + x)h'_-(x) \right]\end{aligned}$$

and

$$(x - w)g_-(x) = (d_0 + 1)h_-(x) + (1 + x)h'_-(x).$$

This shows that

$$\begin{aligned}\Theta(x) &= -\frac{(d_0 + d_\infty + 2)(1 + x)^{d_0+1} h_-(x)}{(1 + x)^{d_0} g_-(x)} \\ &= -(1 + x)\frac{(d_0 + d_\infty + 2)h_-(x)}{g_-(x)},\end{aligned}$$

and hence Θ extends smoothly to $x=-1$ with

$$\Theta(-1)=0, \quad \Theta'(-1)=2.$$

A similar argument shows that Θ extends smoothly to $x=1$ with

$$\Theta(1)=0, \quad \Theta'(1)=-2.$$

Therefore, Θ satisfies all the required conditions, and hence defines an admissible Kähler metric $\omega \in \Omega$. By construction, this metric is a v-soliton. This completes the proof of Theorem 5.8.

6. The Mabuchi constant of Fano admissible manifolds

The purpose of this section is to prove Theorem 1.2. Let $X=\mathbf{P}(\mathcal{O}^{\oplus(d_0+1)} \oplus (\mathcal{L} \otimes \mathcal{O}^{\oplus(d_\infty+1)}))$ be a Fano admissible manifold, and let $\Omega \coloneqq 2\pi c_1(X)$. For $i \in \{0,1,2\}$, set

$$b_i \coloneqq \int_{-1}^{1} x^i p_\Omega(x)\,dx.$$

Since $p_\Omega > 0$ on $(-1,1)$, we have $b_0, b_2 > 0$. Moreover, by the Cauchy–Schwarz inequality, $b_0 b_2 - b_1^2 > 0$.

Using Propositions 4.6 and 4.7 in Section 4.2, we obtain the following.

THEOREM 6.1 ([**8**, Theorem 6.5] FOR CASES $d_0 = d_\infty = 0$). Define $w \coloneqq \frac{d_0-d_\infty}{d_0+d_\infty+2}$. Then

$$M_X = \frac{b_0|b_1-wb_0| - b_1(b_1-wb_0)}{b_0b_2-b_1^2} = 1 + \frac{b_0(|b_1-wb_0|-(b_2-wb_1))}{b_0b_2-b_1^2}.$$

PROOF. Let $\omega \in \Omega$ be an admissible Kähler metric given by (3.4). Since constant functions and z are Killing potentials of (X,ω), Propositions 3.7 and 4.6 imply that

$$\begin{aligned} 0 &= \int_X (1-e^{h_\omega})\frac{\omega^d}{d!} = \int_X \Pi_\omega^G (1-e^{h_\omega})\frac{\omega^d}{d!} \\ &= \int_X (\alpha z+\beta)\frac{\omega^d}{d!} = 2\pi c^d \left(\prod_{a\in\hat{\mathcal{A}}} \mathrm{vol}(Y_a,\omega_a)\right)(\alpha b_1+\beta b_0). \end{aligned}$$

Hence we obtain

$$\alpha b_1 + \beta b_0 = 0. \tag{6.1}$$

Furthermore,

$$\int_X z(1-e^{h_\omega})\frac{\omega^d}{d!} = \int_X z\Pi_\omega^G(1-e^{h_\omega})\frac{\omega^d}{d!} = \int_X (\alpha z^2+\beta z)\frac{\omega^d}{d!}$$

$$= 2\pi c^d \left(\prod_{a \in \hat{\mathcal{A}}} \mathrm{vol}(Y_a, \omega_a) \right) (\alpha b_2 + \beta b_1).$$

On the other hand, by Proposition 4.7, we have

$$\begin{aligned} \int_X z(1 - e^{h_\omega}) \frac{\omega^d}{d!} &= \int_X z \frac{\omega^d}{d!} - w \int_X e^{h_\omega} \frac{\omega^d}{d!} = \int_X z \frac{\omega^d}{d!} - w \int_X \frac{\omega^d}{d!} \\ &= 2\pi c^d \left(\prod_{a \in \hat{\mathcal{A}}} \mathrm{vol}(Y_a, \omega_a) \right) (b_1 - w b_0). \end{aligned}$$

These give us that

$$\alpha b_2 + \beta b_1 = b_1 - w b_0. \tag{6.2}$$

By solving (6.1) and (6.2), we obtain

$$\alpha = \frac{b_0 b_1 - w b_0^2}{b_0 b_2 - b_1^2}, \quad \beta = -\frac{b_1^2 - w b_0 b_1}{b_0 b_2 - b_1^2},$$

and

$$\begin{aligned} M_X &= \max_{x \in [-1,1]} (\alpha z + \beta) = |\alpha| + \beta \\ &= \frac{b_0 |b_1 - w b_0| - b_1 (b_1 - w b_0)}{b_0 b_2 - b_1^2} \\ &= 1 + \frac{b_0 (|b_1 - w b_0| - (b_2 - w b_1))}{b_0 b_2 - b_1^2}, \end{aligned}$$

as required. □

7. Mabuchi solitons on Fano admissible manifolds over $\mathbf{P}^n$

In the final section, we concern the existence and non-existence of Mabuchi solitons on Fano admissible manifolds over $\mathbf{P}^n$. Let X be a Fano admissible manifold over $\mathbf{P}^n$. Then X can be written as

$$X = \mathbf{P}_{\mathbf{P}^n}(\mathcal{O}^{\oplus(d_0+1)} \oplus \mathcal{O}(k)^{\oplus(d_\infty+1)}) \tag{7.1}$$

for some $k \in \mathbf{Z} \setminus \{0\}$ and $d_0, d_\infty \in \mathbf{Z}_{\geq 0}$. In that case,

$$\mathbf{P}_{\mathbf{P}^n}(\mathcal{O}^{\oplus(d_0+1)} \oplus \mathcal{O}(k)^{\oplus(d_\infty+1)}) \cong \mathbf{P}_{\mathbf{P}^n}(\mathcal{O}^{\oplus(d_\infty+1)} \oplus \mathcal{O}(-k)^{\oplus(d_0+1)})$$

and so we may assume $k \in \mathbf{Z}_{\geq 1}$. By Theorem 4.1, X is Fano if and only if $k(d_0+1) < n+1$. In what follows, we assume these conditions. Moreover, by Corollary 5.11, the existence of a Mabuchi soliton on X is equivalent to $M_X < 1$. The purpose of this section is to prove the following theorem.

THEOREM 7.1. The Fano admissible manifold X admits a Mabuchi soliton if and only if

$$(k, d_\infty) = (1, 0) \quad \text{or} \quad (n, k, d_0, d_\infty) = (1, 1, 0, 1).$$

In all other cases, $M_X > 1$.

To prove Theorem 7.1, we compute M_X by using Theorem 6.1. Let

$$\lambda \coloneqq \frac{2(n+1) + k(d_\infty - d_0)}{k(d_0 + d_\infty + 2)}.$$

Then, for each $i \in \{0, 1, 2\}$,

$$b_i = \int_{-1}^{1} (\lambda + x)^n (1+x)^{d_0} (1-x)^{d_\infty} x^i dx.$$

PROPOSITION 7.2. $b_1 - w b_0 > 0$. In particular, we have

$$M_X = 1 + \frac{b_0(b_1 - w b_0 + w b_1 - b_2)}{b_0 b_2 - b_1^2}.$$

PROOF. Define $a \in \mathbf{R}$ by

$$a \coloneqq \frac{n + 1 - k(d_0 + 1)}{k(d_0 + d_\infty + 2)}.$$

By assumption, $a > 0$ and

$$\lambda - 1 = \frac{2[n + 1 - k(d_0 + 1)]}{k(d_0 + d_\infty + 2)} = 2a.$$

Therefore,

$$\begin{aligned} b_1 - w b_0 &= \int_{-1}^{1} (\lambda + x)^n (1+x)^{d_0} (1-x)^{d_\infty} (x - w)\, dx \\ &= \int_{0}^{1} (\lambda + 2u - 1)^n (2u)^{d_0} (2 - 2u)^{d_\infty} (2u - 1 - w)\, du \\ &= 2^{d_0 + d_\infty + n + 2} \int_{0}^{1} (a + u)^n u^{d_0} (1-u)^{d_\infty} \left(u - \frac{d_0 + 1}{d_0 + d_\infty + 2} \right) du. \end{aligned}$$

On the other hand,

$$\begin{aligned} &(d_0 + d_\infty + 2) u^{d_0} (1-u)^{d_\infty} \left(u - \frac{d_0 + 1}{d_0 + d_\infty + 2} \right) \\ &= (d_0 + d_\infty + 2) u^{d_0 + 1} (1-u)^{d_\infty} - (d_0 + 1) u^{d_0} (1-u)^{d_\infty} \\ &= u^{d_0} (1-u)^{d_\infty} [(d_0 + d_\infty + 2) u - (d_0 + 1)] \\ &= u^{d_0} (1-u)^{d_\infty} [(d_0 + 1)(u - 1) + (d_\infty + 1) u] \end{aligned}$$

$$= -(u^{d_0+1}(1-u)^{d_\infty+1})'.$$

Hence, integration by parts yields

$$\begin{aligned} b_1 - wb_0 &= -\frac{2^{d_0+d_\infty+n+2}}{d_0+d_\infty+2}\int_0^1 (a+u)^n (u^{d_0+1}(1-u)^{d_\infty+1})'\,du \\ &= \frac{n2^{d_0+d_\infty+n+2}}{d_0+d_\infty+2}\int_0^1 (a+u)^{n-1}u^{d_0+1}(1-u)^{d_\infty+1}\,du \\ &= \frac{n2^{d_0+d_\infty+n+2}}{d_0+d_\infty+2}\sum_{j=0}^{n-1}\binom{n-1}{j}a^{n-1-j}B(j+d_0+2,d_\infty+2), \end{aligned}$$

where $B(x,y)$ is the beta function. Therefore, $b_1 - wb_0 > 0$ follows from $a > 0$. The second assertion follows immediately from Theorem 6.1. □

Using Corollary 5.10, we obtain the following corollary.

COROLLARY 7.3. For any $k \in \mathbf{Z}_{\geq 1}$ and $d_0, d_\infty \in \mathbf{Z}_{\geq 0}$, $X = \mathbf{P}_{\mathbf{P}^n}(\mathcal{O}^{\oplus(d_0+1)} \oplus \mathcal{O}(k)^{\oplus(d_\infty+1)})$ does not admit a Kähler–Einstein metric.

Now, set

$$I \coloneqq b_1 - wb_0 + wb_1 - b_2.$$

Then

$$I = \int_{-1}^1 (\lambda+x)^n(1+x)^{d_0}(1-x)^{d_\infty+1}(x-w)dx$$

holds. Moreover, the condition $M_X \leq 1$ (respectively $M_X = 1$) is equivalent to $I \leq 0$ (respectively $I = 0$).

LEMMA 7.4. If the following inequality holds, then $M_X > 1$ holds:

$$\frac{n(d_\infty+1)-(d_0+1)}{d_0+d_\infty+4} \geq \frac{n+1-k(d_0+1)}{k(d_0+d_\infty+2)}. \tag{7.2}$$

In particular, in that case X does not admit a Mabuchi soliton.

PROOF. Define $b \in \mathbf{R}$ by

$$b \coloneqq \frac{d_0+1}{d_0+d_\infty+2}.$$

Then $b > 0$, and

$$w + 1 = \frac{2(d_0+1)}{d_0+d_\infty+2} = 2b.$$

Moreover,

$$\begin{aligned} I &= \int_0^1 (\lambda + 2u - 1)^n (2u)^{d_0} (2-2u)^{d_\infty+1} (2u - (w+1)) 2\, du \\ &= 2^{d_0+d_\infty+n+3} \int_0^1 (a+u)^n u^{d_0} (1-u)^{d_\infty+1} (u-b)\, du \end{aligned} \tag{7.3}$$

holds, where a is defined in the proof of Proposition 7.2. Since

$$\begin{aligned} &\int_0^1 (a+u)^n u^{d_0} (1-u)^{d_\infty+1} (u-b) du \\ &= \sum_{j=0}^n \binom{n}{j} a^{n-j} \left[\int_0^1 u^{j+d_0+1} (1-u)^{d_\infty+1} du - b \int_0^1 u^{j+d_0} (1-u)^{d_\infty+1} du \right] \\ &= \sum_{j=0}^n \binom{n}{j} a^{n-j} [B(j+d_0+2, d_\infty+2) - bB(j+d_0+1, d_\infty+2)] \end{aligned}$$

and

$$\begin{aligned} &B(j+d_0+2, d_\infty+2) - bB(j+d_0+1, d_\infty+2) \\ &= \left[\frac{j+d_0+1}{(j+d_0+1)+(d_\infty+2)} - \frac{d_0+1}{d_0+d_\infty+2} \right] B(j+d_0+1, d_\infty+2) \\ &= \frac{(d_\infty+1)j - (d_0+1)}{d_0+d_\infty+2} \frac{B(j+d_0+1, d_\infty+2)}{j+d_0+d_\infty+3} \\ &= \frac{(d_\infty+1)j - (d_0+1)}{d_0+d_\infty+2} \frac{B(j+d_0+1, d_\infty+3)}{d_\infty+2}, \end{aligned}$$

we have

$$I = \frac{2^{d_0+d_\infty+n+3}}{(d_0+2)(d_0+d_\infty+2)} \sum_{j=0}^n \binom{n}{j} a^{n-j} [(d_\infty+1)j - (d_0+1)] B(j+d_0+1, d_\infty+3).$$

Therefore, if we set

$$S_0 \coloneqq \sum_{j=0}^n \binom{n}{j} a^{n-j} B(d_0+1+j, d_\infty+3), \quad S_1 \coloneqq \sum_{j=0}^n j \binom{n}{j} a^{n-j} B(d_0+1+j, d_\infty+3),$$

then, since

$$I = \frac{2^{d_0+d_\infty+n+3}}{(d_0+2)(d_0+d_\infty+2)} [(d_\infty+1)S_1 - (d_0+1)S_0]$$

and $S_0 > 0$, the condition $I \geq 0$ (respectively, $I = 0$) is equivalent to

$$\frac{S_1}{S_0} \geq \frac{d_0+1}{d_\infty+1} \quad \left(\text{respectively, } \frac{S_1}{S_0} = \frac{d_0+1}{d_\infty+1} \right). \tag{7.4}$$

Since

$$\begin{aligned}S_0 &= \sum_{j=0}^{n} \binom{n}{j} a^{n-j} \int_0^1 t^{j+d_0}(1-t)^{d_\infty+2}dt \\ &= \int_0^1 t^{d_0}(1-t)^{d_\infty+2}(a+t)^n dt \\ &= a\int_0^1 t^{d_0}(1-t)^{d_\infty+2}(a+t)^{n-1}dt + \int_0^1 t^{d_0+1}(1-t)^{d_\infty+2}(a+t)^{n-1}dt,\end{aligned}$$

using

$$nx(x+y)^{n-1} = x\frac{\partial}{\partial x}(x+y)^n = \sum_{j=0}^{n} j\binom{n}{j}x^j y^{n-j},$$

we obtain

$$\begin{aligned}S_1 &= \sum_{j=0}^{n} j\binom{n}{j} a^{n-j} \int_0^1 t^{d_0+j}(1-t)^{d_\infty+2}dt \\ &= \int_0^1 t^{d_0}(1-t)^{d_\infty+2} nt(a+t)^{n-1}dt \\ &= n\int_0^1 t^{d_0+1}(1-t)^{d_\infty+2}(a+t)^{n-1}dt.\end{aligned}$$

Now let us set

$$K \coloneqq \int_0^1 t^{d_0+1}(1-t)^{d_\infty+2}(a+t)^{n-1}dt, \quad L \coloneqq \int_0^1 t^{d_0}(1-t)^{d_\infty+2}(a+t)^{n-1}dt.$$

Then $K, L > 0$, and

$$\frac{S_1}{S_0} = \frac{nK}{aL+K}.$$

Define $f\colon (0,+\infty) \to \mathbf{R}$ by

$$f(t) \coloneqq \frac{nt}{aL+t}.$$

Then, since $aL > 0$, f is a strictly increasing function. Next, if we set

$$g(t) \coloneqq t^{d_0}(1-t)^{d_\infty+2}, \quad h(t) \coloneqq (a+t)^{n-1},$$

then

$$\begin{aligned}&K = \int_0^1 tg(t)h(t)\,dt, \quad L = \int_0^1 g(t)h(t)\,dt, \\ &\int_0^1 g(t)\,dt = B(d_0+1, d_\infty+3), \quad \int_0^1 tg(t)\,dt = B(d_0+2, d_\infty+3)\end{aligned}$$

hold. Here, we investigate the sign of

$$B(d_0+1, d_\infty+3)K - B(d_0+2, d_\infty+3)L.$$

- In the case $n = 1$.

  Since $h(t) = 1$, we have $K = B(d_0+2, d_\infty+3), L = B(d_0+1, d_\infty+3)$ and hence $B(d_0+1, d_\infty+3)K - B(d_0+2, d_\infty+3)L = 0$.

- In the case $n \geq 2$.

  Since $h'(t) = (n-1)(a+t)^{n-2} > 0$, h is strictly increasing on $[0,1]$, and

$$\begin{aligned}
&B(d_0+1, d_\infty+3)K - B(d_0+2, d_\infty+3)L \\
&= \int_0^1 tg(t)h(t)\,dt \int_0^1 g(t)\,dt - \int_0^1 g(t)h(t)dt \int_0^1 tg(t)\,dt \\
&= \int_0^1 tg(t)h(t)\,dt \int_0^1 g(s)\,ds - \int_0^1 g(t)h(t)\,dt \int_0^1 sg(s)\,ds \\
&= \int_0^1 \int_0^1 (t-s)h(t)g(s)g(t)\,dsdt
\end{aligned}$$

  holds. By a change of variables, we have

$$\int_0^1 \int_0^1 (t-s)h(t)g(s)g(t)\,dsdt = \int_0^1 \int_0^1 (s-t)h(s)g(t)g(s)\,dsdt$$

  and

$$\begin{aligned}
&B(d_0+1, d_\infty+3)K - B(d_0+2, d_\infty+3)L \\
&= \frac{1}{2}\int_0^1 \int_0^1 ((t-s)h(t) + (s-t)h(s))g(s)g(t)\,dsdt \\
&= \frac{1}{2}\int_0^1 \int_0^1 (h(s)-h(t))(s-t)g(s)g(t)\,dsdt.
\end{aligned}$$

  Therefore, since $(h(s)-h(t))(s-t) \geq 0$, we have $B(d_0+1, d_\infty+3)K - B(d_0+2, d_\infty+3)L \geq 0$.

From the above,

$$K \geq \frac{B(d_0+2, d_\infty+3)}{B(d_0+1, d_\infty+3)}L = \frac{d_0+1}{d_0+d_\infty+4}L$$

holds, and so, by the monotonicity of f,

$$\frac{S_1}{S_0} = \frac{nK}{aL+K} \geq \frac{n(d_0+1)}{a(d_0+d_\infty+4)+d_0+1}$$

follows. Therefore, if (7.2) holds, then

$$\frac{n(d_0+1)}{a(d_0+d_\infty+4)+(d_0+1)} \geq \frac{d_0+1}{d_\infty+1},$$

and so (7.4) follows. In particular, since $I \geq 0$, we have $M_X \geq 1$. Furthermore, the following holds.

(1) In the case $n = 1$.

Since $k(d_0 + 1) < n + 1$, we have $n = k = 1$ and $d_0 = 0$. Hence, it holds

$$a = \frac{n + 1 - k(d_0 + 1)}{k(d_0 + d_\infty + 2)} = \frac{1}{d_\infty + 2}$$

and $B(d_0 + 1, d_\infty + 3)K - B(d_0 + 2, d_\infty + 3)L = 0$. In particular, we have

$$K = \frac{d_0 + 1}{d_0 + d_\infty + 4} L.$$

Here, if we assume $M_X = 1$, then $I = 0$, and so by (7.4) we obtain

$$\frac{d_0 + 1}{d_\infty + 1} = \frac{S_1}{S_0} = \frac{nK}{aL + K} = \frac{n(d_0 + 1)}{a(d_0 + d_\infty + 4) + (d_0 + 1)}.$$

Rearrangement of this gives us

$$d_\infty^2 + d_\infty - 4 = 0,$$

which contradicts $d_\infty \in \mathbf{Z}_{\geq 0}$. Therefore, $M_X > 1$.

(2) In the case $n \geq 2$.

Since h is a strictly increasing function, $s = t$ is the only possibility for $s, t \in [0, 1]$ satisfying $(h(s) - h(t))(s - t) = 0$. Hence the set on which the integrand vanishes has measure zero in $[0, 1] \times [0, 1]$, and therefore

$$\begin{aligned} &B(d_0 + 1, d_\infty + 3)K - B(d_0 + 2, d_\infty + 3)L \\ &= \frac{1}{2} \int_0^1 \int_0^1 (h(s) - h(t))(s - t) g(s) g(t) ds dt > 0 \end{aligned}$$

and hence

$$\frac{S_1}{S_0} = \frac{nK}{aL + K} > \frac{n(d_0 + 1)}{a(d_0 + d_\infty + 4) + (d_0 + 1)} \geq \frac{d_0 + 1}{d_\infty + 1}.$$

This and the equivalence in (7.4) gives us that $M_X > 1$.

From the above, we see that $M_X > 1$ holds in either case. □

7.1. The case $k \geq 2$ or $d_\infty \geq 1$

We first show a non-existence result for Mabuchi solitons.

PROPOSITION 7.5. If $k \geq 2$, or $k = 1$ and $d_\infty \geq 1$, then $M_X > 1$ holds, except in the cases $(n, k, d_0, d_\infty) = (1, 1, 0, 1)$ and $(n, k, d_0, d_\infty) = (2, 1, 0, 1)$. In particular, X does not admit a Mabuchi soliton.

PROOF. First note that the inequality (7.2) in Lemma 7.4 is equivalent to

$$\left[(d_\infty+1)(d_0+d_\infty+2)-\frac{d_0+d_\infty+4}{k}\right]n-\frac{d_0+d_\infty+4}{k}+2(d_0+1)\geq 0. \tag{7.5}$$

Define $f\colon [1,+\infty)\to \mathbf{R}$ by

$$f(t)\coloneqq\left[(d_\infty+1)(d_0+d_\infty+2)-\frac{d_0+d_\infty+4}{k}\right]t-\frac{d_0+d_\infty+4}{k}+2(d_0+1).$$

(1) The case $k\geq 2$.

In this case, we have

$$\begin{aligned}
&(d_\infty+1)(d_0+d_\infty+2)-\frac{d_0+d_\infty+4}{k}\\
&\geq (d_\infty+1)(d_0+d_\infty+2)-\frac{d_0+d_\infty+4}{2}\\
&=\left(d_\infty+\frac{1}{2}\right)(d_0+d_\infty+2)-1\\
&\geq 0.
\end{aligned}$$

Moreover, equality holds if and only if $(k,d_0,d_\infty)=(2,0,0)$.

(a) The case $(k,d_0,d_\infty)=(2,0,0)$.
Since

$$f(t)=\left(1\cdot 2-\frac{4}{2}\right)t-\frac{4}{2}+2=0,$$

the inequality (7.2) holds for any $n\in\mathbf{Z}_{\geq 1}$.

(b) Cases other than (a).
Since $k\geq 2$, we have

$$\begin{aligned}
f(1)&=(d_\infty+1)(d_0+d_\infty+2)-\frac{2(d_0+d_\infty+4)}{k}+2(d_0+1)\\
&\geq (d_\infty+1)(d_0+d_\infty+2)-(d_0+d_\infty+4)+2(d_0+1)\\
&=d_\infty(d_0+d_\infty+2)+2d_0\\
&\geq 0.
\end{aligned}$$

Therefore, since $n\geq 2$ and f is monotonically increasing, the inequality (7.2) is maintained by $f(n)\geq f(1)\geq 0$.

(2) The case $k=1$.

First note that

$$(d_\infty+1)(d_0+d_\infty+2)-(d_0+d_\infty+4)=d_\infty(d_0+d_\infty+2)-2>0$$

and hence f is a monotonically increasing function.

(a) The case $d_0 \geq 1$.
Since

$$\begin{aligned} f(n) > f(1) &= (d_\infty + 1)(d_0 + d_\infty + 2) - 2(d_0 + d_\infty + 4) + 2(d_0 + 1) \\ &= (d_\infty - 1)(d_0 + d_\infty + 2) + 2(d_0 - 1) \geq 0, \end{aligned}$$

the inequality (7.2) holds.

(b) The case $d_0 = 0$.
If $d_\infty \geq 2$, then the inequality (7.2) holds from $f(1) \geq 3 \cdot (0 + 2 + 2) - 2 > 0$. If $d_\infty = 1$, then

$$f(3) = 3\left((1+1)(0+1+2) - \frac{0+1+4}{1}\right) - \frac{0+1+4}{1} + 2(0+1) = 0,$$

and so, except in the cases $(n, d_0, d_\infty) = (1, 0, 1)$ and $(n, d_0, d_\infty) = (2, 0, 1)$, the inequality (7.2) holds.

Therefore, if $k \geq 2$, or $k = 1$ and $d_\infty \geq 1$, then $M_X > 1$ holds, except in the cases $(n, k, d_0, d_\infty) = (1, 1, 0, 1)$ and $(n, k, d_0, d_\infty) = (2, 1, 0, 1)$. In particular, X does not admit a Mabuchi soliton. □

Proposition 7.6. If $(n, k, d_0, d_\infty) = (1, 1, 0, 1)$, then X admits a Mabuchi soliton. On the other hand, when $(n, k, d_0, d_\infty) = (2, 1, 0, 1)$, $M_X > 1$ and hence X does not admit a Mabuchi soliton.

Proof. First, assume that $(n, k, d_0, d_\infty) = (1, 1, 0, 1)$. Then, since

$$\lambda = \frac{2(1+1)/1 + 1 - 0}{0 + 1 + 2} = \frac{5}{3}, \quad w = \frac{0 - 1}{0 + 1 + 2} = -\frac{1}{3},$$

we have

$$\begin{aligned} I &= \int_{-1}^{1} \left(\frac{5}{3} + x\right)(1 - x)^2 \left(x + \frac{1}{3}\right) dx = \int_{-1}^{1} \left(x^4 - \frac{22}{9}x^2 + \frac{8}{9}x + \frac{5}{9}\right) dx \\ &= \left[\frac{1}{5}x^5 - \frac{22}{27}x^3 + \frac{5}{9}x\right]_{-1}^{1} = -\frac{16}{135} < 0. \end{aligned}$$

In particular, $M_X < 1$ and X admits a Mabuchi soliton by Corollary 5.11. Next, if $(n, k, d_0, d_\infty) = (2, 1, 0, 1)$, then, since

$$\lambda = \frac{2(2+1)/1 + 1 - 0}{0 + 1 + 2} = \frac{7}{3}, \quad w = \frac{0 - 1}{0 + 1 + 2} = -\frac{1}{3},$$

we have

$$I = \int_{-1}^{1} \left(\frac{7}{3} + x\right)^2 (1 - x)^2 \left(x + \frac{1}{3}\right) dx$$

$$
\begin{aligned}
&= \int_{-1}^{1} \left(x^5 + 3x^4 - 2x^3 - \frac{194}{27}x^2 + \frac{91}{27}x + \frac{49}{27} \right) dx \\
&= \left[\frac{3}{5}x^5 - \frac{194}{81}x^3 + \frac{49}{27}x \right]_{-1}^{1} \\
&= \frac{16}{405} \\
&> 0.
\end{aligned}
$$

This shows $M_X > 1$ and that X does not admit a Mabuchi soliton. □

7.2. The case $k = 1$ and $d_\infty = 0$

Next we show a result of existence for Mabuchi solitons. The following proposition completes the proof of Theorem 7.1.

PROPOSITION 7.7. If $k = 1$ and $d_\infty = 0$, then X admits a Mabuchi soliton.

PROOF. Let a and b be real numbers defined in the proofs of Proposition 7.2 and Lemma 7.4, respectively. Then, by (7.3) in Lemma 7.4, we have

$$
\begin{aligned}
I &= 2^{d_0+d_\infty+n+3} \int_0^1 (a+u)^n u^{d_0} (1-u)^{d_\infty+1} (u-b) du \\
&= 2^{d_0+n+3} \int_0^1 (a+u)^n u^{d_0} (1-u)(u-b) du, \\
a &= \frac{(n+1)-(d_0+1)}{d_0+2} = \frac{n-d_0}{d_0+2}, \quad b = \frac{d_0+1}{d_0+2}.
\end{aligned}
$$

Here, by setting

$$
\alpha := -\frac{1}{2(d_0+2)}, \quad \beta := -\frac{d_0+1}{2(d_0+2)},
$$

we obtain

$$
u^{d_0}(1-u)(u-b) = \alpha \left(\frac{d}{du} u^{d_0+1}(1-u)^2 \right) + \beta u^{d_0}(1-u)^2.
$$

Therefore, by integration by parts,

$$
\begin{aligned}
I &= 2^{d_0+n+3} \int_0^1 (a+u)^n \left[\alpha \Big(\frac{d}{du} u^{d_0+1}(1-u)^2 \Big) + \beta u^{d_0}(1-u)^2 \right] du \\
&= 2^{d_0+n+3} \left\{ \left[\alpha (a+u)^n u^{d_0+1}(1-u)^2 \right]_0^1 - \int_0^1 \alpha n (a+u)^{n-1} u^{d_0+1}(1-u)^2 du \right\} \\
&\quad + 2^{d_0+n+3} \int_0^1 \beta (a+u)^n u^{d_0}(1-u)^2 du \\
&= 2^{d_0+n+3} \int_0^1 (a+u)^{n-1} u^{d_0}(1-u)^2 (-\alpha n u + \beta(a+u)) du
\end{aligned}
$$

$$= \frac{2^{d_0+n+2}}{d_0+2} \int_0^1 (a+u)^{n-1} u^{d_0} (1-u)^2 [(n-d_0-1)u - a(d_0+1)] du.$$

Thus, when $n - d_0 - 1 = 0$, we have

$$I = -\frac{2^{d_0+n+2}}{d_0+2} \int_0^1 (a+u)^{n-1} u^{d_0} (1-u)^2 a(d_0+1) du \ < 0$$

and hence $M_X < 1$, which shows that X admits a Mabuchi soliton. From now on, assume $n - d_0 - 1 > 0$. If we set

$$c \coloneqq \frac{a(d_0+1)}{n-d_0-1} = \frac{(n-d_0)(d_0+1)}{(n-d_0-1)(d_0+2)},$$

we have $c > 0$ and

$$I = \frac{2^{d_0+n+2}(n-d_0-1)}{d_0+2} \int_0^1 (a+u)^{n-1} u^{d_0} (1-u)^2 (u-c)\, du.$$

Hence, by setting

$$J \coloneqq \int_0^1 (a+u)^{n-1} u^{d_0+1} (1-u)^2 du, \quad K \coloneqq \int_0^1 (a+u)^{n-1} u^{d_0} (1-u)^2 du,$$

we have $I = J - cK$, and $I \leq 0$ (respectively, $I = 0$) if and only if

$$\frac{J}{K} \leq c \quad \left(\text{respectively, } \frac{J}{K} = c \right). \tag{7.6}$$

In what follows, we show that $\dfrac{J}{K} < c$. First,

$$\begin{aligned}
J &= \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j} \int_0^1 u^{j+d_0+1} (1-u)^2 du \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j} B(j+d_0+2, 3) \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j} \frac{2}{(j+d_0+2)(j+d_0+3)(j+d_0+4)}, \\
K &= \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j} \int_0^1 u^{j+d_0} (1-u)^2 du \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j} B(j+d_0+1, 3) \\
&= \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j} \frac{2}{(j+d_0+1)(j+d_0+2)(j+d_0+3)}.
\end{aligned}$$

Hence, by setting

$$p_j \coloneqq \sum_{j=0}^{n-1} \binom{n-1}{j} a^{n-1-j}, \quad q_j \coloneqq \frac{2}{(j+d_0+1)(j+d_0+2)(j+d_0+3)}$$

we have

$$J = \sum_{j=0}^{n-1} p_j q_j \frac{j+d_0+1}{j+d_0+4}, \quad K = \sum_{i=0}^{n-1} p_i q_i.$$

Next, if we set $r_j \coloneqq \dfrac{1}{K} p_j q_j$, then $\sum\limits_{j=0}^{n-1} r_j = 1$ and

$$\frac{J}{K} = \sum_{j=0}^{n-1} r_j \frac{j+d_0+1}{j+d_0+4}.$$

Define $f\colon [0,+\infty) \to \mathbf{R}$ by $f(t) \coloneqq \dfrac{t+d_0+1}{t+d_0+4}$. Then f is a strictly increasing concave function. Hence we have

$$\frac{J}{K} = \sum_{j=0}^{n-1} r_j f(j) \leq f\left(\sum_{j=0}^{n-1} r_j j \right).$$

Here, we point out that

$$\sum_{j=0}^{n-1} r_j j \leq \frac{\sum\limits_{j=0}^{n-1} p_j j}{\sum\limits_{i=0}^{n-1} p_i}$$

holds. In fact, since $p_j \geq 0$ and q_j is monotonically decreasing in j, we have

$$\begin{aligned}
\sum_{j=0}^{n-1} p_j q_j j \sum_{i=0}^{n-1} p_i - \sum_{j=0}^{n-1} p_j j \sum_{i=0}^{n-1} p_i q_i &= \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} j(q_j - q_i) p_j p_i \\
&= \sum_{i<j} j(q_j - q_i) p_j p_i + \sum_{j<i} j(q_j - q_i) p_j p_i \\
&= \sum_{i<j} j(q_j - q_i) p_j p_i + \sum_{i<j} i(q_i - q_j) p_i p_j \\
&= \sum_{i<j} (j-i)(q_j - q_i) p_j p_i \leq 0.
\end{aligned}$$

Hence,

$$\sum_{j=0}^{n-1} r_j j = \frac{\sum\limits_{j=0}^{n-1} p_j q_j j}{\sum\limits_{i=0}^{n-1} p_i q_i} \leq \frac{\sum\limits_{j=0}^{n-1} p_j j}{\sum\limits_{i=0}^{n-1} p_i}.$$

Furthermore, since

$$\sum_{i=0}^{n-1} p_i = \sum_{i=0}^{n-1} \binom{n-1}{j} a^{n-1-i} = (1+a)^{n-1},$$
$$\sum_{j=0}^{n-1} p_j j = \sum_{j=0}^{n-1} \binom{n-1}{j} j \cdot a^{n-1-j} = (n-1)(1+a)^{n-2}$$

and f is monotonically increasing,

$$\frac{J}{K} \leq f\left(\sum_{j=0}^{n-1} r_j j\right) \leq f\left(\frac{\sum\limits_{j=0}^{n-1} p_j j}{\sum\limits_{i=0}^{n-1} p_i}\right) = f\left(\frac{n-1}{1+a}\right).$$

Thus, we obtain

$$\begin{aligned} f\left(\frac{n-1}{1+a}\right) &= f\left(\frac{(d_0+2)(n-1)}{n+2}\right) \\ &= \frac{(2d_0+3)n+d_0}{2(d_0+3)n+d_0+6} \end{aligned}$$

and

$$\frac{J}{K} \leq \frac{(2d_0+3)n+d_0}{2(d_0+3)n+d_0+6}. \tag{7.7}$$

Finally, let us show

$$\frac{(2d_0+3)n+d_0}{2(d_0+3)n+(d_0+6)} < c. \tag{7.8}$$

First, by a direct computation, the condition (7.8) is equivalent to

$$d_0 n^2 + (d_0^2 + 12d_0 + 12)n - 4d_0(d_0+1) > 0.$$

Moreover, since $d_0 + 1 < n$, we have in fact

$$\begin{aligned} d_0 n^2 + (d_0^2 + 12d_0 + 12)n - 4d_0(d_0+1) &> d_0 n^2 + (d_0^2 + 12d_0 + 12)n - 4d_0 n \\ &= n(d_0 n + d_0^2 + 8d_0 + 12) \\ &> 0. \end{aligned}$$

Therefore, by (7.8) and (7.7) we obtain

$$\frac{J}{K} \leq \frac{(2d_0+3)n+d_0}{2(d_0+3)n+(d_0+6)} < c.$$

Hence, by (7.6), we obtain $I < 0$, namely $M_X < 1$. □

REMARK 7.8. We note that the manifold of the form

$$\mathbf{P}_{\mathbf{P}^n}(\mathcal{O}^{\oplus(d_0+1)} \oplus \mathcal{O}(k)^{\oplus(d_\infty+1)})$$

considered in this section is toric. Yao gave a formula for the Mabuchi constant of toric Fano manifolds in [**14**, Section 5.1] as follows. Let P be its anticanonical moment polytope, and let θ_P denote the affine function on P characterized by

$$-f(0) + \frac{1}{V_P}\int_P f(x)\,dx = \frac{1}{V_P}\int_P f(x)\theta_P(x)\,dx$$

for every affine function f on P, where V_P is the volume of P. The Mabuchi constant is then given by

$$M_X = \max_P \theta_P.$$

Concerning the relation between Yao's formula and Theorem 6.1, a key ingredient is the identity

$$\theta_P(\mu_\omega) = \frac{1}{2}\Pi_\omega^G(\mathrm{Scal}(\omega) - 2n),$$

which holds for the normalized moment map μ_ω described in Section 5.1 (see also [**14**, Section 2.1]). Then, it follows from Mabuchi's result ([**9**, Theorem 9.1]) that

$$\theta_P(\mu_\omega) = \Pi_\omega^G(1 - e^{h_\omega}) = \alpha z + \beta.$$

Therefore,

$$\max_P \theta_P = \max_X \theta_P(\mu_\omega) = \max_X(\alpha z + \beta),$$

which clarifies the relation between them. Using Proposition 3.7 and 4.7, the explicit formula for the Mabuchi constant can then be recovered.

However, we emphasize that Theorem 6.1 itself applies to arbitrary Fano admissible manifolds, which are not necessarily toric. Thus the theorem, in its full generality, is not a direct consequence of Yao's result.

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