Submitted to
*Journal of the Mathematical Society of Japan*

# Euler characteristics of fixed-point sets of smooth finite group actions on $S^5$

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(Received Feb. 25, 2026)

**Abstract.** Among smooth actions of finite groups on the 5-sphere $S^5$, we determine all possible values of the Euler characteristics of fixed-point sets. More precisely, we prove that, for every smooth action of a finite group $G$ on $S^5$, the Euler characteristic of $(S^5)^G$ is a nonnegative even integer, and conversely that every nonnegative even integer $m$ is realized by a smooth action of a fixed dihedral group on $S^5$ whose fixed-point set consists of exactly $m$ points. We also show that for any smooth finite group action on a $\mathbb{Z}_2$-homology 5-sphere, the Euler characteristic of the fixed-point set is even. In contrast, for each odd prime $p$ and each integer $m \geq 0$ with $m \neq 1$, we construct $\mathbb{Z}_p$-homology 5-spheres admitting smooth actions of certain finite groups with exactly $m$ fixed points.

## 1. Introduction

Throughout this paper, $G$ denotes a finite group unless otherwise stated, and all $G$-actions on manifolds are assumed to be smooth. Let $M$ be a manifold with $G$-action. For a subgroup $H$ of $G$, let $M^H$ denote the $H$-fixed-point set of $M$. If $M$ is a closed manifold, then $M^H$ is a closed submanifold of $M$. For a nonnegative integer $m$, we call the $G$-action on $M$ an *$m$-fixed-point $G$-action* if $|M^G| = m$. In particular, the case $m = 0$ means $M^G = \emptyset$. When $m = 1$ (resp. $m$ is odd), we call it a *one-fixed-point* (resp. an *odd-fixed-point*) *$G$-action*.

Finite group actions on spheres, and the topology of their fixed-point sets, have been studied extensively. A central problem is to determine, given a nonnegative integer $m$, a finite group $G$, and a nonnegative integer $n$, whether there exists an $m$-fixed-point $G$-action on $S^n$. A finite group $G$ is called an *Oliver group* if there exists a contractible finite $G$-CW-complex without $G$-fixed points [**27**]. It is known that every nonsolvable group is an Oliver group (see [**27**]). E. Laitinen–M. Morimoto–K. Pawałowski [**16**] proved that if $G$ is a finite nonsolvable group such that $|G/K|$ is odd, where $K$ denotes the unique maximal perfect normal subgroup of $G$, then for each integer $m \geq 1$, there exists an $m$-fixed-point $G$-action on $S^n$ for some sufficiently large $n$. Subsequently, E. Laitinen–M. Morimoto [**15**] showed that for any integer $m \geq 1$, there exists an $m$-fixed-point $G$-action on a sphere if and only if $G$ is an Oliver group. More recently, for specific nonsolvable Oliver groups, several authors have investigated the dimensions $n$ for which $S^n$ admits (or does not admit) one- or odd-fixed-point actions; see, for instance, [**21, 24, 25, 23, 26, 29**]. On the other hand, if the acting group is allowed to vary, then there exists a one-fixed-

2020 *Mathematics Subject Classification.* Primary 57S17; Secondary 55M35.
*Key Words and Phrases.* smooth action, fixed point, tangential representation.

point action of a finite group on $S^n$ if and only if $n \geq 6$; see [**1, 2, 6, 22**]. In lower dimensions, the situation is much more rigid. If a finite group $G$ acts on a $\mathbb{Z}$-homology sphere of dimension $\leq 4$, then the Euler characteristic of the $G$-fixed-point set is either 0, 1, or 2. Although this consequence is standard and can be extracted from the approaches of S. DeMichelis [**7**] and N. P. Buchdahl–S. Kwasik–R. Schultz [**6**], we have not found a convenient explicit statement in the literature. For the reader's convenience, we state the result below as Proposition 1.1 and provide a proof, using standard methods from transformation group theory, in Section 6.

PROPOSITION 1.1. *If a finite group $G$ acts effectively on a $\mathbb{Z}$-homology sphere $\Sigma$ of dimension $\leq 4$, then the Euler characteristic of the $G$-fixed-point set is either 0, 1, or 2. In particular, if $\chi(\Sigma^G) = 1$, then $G$ is isomorphic to the alternating group $A_5$, $\dim \Sigma = 3$, and $|\Sigma^G| = 1$.*

The case $\chi(\Sigma^G) = 1$ in Proposition 1.1 is realizable: there exist $\mathbb{Z}$-homology 3-spheres admitting one-fixed-point $A_5$-actions; see, e.g., [**9, 14**]. On the other hand, it follows from [**6**, Theorem I.1] that $A_5$ cannot act on $S^3$ with exactly one fixed point. Consequently, we obtain:

COROLLARY 1.2. *For every effective action of a finite group $G$ on $S^n$ with $n \leq 4$, the Euler characteristic of the $G$-fixed-point set is either 0 or 2.*

For a finite group $G$, let

$$X_{S^n}(G) = \left\{ \chi((S^n)^G) \,\middle|\, G \text{ acts smoothly on } S^n \right\}.$$

We also put

$$X_{S^n} = \bigcup_G X_{S^n}(G),$$

where $G$ runs over all finite groups. With this notation, the discussion above yields the following two conclusions. First, if $n \leq 4$, then

$$X_{S^n} = \{0, 2\}.$$

Second, if $n \geq 6$, then there exists a finite group acting on $S^n$ with exactly one fixed point, and hence

$$1 \in X_{S^n}.$$

This leads us to a natural question for the intermediate case $n = 5$: whether $X_{S^5}$ is restricted to $\{0, 2\}$, as in dimensions $n \leq 4$, or contains odd values, as in dimensions $n \geq 6$.

In this paper, we answer this question by determining all possible values of the Euler characteristics of fixed-point sets of finite group actions on $S^5$. Our main result is as follows:

THEOREM 1.3. *The equality*

$$X_{S^5} = 2\mathbb{N} \cup \{0\}$$

*holds. More precisely, the following assertions hold:*

*(1) For every finite group $G$,*

$$X_{S^5}(G) \subset 2\mathbb{N} \cup \{0\}.$$

*(2) There exists a finite group $G$ such that*

$$X_{S^5}(G) = 2\mathbb{N} \cup \{0\}.$$

In related work, E. Laitinen and P. Traczyk [**17**] proved that if a finite group $G$ acts on $S^5$ under the additional hypothesis that the $H$-fixed-point set of every nontrivial subgroup $H$ of $G$ has dimension at most 2, then the $G$-fixed-point set is either empty or a sphere of dimension at most 2. Thus, under their hypothesis, the diffeomorphism type of $(S^5)^G$ is determined. In the present paper, we remove this hypothesis and consider finite group actions on $\mathbb{Z}$-homology 5-spheres $\Xi$. Our aim is to determine the possible values of $\chi(\Xi^G)$.

Theorem 1.3 follows from the next two results: Theorem 1.4 provides the necessary restriction, while Theorem 1.5 shows that all the remaining possibilities actually occur.

Hereafter, $C_n$ and $D_{2n}$ denote a cyclic group of order $n$ and a dihedral group of order $2n$, respectively. Additionally, $A_n$ and $S_n$ denote the alternating and symmetric groups on $n$ letters, respectively.

THEOREM 1.4. *Let $\Xi$ be a $\mathbb{Z}_2$-homology 5-sphere with $G$-action. Then the Euler characteristic of the $G$-fixed-point set $\Xi^G$ is an even integer, i.e. $\chi(\Xi^G) \equiv 0 \mod 2$. Moreover, if $\Xi$ is a $\mathbb{Z}$-homology 5-sphere, then $\chi(\Xi^G)$ is a nonnegative even integer, i.e. $\chi(\Xi^G) = 2m$ for some integer $m \geq 0$.*

THEOREM 1.5 (CF. PROPOSITION 4.7). *Let $a$ and $b$ be positive integers $\geq 2$ with $\gcd(a, b) = 1$. For every nonnegative integer $k$, there exists an effective $2k$-fixed-point action of the dihedral group $D_{2ab}$ on $S^5$.*

COMMENT 1.6. In Theorem 1.5, in the case where $G$ is not a dihedral group, the author does not know whether there exists an effective $G$-action on a $\mathbb{Z}$-homology 5-sphere $\Xi$ such that $\chi(\Xi^G) = 2m \geq 4$. However, Proposition 3.4 shows that, if $\chi(\Xi^G) = 2m \geq 4$, then $G$ has a nontrivial cyclic normal subgroup $C$ such that $G/C$ is a nontrivial 2-group (in fact $|G/C| = 2^k, 1 \leq k \leq 4$), and $G$ is not isomorphic to the direct product of $C$ and $G/C$.

The mod 2 conclusion in Theorem 1.4 is best possible with respect to the homological coefficient ring: it does not hold if $\mathbb{Z}_2$ is replaced by $\mathbb{Z}_p$ for any odd prime $p$. Indeed, for any odd prime $p$ and any nonnegative integer $m$ with $m \neq 1$, there exist $\mathbb{Z}_p$-homology 5-spheres admitting $m$-fixed-point actions of certain finite groups.

THEOREM 1.7 (CF. PROPOSITION 5.9). *Let $G$ be either $A_4$, $S_4$, or $A_5$, and let $p$ be an odd prime. For every nonnegative integer $m$ with $m \neq 1$, there exists a $\mathbb{Z}_p$-homology 5-sphere admitting an $m$-fixed-point $G$-action. Moreover, if $p \neq 3$ and $G \in \{S_4, A_5\}$, then we can choose $m = 1$.*

In Section 2, we collect basic results and notation used throughout this paper. Theorem 1.4 is proved in Section 3. In Section 4, we construct certain dihedral actions on $S^5$ arising from Brieskorn manifolds and prove Theorem 1.5. In Section 5, we study certain finite group actions on the homogeneous space $SU(3)/SO(3)$, and use them to construct actions of certain finite groups on $\mathbb{Z}_p$-homology 5-spheres, which proves Theorem 1.7. Finally, in Section 6, we prove Proposition 1.1.

**Acknowledgements**

The author would like to thank Professor Masaharu Morimoto and Dr. Piotr Mizerka for helpful discussions and valuable comments on this work. The author is also grateful to Dr. Kamil Śmietaniak for further valuable comments which helped to improve the manuscript.

## 2. Preliminaries

In this section we collect basic results and notation that will be used in the rest of the paper.

We recall the following three standard results: Lemma 2.1, Lemma 2.2, and Proposition 2.3.

LEMMA 2.1 ([**4**, Chapter III, Theorem 4.3]). *Let $p$ be a prime. If a $p$-group $P$ acts on a compact manifold $M$, then $\chi(M)$ is congruent to $\chi(M^P)$ modulo $p$.*

LEMMA 2.2 (SMITH'S THEOREM [**28**], [**4**, Chapter III, Theorem 5.1]). *Let $p$ be a prime and $P$ a $p$-group. If $P$ acts on a $\mathbb{Z}_p$-homology sphere $\Sigma$, then the $P$-fixed-point set $\Sigma^P$ is itself a $\mathbb{Z}_p$-homology sphere (possibly empty).*

Note that the classification of closed connected manifolds of dimensions $\leq 2$ shows that any $\mathbb{Z}_p$-homology sphere of dimension $\leq 2$ is diffeomorphic to the sphere of the same dimension.

PROPOSITION 2.3. *If a finite cyclic group $C$ acts on a $\mathbb{Q}$-homology $n$-sphere $X$, $n \geq 0$, then $\chi(X^C) = 0$ or 2.*

PROOF. If $n = 0$ then the claim is obvious. Thus we may assume that $n \geq 1$. Let $g$ be a generator of $C$. Then we have $X^C = X^g$. By the Lefschetz fixed point formula ([**31**, Exercise (6.17), 3, p.225]), $\chi(X^g)$ coincides with the Lefschetz number

$$L(g, X) = \sum_{k=0}^{n} (-1)^k \operatorname{Trace}\left(g_* : H_k(X; \mathbb{Q}) \to H_k(X; \mathbb{Q})\right).$$

Since $X$ is a $\mathbb{Q}$-homology $n$-sphere, we have $\chi(X^C) = \chi(X^g) = 1 + (-1)^n \deg g$. This shows that $\chi(X^C)$ lies in $\{0, 2\}$, because $g : X \to X$ is a diffeomorphism (hence $\deg g = \pm 1$). □

Let $G$ act on a manifold $M$ of positive dimension and let $x \in M^G$. The $G$-action on $M$ induces a linear $G$-action on $T_x(M)$, i.e. $T_x(M)$ is an $\mathbb{R}[G]$-module. We call $T_x(M)$ a *tangential* $\mathbb{R}[G]$*-module*. For a subgroup $H$ of $G$, let $M_x^H$ denote the connected component of $M^H$ containing $x \in M^G$. If $H$ is normal in $G$ and $M_x^H$ is connected, then $G/H$ acts on $M_x^H$.

The following three results will be used repeatedly in the proofs of Theorem 1.4 and Proposition 1.1.

LEMMA 2.4 ([**30**, Lemma 2.4]). *Let $G$ act on a closed manifold $M$ with a $G$-fixed point. If there exists a normal subgroup $H$ of $G$ such that $M^H$ is a sphere of dimension $\leq 2$, then $M^G$ is a sphere of dimension $\leq 2$.*

PROOF. If a finite group acts on a sphere of dimension $\leq 2$, then the (nonempty) fixed-point set is a sphere of dimension $\leq 2$. Therefore $M^G = (M^H)^{G/H}$ is a sphere of dimension $\leq 2$. □

LEMMA 2.5 ([**30**, Lemma 2.5]). *Let $G$ act on a closed manifold $M$ with a $G$-fixed point. If there exists a normal subgroup $H$ of $G$ such that $\dim T_x(M)^H = 1$ for each $x \in M^G$, then $\chi(M^G)$ is a nonnegative even integer.*

PROOF. By the hypothesis that $\dim T_x(M)^H = 1$ for every $x \in M^G$, each connected component $M_x^H$ is a connected closed manifold of dimension 1; hence $M_x^H \cong S^1$. Since $G/H$ acts on $M_x^H$, $(M_x^H)^{G/H}$ is either $S^1$ or $S^0$. Moreover, $M^G$ is a disjoint union of these sets $(M_x^H)^{G/H}$. Therefore we obtain $\chi(M^G) = 2m$ for some integer $m \geq 0$. □

PROPOSITION 2.6. *Let $G$ act on a closed manifold $M$ with $x \in M^G$. Let $H$ and $K$ be subgroups of $G$ with $H \leq K$. If $M^H$ is connected and $\dim M^H = \dim M_x^K$, then we have $M^H = M^K$.*

PROOF. As $H \leq K$, we have $M_x^K \subset M^K \subset M^H$. Since $M_x^K$ is a closed connected submanifold of $M^H$ of the same dimension, we see that $M_x^K$ is both open and closed in the connected manifold $M^H$. This implies that $M_x^K = M^H$, and therefore $M^K = M^H$. □

The following lemma will be used to check whether a given action is effective.

LEMMA 2.7. *Let a finite group $G$ act on a closed connected manifold $M$ of dimension $n > 0$. Suppose that the $G$-fixed-point set $M^G$ is nonempty. Then the $G$-action on $M$ is effective if and only if $T_x(M)$ is faithful for some $x \in M^G$ (equivalently, for every $x \in M^G$).*

PROOF. Let $\Phi : G \to \mathrm{Diff}(M)$ be the $G$-action on $M$, and let $\rho_x : G \to GL(T_x(M))$ be the tangential representation at $x \in M^G$, which is defined by $\rho_x(g) = d\Phi(g)_x$ for $g \in G$.

First suppose that $\rho_x$ is faithful for some $x \in M^G$, i.e. $\ker \rho_x = \{e\}$. Let $g \in \ker \Phi$. Then we have $d\Phi(g)_x = d(\mathrm{id}_M)_x = \mathrm{id}_{T_x(M)}$. As $d\Phi(g)_x = \rho_x(g)$, we have $\rho_x(g) = \mathrm{id}_{T_x(M)}$ (hence $g \in \ker \rho_x$). Since $\ker \rho_x = \{e\}$, it follows that $g = e$, which means that $\ker \Phi = \{e\}$.

Conversely, suppose that $\Phi$ is effective, i.e. $\ker \Phi = \{e\}$. Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$. Define a $G$-invariant Riemannian metric $\langle \cdot, \cdot \rangle^G$ on $M$ by

$$\langle v, w \rangle_p^G = \frac{1}{|G|} \sum_{h \in G} \langle d\Phi(h)_p(v),\, d\Phi(h)_p(w) \rangle_{\Phi(h)(p)} \quad (p \in M,\ v, w \in T_p(M)).$$

Then, for each $g \in G$, the diffeomorphism $\Phi(g)$ is an isometry of $(M, \langle \cdot, \cdot \rangle^G)$. Let $x \in M^G$ and suppose that $\rho_x(g) = \mathrm{id}_{T_x(M)}$. Then we have $\Phi(g)(x) = x = \mathrm{id}_M(x)$ and $d\Phi(g)_x = \rho_x(g) = \mathrm{id}_{T_x(M)}$. Since both $\Phi(g)$ and $\mathrm{id}_M$ are local isometries of the connected Riemannian manifold $(M, \langle \cdot, \cdot \rangle^G)$, it follows from [**18**, Proposition 5.22] that $\Phi(g) = \mathrm{id}_M$ on $M$. As $\Phi$ is effective, it follows that $g = e$, and hence $\ker \rho_x = \{e\}$. □

Let $M$ be a closed manifold equipped with $m$-fixed-point $G$-action $\Phi$ for some $m \in \mathbb{N}$. Let $\mathbb{V}_\Phi$ denote the set of isomorphism classes of the tangential $\mathbb{R}[G]$-modules at all $G$-fixed points.

LEMMA 2.8. *Let $m$ and $n$ be positive integers. Let $M$ and $N$ be closed connected manifolds of the same dimension $k \geq 2$ equipped with $m$- and $n$-fixed-point $G$-actions $\Phi$ and $\Psi$, respectively. If $\mathbb{V}_\Phi \cap \mathbb{V}_\Psi \neq \emptyset$, then there exists an $(m+n-2)$-fixed-point $G$-action on a connected sum of $M$ and $N$. Moreover, for a commutative ring $R$ with unity, if $M$ and $N$ are $R$-homology $k$-spheres, then the resulting connected sum is an $R$-homology $k$-sphere.*

PROOF. As $\mathbb{V}_\Phi \cap \mathbb{V}_\Psi \neq \emptyset$, there exist $G$-fixed points $x \in M^G$ and $y \in N^G$ such that $T_x(M)$ is isomorphic to $T_y(N)$ as $\mathbb{R}[G]$-modules. Fix a $G$-linear isomorphism

$$L : T_x(M) \xrightarrow{\cong} T_y(N)$$

and choose $G$-invariant inner products on $T_x(M)$ and $T_y(N)$ so that $L$ is an isometry. By the equivariant tubular neighborhood theorem, we can find $G$-embeddings of the unit disks

$$i : D(T_x(M)) \to M \quad \text{and} \quad j : D(T_y(N)) \to N$$

centered at $x$ and $y$, respectively, such that their images contain no other $G$-fixed points (this is possible since $x$ and $y$ are isolated $G$-fixed points). Since $L$ is an isometry, we can define the $G$-diffeomorphism

$$h = j \circ L \circ i^{-1} : i(D(T_x(M))) \to j(D(T_y(N)))$$

and its restriction to the boundaries, which we denote by

$$\partial h : \partial(i(D(T_x(M)))) \to \partial(j(D(T_y(N)))),$$

is also a $G$-diffeomorphism. Since $k \geq 2$, the complements of embedded $k$-disks by $i$ and $j$ are connected, and therefore this gluing yields the usual connected sum

$$M\#N = \big(M \setminus \mathrm{int}(i(D(T_x(M))))\big) \cup_{\partial h} \big(N \setminus \mathrm{int}(j(D(T_y(N))))\big).$$

The $G$-action on the resulting connected sum $M\#N$ is well-defined. Then we have

$$(M\#N)^G = (M^G \setminus \{x\}) \sqcup (N^G \setminus \{y\}),$$

and hence $|(M\#N)^G| = m + n - 2$.

Now assume that $M$ and $N$ are $R$-homology $k$-spheres. By using the Mayer–Vietoris sequence with coefficients in $R$, we see that the resulting connected sum $M\#N$ is also an $R$-homology $k$-sphere. □

The connected-sum construction in this lemma will be used in Sections 4 and 5 to increase the number of fixed points.

PROPOSITION 2.9. *Let $P_2$ be a 2-group and $M$ a $\mathbb{Q}$-homology 3-sphere with $P_2$-action. Then $M^{P_2}$ has a nonnegative even Euler characteristic, i.e. $\chi(M^{P_2}) = 2m$ for some integer $m \geq 0$.*

PROOF. As $M$ is a $\mathbb{Q}$-homology 3-sphere, we have $\chi(M) = 0$. If the $P_2$-action on $M$ is trivial (including $|P_2| = 1$), then $\chi(M^{P_2}) = 0$. Suppose that the $P_2$-action on $M$ is nontrivial. By Lemma 2.1, we have $\chi(M^{P_2}) \equiv 0 \bmod 2$. It remains to prove that $\chi(M^{P_2}) \geq 0$. Let $K$ be the kernel of the $P_2$-action on $M$, and then $P_2/K$ acts effectively on $M$. If $|P_2/K| = 2$ then we get $\chi(M^{P_2}) = 0$ or 2 by Proposition 2.3. If $|P_2/K| \geq 4$ then it holds that $\dim T_x(M)^{P_2} \leq 1$ for each $x \in M^{P_2}$, because $P_2/K$ cannot be contained in $O(1)$ as a subgroup. This implies that each connected component of $M^{P_2}$ has dimension $\leq 1$. Therefore $M^{P_2}$ is a disjoint union of some circles and some isolated points, which shows that $\chi(M^{P_2}) \geq 0$. □

REMARK 2.10. For $\mathbb{Z}$-homology 3-spheres $\Sigma$ with $G$-actions, we have $\chi(\Sigma^G) \in \{0, 1, 2\}$ (Proposition 1.1). In contrast, for $\mathbb{Q}$-homology 3-spheres with $G$-actions, this set is unbounded (already for 2-groups). Indeed, one can realize $2m$ fixed points by iterating Lemma 2.8 starting from the 4-fixed-point $D_4$-action on $\mathbb{R}P^3$ (Proposition 4.1).

Let $\rho : G \to SO(n)$ be a faithful real representation of $G$ and $V$ the corresponding $n$-dimensional $\mathbb{R}[G]$-module (with a $G$-invariant inner product). For a normal subgroup $L$ of $G$, $V$ decomposes into the $L$-fixed-point set $V^L$ and its orthogonal complement $V_L$ as $\mathbb{R}[G]$-modules, i.e. $V = V^L \oplus V_L$. Let $\rho^L$ and $\rho_L$ denote the corresponding subrepresentations of $G$ with $V^L$ and $V_L$, respectively. In this situation, the following proposition holds.

PROPOSITION 2.11 ([**30**, Proposition 2.6]). *Let $L$ be a normal subgroup of $G$ and $\rho : G \to SO(n)$ a faithful real representation of degree $n$. Let*

$$\rho^L : G \to O(\ell) \quad \textit{and} \quad \rho_L : G \to O(m)$$

*be the subrepresentations of $\rho$, where $\ell = \dim V^L$ and $m = \dim V_L$. Let $H$ and $K$ be the kernels of $\rho^L$ and $\rho_L$, respectively. Then the following holds.*

*(1) The intersection of $H$ and $K$ is trivial, i.e. $H \cap K = \{e\}$.*

*(2) $G$ is isomorphic to a subgroup of $G/H \times G/K$.*

*(3) $H$ is isomorphic to a subgroup of $SO(m)$ with $L \subset H$.*

*(4) $K$ is isomorphic to a subgroup of $SO(\ell)$.*

*(5) $G/H$ is isomorphic to a subgroup of $O(\ell)$.*

*(6) $G/K$ is isomorphic to a subgroup of $O(m)$.*

For a finite group $G$, let $F(G)$ denote the unique largest nilpotent normal subgroup of $G$, which is often called the *Fitting subgroup* of $G$. Also, we often use the well-known fact that a nontrivial finite subgroup of $SO(3)$ is isomorphic to

$$C_n,\ D_{2n},\ A_4,\ S_4,\ \text{or } A_5 \quad (n \geq 2).$$

PROPOSITION 2.12. *Let $G$ be a finite subgroup of $SO(3) \times SO(3)$ such that the Fitting subgroup $F(G)$ is cyclic. Then one of the following holds.*

*(1) $G$ is isomorphic to $A_5 \times L$ for some finite subgroup of $SO(3)$ with cyclic $F(L)$.*

*(2) $[G : F(G)] = 2^r$, where $r = 0$, 1, or 2.*

PROOF. As shown in the proof of [**30**, Proposition 2.10], we have:

(1) If $G$ is nonsolvable then $G$ is isomorphic to $A_5 \times L$ for some finite subgroup of $SO(3)$ with cyclic $F(L)$.

(2) If $G$ is solvable then $G$ is isomorphic to a subgroup of $D_{2m} \times D_{2n}$ for some positive integers $m$ and $n$.

It suffices to consider the case where $G$ is solvable. Let $D = D_{2m} \times D_{2n}$. Then it holds that $[D : F(D)] = 1$, 2, or 4. By the second isomorphism theorem, we have an isomorphism $GF(D)/F(D) \cong G/(F(D) \cap G)$. This means that $G$ is an extension of a group of order 1, 2, or 4 by a normal nilpotent group $F(D) \cap G$. By the maximality of $F(G)$, $F(D) \cap G$ is contained in $F(G)$. Therefore we obtain $[G : F(G)] = 2^r$ for some $r = 0$, 1, or 2. □

At the end of this section, we list certain subgroups of the alternating groups $A_5$ and $A_6$, irreducible real modules of $A_5$ and $A_6$, and tables of the dimensions of the fixed-point sets for these subgroups. They will be used in Section 3 and Appendix 6.

We recall the irreducible real representations of $A_5$ and $A_6$ and fix notation. There are five irreducible $\mathbb{R}[A_5]$-modules (up to isomorphism), which are denoted by

$$\mathbb{R}_{A_5},\ U_{3.1},\ U_{3.2},\ U_4, \text{ and } U_5.$$

Here, $\mathbb{R}_{A_5}$ is the trivial one and $U_i$ (resp. $U_{j.k}$) indicates $\dim U_i = i$ (resp. $\dim U_{j.k} = j$). We fix the six subgroups of $A_5 = \langle\, (1,2,3,4,5),\ (3,4,5)\,\rangle$ as follows:

$$\begin{aligned}
C_2 &= \langle\, (2,3)(4,5)\,\rangle && \text{(a cyclic group of order 2)},\\
C_3 &= \langle\, (1,2,3)\,\rangle && \text{(a cyclic group of order 3)},\\
D_4 &= \langle\, (2,3)(4,5),\ (2,5)(3,4)\,\rangle && \text{(a dihedral group of order 4)},\\
C_5 &= \langle\, (1,2,3,4,5)\,\rangle && \text{(a cyclic group of order 5)},\\
D_6 &= \langle\, (2,3)(4,5),\ (1,2,3)\,\rangle && \text{(a dihedral group of order 6), and}\\
A_4 &= \langle\, (2,3)(4,5),\ (3,4,5)\,\rangle && \text{(an alternating group of order 12)}.
\end{aligned}$$

Note that these subgroups of $A_5$ satisfy

$$C_3 \trianglelefteq D_6,\ D_4 \trianglelefteq A_4,\ C_2 = D_4 \cap D_6, \text{ and } \langle D_4, D_6\rangle = A_5.$$

There are seven irreducible $\mathbb{R}[A_6]$-modules (up to isomorphism), which are denoted by

$$\mathbb{R}_{A_6}, V_{5.1}, V_{5.2}, V_{8.1}, V_{8.2}, V_9, \text{ and } V_{10}.$$

Here, $\mathbb{R}_{A_6}$ is the trivial one and $V_i$ (resp. $V_{j.k}$) indicates $\dim V_i = i$ (resp. $\dim V_{j.k} = j$). We will only use $V_{5.1}$ and $V_{5.2}$ in this paper. We fix the four subgroups of $A_6 = \langle\, (1,2,3,4,5),\ (4,5,6)\,\rangle$ as follows:

$$\begin{aligned}
\mathcal{D}_4 &= \langle\, (3,5)(4,6),\ (3,4)(5,6)\,\rangle && \text{(a dihedral group of order 4)},\\
\mathfrak{D}_4 &= \langle\, (1,2)(4,6),\ (3,5)(4,6)\,\rangle && \text{(a dihedral group of order 4)},\\
\mathcal{S}_4 &= \langle\, (1,2)(3,4),\ (3,4)(5,6),\ (4,5,6)\,\rangle && \text{(a symmetric group of order 24), and}\\
\mathfrak{S}_4 &= \langle\, (1,2)(4,6),\ (3,4)(5,6),\ (1,4,5)(2,6,3)\,\rangle && \text{(a symmetric group of order 24)}.
\end{aligned}$$

Here, $\mathfrak{D}_4$ (resp. $\mathfrak{S}_4$) denotes the subgroup of $A_6$ isomorphic to, but not conjugate to $\mathcal{D}_4$ (resp. $\mathcal{S}_4$) in $A_6$. Note that $\mathcal{D}_4$ and $\mathfrak{D}_4$ are normal subgroups of $\mathcal{S}_4$ and $\mathfrak{S}_4$, respectively.

The dimensions of the fixed-point sets $\dim U^H$ and $\dim V^H$ for the subgroups $H$ listed above are summarized in Tables 1 and 2; see [**22**, Table 1.1] and [**29**, Table 3.1].

Table 1. Table of $\dim U_*^H$.

| | $C_2$ | $C_3$ | $D_4$ | $C_5$ | $D_6$ | $A_4$ |
|---|---|---|---|---|---|---|
| $U_{3.1}$ | 1 | 1 | 0 | 1 | 0 | 0 |
| $U_{3.2}$ | 1 | 1 | 0 | 1 | 0 | 0 |
| $U_4$ | 2 | 2 | 1 | 0 | 1 | 1 |
| $U_5$ | 3 | 1 | 2 | 1 | 1 | 0 |

Table 2. Table of $\dim V_*^H$.

| | $\mathcal{D}_4$ | $\mathfrak{D}_4$ | $\mathcal{S}_4$ | $\mathfrak{S}_4$ |
|---|---|---|---|---|
| $V_{5.1}$ | 2 | 2 | 0 | 1 |
| $V_{5.2}$ | 2 | 2 | 1 | 0 |

## 3. Proof of Theorem 1.4

Throughout this section, let $\Xi$ be a $\mathbb{Z}_2$-homology 5-sphere with effective $G$-action. Since $H^1(\Xi;\mathbb{Z}_2) = 0$, $\Xi$ is orientable (because the first Stiefel-Whitney class $w_1(\Xi)$ vanishes). The $G$-action on $\Xi$ is called *orientation-preserving* if for every $g \in G$, the induced

diffeomorphism $\Phi(g)$ preserves an orientation of $\Xi$; otherwise it is called *orientation-reversing*. Let $G^+$ be the subgroup of $G$ consisting of elements $g \in G$ such that the induced diffeomorphisms $\Phi(g)$ preserve its orientation. If the $G$-action on $\Xi$ is orientation-preserving (resp. orientation-reversing), then $[G : G^+] = 1$ (resp. $[G : G^+] = 2$). If $\Xi^G = \emptyset$ then $\chi(\Xi^G) = 0$. If $G^+$ is trivial (i.e. $|G| = 1$ or 2), then it follows that $\chi(\Xi^G) = 0$ or 2 by Proposition 2.3 (since $\Xi$ is a $\mathbb{Q}$-homology 5-sphere). Henceforth, we assume that the $G$-action on $\Xi$ has a $G$-fixed point $x \in \Xi^G$ and that $G^+$ is nontrivial. Then the restricted $G^+$-action on $T_x(\Xi)$ is orientation-preserving and effective, and hence $G^+$ is isomorphic to a nontrivial finite subgroup of $SO(5)$.

If $G^+$ contains a normal subgroup isomorphic to $A_5$ or $A_6$, then Proposition 3.1 settles this case. Otherwise, we will show that $F(G^+)$ is nontrivial. We then distinguish the cases where $F(G^+)$ is noncyclic and cyclic, which are handled by Propositions 3.2 and 3.3, respectively. Therefore, Theorem 1.4 follows from these three propositions.

PROPOSITION 3.1. *If $G^+$ contains a normal subgroup $L$ isomorphic to $A_5$ or $A_6$, then $\chi(\Xi^G)$ is an even integer. In particular, if $\Xi$ is a $\mathbb{Z}$-homology 5-sphere, then $\chi(\Xi^G)$ is a nonnegative even integer.*

PROOF. In this proof, we identify $L$ with either $A_5$ or $A_6$, and we use the notation for the subgroups of $L$ and the $\mathbb{R}[L]$-modules fixed in Section 2. Note that we have $\Xi^{G^+} = (\Xi^L)^{G^+/L}$ and $\Xi^G = (\Xi^{G^+})^{G/G^+}$.

First suppose $L = A_5$. By Lemma 2.1, we have $\chi(\Xi^G) \equiv \chi(\Xi^{G^+}) \bmod 2$. Thus it suffices to show that $\chi(\Xi^{G^+}) \equiv 0 \bmod 2$ in the following three cases (I), (II), and (III).

(I) There exists $x \in \Xi^{G^+}$ such that $\mathrm{Res}^{G^+}_{A_5} T_x(\Xi) \cong \mathbb{R}^{\oplus 2}_{A_5} \oplus U_{3.i}$ ($i = 1$ or 2).

(II) There exists $x \in \Xi^{G^+}$ such that $\mathrm{Res}^{G^+}_{A_5} T_x(\Xi) \cong U_5$.

(III) For every $x \in \Xi^{G^+}$, we have $\mathrm{Res}^{G^+}_{A_5} T_x(\Xi) \cong \mathbb{R}_{A_5} \oplus U_4$.

Case (I). In this case, we have $\dim T_x(\Xi)^{D_4} = \dim T_x(\Xi)^{A_5} = 2$. Lemma 2.2 shows that $\Xi^{D_4}$ is a sphere of dimension 2. By Proposition 2.6, $\Xi^{A_5}$ coincides with $\Xi^{D_4}$, and hence $\Xi^{A_5} \cong S^2$. It follows from Lemma 2.4 that $\Xi^{G^+}$ is a sphere of dimension $\leq 2$.

Case (II). In this case, we have $\dim T_x(\Xi)^{D_4} = 2$ and $\dim T_x(\Xi)^{A_4} = 0$. It follows from Lemmas 2.2 and 2.4 that $\Xi^{D_4} \cong S^2$ and $\Xi^{A_4} = (\Xi^{D_4})^{A_4/D_4} \cong S^0$. Therefore, $\Xi^{A_5}$ consists of exactly one or two points, i.e. $|\Xi^{A_5}| = 1$ or 2. Suppose that $\Xi^{A_5} = \{x\}$. By $\dim T_x(\Xi)^{D_6} = 1$ and $\dim T_x(\Xi)^{C_2} = 3$, $\Xi^{D_6}_x$ is a connected closed manifold of dimension 1, i.e. $\Xi^{D_6}_x \cong S^1$, and $\Xi^{C_2}$ is a $\mathbb{Z}_2$-homology 3-sphere (Lemma 2.2). Since $\Xi^{D_6} \cap \Xi^{D_4} = \Xi^{\langle D_4, D_6\rangle} = \Xi^{A_5}$, we have $\Xi^{D_6}_x \cap \Xi^{D_4} = \{x\}$. Thus we see that the submanifolds $\Xi^{D_6}_x$ and $\Xi^{D_4}$ of $\Xi^{C_2}$ intersect at $x$, transversely. On the other hand, since $H_1(\Xi^{C_2}; \mathbb{Z}_2) = H_2(\Xi^{C_2}; \mathbb{Z}_2) = 0$, the $\mathbb{Z}_2$-intersection number of $\Xi^{D_6}_x$ and $\Xi^{D_4}$ in $\Xi^{C_2}$ is trivial. This contradicts our assumption that $|\Xi^{A_5}| = 1$. Therefore we obtain $|\Xi^{A_5}| = 2$, i.e. $\Xi^{A_5} \cong S^0$. By Lemma 2.4, we have $\Xi^{G^+} \cong S^0$.

Case (III). In this case, we have $\dim T_x(\Xi)^{A_5} = 1$ for each $x \in \Xi^{G^+}$. Therefore it follows from Lemma 2.5 that $\chi(\Xi^{G^+}) \equiv 0 \bmod 2$. We next consider the case that $\Xi$ is a

$\mathbb{Z}$-homology 5-sphere. Lemma 2.2 shows that $\Xi^{C_5}$ is a circle $S^1$. By Proposition 2.6, we have $\Xi^{C_5} = \Xi^{A_5} \cong S^1$, and Lemma 2.4 shows that $\Xi^{G^+}$ is a sphere of dimension $\leq 1$.

In the case where $\Xi$ is a $\mathbb{Z}$-homology 5-sphere, we proved that $\Xi^{G^+}$ is a sphere of dimension $\leq 2$. Therefore, by Lemma 2.4, $\Xi^G$ is also a sphere of dimension $\leq 2$; hence $\chi(\Xi^G) = 0$ or 2.

Next suppose $L = A_6$. Then the restriction $\mathrm{Res}^G_{A_6} T_x(\Xi)$, $x \in \Xi^G$, is isomorphic to either $V_{5.1}$ or $V_{5.2}$. We consider the case $\mathrm{Res}^G_{A_6} T_x(\Xi) \cong V_{5.1}$. By $\dim V_{5.1}^{\mathcal{D}_4} = 2$ and $\dim V_{5.1}^{\mathcal{S}_4} = 0$, we have $\Xi^{\mathcal{D}_4} \cong S^2$ and $\Xi^{\mathcal{S}_4} \cong S^0$ by Lemmas 2.2 and 2.4; hence $|\Xi^{A_6}| = 1$ or 2. In fact, it follows from [**29**, Lemma 3.1] that $|\Xi^{A_6}| = 2$. Similarly, in the case $\mathrm{Res}^G_{A_6} T_x(\Xi) \cong V_{5.2}$, we can get $|\Xi^{A_6}| = 2$ by replacing $\mathcal{D}_4$ and $\mathcal{S}_4$ with $\mathfrak{D}_4$ and $\mathfrak{S}_4$, respectively. Applying Lemma 2.4 twice, we obtain $\Xi^{A_6} = \Xi^{G^+} = \Xi^G \cong S^0$. □

Note that $F(G^+)$ is characteristic in $G^+$ and hence normal in $G$. By [**10**, Theorems 1.4 and 1.5], every minimal normal subgroup of a finite group is a direct product of isomorphic finite simple groups. If $F(G^+)$ is trivial and $G^+$ is nontrivial, then $G^+$ has no nontrivial abelian normal subgroup; in particular, any minimal normal subgroup of $G^+$ is nonabelian. Moreover, any nonabelian finite simple subgroup of $SO(5)$ is isomorphic to either $A_5$ or $A_6$, and $SO(5)$ contains no subgroup isomorphic to a direct product of two or more nontrivial nonabelian finite simple groups, see [**19**, Corollary 2]. Therefore, if $F(G^+)$ is trivial and $G^+$ is nontrivial, then $G^+$ contains a normal subgroup isomorphic to either $A_5$ or $A_6$. Consequently, we may assume without loss of generality that $F(G^+)$ is nontrivial.

PROPOSITION 3.2. *If $F(G^+)$ is noncyclic, then $\chi(\Xi^G)$ is a nonnegative even integer. In particular, if $\Xi$ is a $\mathbb{Z}$-homology 5-sphere, then $\Xi^G$ is a sphere of dimension $\leq 2$.*

PROOF. As $F(G^+)$ is noncyclic, there exists a noncyclic Sylow $p$-subgroup $P$ of $F(G^+)$. For each $x \in \Xi^G$, we have $\dim T_x(\Xi)^P \leq 2$. Indeed, if $\dim T_x(\Xi)^P \geq 3$, then $P$ acts effectively and orientation-preservingly on a $P$-invariant subspace of $T_x(\Xi)$ of dimension at most 2 (which is impossible since any finite subgroup of $SO(2)$ is cyclic). Since $P$ is characteristic in $F(G^+)$ and $F(G^+)$ is characteristic in $G^+$, $P$ is characteristic in $G^+$, and therefore $P$ is normal in $G$. If $P$ is of 2-power order, then it follows from Lemmas 2.2 and 2.4 that $\Xi^G \cong S^k$ for some $0 \leq k \leq 2$. If $P$ is of odd order, then the codimension of $T_x(\Xi)^P$ in $T_x(\Xi)$ is even for each $x \in \Xi^G$. As $\dim T_x(\Xi)^P = 2m + 1 \leq 2$ for each $x \in \Xi^G$, we have $\dim T_x(\Xi)^P = 1$ for each $x \in \Xi^G$. Therefore it follows from Lemma 2.5 that $\chi(\Xi^G) \equiv 0 \bmod 2$.

We next assume that $\Xi$ is a $\mathbb{Z}$-homology 5-sphere. Then we have $\dim T_x(\Xi)^P \leq 2$ for some noncyclic Sylow subgroup $P$ of $F(G^+)$. It follows from Lemmas 2.2 and 2.4 that $\Xi^G = (\Xi^P)^{G/P}$ is a sphere of dimension $\leq 2$. □

PROPOSITION 3.3. *If $F(G^+)$ is a nontrivial cyclic group, then $\chi(\Xi^G)$ is an even integer. In particular, if $\Xi$ is a $\mathbb{Z}$-homology 5-sphere, then $\chi(\Xi^G)$ is a nonnegative even integer.*

PROOF. We have already proved the assertion in the case where $G^+$ has some nonabelian minimal normal subgroup. Thus we can assume that $G^+$ has no nonabelian minimal normal subgroups. Since the cyclic $F(G^+)$-action on $T_x(\Xi)$ is orientation-preserving and effective, one of the following cases occurs:

(I) For every $x \in \Xi^G$, we have $\dim T_x(\Xi)^{F(G^+)} = 1$.

(II) There exists $x \in \Xi^G$ such that $\dim T_x(\Xi)^{F(G^+)} = 3$.

Case (I). It follows from Lemma 2.5 that $\chi(\Xi^G)$ is a nonnegative even integer.

Case (II). Let $\rho : G^+ \to SO(5)$ be the restriction to $G^+$ of the tangential representation of $G$ at $x \in \Xi^G$ and let

$$\rho^{F(G^+)} : G^+ \to O(3) \quad \text{and} \quad \rho_{F(G^+)} : G^+ \to O(2)$$

be two subrepresentations of $\rho$ corresponding to $T_x(\Xi)^{F(G^+)}$ and $T_x(\Xi)_{F(G^+)}$, respectively. Put $K = \ker \rho_{F(G^+)}$. Then, by Proposition 2.11, the following holds.

(1) The intersection $F(G^+) \cap K$ is trivial.

(2) $K$ is isomorphic to a subgroup of $SO(3)$.

(3) $G^+/K$ is isomorphic to a subgroup of $O(2)$.

By [**10**, Lemma 1.1, Chapter 6, p.217], the product $F(G^+)F(K)$ is again a nilpotent normal subgroup of $G^+$. The maximality of $F(G^+)$ implies that $F(G^+)F(K) = F(G^+)$. As $F(G^+) \cap F(K) = \{e\}$, we see that $F(K)$ must be trivial. If $K$ is isomorphic to $A_5$, then this contradicts our assumption that $G^+$ has no nonabelian minimal normal subgroups. If $K$ is isomorphic to $C_n$, $D_{2n}$, $A_4$, or $S_4$, where $n \geq 2$, then $F(K)$ is nontrivial. Therefore we see that $K$ is trivial, and $G^+$ is isomorphic to a finite subgroup of $O(2)$, which means that $G^+$ is cyclic or dihedral. By Proposition 2.3, we have $\chi(\Xi^{F(G^+)}) = 0$ or 2. Since $[G : F(G^+)] = 2^k$ for some $0 \leq k \leq 2$, it follows from Lemma 2.1 that $\chi(\Xi^G) \equiv 0 \bmod 2$.

In Case (II), we assume that $\Xi$ is a $\mathbb{Z}$-homology 5-sphere. Then there exists a Sylow $p$-subgroup $P$ of $F(G^+)$ such that $\dim T_x(\Xi)^{F(G^+)} = \dim T_x(\Xi)^P = 3$. By Lemma 2.2, $\Xi^P$ is a $\mathbb{Z}_p$-homology 3-sphere (thus connected). Therefore, by Proposition 2.6, $\Xi^{F(G^+)}$ coincides with $\Xi^P$. Hence $\Xi^{F(G^+)}$ is a $\mathbb{Z}_p$-homology 3-sphere (thus a $\mathbb{Q}$-homology 3-sphere). Since $[G : F(G^+)] = 2^k$ for some $0 \leq k \leq 2$, it follows from Proposition 2.9 that $\chi(\Xi^G)$ is a nonnegative even integer. □

This completes the proof of Theorem 1.4.

If $\Xi$ is a $\mathbb{Z}$-homology 5-sphere and $\chi(\Xi^G)$ is neither 0 nor 2 (by Theorem 1.4, equivalently $\chi(\Xi^G) = 2m \geq 4$ for some integer $m$), the group $G$ is further constrained as follows:

PROPOSITION 3.4. *If $\Xi$ is a $\mathbb{Z}$-homology 5-sphere and $\chi(\Xi^G)$ is neither 0 nor 2, then the group $G$ satisfies the following three conditions.*

*(1) $F(G^+)$ is a nontrivial cyclic group.*

*(2)* $[G : F(G^+)] = 2^k$, *where* $1 \leq k \leq 4$.

*(3)* $G$ *is not isomorphic to the direct product of* $F(G^+)$ *and* $G/F(G^+)$.

Proof. By Propositions 3.1 and 3.2, we may assume that every minimal normal subgroup of $G^+$ is abelian and $F(G^+)$ is a nontrivial cyclic group (if not it follows that $\chi(\Xi^G) = 0$ or 2). Moreover, it suffices to prove the assertion in the following situations.

(I) For every $x \in \Xi^G$, we have $\dim T_x(\Xi)^{F(G^+)} = 1$.

(II) There exists $x \in \Xi^G$ such that $\dim T_x(\Xi)^{F(G^+)} = 3$.

If $G = F(G^+)$ then it follows from Proposition 2.3 that $\chi(\Xi^G) = 0$ or 2. Therefore we moreover assume that $G \neq F(G^+)$. We next prove the assertion (2) in the cases (I) and (II).

Case (I). Let $\rho : G^+ \to SO(5)$ be the representation of $G^+$ that appeared in the proof of Proposition 3.3. Applying Proposition 2.11 to $\rho : G^+ \to SO(5)$ with $\dim T_x(\Xi)^{F(G^+)} = 1$, we see that $K = \ker \rho_{F(G^+)}$ is trivial and that $G^+/K (\cong G^+)$ is isomorphic to a subgroup of $O(4)$. We identify $G^+$ with its image in $O(4)$. Put $L^+ = G^+ \cap SO(4)$, and then $[G^+ : L^+] \leq 2$. Since $L^+$ is normal in $G^+$ and $F(L^+)$ is characteristic in $L^+$, $F(L^+)$ is normal in $G^+$. As $F(L^+)$ is nilpotent, the maximality of $F(G^+)$ shows that $F(L^+) \leq F(G^+) \cap L^+$. Also, the intersection $F(G^+) \cap L^+$ is a nilpotent normal subgroup of $L^+$; hence $F(G^+) \cap L^+ \leq F(L^+)$. Therefore $F(L^+)$ coincides with $F(G^+) \cap L^+$, and $[F(G^+) : F(L^+)] \leq 2$. Suppose that the matrix $-I$ in $SO(4)$ lies in $L^+$, where $I$ denotes the identity matrix in $SO(4)$. Note that $\langle -I \rangle$ is a nilpotent normal subgroup of $G^+$ and it is contained in $F(G^+)$. By our assumption that $F(G^+)$ is cyclic, $\langle -I \rangle$ is characteristic in $F(G^+)$. The transitivity of characteristic subgroups says that $\langle -I \rangle$ is characteristic in $G^+$. Therefore, $\langle -I \rangle$ is normal in $G$. By Lemma 2.2, $\Xi^{\langle -I \rangle}$ is a sphere of dimension $\leq 1$. By Lemma 2.4, $\Xi^G$ is also a sphere of dimension $\leq 1$. This contradicts our hypothesis that $\chi(\Xi^G) \notin \{0, 2\}$. Therefore $-I$ does not lie in $L^+$. Let $\pi : SO(4) \to SO(3) \times SO(3)$ be the canonical double covering homomorphism. Its kernel is $\{\pm I\} = \langle -I \rangle$. As $-I \notin L^+$, the restriction $\pi|_{L^+}$ is injective. Thus we may assume that $L^+$ is a subgroup of $SO(3) \times SO(3)$. As $F(L^+)$ is a subgroup of $F(G^+)$, $F(L^+)$ is a cyclic group. Therefore, by Proposition 2.12, one of the following holds.

(A) $L^+$ contains a characteristic subgroup isomorphic to $A_5$.

(B) $L^+$ is isomorphic to $A_5 \times A_5$.

(C) $L^+$ is an extension of a 2-group by $F(L^+)$ with $[L^+ : F(L^+)] = 2^\ell$, where $0 \leq \ell \leq 2$.

By our assumption that every minimal normal subgroup of $G^+$ is abelian, we conclude that $L^+$ is an extension of a 2-group by $F(L^+)$ with $[L^+ : F(L^+)] = 2^\ell$, where $0 \leq \ell \leq 2$. Therefore we obtain the equations

$$[G : F(G^+)] = [G : G^+] \cdot [G^+ : F(G^+)] \quad \text{and} \quad [G^+ : F(G^+)] = \frac{[G^+ : L^+] \cdot [L^+ : F(L^+)]}{[F(G^+) : F(L^+)]}.$$

These equations show that $[G : F(G^+)] = 2^k$, where $1 \leq k \leq 4$ (because $G \neq F(G^+)$).

Case (II). In this case we have already seen that $G^+$ is either cyclic or dihedral (in the proof of Case (II) in Proposition 3.3). Therefore we have $[G : F(G^+)] = [G : G^+] \cdot [G^+ : F(G^+)] = 2$ or 4 (because $G \neq F(G^+)$).

Finally we show that $G$ is not isomorphic to the direct product of $F(G^+)$ and $G/F(G^+)$. Suppose that $G$ can be identified with a direct product of a nontrivial cyclic group $C$ and a nontrivial 2-group $P_2$. By Lemma 2.2, $\Xi^{P_2}$ is a $\mathbb{Z}_2$-homology sphere (thus a $\mathbb{Q}$-homology sphere). As $\Xi^G = (\Xi^{P_2})^C$, it follows from Proposition 2.3 that $\chi(\Xi^G) = 0$ or 2. This contradicts our hypothesis that $\chi(\Xi^G) \notin \{0, 2\}$. Therefore $G$ is not isomorphic to the direct product of $F(G^+)$ and $G/F(G^+)$. □

## 4. Dihedral group actions arising from Brieskorn spheres

In this section, we prove Theorem 1.5. We begin by recalling the definition and some basic properties of Brieskorn manifolds. We then construct explicit dihedral group actions on certain Brieskorn manifolds.

Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, $n \geq 2$, be an $n$-tuple of integers with $a_j \geq 2$ for all $j$. The Brieskorn manifold $\Sigma(\boldsymbol{a})$ is defined by

$$\Sigma(\boldsymbol{a}) = \Sigma(a_1, a_2, \ldots, a_n) = \left\{ (z_1, z_2, \ldots, z_n) \in S(\mathbb{C}^n) \,|\, z_1^{a_1} + z_2^{a_2} + \cdots + z_n^{a_n} = 0 \right\},$$

where $S(\mathbb{C}^n)$ denotes the unit sphere in the complex vector space $\mathbb{C}^n$. The Brieskorn manifold $\Sigma(\boldsymbol{a})$ is a smooth closed manifold of dimension $2n-3$. If $n = 2$ then $\Sigma(\boldsymbol{a})$ is a torus link, which has $d$ components, where $d = \gcd(a_1, a_2)$, see [**20**, Assertion (Brauner), p.4]. If $n \geq 3$ then $\Sigma(\boldsymbol{a})$ is $(n-3)$-connected, see [**20**, Theorem 5.2, p.45]. A Brieskorn manifold that is a homotopy sphere is often called a *Brieskorn sphere*. In general, a Brieskorn sphere need not be diffeomorphic to the standard sphere of the same dimension. Since the group $\Theta_5$ of oriented diffeomorphism classes of homotopy 5-spheres is trivial, every homotopy 5-sphere is diffeomorphic to $S^5$ (e.g., see [**13**]). Thus, in the rest of this section, any homotopy 5-sphere will be identified with $S^5$.

We now give a $D_4$-action on $\mathbb{R}P^3$ with four fixed points, as described in Remark 2.10.

PROPOSITION 4.1. *The real projective space $\mathbb{R}P^3$ admits an effective 4-fixed-point $D_4$-action.*

PROOF. It is known that the Brieskorn manifold $\Sigma(2,2,2)$ is diffeomorphic to $\mathbb{R}P^3$ (e.g., [**20**, Example 9.8, p.80]) and thus we identify $\mathbb{R}P^3$ with $\Sigma(2,2,2)$. Let $D_4 = \langle a \rangle \times \langle b \rangle$, where $a^2 = b^2 = e$. For generators $a$ and $b$ of $D_4$, define the diffeomorphisms of $\Sigma(2,2,2)$ by

$$\Phi(a)(z_1, z_2, z_3) = (-z_1, z_2, z_3) \quad \text{and} \quad \Phi(b)(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, -\overline{z_3}).$$

It is immediate that $\Phi(a)$ and $\Phi(b)$ are nontrivial involutions and commute with each other. Therefore $\Phi(a)$ and $\Phi(b)$ define an effective $D_4$-action $\Phi$ on $\Sigma(2,2,2)$. By solving the equations $\Phi(a)(\boldsymbol{z}) = \boldsymbol{z}$ and $\Phi(b)(\boldsymbol{z}) = \boldsymbol{z}$ on $\Sigma(2,2,2)$, where $\boldsymbol{z} \in \Sigma(2,2,2)$, we obtain

$$\Sigma(2,2,2)^{D_4} = \left\{ \left(0, \tfrac{1}{\sqrt{2}}, \pm\tfrac{i}{\sqrt{2}}\right), \left(0, -\tfrac{1}{\sqrt{2}}, \pm\tfrac{i}{\sqrt{2}}\right) \right\},$$

which consists of exactly four points. □

For $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, let $\mathbb{G}(\boldsymbol{a})$ denote a weighted graph consisting of vertices and edges defined as follows. It has $n$ vertices weighted by $a_1, \ldots, a_n$. Two vertices $a_j$ and $a_k$, $j \neq k$, are joined if $\gcd(a_j, a_k) > 1$. By the definition, the connected components and the isolated vertices of $\mathbb{G}(\boldsymbol{a})$ are well-defined.

The following classical result provides a criterion for determining whether a given Brieskorn manifold is a $\mathbb{Z}$-homology sphere.

THEOREM 4.2 ([**5**, Brieskorn's criterion, Satz 1]). *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, $n \geq 3$, be an $n$-tuple of integers with $a_j \geq 2$ for all $j$. Then the following two conditions are equivalent.*

*(1) $\Sigma(\boldsymbol{a})$ is a $\mathbb{Z}$-homology $(2n-3)$-sphere. In particular, if $n \geq 4$, then $\Sigma(\boldsymbol{a})$ is a Brieskorn $(2n-3)$-sphere.*

*(2) The graph $\mathbb{G}(\boldsymbol{a})$ satisfies one of the two following conditions:*

- (a) *$\mathbb{G}(\boldsymbol{a})$ has at least two isolated vertices.*
- (b) *$\mathbb{G}(\boldsymbol{a})$ has exactly one isolated vertex and one component $K$ which consists of an odd number of vertices and it holds that $\gcd(a_j, a_k) = 2$ for any $a_j, a_k \in K$ with $j \neq k$.*

PROPOSITION 4.3. *Let $n$ and $m$ be integers with $n \geq 2$, $m \geq 2$. Let $\boldsymbol{a} = (a_1, \ldots, a_n, m, m)$ be an $(n+2)$-tuple of integers such that $a_j \geq 2$ for all $j$ and $\gcd(a_j, a_k) = 1$ for all $1 \leq j < k \leq n$. Assume that at least one of the following holds:*

*(1) at least two of $\{a_1, \ldots, a_n\}$ are coprime to $m$, and*

*(2) $n = 2$ and $m = 2$.*

*Then $\Sigma(\boldsymbol{a})$ is a Brieskorn $(2n+1)$-sphere.*

PROOF. We apply Theorem 4.2. In case (1), the graph $\mathbb{G}(\boldsymbol{a})$ has at least two isolated vertices, hence the assertion follows from the condition (2)(a). In case (2) (i.e. $n = 2$ and $m = 2$), if both $a_1$ and $a_2$ are odd, then the condition (2)(a) applies. If exactly one of $a_1, a_2$ is even, then $\mathbb{G}(\boldsymbol{a})$ satisfies the condition (2)(b), and the assertion follows. □

PROPOSITION 4.4. *Let $\boldsymbol{a} = (a_1, \ldots, a_n, m, m)$, $n \geq 2$, be an $(n+2)$-tuple as in Proposition 4.3 and $\ell = \operatorname{lcm}(a_1, \ldots, a_n)$. Then there exists an effective action of a cyclic group $C_\ell$ on the Brieskorn $(2n+1)$-sphere $\Sigma(\boldsymbol{a})$ such that the $C_\ell$-fixed-point set is a torus link with $m$ components, i.e. a disjoint union of $m$ circles.*

PROOF. First note that $\ell$ is equal to the product $a_1 \cdots a_n$, since $\gcd(a_j, a_k) = 1$ for all $1 \leq j < k \leq n$. Let $C$ be a cyclic group of order $\ell$ which has a generator $\xi$, i.e. $C = \langle\, \xi \,|\, \xi^\ell = e \,\rangle$. We define the $C$-action $\Phi$ on $\Sigma(\boldsymbol{a})$ by

$$\Phi(\xi)(z_1, \ldots, z_n, z_{n+1}, z_{n+2}) = \left( e^{\frac{2\pi i}{a_1}} z_1, \ldots, e^{\frac{2\pi i}{a_n}} z_n, z_{n+1}, z_{n+2} \right).$$

One checks directly that the action $\Phi$ is well-defined. Moreover, if $\Phi(\xi^\lambda)$ is the identity map for some integer $\lambda$, then $e^{2\pi i\lambda/a_j} = 1$ for all $j$, and hence $\ell = \mathrm{lcm}(a_1, \ldots, a_n)$ divides $\lambda$. Hence the action $\Phi$ is effective. It is easy to see that the $C$-fixed-point set is given by

$$\Sigma(\boldsymbol{a})^C = \{\, (0, \ldots, 0, z_{n+1}, z_{n+2}) \in S(\mathbb{C}^{n+2}) \,|\, z_{n+1}^m + z_{n+2}^m = 0 \,\},$$

and $\Sigma(\boldsymbol{a})^C$ can be identified with a torus link $\Sigma(m, m)$ of $m$ components. □

PROPOSITION 4.5. *Let $\boldsymbol{a} = (a_1, \ldots, a_n, 2, 2)$ be an $(n+2)$-tuple as in Proposition 4.3 and $\ell = \mathrm{lcm}(a_1, \ldots, a_n)$. Then there exists an effective action of a dihedral group $D_{2\ell}$ on the Brieskorn $(2n+1)$-sphere $\Sigma(\boldsymbol{a})$ such that the $D_{2\ell}$-fixed-point set consists of exactly four points.*

PROOF. Let $D$ be the dihedral group of order $2\ell$ defined by

$$D = \langle a, b \,|\, a^\ell = b^2 = baba = e \,\rangle.$$

Define the $D$-action $\Phi$ on $\Sigma(\boldsymbol{a})$ by

$$\Phi(a)(z_1, \ldots, z_n, z_{n+1}, z_{n+2}) = \left(e^{\frac{2\pi i}{a_1}} z_1, \ldots, e^{\frac{2\pi i}{a_n}} z_n, z_{n+1}, z_{n+2}\right) \text{ and}$$
$$\Phi(b)(z_1, \ldots, z_n, z_{n+1}, z_{n+2}) = (\overline{z_1}, \ldots, \overline{z_n}, \overline{z_{n+1}}, -\overline{z_{n+2}}).$$

A direct computation shows that the action $\Phi$ is well-defined. Put $C = \langle a \rangle \cong C_\ell$. By Proposition 4.4, the restricted $C$-action is effective and

$$\Sigma(\boldsymbol{a})^C = \{\, (0, \ldots, 0, z_{n+1}, z_{n+2}) \in S(\mathbb{C}^{n+2}) \,|\, z_{n+1}^2 + z_{n+2}^2 = 0 \,\}.$$

Since $C$ is normal in $D$, we have $\Sigma(\boldsymbol{a})^D = (\Sigma(\boldsymbol{a})^C)^{\langle b \rangle}$. The involution $\Phi(b)$ maps $(z_{n+1}, z_{n+2})$ to $(\overline{z_{n+1}}, -\overline{z_{n+2}})$, and therefore we have

$$\Sigma(\boldsymbol{a})^D = \left\{ \left(0, \ldots, 0, \tfrac{1}{\sqrt{2}}, \pm\tfrac{i}{\sqrt{2}}\right), \left(0, \ldots, 0, -\tfrac{1}{\sqrt{2}}, \pm\tfrac{i}{\sqrt{2}}\right) \right\},$$

which consists of exactly four points.

Finally, we show that the $D$-action is effective. Put $N = \ker \Phi$. Since the restricted $C$-action is effective, we have $N \cap C = \{e\}$. Suppose that $N$ is not contained in $C$. Then there is a positive integer $\lambda$ such that $a^\lambda b \in N$. By the relation $baba = e$, we have $a^\lambda b = ba^{-\lambda} \in N$. Since $N$ is normal in $D$, we have $a(a^\lambda b)a^{-1} = a^{\lambda+2}b \in N$. We now obtain $(a^{\lambda+2}b)(ba^{-\lambda}) = a^2 \in N \cap C$. This is a contradiction to $a^2 \neq e$ (since $\ell = \mathrm{lcm}(a_1, \ldots, a_n) \geq 6$). Therefore $N$ is contained in $C$; hence $N \cap C = N = \{e\}$. It follows that the action $\Phi$ is effective. □

COROLLARY 4.6. *Let $a$ and $b$ be integers $\geq 2$ with $\gcd(a, b) = 1$. Then there exists an effective action of a dihedral group $D_{2ab}$ of order $2ab$ on the Brieskorn 5-sphere $\Sigma(a, b, 2, 2)$ such that the $D_{2ab}$-fixed-point set consists of exactly four points. In particular, $D_{2ab}$ can act on $S^5$ with exactly four fixed points.*

PROPOSITION 4.7. *Let $a$ and $b$ be integers $\geq 2$ with $\gcd(a, b) = 1$ and $D_{2ab}$ a*

*dihedral group of order* $2ab$*. For a given nonnegative integer* $r$*, there exists an effective* $D_{2ab}$*-action on* $S^5$ *with* $\left|(S^5)^{D_{2ab}}\right| = 2r$ *(including the case* $r = 0$*).*

PROOF. Put $D = D_{2ab}$. First we construct linear examples realizing the cases $r = 0$ and $r = 1$. Let $W$ be the standard faithful $\mathbb{R}[D]$-module of dimension 2, which is obtained by the inclusion $D \subset O(2)$, and $\mathbb{R}_\pm$ the nontrivial $\mathbb{R}[D]$-module of dimension 1. Then the 6-dimensional $\mathbb{R}[D]$-modules

$$U = W^{\oplus 3} \quad \text{and} \quad V = \mathbb{R} \oplus \mathbb{R}_\pm \oplus W^{\oplus 2}$$

are faithful and satisfy $\dim U^D = 0$ and $\dim V^D = 1$. Then the 5-sphere $S(U)$ (resp. $S(V)$) gives an effective linear $D$-action with $|S(U)^D| = 0$ (resp. $|S(V)^D| = 2$).

Next we consider the case $r \geq 2$. By Corollary 4.6, there is an effective 4-fixed-point $D$-action $\Phi$ on $S^5$. Let $\mathbb{V}_\Phi$ be the set of isomorphism classes of tangential $\mathbb{R}[D]$-modules at $D$-fixed points of the action $\Phi$. Suppose that we have constructed an effective $2r$-fixed-point $D$-action $\Psi$ on $S^5$ such that $\mathbb{V}_\Psi \subset \mathbb{V}_\Phi$. Clearly, the case $r = 2$ is realized by $\Phi$ itself. Since $\mathbb{V}_\Psi \cap \mathbb{V}_\Phi \neq \emptyset$, by applying Lemma 2.8 to $\Psi$ and $\Phi$, we get a $2(r+1)$-fixed-point $D$-action $\Omega$ on $S^5$ satisfying $\mathbb{V}_\Omega \subset \mathbb{V}_\Phi$. As $\Phi$ is effective, Lemma 2.7 implies that the tangential representation of $D$ at every $D$-fixed point of $\Phi$ is faithful. In particular, every class in $\mathbb{V}_\Phi$ is faithful. As $\emptyset \neq \mathbb{V}_\Omega \subset \mathbb{V}_\Phi$, there exists a $D$-fixed point $y$ of $\Omega$ such that the tangential representation of $D$ at $y$ is faithful. Using Lemma 2.7 again, we conclude that the $D$-action $\Omega$ is itself effective. This completes the induction for $r \geq 2$, and the proposition holds for all $r \geq 0$. □

## 5. Finite group actions arising from $SU(3)/SO(3)$

In this section, we prove Theorem 1.7 by studying group actions on the symmetric space $SU(3)/SO(3)$ and its quotient.

Throughout this section, we write $G = SU(3)$ (so $G$ is not necessarily finite). Let $\tau : G \to G$ be the involution defined by $\tau(g) = {}^t g^{-1}$, where ${}^t g$ denotes the transpose of the matrix $g \in G$. Then we have

$$G^\tau = \{\, g \in G \mid \tau(g) = g \,\} = \{\, g \in G \mid {}^t g = g^{-1} \,\}.$$

In the rest of this section, we denote $G^\tau$ by $H$. Note that $H$ is a subgroup of $G$ isomorphic to $SO(3)$. The symmetric space $G/H$ is diffeomorphic to the manifold $X_{-1}$ in Barden's notation (see [**3**, Remark (2), p. 374]), and hence it is a 1-connected closed manifold of dimension 5 with $H_2(G/H;\mathbb{Z}) \cong \mathbb{Z}_2$ (see [**3**, Lemma 1.1]). It follows from the Poincaré duality and the universal coefficient theorem that

$$H_*(G/H;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & (* = 0,\, 5), \\ \mathbb{Z}_2 & (* = 2), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $G/H$ is a $\mathbb{Z}_p$-homology 5-sphere for any odd prime $p$. The center $Z$ of $G$ is the cyclic group of order 3 generated by $\omega I_3$, where $I_3$ is the identity matrix in $G$ and

$\omega = \exp(2\pi i/3)$.

We fix subgroups $\mathbb{T}$, $\mathbb{O}$, and $\mathbb{I}$ of $H$ isomorphic to $A_4$, $S_4$, and $A_5$, respectively. We first prove the following proposition.

PROPOSITION 5.1. *Let $K$ be one of $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I}$, or $H$. Then the fixed-point set of the action of $K$ on $G/H$ by left multiplication coincides with the set $\{H, \omega H, \omega^2 H\}$.*

To prove Proposition 5.1, we recall the Cartan embedding of the symmetric space $G/H$. Define the map $\Psi : G/H \to G$ by

$$\Psi(gH) = g\,\tau(g^{-1}),$$

and thus $\Psi(gH) = g\,{}^t g$. Then the map $\Psi$ is a (smooth) embedding called the Cartan embedding, see [**12**, Chap. VII, Exercises and Further Results (2), p. 347].

LEMMA 5.2. *The embedding $\Psi : G/H \to G$ is equivariant with respect to the action of $G$ on $G/H$ by left multiplication and the action $\alpha$ of $G$ on $G$ given by $\alpha(a, g) = a\,g\,{}^t a$. Moreover, for a subgroup $K$ of $H$, the $K$-fixed-point set $(G/H)^K$ can be identified with $N \cap C_G(K)$, where $N = \operatorname{Im}\Psi$ and $C_G(K)$ denotes the centralizer of $K$ in $G$.*

PROOF. For any $a$, $g \in G$, we have

$$\Psi(agH) = ag^t(ag) = a(g^t g)^t a = \alpha(a, \Psi(gH)).$$

Therefore the embedding $\Psi$ is equivariant. Since $\Psi$ is a $G$-embedding, $(G/H)^K$ is identified with $N^K$. As ${}^t k = k^{-1}$ for any $k \in K \leq H$, the lemma follows from

$$N^K = \{\, g \in N \mid kg^t k = g,\ {}^\forall k \in K \,\} = \{\, g \in N \mid kg = gk,\ {}^\forall k \in K \,\} = N \cap C_G(K).$$

□

PROPOSITION 5.3. *Let $K$ be one of $\mathbb{T}$, $\mathbb{O}$, $\mathbb{I}$, or $H$. Then $C_G(K)$ coincides with $Z$.*

PROOF. The inclusion $K \subset G \subset GL(3, \mathbb{C})$ induces a faithful complex representation $\rho : K \to GL(3, \mathbb{C})$ of degree 3, which is in fact irreducible. This follows from [**8**, Chapter 4.8] for $K \in \{\mathbb{T}, \mathbb{O}, \mathbb{I}\}$. For $K = H$, it follows from the fact that all irreducible complex representations of $SO(3)$ are odd-dimensional and that every one-dimensional complex representation is trivial; hence any nontrivial 3-dimensional complex representation is irreducible (see, e.g., [**11**, Propositions 4.35 and 10.8]). By Schur's lemma, each matrix $X \in C_G(K)$ is a scalar matrix $\lambda I_3$ for some scalar $\lambda \in \mathbb{C} \setminus \{0\}$. Since $X = \lambda I_3 \in SU(3)$, we have $\det X = 1$, which shows that the scalar $\lambda$ satisfies $\lambda^3 = 1$ (hence $C_G(K) \subset Z$). Since $Z$ is the center of $G$, we have $Z \subset C_G(K)$. Therefore we obtain $C_G(K) = Z$. □

PROOF OF PROPOSITION 5.1. We first prove that $(G/H)^K$ consists of exactly three points. Let $\Psi : G/H \to G$ be the Cartan embedding and put $N = \operatorname{Im}\Psi$. It follows from Lemma 5.2 and Proposition 5.3 that $(G/H)^K$ is identified with $N \cap Z$. Since $\Psi(zH) = z\,{}^t z = z^2 = z^{-1}$ for each $z \in Z$, $Z$ is contained in $N$ (hence

$N \cap Z = Z$). Therefore, $(G/H)^K$ consists of exactly three points. On the other hand, it is clear that the set $\{ H, \omega H, \omega^2 H \}$ is fixed by the action of $K$ on $G/H$ by left multiplication. Since $|(G/H)^K| = 3$ and $\{ H, \omega H, \omega^2 H \} \subset (G/H)^K$, we obtain $(G/H)^K = \{ H, \omega H, \omega^2 H \}$. □

Recall that $G = SU(3)$ and $G/Z$ is the projective special unitary group $PSU(3)$. In the rest of this section, we put $\overline{G} = G/Z$, and let $q : G \to \overline{G}$ denote the canonical projection. It is known that $N_G(H) = ZH$ and $Z \cap H = I_3$, e.g., see [**32**, Lemma 2.2 (b)]. Therefore, for a subgroup $K$ of $H$, the restriction $q$ to $K$ is injective, and hence $q(K)$ coincides with $ZK/Z \cong K$. Thus we denote $q(K)$ by $\overline{K}$. Let $\tau_Z : \overline{G} \to \overline{G}$ be the involution defined by $\tau_Z(gZ) = \tau(g)Z = {}^t g^{-1} Z$ for $gZ \in \overline{G}$. This involution is well-defined. The fixed-point set is given by

$$\overline{G}^{\tau_Z} = \{\ gZ \in \overline{G} \,|\, \tau_Z(gZ) = gZ, \} = \{ gZ \in \overline{G} \,|\, g^t g \in Z \}.$$

As shown in Lemma 5.4 below, $\overline{G}^{\tau_Z}$ coincides with $\overline{H}$. We define the map $\Psi_Z : \overline{G}/\overline{H} \to \overline{G}$ by

$$\Psi_Z(gZ\overline{H}) = (gZ)\, \tau_Z((gZ)^{-1}),$$

and thus $\Psi_Z(gZ\overline{H}) = g\, {}^t g Z$. This is well-defined. Indeed, if $g_1 Z\overline{H} = g_2 Z\overline{H}$, then there are $h \in H$ and $z \in Z$ such that $g_2 = g_1 hz$. Therefore we have

$$\Psi_Z(g_2 Z\overline{H}) = g_2\, {}^t g_2 Z = (g_1 hz)^t (g_1 hz) Z = g_1 (hz\, {}^t(hz))^t g_1 Z = g_1\, {}^t g_1 Z = \Psi_Z(g_1 Z\overline{H}),$$

where we used ${}^t h = h^{-1}$ and $hz\, {}^t(hz) = z^2 \in Z$. Then $\Psi_Z$ is the Cartan embedding of the symmetric space $\overline{G}/\overline{H}$. In particular, $\Psi_Z$ is a (smooth) embedding (see [**12**, Chap. VII, Exercises and Further Results (2), p. 347]). Here we note that $\overline{G}/\overline{H}$ can be canonically identified with $G/ZH$ as $G$-manifolds, via the $G$-diffeomorphism $\Lambda$ from $\overline{G}/\overline{H}$ to $G/ZH$ defined by $\Lambda(gZ\overline{H}) = gZH$, where $G$ acts on both spaces $\overline{G}/\overline{H}$ and $G/ZH$ by left multiplication. Therefore we may regard the Cartan embedding $\Psi_Z : G/ZH \to \overline{G}$ defined by $\Psi_Z(gZH) = g^t g Z$.

Lemma 5.4. *The following holds.*

*(1) The fixed-point set $\overline{G}^{\tau_Z}$ coincides with $\overline{H}$.*

*(2) The embedding $\Psi_Z : G/ZH \to \overline{G}$ is equivariant with respect to the action of $G$ on $G/ZH$ by left multiplication and the action $\alpha_Z$ of $G$ on $\overline{G}$ defined by $\alpha_Z(a, gZ) = a\, g\, {}^t a Z$ for $a \in G$.*

*(3) The $K$-fixed-point set $(G/ZH)^K$ is identified with $\overline{N} \cap C_{\overline{G}}(\overline{K})$, where $\overline{N} = \operatorname{Im} \Psi_Z$.*

Proof. Proof of (1). Suppose that $gZ \in \overline{H}$, i.e. there are $z \in Z$ and $h \in H$ such that $g = zh$. Then we have

$$(zh)^t (zh) = zh({}^t h\, {}^t z) = z(h\, {}^t h)z = z(hh^{-1})z = z^2 \in Z,$$

where we used ${}^t h = h^{-1}$ for any $h \in H$. This implies that $gZ$ lies in $\overline{G}^{\tau_Z}$. Next suppose that $gZ \in \overline{G}^{\tau_Z}$. Then it holds that ${}^t g = z\, g^{-1}$ for some $z \in Z$. Since we have

$$ {}^t(gz) = {}^t z\, {}^t g = z(zg^{-1}) = z^2 g^{-1} = z^{-1} g^{-1} = (gz)^{-1}, $$

$gz$ lies in $H$, and hence $gZ$ lies in $\overline{H} = ZH/Z$. Therefore we obtain $\overline{G}^{\tau_Z} = \overline{H}$.

The proofs of (2) and (3) are completely analogous to the proof of Lemma 5.2; for completeness, we briefly indicate the argument.

Proof of (2). For any $a, g \in G$, we have

$$ \Psi_Z(agZH) = ag\, {}^t(ag)\, Z = a(g\, {}^t g)\, {}^t a\, Z = \alpha_Z\big(a, \Psi_Z(gZH)\big). $$

Proof of (3). Put $\overline{N} = \operatorname{Im} \Psi_Z$. Since $\Psi_Z$ is equivariant, $(G/ZH)^K$ is identified with $\overline{N}^{\,\overline{K}}$. Identifying $\overline{K}$ (resp. $\overline{H}$) with $K$ (resp. $H$), we have ${}^t k = k^{-1}$ for all $k \in \overline{K} \le \overline{H}$. Therefore the assertion (3) follows from

$$ \overline{N}^{\overline{K}} = \{\, x \in \overline{N} \mid kx\, {}^t k = x,\ \forall\, k \in \overline{K} \,\} = \{\, x \in \overline{N} \mid kx = xk,\ \forall\, k \in \overline{K} \,\} = \overline{N} \cap C_{\overline{G}}(\overline{K}). $$

□

For compact Lie groups $A$ and $B$, we write $\operatorname{Hom}(A, B)$ for the set of all smooth group homomorphisms from $A$ to $B$. When both $A$ and $B$ are finite, this agrees with the usual notion of a group homomorphism. If $B$ is abelian, then $\operatorname{Hom}(A, B)$ is an abelian group with respect to the operation $(\varphi \cdot \psi)(a) = \varphi(a)\psi(a)$ for $a \in A$ and $\varphi, \psi \in \operatorname{Hom}(A, B)$.

LEMMA 5.5. *Let $K$ be a subgroup of $H$. If $\operatorname{Hom}(K, Z)$ is trivial and $C_G(K) = Z$, then $C_{\overline{G}}(\overline{K}) = \{Z\}$.*

PROOF. First recall that $\overline{K}$ can be identified with $K$ via the projection $q$. Thus we regard $k \in K$ as $q(k) \in \overline{K}$. Note that a coset $gZ \in \overline{G}$ belongs to $C_{\overline{G}}(\overline{K})$ if and only if it holds that $gkg^{-1}k^{-1} \in Z$ for every $k \in K$. Therefore, for $gZ \in C_{\overline{G}}(\overline{K})$, we can define the map $\chi_{gZ} : K \to Z$ by $\chi_{gZ}(k) = gkg^{-1}k^{-1} \in Z$. Since $Z$ is the center of $G$, the map $\chi_{gZ}$ depends only on $gZ \in C_{\overline{G}}(\overline{K})$ (i.e. $\chi_{g_1 Z} = \chi_{g_2 Z}$ whenever $g_1 Z = g_2 Z$ in $C_{\overline{G}}(\overline{K})$). For $k_1, k_2 \in K$, it holds that

$$ \begin{aligned} \chi_{gZ}(k_1 k_2) &= gk_1k_2g^{-1}(k_1k_2)^{-1} \\ &= gk_1k_2g^{-1}k_2^{-1}k_1^{-1} \\ &= gk_1g^{-1}(gk_2g^{-1}k_2^{-1})k_1^{-1} \\ &= gk_1g^{-1}\chi_{gZ}(k_2)k_1^{-1} \\ &= gk_1g^{-1}k_1^{-1}\chi_{gZ}(k_2) \quad (\text{since } \chi_{gZ}(k_2) \in Z) \\ &= \chi_{gZ}(k_1)\, \chi_{gZ}(k_2). \end{aligned} $$

Thus the defined map $\chi_{gZ} : K \to Z$ is a group homomorphism, and therefore $\chi_{gZ}$ lies in $\operatorname{Hom}(K, Z)$ for any $gZ \in C_{\overline{G}}(\overline{K})$. By our hypothesis that $\operatorname{Hom}(K, Z)$ is trivial, we see

that $gZ$ lies in $C_{\overline{G}}(\overline{K})$ if and only if $g$ lies in $C_G(K)$ (because $\chi_{gZ}(k)$ is trivial for each $k \in K$). Therefore, by our hypothesis that $C_G(K) = Z$, we obtain $C_{\overline{G}}(\overline{K}) = \{Z\}$. □

PROPOSITION 5.6. *Let $K$ be one of $\mathbb{O}$, $\mathbb{I}$, or $H$. Then the fixed-point set of the $K$-action on $G/ZH$ by left multiplication coincides with the set $\{ZH\}$.*

PROOF. It is known that the abelianizations of $S_4$ and $A_5$ are isomorphic to $C_2$ and the trivial group, respectively. Therefore we see that $\mathrm{Hom}(\mathbb{O}, Z)$ and $\mathrm{Hom}(\mathbb{I}, Z)$ are trivial. Moreover, since $H \cong SO(3)$ is connected and $Z$ is finite, any smooth group homomorphism from $H$ to $Z$ must be constant (hence trivial). Consequently, $\mathrm{Hom}(K, Z)$ is trivial for $K \in \{\mathbb{O}, \mathbb{I}, H\}$. It follows from Proposition 5.3, Lemma 5.4 (3) and Lemma 5.5 that $(G/ZH)^K$ is identified with $\overline{N} \cap \{Z\} = \{Z\}$. In particular, $(G/ZH)^K$ consists of exactly one point. As $ZH$ lies in $(G/ZH)^K$, we obtain $(G/ZH)^K = \{ZH\}$. □

COMMENT 5.7. Proposition 5.6 does not hold for $K = \mathbb{T}$. Indeed, if we choose the subgroup $\mathbb{T}$ of $H$ as

$$\mathbb{T} = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle,$$

then one can check directly that $(G/ZH)^{\mathbb{T}}$ contains the set $\{ZH, aZH, a^2ZH\}$, where $a = \mathrm{diag}(1, \omega^2, \omega)$. On the other hand, for $gZ \in C_{\overline{G}}(\overline{\mathbb{T}})$, set $\chi_{gZ}(k) = gkg^{-1}k^{-1} \in Z$ ($k \in \mathbb{T}$), and define a (well-defined) group homomorphism $\chi : C_{\overline{G}}(\overline{\mathbb{T}}) \to \mathrm{Hom}(\mathbb{T}, Z)$ by $\chi(gZ) = \chi_{gZ}$. As $C_G(\mathbb{T}) = Z$ (Proposition 5.3), it follows that $\ker(\chi) = \{Z\}$. Since $\mathrm{Hom}(\mathbb{T}, Z) \cong C_3$, we have $|C_{\overline{G}}(\overline{\mathbb{T}})| \leq |\mathrm{Hom}(\mathbb{T}, Z)| = 3$. Furthermore, it follows from Lemma 5.4 (3) that $|(G/ZH)^{\mathbb{T}}| = |\overline{N} \cap C_{\overline{G}}(\overline{\mathbb{T}})| \leq |C_{\overline{G}}(\overline{\mathbb{T}})| \leq 3$. Therefore we obtain $(G/ZH)^{\mathbb{T}} = \{ZH, aZH, a^2ZH\}$.

PROPOSITION 5.8. *Let $p$ be an odd prime with $p \neq 3$. Then $G/ZH$ is a $\mathbb{Z}_p$-homology 5-sphere.*

PROOF. Since the $Z$-action on $G/H$ is free and $|Z| = 3$ is coprime to $p$, the transfer homomorphism, see [**4**, Chapter III, Theorem 2.4, p.120], gives an isomorphism:

$$H_*(G/ZH; \mathbb{Z}_p) \cong H_*(G/H; \mathbb{Z}_p)^Z \subset H_*(G/H; \mathbb{Z}_p).$$

Since $G/H$ is a $\mathbb{Z}_p$-homology 5-sphere, $H_i(G/ZH; \mathbb{Z}_p)$ is trivial for all $1 \leq i \leq 4$. The $Z$-action on the orientable closed manifold $G/H$ is orientation-preserving, and therefore the quotient $G/ZH$ is also orientable. Since $G/ZH$ is a connected, closed, orientable manifold of dimension 5, it follows that $H_0(G/ZH; \mathbb{Z}) \cong H_5(G/ZH; \mathbb{Z}) \cong \mathbb{Z}$. The universal coefficient theorem implies that $H_0(G/ZH; \mathbb{Z}_p) \cong H_5(G/ZH; \mathbb{Z}_p) \cong \mathbb{Z}_p$. Therefore $G/ZH$ is a $\mathbb{Z}_p$-homology 5-sphere. □

Let $K$ be one of $\mathbb{T}$, $\mathbb{O}$, or $\mathbb{I}$. Recall that the action $\Phi$ of $K$ on $G/H$ by left multiplication has the fixed-point set $\{H, \omega H, \omega^2 H\}$ by Proposition 5.1. For $\omega I_3 \in Z \leq C_G(K)$, the diffeomorphism $\Phi(\omega I_3)$ is $K$-equivariant, i.e. a $K$-diffeomorphism. Therefore the tangential $\mathbb{R}[K]$-modules at the three fixed points $H$, $\omega H$, $\omega^2 H$ are mutually isomorphic as $\mathbb{R}[K]$-modules. In particular, we have $\mathbb{V}_\Phi = \{[T_H(G/H)]\}$.

PROPOSITION 5.9. *Let $K$ be one of $\mathbb{T}$, $\mathbb{O}$, or $\mathbb{I}$, and let $p$ be an odd prime. For every nonnegative integer $m$ with $m \neq 1$, there exists a $\mathbb{Z}_p$-homology 5-sphere admitting an $m$-fixed-point $K$-action. Moreover, if $p \neq 3$ and $K \in \{\mathbb{O}, \mathbb{I}\}$, then we can choose $m = 1$.*

PROOF. **Case** $m = 0$. Let $U$ be the 3-dimensional $\mathbb{R}[K]$-module obtained from the restriction to $K$ of the standard 3-dimensional representation of $H \cong SO(3)$. Then the linear $K$-action on $S(U^{\oplus 2})$ has no $K$-fixed points.

Let $\Phi$ be the $K$-action on $G/H$ obtained by Proposition 5.1, and put $V = T_H(G/H)$.

**Case** $m = 2$. As $V^K = \{0\}$, the linear $K$-action on $S(\mathbb{R} \oplus V)$ has exactly two $K$-fixed points, where $\mathbb{R}$ denotes the trivial $\mathbb{R}[K]$-module of dimension 1.

**Case** $m \geq 3$. Let $m \geq 3$. We can construct a $K$-action on a $\mathbb{Z}_p$-homology 5-sphere with exactly $m$ fixed points by taking the connected sum of $(m-2)$ copies of $G/H$ (using Lemma 2.8 repeatedly), starting with the action $\Phi$ on $G/H$.

**Case** $m = 1$**.** Suppose that $p \neq 3$ and $K \in \{\mathbb{O}, \mathbb{I}\}$. Then the assertion follows from Propositions 5.6 and 5.8. □

## 6. Appendix

Throughout this appendix, let $\Sigma$ be a $\mathbb{Z}$-homology sphere with effective $G$-action and let $n$ be the dimension of $\Sigma$.

The object of this appendix is to prove Proposition 1.1. For the convenience of the reader, we restate it here.

PROPOSITION 6.1. *If $n \leq 4$, then $\chi(\Sigma^G)$ lies in $\{0, 1, 2\}$. Moreover, if $\chi(\Sigma^G) = 1$, then $G \cong A_5$, $n = 3$, and $|\Sigma^G| = 1$.*

This statement should follow readily from the approaches of DeMichelis [**7**] and Buchdahl–Kwasik–Schultz [**6**]. Since we have not found it stated explicitly in those references, we include a proof using standard methods from transformation group theory. We begin with the following case:

PROPOSITION 6.2. *Suppose that the $G$-action on $\Sigma$ is nontrivial and orientation-preserving and that $\Sigma^G \neq \emptyset$. If $2 \leq n \leq 4$, then $\Sigma^G$ is either a sphere $S^k$ for some $0 \leq k \leq 2$ or a singleton. Moreover, if $\Sigma^G$ is a singleton, then $G \cong A_5$ and $n = 3$.*

PROOF. Since the $G$-action on $\Sigma$ is effective and it has a $G$-fixed point $x \in \Sigma^G$, the $G$-action on $T_x(\Sigma)$ is effective and orientation-preserving (by Lemma 2.7), and therefore $G$ is isomorphic to a nontrivial finite subgroup of $SO(n)$. Suppose that $F(G)$ is nontrivial. Then some Sylow $p$-subgroup $P$ of $F(G)$ is nontrivial. Note that $P$ is normal in $G$, since $P$ is characteristic in $F(G)$. By Lemma 2.2, $\Sigma^P$ is a $\mathbb{Z}_p$-homology sphere with $G/P$-action. Since the restricted $P$-action on $T_x(\Sigma)$ is effective and orientation-preserving and $n \leq 4$, the codimension of $T_x(\Sigma)^P$ in $T_x(\Sigma)$ is at least 2, i.e. $\dim T_x(\Sigma)^P \leq 2$. Consequently, by Lemma 2.2, $\Sigma^P$ is a $\mathbb{Z}_p$-homology sphere of dimension $\leq 2$ (thus $\Sigma^P \cong S^k$ for some $0 \leq k \leq 2$). By Lemma 2.4, $\Sigma^G = (\Sigma^P)^{G/P}$ is a sphere of dimension $\leq 2$. Now we proved that if $2 \leq n \leq 4$ and $F(G)$ is nontrivial, then $\Sigma^G$ is a sphere $S^k$, where $0 \leq k \leq 2$.

**Case** $n = 2$**.** Since any nontrivial finite subgroup of $SO(2)$ is a nontrivial cyclic group, $F(G)$ is nontrivial, and hence $\Sigma^G$ is a sphere of dimension $\leq 2$.

For the rest of this proof, we may assume that $F(G)$ is trivial. Furthermore, we use the notation introduced in Section 2 for the chosen subgroups of $A_5$ and the irreducible $\mathbb{R}[A_5]$-modules, together with the corresponding fixed-point dimensions (Table 1).

**Case** $n = 3$**.** If $F(G)$ is trivial then $G$ is isomorphic to $A_5$, because a nontrivial finite subgroup of $SO(3)$ is isomorphic to $C_n$, $D_{2n}$, $A_4$, $S_4$, or $A_5$, $n \geq 2$. We identify $G$ with $A_5$. Then the tangential $\mathbb{R}[G]$-module $T_x(\Sigma)$ is isomorphic to $U_{3.1}$ or $U_{3.2}$. By Lemma 2.2 and by $\dim T_x(\Sigma)^{D_4} = 0$ for $D_4 \leq A_5$, we obtain $\Sigma^{A_5} \subset \Sigma^{D_4} \cong S^0$. Therefore it follows that $|\Sigma^{A_5}| = 1$ or 2.

**Case** $n = 4$**.** Since $G$ is a nontrivial subgroup of $SO(4)$ with trivial Fitting subgroup, $G$ does not contain the matrix $-I$, where $I$ is the identity in $SO(4)$. The double covering map $SO(4) \to SO(3) \times SO(3)$, whose kernel is $\langle -I \rangle$, maps $G$ isomorphically onto a finite subgroup of $SO(3) \times SO(3)$. Thus we may assume that $G$ is identified with a finite subgroup of $SO(3) \times SO(3)$ with trivial Fitting subgroup. By Proposition 2.12, $G$ is isomorphic to $A_5$ or $A_5 \times A_5$. Since $A_5 \times A_5$ has no faithful real representations of degree $\leq 4$, it follows that $G$ is isomorphic to $A_5$. Thus we identify $G$ with $A_5$. Then one of the following occurs:

(1) There exists $x \in \Sigma^{A_5}$ such that $T_x(\Sigma)$ is isomorphic to $\mathbb{R} \oplus U_{3.i}$, where $i = 1$ or 2.

(2) For every $x \in \Sigma^{A_5}$, $T_x(\Sigma)$ is isomorphic to $U_4$.

We first suppose that $T_x(\Sigma) \cong \mathbb{R} \oplus U_{3.i}$ for some $x \in \Sigma^{A_5}$. By Lemma 2.2, we have $\Sigma^{D_4} \cong S^1$ for $D_4 \leq A_5$. Since $\dim T_x(\Sigma)^{D_4} = \dim T_x(\Sigma)^{A_5} = 1$, Proposition 2.6 implies that $\Sigma^{A_5} = \Sigma^{D_4}$; hence $\Sigma^{A_5} \cong S^1$. Next suppose that $T_x(\Sigma) \cong U_4$ for each $x \in \Sigma^{A_5}$. By Lemma 2.2, we obtain $\Sigma^{C_5} \cong S^0$ for $C_5 \leq A_5$. Hence $|\Sigma^{A_5}| = 1$ or 2. Recall that the subgroups $C_2$, $D_4$, and $D_6$ of $A_5$ satisfy $\langle D_4, D_6 \rangle = A_5$ and $C_2 = D_4 \cap D_6$. Since the Fitting subgroups of $D_4$, $D_6$, and $C_2$ are nontrivial, $\Sigma^{D_4}$, $\Sigma^{D_6}$, and $\Sigma^{C_2}$ are diffeomorphic to $S^1$, $S^1$, and $S^2$, respectively (where we also used Table 1). The submanifolds $\Sigma^{D_4}$ and $\Sigma^{D_6}$ of $\Sigma^{C_2}$ intersect at one or two $A_5$-fixed points, transversely. Since the $\mathbb{Z}_2$-intersection number of $\Sigma^{D_4}$ and $\Sigma^{D_6}$ in $\Sigma^{C_2}$ is trivial, we conclude that $\Sigma^{A_5} = \Sigma^{D_4} \cap \Sigma^{D_6}$ is not a singleton, i.e. $\Sigma^{A_5} \cong S^0$. □

Proof of Proposition 6.1. If the $G$-action on $\Sigma$ is trivial or $\Sigma^G = \emptyset$, then it is immediate that $\chi(\Sigma^G) \in \{0, 2\}$. Thus we may assume the $G$-action is nontrivial and it has a $G$-fixed point. If $n = 0$, then $\Sigma \cong S^0$. Our hypothesis that $\Sigma^G \neq \emptyset$ forces the action to be trivial. If $n = 1$, then $\Sigma \cong S^1$. Any finite group action on $S^1$ is conjugate to an orthogonal action. Therefore, if $0 \leq n \leq 1$ and $\Sigma^G \neq \emptyset$, then $\Sigma^G \cong S^k$ for some $0 \leq k \leq 1$. We consider the case $2 \leq n \leq 4$. Let $G^+$ denote the subgroup of $G$ consisting of the elements $g \in G$ such that the induced diffeomorphisms of $\Sigma$ preserve its orientation. If the $G$-action on $\Sigma$ is orientation-preserving, then the assertion follows directly from Proposition 6.2. Thus we may assume that the $G$-action on $\Sigma$ is orientation-reversing, i.e. $[G : G^+] = 2$.

First suppose that $\chi(\Sigma^G) \equiv 1 \bmod 2$. By Lemma 2.1, we have $\chi(\Sigma^{G^+}) \equiv 1 \bmod 2$ (hence $\Sigma^{G^+} \neq \emptyset$). Proposition 6.2 shows that $G^+$ is isomorphic to $A_5$, $n = 3$, and

$|\Sigma^{G^+}| = 1$. As $|\mathrm{Out}(A_5)| = 2$ and $[G : G^+] = 2$, we see that $G$ is isomorphic to the symmetric group $S_5$ or the direct product $A_5 \times Z_2$, where $Z_2$ is a group of order 2. However, $S_5$ has no faithful 3-dimensional real representations (e.g., [**26**, Section 4]). Therefore $G$ is isomorphic to $A_5 \times Z_2$. Then it holds that $\dim \Sigma^{Z_2} \leq 2$. Lemmas 2.2 and 2.4 show that $\Sigma^G$ is a sphere of dimension $\leq 2$; hence $\chi(\Sigma^G) = 0$ or 2. Therefore if $\chi(\Sigma^G) \equiv 1 \bmod 2$, then the $G$-action on $\Sigma$ must be orientation-preserving.

Next suppose that $\chi(\Sigma^G) \equiv 0 \bmod 2$, and hence $\chi(\Sigma^{G^+}) \equiv 0 \bmod 2$ (by Lemma 2.1). If $G^+$ is nontrivial, then Proposition 6.2 shows that $\Sigma^{G^+}$ is a sphere of dimension $\leq 2$. By Lemma 2.4, $\Sigma^G$ is also a sphere of dimension $\leq 2$ (hence $\chi(\Sigma^G) = 0$ or 2). If $G^+$ is trivial, then $G$ is a cyclic group of order 2. By Proposition 2.3, we obtain $\chi(\Sigma^G) = 0$ or 2 (since $\Sigma$ is a $\mathbb{Q}$-homology sphere). □

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