TILTING THEORY FOR HYPERSURFACE SINGULARITIES OF DIMENSION ONE

OSAMU IYAMA AND JUNYANG LIU

Abstract. Any $\mathbb{N}$-graded commutative Gorenstein ring R of Krull dimension one with R_0 a field admits a standard silting object V in the stable category $\underline{\mathrm{CM}}_0^{\mathbb{Z}}R$, and the object V is tilting if and only if the a-invariant a is nonnegative, as shown by Buchweitz, the first author, and Yamaura. In this article, under the additional assumption that R is a hypersurface singularity, we prove that the endomorphism algebra of V is Iwanaga–Gorenstein of self-injective dimension at most 2, and we give its explicit presentation in terms of a quiver with relation. In the case where a is negative, we prove that the dg endomorphism algebra of V is Gorenstein, and we give its explicit presentation in terms of a dg path algebra. We illustrate our results with several examples including numerical semigroup algebras generated by two elements. Moreover, for each singularity of finite or countable Cohen–Macaulay representation type, we include the Auslander–Reiten quiver of the category $\mathrm{CM}_0^{\mathbb{Z}}R$ with the position of the standard silting object. As a step in the proof of our results, we give a characterization of Gorensteinness of homologically finite dg algebras in terms of Serre functors.

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1. Introduction

The study of (maximal) Cohen–Macaulay modules is one of the central subjects in commutative algebra and representation theory, cf. [3, 6, 35, 33, 28]. It has strong connections with algebraic geometry, singularity theory, and mathematical physics. Tilting theory is one of the central tools in modern representation theory, including Cohen–Macaulay representations, and a number of important results are obtained by applying tilting theory to singularity categories associated with commutative and noncommutative Gorenstein rings, cf. [20, 9, 27, 13, 30, 29, 5, 16, 15, 11, 26, 17]. For example, a milestone of Cohen–Macaulay representation theory is Buchweitz–Greuel–Schreyer's classification of commutative complete local Gorenstein rings of finite Cohen–Macaulay representation type as ADE singularities, see [35, 28]. This result gives a commutative counterpart of Gabriel's famous classification of quivers of finite representation type as Dynkin quivers, and the connection between these two classifications is given by tilting theory, see, e.g. , [10, 19, 5] and Section 6 below.

For a commutative Gorenstein ring R, the category $\mathrm{CM}\, R$ of Cohen–Macaulay modules is a Frobenius category. Its stable category is triangle equivalent to the singularity category associated with R introduced by Buchweitz [4] and Orlov [31]. For a $\mathbb{Z}$-graded commutative Gorenstein ring R, we consider the $\mathbb{Z}$-graded version $\mathrm{CM}^{\mathbb{Z}}R$ and its subcategory $\mathrm{CM}^{\mathbb{Z}}_0R$, both of which are Frobenius categories. The latter enjoys Auslander–Reiten–Serre duality [3]. Buchweitz–Iyama–Yamaura proved [5] that, when R is $\mathbb{N}$-graded and of Krull dimension one such that R_0 is a field, the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}_0R$ admits a silting object

$$\bigoplus_{i=1}^{N} R(i)_{\geq 0}$$

for $N \gg 0$, whose basic part V is called the *standard silting object*. Moreover, it is tilting if and only if either R is regular or the a-invariant of R is nonnegative. Since we have a triangle equivalence

$$\underline{\mathrm{CM}}^{\mathbb{Z}}_0R \xrightarrow{\sim} \mathrm{per}\, \Gamma$$

for Γ the dg(=differential graded) endomorphism algebra of V, it is important to study Γ. The aim of this article is to study Γ for hypersurface singularities $R = k[x,y]/(f)$. The case of the standard grading $\deg x = \deg y = 1$ over algebraically closed fields k was discussed in [5]. In this article, we study the case of arbitrary gradings and arbitrary fields k as follows.

Assumption 1.1. *Let k be a field and $k[x,y]$ a $\mathbb{Z}$-graded polynomial algebra such that x and y have positive degrees m and n, respectively. Let f be a homogeneous element of $k[x,y]$ with $\deg f > 0$ and set $R = k[x,y]/(f)$. Let $a = \deg f - m - n$ be the a-invariant of R.*

In this case, we prove some homological properties of the endomorphism algebra of V and describe it in terms of a quiver. We divide our results into two cases: Theorem A (nonnegative a-invariant) and Theorem C (negative a-invariant).

Theorem A (see Theorem 4.4 for details)**.** *Under Assumption 1.1, suppose that we have $a \geq 0$. Denote the endomorphism algebra of the standard tilting object V in $\underline{\mathrm{CM}}^{\mathbb{Z}}_0R$ by Γ.*

a) The algebra Γ is finite-dimensional and Iwanaga–Gorenstein of self-injective dimension less than or equal to 2. Moreover, its global dimension is finite if and only if it is less than or equal to 2 if and only if R is reduced.

b) The algebra Γ admits a presentation by a quiver with relations given in Definition 4.3.

The generality of Theorem A for arbitrary grading is important because it includes many notable examples, e.g. , the ADE singularities, the T_{pq} singularities and the numerical semigroup algebras generated by two elements as follows.

Corollary B (=Corollary 4.6)**.** *Let $S \subset \mathbb{N}$ be a cofinite proper submonoid which is generated by two elements. Let $k[S] \subset k[x]$ be the numerical semigroup algebra. Then the global dimension of the endomorphism algebra Γ_S of the standard tilting object V_S in $\underline{\mathrm{CM}}^{\mathbb{Z}} k[S]$ is less than or equal to 2.*

When the a-invariant is negative, the dg endomorphism algebra of V is described in Theorem C.

Theorem C (see Theorem 4.14 for details)**.** *Under Assumption 1.1, suppose that we have $a < 0$ and that R is not regular. Denote the dg endomorphism algebra of the standard silting object V in $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ by Γ.*

a) The dg algebra Γ is proper and Gorenstein (see Section 3 for the definitions).
b) The dg algebra Γ admits a presentation by a dg path algebra given in Definition 4.13.

The article is organized as follows: In Section 2, we recall $\mathbb{Z}$-graded commutative Gorenstein rings and the category $\mathrm{CM}_0^{\mathbb{Z}} R$.

In Section 3, we recall homologically finite and Gorenstein dg algebras and prove the following results.

Proposition D (=Proposition 3.1 and Corollary 3.2)**.** *Let A be a homologically finite dg algebra (see Section 3 for the definition).*

a) We have a canonical isomorphism

$$D\mathrm{RHom}_A(X, Y) \xrightarrow{\sim} \mathrm{RHom}_A(Y, X \overset{\mathrm{L}}{\otimes}_A DA)$$

bifunctorial in $X \in \mathrm{per}\, A$ and $Y \in \mathcal{D}(A)$.
b) The dg algebra A is Gorenstein if and only if the triangulated category $\mathrm{per}\, A$ has a Serre functor.

In Section 4, we consider hypersurface singularities with arbitrary positive grading. In Section 4.1, we introduce a quiver with relations and state the main results in the case where the a-invariant of R is nonnegative. In Section 4.2, we study the case of numerical semigroup algebras and state the results. We then give examples involving nonstandard gradings, nonreduced rings, and fields that are not algebraically closed to illustrate the construction of the quivers with relations, cf. Section 4.3. In Section 4.4, we introduce a dg path algebra and state the main results in the case where the a-invariant of R is negative. Then we give an example to illustrate the construction of the dg path algebras, cf. Section 4.5.

In Section 5, we prove our results. In Section 5.1, we discuss the structures of the morphism spaces, radical morphism spaces, and the square of radical morphism spaces between direct summands of V. In Section 5.2, we use them to prove Theorem A. In Section 5.3, we prove Theorem C.

In Section 6, we draw the Auslander–Reiten quivers of the categories $\mathrm{CM}_0^{\mathbb{Z}} R$ for finite and countable Cohen–Macaulay representation types in the $\mathbb{Z}$-graded case. Our Auslander–Reiten quivers of $\mathrm{CM}_0^{\mathbb{Z}} R$ are displayed in a standard way following the theory of translation quivers due to Gabriel and Riedtmann, e.g. , [32, 2]. Then we illustrate the explicit positions of the standard silting objects V in the Auslander–Reiten quivers.

In Appendix A, we give cofibrant dg replacements of self-injective Nakayama algebras. In Section A.1, we recall A_∞-algebras. In Section A.2, we prove our result.

Notation 1.2. *The following notation is used throughout the article: We let k be a field. For a k-vector space V, we denote its k-dual space* $\mathrm{Hom}_k(V,k)$ *by DV. Algebras have units and morphisms of algebras preserve the units. Modules are unital right modules. For a ring R, we denote the category of finitely generated R-modules by* $\operatorname{mod} R$. *The degree of a homogeneous element a in a $\mathbb{Z}$-graded vector space is denoted by* $\deg a$. *We denote the shift functor of $\mathbb{Z}$-graded vector spaces by* (1) *so that we have $V(1)_n = V_{n+1}$. We denote the truncation of a $\mathbb{Z}$-graded vector space V in nonnegative degrees by $V_{\geq 0}$.*

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2. Preliminaries

A commutative noetherian ring of finite Krull dimension is *Gorenstein* if it has finite injective dimension as a module over itself. For a Gorenstein ring R, a finitely generated R-module M is *(maximal) Cohen–Macaulay* if $\mathrm{Ext}^i_R(M,R)$ vanishes for all positive integers i. For a $\mathbb{Z}$-graded Gorenstein ring R, we write $\mathrm{CM}^{\mathbb{Z}}R$ for the category whose objects are $\mathbb{Z}$-graded R-modules which are Cohen–Macaulay and whose morphisms are R-homomorphisms which are homogeneous of degree 0. The category $\mathrm{CM}^{\mathbb{Z}}_0 R$ is defined to be the full subcategory of $\mathrm{CM}^{\mathbb{Z}}R$ whose objects are the $\mathbb{Z}$-graded R-modules M such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is projective for every nonmaximal prime ideal $\mathfrak{p}$ of R. Both categories $\mathrm{CM}^{\mathbb{Z}}R$ and $\mathrm{CM}^{\mathbb{Z}}_0 R$ are Frobenius. They coincide if and only if R has at worst isolated singularities. If R is an $\mathbb{N}$-graded Gorenstein ring of Krull dimension one with $R_0 = k$ a field, then there exists a unique integer a such that the $\mathbb{Z}$-graded R-module $\mathrm{Ext}^1_R(k, R(a))$ is isomorphic to k. We call a the *a-invariant* of R.

3. Homologically finite dg algebras and Serre functors

Let A be a dg algebra. We write $\mathcal{D}(A)$ for the (unbounded) derived category of A. Its thick subcategory generated by the free dg A-module of rank one is the *perfect derived category* $\operatorname{per} A$. It consists of compact objects in $\mathcal{D}(A)$. Recall that A is *proper* if its underlying complex is perfect over k. Equivalently, its homology $H^p(A)$ is finite-dimensional for all integers p and vanishes for all $|p| \gg 0$. The dg algebra A is *homologically finite* if its homology $H^p(A)$ is finite-dimensional for all integers p. Following Section 0 of [18], a dg algebra A is *Gorenstein* if A and DA generate the same thick subcategory of $\mathcal{D}(A)$. The following proposition is stated in Section I.4.6 of [12] and Section 10.4 of [23], and the Serre duality in Lemma 4.1 of [25] is its dual version. We include our proof for the convenience of the reader.

Proposition 3.1. *Let A be a homologically finite dg algebra. Then we have a canonical isomorphism*

$$D\mathrm{RHom}_A(X,Y) \xrightarrow{\sim} \mathrm{RHom}_A(Y, X \overset{\mathrm{L}}{\otimes}_A DA)$$

bifunctorial in $X \in \operatorname{per} A$ and $Y \in \mathcal{D}(A)$.

Proof. Since the dg A-module X is perfect, we have the isomorphism

$$Y \overset{\mathrm{L}}{\otimes}_A \mathrm{RHom}_A(X,A) \xrightarrow{\sim} \mathrm{RHom}_A(X,Y)\,.$$

It follows that we have the composite isomorphism
(3.1)

$$D\mathrm{RHom}_A(X,Y) \xrightarrow{\sim} D(Y \overset{\mathrm{L}}{\otimes}_A \mathrm{RHom}_A(X,A)) \xrightarrow{\sim} \mathrm{RHom}_A(Y, D\mathrm{RHom}_A(X,A)) \,.$$

Since the dg algebra A is homologically finite, we have the isomorphism $A \xrightarrow{\sim} DDA$ in $\mathcal{D}(A)$. It follows that we have the composite isomorphism

$$D(X \overset{\mathrm{L}}{\otimes}_A DA) \xrightarrow{\sim} \mathrm{RHom}_A(X, DDA) \xleftarrow{\sim} \mathrm{RHom}_A(X,A) \,.$$

If we apply the functor D to it, we obtain the isomorphism

$$X \overset{\mathrm{L}}{\otimes}_A DA \xrightarrow{\sim} D\mathrm{RHom}_A(X,A) \,. \tag{3.2}$$

Combining the isomorphisms (3.1) and (3.2), we obtain the desired isomorphism. Its bifunctoriality follows from that of the above isomorphisms. √

Corollary 3.2. *Let A be a homologically finite dg algebra. Then it is Gorenstein if and only if the triangulated category* per A *has a Serre functor.*

Proof. For the necessity, by Proposition 3.1, the functor $? \overset{\mathrm{L}}{\otimes}_A DA$ is a Serre functor of the triangulated category per A. We now prove the sufficiency. By Proposition 3.1, we have the natural isomorphism

$$\mathrm{Hom}_{\mathrm{per}\,A}(A, ?) \xrightarrow{\sim} D\mathrm{Hom}_{\mathrm{per}\,A}(?, DA) \,.$$

On the other hand, let $\mathbb{S}$ be a Serre functor on per A. Then we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{per}\,A}(A, ?) \xrightarrow{\sim} D\mathrm{Hom}_{\mathrm{per}\,A}(?, \mathbb{S}A) \,.$$

By the Yoneda Lemma, we deduce that $DA \xrightarrow{\sim} \mathbb{S}A$ lies in per A and hence the thick subcategory of $\mathcal{D}(A)$ generated by DA is contained in per A. Since the dg algebra A is homologically finite, by applying duality we deduce that per A is also contained in the thick subcategory of $\mathcal{D}(A)$ generated by DA. √

4. Our results for hypersurface singularities

We use the following setting throughout this section. Let k be a field and $k[x,y]$ a $\mathbb{Z}$-graded polynomial algebra such that x and y are of positive degrees m and n, respectively. Without loss of generality, we may and will assume that m and n are coprime and that we have $m \leq n$. Let f be a homogeneous element of $k[x,y]$ of positive degree. Without loss of generality, we may and will assume that it is monic. Then we decompose f as

$$f = x^{n_x} y^{n_y} \prod_{j=1}^{q} f_j^{n_j} \,,$$

where n_x, n_y and n_j are nonnegative integers and f_j are distinct monic irreducible polynomials which are distinct from x and y. The $\mathbb{N}$-graded commutative ring

$$R = k[x,y]/(f)$$

is Gorenstein with a-invariant $a = \deg f - m - n$. Denote the stable category $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ by $\mathcal{T}$.

Proposition 4.1. *Let X be an object in $\mathcal{T}$ such that the thick subcategory generated by it coincides with $\mathcal{T}$. Then the dg algebra* $\mathrm{RHom}_{\mathcal{T}}(X,X)$ *is a proper Gorenstein dg algebra.*

Proof. The Gorensteinness of the dg algebra $A = \mathrm{RHom}_{\mathcal{T}}(X, X)$ follows from Proposition 4.6 of [5] and the sufficiency in Corollary 3.2. We now prove that the dg algebra A is proper. Since each homogeneous component of R is finite-dimensional, the category of finitely generated graded R-modules is Hom-finite and hence so is $\mathcal{T}$. It follows that A is homologically finite. So it suffices to show its homology is concentrated in bounded degrees. Denote the shift functor of $\mathcal{T}$ by Σ and the object $X \oplus \Sigma X$ by Y. Since the graded R-module Y is finitely generated, the morphism space $\mathrm{Hom}_R(Y, Y(p))_0$ vanishes for all $p \ll 0$ and hence so does $\mathrm{Hom}_{\mathcal{T}}(Y, Y(p))$. By Proposition 4.6 of [5], the morphism space $\mathrm{Hom}_{\mathcal{T}}(Y, Y(p))$ also vanishes for all $p \gg 0$. Since R is a hypersurface, by the graded version of the sufficiency in part (ii) of Theorem 6.1 of [8], we have $\Sigma^2 \simeq (\deg f)$. Therefore, the morphism space $\mathrm{Hom}_{\mathcal{T}}(Y, \Sigma^{2p} Y)$ vanishes for all $|p| \gg 0$. Then the statement follows. $\surd$

4.1. **The structure theorem for nonnegative a-invariants.** Suppose that a is nonnegative. For any positive integer i, we define graded R-modules

$$V^i = R(i)_{\geq 0}\,,\ V^{[1,a]} = \bigoplus_{i=1}^{a} V^i\,.$$

Let K be the graded total quotient algebra of R. It decomposes as $K = K^x \times K^y \times \prod_{j=1}^{q} K^j$, where we set $K^x = k[x, y^{\pm 1}]/(x^{n_x})$, $K^y = k[x^{\pm 1}, y]/(y^{n_y})$, and $K^j = k[x^{\pm 1}, y^{\pm 1}]/(f_j^{n_j})$. Denote by 1_x, 1_y and 1_j the corresponding central idempotents of K. If we have $i \geq a+1$, then V^i is a graded K-module and we define graded R-modules

$$V^{i,x} = 1_x \cdot V^i \xrightarrow{\sim} K^x(i)_{\geq 0}\,,\ V^x = \bigoplus_{i=a+1}^{a+n} V^{i,x}\,,$$

$$V^{i,y} = 1_y \cdot V^i \xrightarrow{\sim} K^y(i)_{\geq 0}\,,\ V^y = \bigoplus_{i=a+1}^{a+m} V^{i,y}\,,$$

$$V^{i,f_j} = 1_j \cdot V^i \xrightarrow{\sim} K^j(i)_{\geq 0}\,,\ V^{f_j} = V^{a+1,f_j} \text{ for } 1 \leq j \leq q\,.$$

Set $V = V^{[1,a]} \oplus V^x \oplus V^y \oplus \bigoplus_{j=1}^{q} V^{f_j}$. By parts (a) and (b) of Lemma 4.11 of [5], we have $\mathrm{add}\, V = \mathrm{add} \bigoplus_{i=1}^{N} R(i)_{\geq 0}$ for all integers $N \geq a+n$. From Theorem 1.4 of [5], we deduce that V is a tilting object in the stable category $\mathcal{T} = \underline{\mathrm{CM}}_0^{\mathbb{Z}} R$. We call it the *standard tilting object*.

Lemma 4.2. *We have an isomorphism*

$$\mathrm{End}_{\mathcal{T}}(V) \xleftarrow{\sim} \mathrm{End}_R(V)_0 \simeq \begin{bmatrix} {}_R(V^{[1,a]})_0 & 0 & 0 & 0 & \cdots & 0 \\ {}_R(V^{[1,a]}, V^x)_0 & {}_R(V^x)_0 & 0 & 0 & \cdots & 0 \\ {}_R(V^{[1,a]}, V^y)_0 & 0 & {}_R(V^y)_0 & 0 & \cdots & 0 \\ {}_R(V^{[1,a]}, V^{f_1})_0 & 0 & 0 & {}_R(V^{f_1})_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_R(V^{[1,a]}, V^{f_q})_0 & 0 & 0 & 0 & \cdots & {}_R(V^{f_q})_0 \end{bmatrix}$$

of algebras, where we write ${}_R(?,-)_0$ *(respectively,* ${}_R(?)_0$*) for* $\mathrm{Hom}_R(?,-)_0$ *(respectively,* $\mathrm{End}_R(?)_0$*). The entries are described via the bijections*

$$\begin{aligned}
\mathrm{Hom}_R(V^i, V^{i'})_0 &\xrightarrow{\sim} R_{i'-i} \quad \text{for all } 1 \le i, i' \le a\,,\\
\mathrm{Hom}_R(V^{i,x}, V^{i',x})_0 &\xrightarrow{\sim} K^x_{i'-i} \quad \text{for all } a+1 \le i, i' \le a+n\,,\\
\mathrm{Hom}_R(V^{i,y}, V^{i',y})_0 &\xrightarrow{\sim} K^y_{i'-i} \quad \text{for all } a+1 \le i, i' \le a+m\,,\\
\mathrm{End}_R(V^{a+1,f_j})_0 &\xrightarrow{\sim} K^j_0 \quad \text{for all } 1 \le j \le q\,,\\
\mathrm{Hom}_R(V^i, V^{i',x})_0 &\xrightarrow{\sim} K^x_{i'-i} \quad \text{for all } 1 \le i \le a\,,\ a+1 \le i' \le a+n\,,\\
\mathrm{Hom}_R(V^i, V^{i',y})_0 &\xrightarrow{\sim} K^y_{i'-i} \quad \text{for all } 1 \le i \le a\,,\ a+1 \le i' \le a+m\,,\\
\mathrm{Hom}_R(V^i, V^{a+1,f_j})_0 &\xrightarrow{\sim} K^j_{a+1-i} \quad \text{for all } 1 \le i \le a\,,\ 1 \le j \le q\,.
\end{aligned}$$

Proof. The statements follow from Lemma 5.2 and the proof of Lemma 5.3. √

To describe the endomorphism algebra $\mathrm{End}_{\mathcal{T}}(V)$, we introduce the following quiver with relations $(Q, I_{r,s})$.

Definition 4.3. The set Q_0 of the vertices of Q is the union

$$\begin{aligned}
&\{i \mid 1 \le i \le a\}\\
\cup&\{(i,x) \mid 0 \le i < n, n_x \ne 0\} \cup \{(i,y) \mid 0 \le i < m, n_y \ne 0\} \cup \{(0,j) \mid 1 \le j \le q\}\,.
\end{aligned}$$

Note that the power of x in each term of f_j must be a multiple of n. Put $g_j(t_j) = f_j(t_j^{\frac{1}{n}}, 1)$.

The set Q_1 of the arrows of Q is the union

$$\begin{aligned}
&\{x = x_i \colon i \to i+m \mid 1 \le i \le a-m\}\\
\cup&\{y = y_i \colon i \to i+n \mid 1 \le i \le a-n\}\\
\cup&\{a_x = a_{i,x} \colon a-n+1+i \to (i,x) \mid \max\{n-a,0\} \le i < n\}\\
\cup&\{a_y = a_{i,y} \colon a-m+1+i \to (i,y) \mid \max\{m-a,0\} \le i < m\}\\
\cup&\{a_j = a_{i,j} \colon a-m+1+i \to (0,j) \mid \max\{m-a,0\} \le i < m\,,\ 1 \le j \le q\}\\
\cup&\{b_x = b_{i,x} \colon (i,x) \to (i+m,x) \mid 0 \le i < n\}\\
\cup&\{b_y = b_{i,y} \colon (i,y) \to (i+n,y) \mid 0 \le i < m\}\\
\cup&\{b_j = b_{0,j} \colon (0,j) \to (0,j) \mid 1 \le j \le q\}\,,
\end{aligned}$$

where (i,x) and (i,y) denote $(i - \lfloor \frac{i}{n} \rfloor n, x)$ and $(i - \lfloor \frac{i}{m} \rfloor m, y)$, respectively, for all integers i. We define I to be the ideal of kQ generated by

$$b_x^{n_x}\,,\ b_y^{n_y}\,,\ g_j(b_j)^{n_j}\,,\ xy - yx\,,\ b_{i,x}a_{i,x}y^{\lfloor \frac{i+m}{n} \rfloor} - a_{i+m,x}x\,,\ b_{i,y}a_{i,y}x^{\lfloor \frac{i+n}{m} \rfloor} - a_{i+n,y}y\,.$$

To define other ideals, let us introduce the following notation. We choose integers r and s satisfying $rm + sn = 1$. For any $1 \le j \le q$, set $N^j = k[b_{0,j}]/(g_j(b_{0,j})^{n_j})$. Let F^j be the free $k[x,y] \otimes_k N^j$-module with the basis $(a_{i,j})_{\max\{m-a,0\} \le i < m}$, where N^j acts on the left and $k[x,y]$ acts on the right. Let F^j_{xy} be its localization with respect to the multiplicative set $\{(xy)^i \mid i \in \mathbb{N}\} \subset k[x,y]$. Let $(F^j_{xy})_{r,s}$ be the $k[x^{\pm 1}, y^{\pm 1}] \otimes_k N^j$-submodule of F^j_{xy} generated by

$$b_{0,j}a_{m-1,j} - a_{m-1,j}\frac{x^n}{y^m} \quad \text{and} \quad a_{i,j} - a_{i+1,j}x^r y^s\,.$$

Denote the intersection of F^j and $(F^j_{xy})_{r,s}$ by $F^j_{r,s}$. Let $\widetilde{Q}$ be the infinite quiver obtained from Q by adding vertices i for $i \le 0$ and arrows $x = x_i \colon i \to i+m$, $y = y_i \colon i \to i+n$ for

$i \leq 0$. The vector space

$$\bigoplus_{i \leq a} e_{(0,j)}(k\widetilde{Q}/(xy - yx, g_j(b_{0,j})^{n_j}))e_i$$

admits a natural $k[x,y] \otimes_k N^j$-module structure such that there is a canonical isomorphism $\psi^j_{r,s}$ from F^j to it which maps $a_{i,j}$ to $a_{i,j}$. Set $\overline{I}^j_{r,s} = \psi^j_{r,s}(F^j_{r,s}) \sum_{i=1}^a e_i$. We define $I^j_{r,s}$ to be the preimage of $\overline{I}^j_{r,s}$ under the canonical surjection

$$\textstyle\bigoplus_{i=1}^a e_{(0,j)}(kQ)e_i \longrightarrow \bigoplus_{i=1}^a e_{(0,j)}(kQ/(xy - yx, g_j(b_{0,j})^{n_j}))e_i \,.$$

We define $I_{r,s}$ to be the ideal of kQ generated by I and $I^j_{r,s}$ for $1 \leq j \leq q$.

Recall that a (not necessarily commutative) noetherian ring is *Iwanaga–Gorenstein* if it has finite injective dimension as both a left and a right module over itself.

Theorem 4.4. *Let R and $(Q, I_{r,s})$ be as above. Denote the endomorphism algebra of the standard tilting object V in $\underline{\mathrm{CM}}^{\mathbb{Z}}_0 R$ by Γ.*

a) The algebra Γ is finite-dimensional and Iwanaga–Gorenstein of self-injective dimension less than or equal to 2. Moreover, its global dimension is finite if and only if it is less than or equal to 2 if and only if R is reduced.

b) The algebra Γ is isomorphic to $kQ/I_{r,s}$. In particular, the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}_0 R$ is triangle equivalent to $\mathrm{per}\,(kQ/I_{r,s})$.

The isomorphism in part b) of Theorem 4.4 is induced by the surjective morphism $\varphi_{r,s} \colon kQ \to \Gamma$ of algebras given as follows. The images of the idempotents corresponding to the vertices i, (i,x), (i,y), and $(0,j)$ under $\varphi_{r,s}$ are the idempotents of Γ corresponding to the indecomposable direct summands V^i, $V^{a+1+i,x}$, $V^{a+1+i,y}$, and V^{a+1,f_j}, respectively. The images of the elements x_i, y_i, $a_{i,x}$, $a_{i,y}$, $a_{i,j}$, $b_{i,x}$, $b_{i,y}$, and $b_{0,j}$ under $\varphi_{r,s}$ are given as follows. The elements $\varphi_{r,s}(x_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+m)_{\geq 0}$$

given by multiplication by x. The elements $\varphi_{r,s}(y_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+n)_{\geq 0}$$

given by multiplication by y. The elements $\varphi_{r,s}(a_{i,x})$ are the morphisms

$$R(a+1+i-n)_{\geq 0} \longrightarrow K^x(a+1+i)_{\geq 0}$$

given by multiplication by y. The elements $\varphi_{r,s}(a_{i,y})$ are the morphisms

$$R(a+1+i-m)_{\geq 0} \longrightarrow K^y(a+1+i)_{\geq 0}$$

given by multiplication by x. The elements $\varphi_{r,s}(a_{i,j})$ are the morphisms

$$R(a+1+i-m)_{\geq 0} \longrightarrow K^j(a+1)_{\geq 0}$$

given by multiplication by $(x^r y^s)^{m-i}$. The elements $\varphi_{r,s}(b_{i,x})$ are the morphisms

$$K^x(a+1+i)_{\geq 0} \longrightarrow K^x(a+1+i+m-\lfloor \tfrac{i+m}{n} \rfloor n)_{\geq 0}$$

given by multiplication by $\frac{x}{y^{\lfloor \frac{i+m}{n} \rfloor}}$. The elements $\varphi_{r,s}(b_{i,y})$ are the morphisms

$$K^y(a+1+i)_{\geq 0} \longrightarrow K^y(a+1+i+n-\lfloor \tfrac{i+n}{m} \rfloor m)_{\geq 0}$$

given by multiplication by $\frac{y}{x^{\lfloor \frac{i+n}{m} \rfloor}}$. The elements $\varphi_{r,s}(b_{0,j})$ are the morphisms

$$K^j(a+1)_{\geq 0} \longrightarrow K^j(a+1)_{\geq 0}$$

given by multiplication by $\frac{x^n}{y^m}$. Then the kernel of $\varphi_{r,s}$ coincides with $I_{r,s}$.

Remark 4.5. *The quiver Q is uniquely determined by R up to isomorphism. But the ideal $I_{r,s}$ and the morphism $\varphi_{r,s}$ depend on choices of the pair (r,s) in general.*

4.2. **Numerical semigroup algebras.** For a cofinite submonoid $S \subseteq \mathbb{N}$, the stable category $\mathcal{T} = \underline{\mathrm{CM}}^{\mathbb{Z}} k[S]$ contains a standard tilting object $V_S = \bigoplus_{i=1}^{a+1} k[S](i)_{\geq 0}$ given in Theorem 1.2 of [5]. The following result is an application of Theorem 4.4.

Corollary 4.6. *Let $S \subset \mathbb{N}$ be a cofinite proper submonoid which is generated by two elements. Let $k[S] \subset k[x]$ be the numerical semigroup algebra. Then the global dimension of the endomorphism algebra Γ_S of the standard tilting object V_S in $\underline{\mathrm{CM}}^{\mathbb{Z}} k[S]$ is less than or equal to 2.*

Proof. Assume that S is generated by m and n. Since m and n are greater than 1, the a-invariant $a = mn - m - n$ is nonnegative. The standard tilting object V_S is isomorphic to the object V defined in Section 4.1. Then the statement follows by applying part a) of Theorem 4.4 to $f = x^n - y^m$. √

We give a family of examples to illustrate quivers with relations $(Q, I_{r,s})$ in this case. For simplicity, if we have $n_x = 1$, $n_y = 1$, or $n_j \deg g_j = 1$, then we omit loops b_x, b_y, or b_j, respectively.

Example 4.7. Let k be an arbitrary field. For $R = k[x,y]/(x^{m+1} - y^m)$ with $\deg x = m \geq 2$ and $\deg y = m+1$, we have $a = m^2 - m - 1$. The quiver Q is shown in Figure 1.

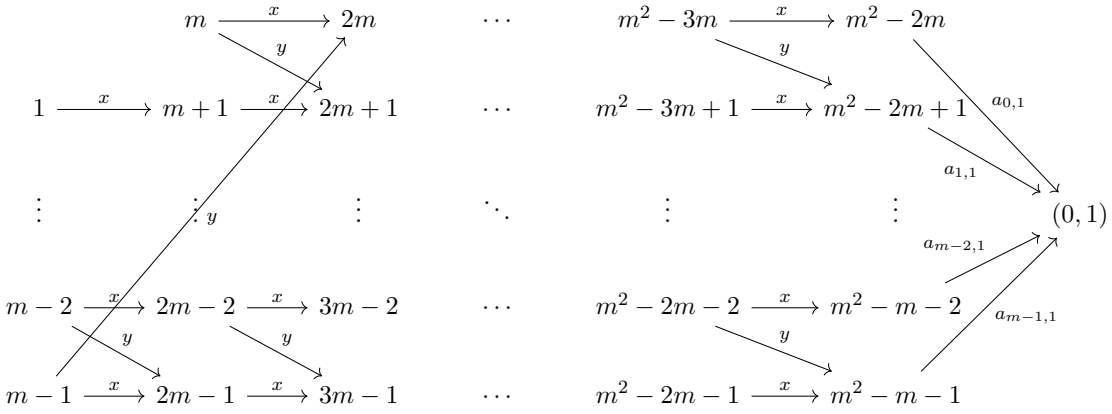


FIGURE 1. The quiver Q for $R = k[x,y]/(x^{m+1} - y^m)$ with $\deg x = m$ and $\deg y = m+1$.

The ideal I is generated by $xy - yx$. For a choice of the pair (r,s), the module $(F^1_{xy})_{r,s}$ over $k[x^{\pm 1}, y^{\pm 1}] \otimes_k k$ is generated by $a_{m-1,1}\frac{x^{m+1}-y^m}{y^m}$ and $a_{i,1} - a_{i+1,1}x^r y^s$. The ideal $\overline{I}^1_{r,s}$ is generated by $a_{i,1}x - a_{i+1,1}y$. Therefore, for any choice of (r,s), the ideal $I_{r,s}$ is generated by $xy - yx$ and $a_{i,1}x - a_{i+1,1}y$.

The surjective morphism $\varphi_{r,s} \colon kQ \to \Gamma_S$ of algebras is given as follows. The images of the idempotents corresponding to the vertices i (respectively, $(0,1)$) under $\varphi_{r,s}$ are the

idempotents of Γ_S corresponding to the indecomposable direct summands V^i (respectively, V^{m^2-m,f_1}). The images of the elements x_i, y_i, and $a_{i,1}$ under $\varphi_{r,s}$ are given as follows. The elements $\varphi_{r,s}(x_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+m)_{\geq 0}$$

given by multiplication by x. The elements $\varphi_{r,s}(y_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+m+1)_{\geq 0}$$

given by multiplication by y. The elements $\varphi_{r,s}(a_{i,1})$ are the morphisms

$$R(m^2-2m+i)_{\geq 0} \longrightarrow (k[x^{\pm 1}, y^{\pm 1}]/(x^{m+1}-y^m))(m^2-m)_{\geq 0}$$

given by multiplication by $(x^r y^s)^{m-i}$. Then the kernel of $\varphi_{r,s}$ coincides with $I_{r,s}$.

It is natural to ask whether, if a cofinite submonoid $S \subset \mathbb{N}$ is not generated by two elements, the global dimension of the endomorphism algebra Γ_S of the standard tilting object V_S is also less than or equal to 2. The following counterexample in the case where S is generated by three elements shows that this is not the case.

Example 4.8. Let k be an arbitrary field and $S \subset \mathbb{N}$ the submonoid generated by 15, 21, and 35. Let $k[S] \subset k[x]$ be the numerical semigroup algebra. We have $a = 139$. Denote the Gabriel quiver of Γ_S by Q. The set Q_0 of its vertices is $\{i \mid 1 \leq i \leq 140\}$. The set Q_1 of its arrows is the union

$$\begin{aligned} &\{x = x_i \colon i \to i+15 \mid 1 \leq i \leq 124\} \\ \cup &\{y = y_i \colon i \to i+21 \mid 1 \leq i \leq 118\} \\ \cup &\{z = z_i \colon i \to i+35 \mid 1 \leq i \leq 104\} \\ \cup &\{a = a_i \colon i+125 \to 140 \mid 0 \leq i < 15\}\,. \end{aligned}$$

The arrows of Q make Q_0 into a poset, and the algebra Γ_S is isomorphic to the incidence algebra of Q_0. We have the minimal projective resolution

$$0 \longrightarrow P_{140}^{\oplus 2} \xrightarrow{\begin{bmatrix} -ax^4 & ax^4 \\ 0 & -2ax \\ ax & ax \\ a & 0 \\ 0 & a \end{bmatrix}} P_{72} \oplus P_{121}^{\oplus 2} \oplus P_{127}^{\oplus 2} \xrightarrow{\begin{bmatrix} z & y^4 & 0 & x^6 & x^6 \\ y & z^2 & z^2 & 0 & 0 \\ x & 0 & 0 & z^2 & -z^2 \\ 0 & 0 & x & -y & -y \\ 0 & x & x & -y & y \end{bmatrix}} P_{37} \oplus P_{51} \oplus P_{57} \oplus P_{106}^{\oplus 2}$$

$$\xrightarrow{\begin{bmatrix} y & -z & 0 & x^6 & 0 \\ -x & 0 & z & -y^4 & y^4 \\ 0 & x & -y & 0 & -z^2 \end{bmatrix}} P_{16} \oplus P_{22} \oplus P_{36} \xrightarrow{[x\ y\ z]} P_1 \longrightarrow S_1 \longrightarrow 0$$

of the simple Γ_S^{op}-module S_1. The global dimension of Γ_S is 4.

More generally, fix an integer $n > 2$ and consider cofinite submonoids $S \subset \mathbb{N}$ whose minimal number of generators is n, and the endomorphism algebra Γ_S of the standard tilting object V_S. We believe that the global dimension of Γ_S can be arbitrarily large as S varies.

4.3. **Examples of Theorem 4.4.** The following examples illustrate the construction of the quivers with relations $(Q, I_{r,s})$ in settings involving nonstandard gradings, nonreduced rings R, and fields k that are not algebraically closed.

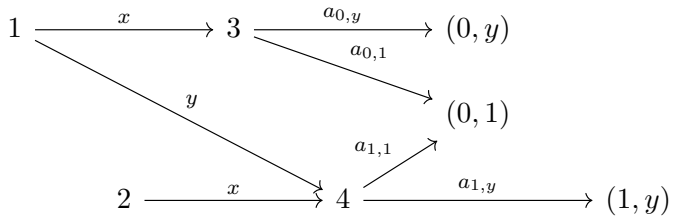


FIGURE 2. The quiver Q for the type E_7 example with $\deg x = 2$ and $\deg y = 3$.

Example 4.9. The classification of ADE singularities $R = k[x,y]/(f)$ is given in Section 6. It shows that almost all ADE singularities have nonstandard gradings. In the case of type E_7, we have $f = y(x^3 - y^2)$ with $\deg x = 2$ and $\deg y = 3$. So we have $a = 4$. The quiver Q is shown in Figure 2.

The ideal I is generated by $a_{1,y}y$. For a choice of the pair (r,s), the module $(F^1_{xy})_{r,s}$ over $k[x^{\pm 1}, y^{\pm 1}] \otimes_k k$ is generated by $a_{1,1}\frac{x^3-y^2}{y^2}$ and $a_{0,1} - a_{1,1}x^r y^s$. The ideal $\overline{I}^1_{r,s}$ is generated by $a_{0,1}x - a_{1,1}y$. Therefore, for any choice of (r,s), the ideal $I_{r,s}$ is generated by $a_{1,y}y$ and $a_{0,1}x - a_{1,1}y$.

The surjective morphism $\varphi_{r,s}\colon kQ \to \Gamma$ of algebras is given as follows. The images of the idempotents corresponding to the vertices i, (i,y), and $(0,1)$ under $\varphi_{r,s}$ are the idempotents of Γ corresponding to the indecomposable direct summands V^i, $V^{i+5,y}$, and V^{5,f_1}, respectively. The images of the elements x_i, y_1, $a_{i,y}$, and $a_{i,1}$ under $\varphi_{r,s}$ are given as follows. The elements $\varphi_{r,s}(x_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+2)_{\geq 0}$$

given by multiplication by x. The element $\varphi_{r,s}(y_1)$ is the morphism

$$R(1)_{\geq 0} \longrightarrow R(4)_{\geq 0}$$

given by multiplication by y. The elements $\varphi_{r,s}(a_{i,y})$ are the morphisms

$$R(i+3)_{\geq 0} \longrightarrow k[x^{\pm 1}](i+5)_{\geq 0}$$

given by multiplication by x. The elements $\varphi_{r,s}(a_{i,1})$ are the morphisms

$$R(i+3)_{\geq 0} \longrightarrow (k[x^{\pm 1}, y^{\pm 1}]/(x^3 - y^2))(5)_{\geq 0}$$

given by multiplication by $(x^r y^s)^{2-i}$. Then the kernel of $\varphi_{r,s}$ coincides with $I_{r,s}$.

For other ADE singularities, the corresponding quivers with relations can be illustrated similarly.

Example 4.10. We give a family of examples in the case where R is not reduced. Let k be an arbitrary field and $n_y \geq 2$ an integer. For $R = k[x,y]/(y^{n_y})$ with $\deg x = 2$ and $\deg y = 3$, we have $a = 3n_y - 5$. If n_y is even, then the quiver Q is shown in Figure 3.

If n_y is odd, then the quiver Q is shown in Figure 4.

In both cases, the ideal I is generated by $b_y^{n_y}$, $xy - yx$, $b_{0,y}a_{0,y}x - a_{1,y}y$, and $b_{1,y}a_{1,y}x^2 - a_{0,y}y$. Since we have $q = 0$ in this case, the ideal $I_{r,s}$ coincides with I for every choice of the pair (r,s).

The surjective morphism $\varphi_{r,s}\colon kQ \to \Gamma$ of algebras is given as follows. The images of the idempotents corresponding to the vertices i (respectively, (i,y)) under $\varphi_{r,s}$ are the idempotents of Γ corresponding to the indecomposable direct summands V^i (respectively,

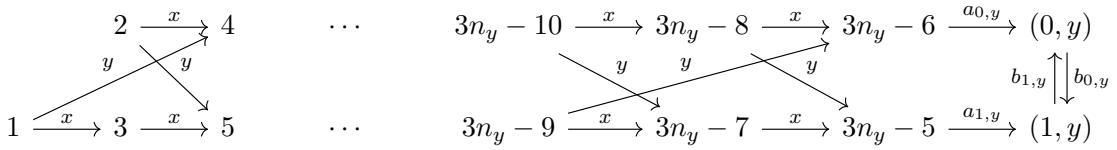


FIGURE 3. The quiver Q for $R = k[x,y]/(y^{n_y})$ with $\deg x = 2$, $\deg y = 3$, and n_y even.

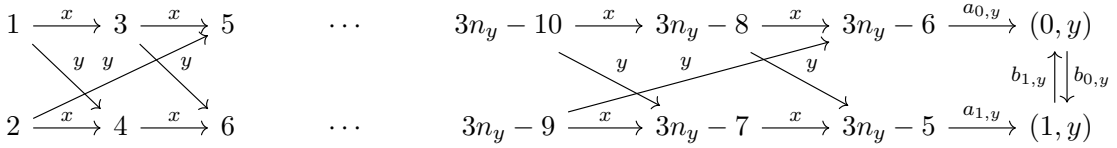


FIGURE 4. The quiver Q for $R = k[x,y]/(y^{n_y})$ with $\deg x = 2$, $\deg y = 3$, and n_y odd.

$V^{3n_y-4+i,y}$). The images of the elements x_i, y_i, $a_{i,y}$, and $b_{i,y}$ under $\varphi_{r,s}$ are given as follows. The elements $\varphi_{r,s}(x_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+2)_{\geq 0}$$

given by multiplication by x. The elements $\varphi_{r,s}(y_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i+3)_{\geq 0}$$

given by multiplication by y. The elements $\varphi_{r,s}(a_{i,y})$ are the morphisms

$$R(3n_y-6+i)_{\geq 0} \longrightarrow (k[x^{\pm 1},y]/(y^{n_y}))(3n_y-4+i)_{\geq 0}$$

given by multiplication by x. The element $\varphi_{r,s}(b_{0,y})$ is the morphism

$$(k[x^{\pm 1},y]/(y^{n_y}))(3n_y-4)_{\geq 0} \longrightarrow (k[x^{\pm 1},y]/(y^{n_y}))(3n_y-3)_{\geq 0}$$

given by multiplication by $\frac{y}{x}$. The element $\varphi_{r,s}(b_{1,y})$ is the morphism

$$(k[x^{\pm 1},y]/(y^{n_y}))(3n_y-3)_{\geq 0} \longrightarrow (k[x^{\pm 1},y]/(y^{n_y}))(3n_y-4)_{\geq 0}$$

given by multiplication by $\frac{y}{x^2}$. Then the kernel of $\varphi_{r,s}$ coincides with $I_{r,s}$.

Example 4.11. To give an example in the case where k is not algebraically closed, we consider $R = \mathbb{Q}[x,y]/(x^6+y^4)$ with $\deg x = 2$ and $\deg y = 3$. Then we have $a = 7$. The quiver Q is shown in Figure 5. The ideal I is generated by $xy - yx$ and $b_{0,1}^2 + e_{0,1}$. For a choice of the pair (r,s), the module $(F_{xy}^1)_{r,s}$ over $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}] \otimes_{\mathbb{Q}} \mathbb{Q}[b_{0,1}]/(b_{0,1}^2+1)$ is generated by $b_{0,1}a_{1,1} - a_{1,1}\frac{x^3}{y^2}$ and $a_{0,1} - a_{1,1}x^r y^s$. The pair (r,s) is in one of the following four cases.

If we have $(r,s) = (12t-1, -8t+1)$ for an integer t, then the ideal $\overline{I}_{r,s}^1$ is generated by $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$ and $a_{0,1}x - a_{1,1}y$. Therefore, the ideal $I_{r,s}$ is generated by $xy - yx$, $b_{0,1}^2 + e_{0,1}$, $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$, and $a_{0,1}x - a_{1,1}y$.

If we have $(r,s) = (12t+2, -8t-1)$ for an integer t, then the ideal $\overline{I}_{r,s}^1$ is generated by $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$ and $a_{0,1}y - a_{1,1}x^2$. Therefore, the ideal $I_{r,s}$ is generated by $xy - yx$, $b_{0,1}^2 + e_{0,1}$, $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$, and $a_{0,1}y - a_{1,1}x^2$.

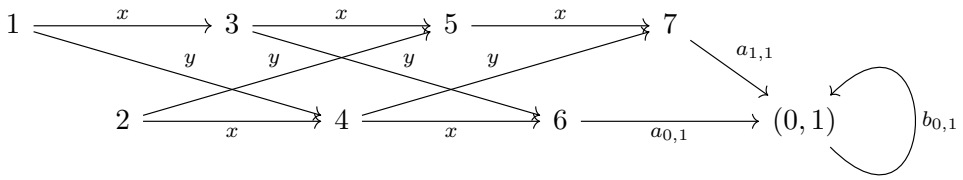


FIGURE 5. The quiver Q for $R = \mathbb{Q}[x, y]/(x^6 + y^4)$ with $\deg x = 2$ and $\deg y = 3$.

If we have $(r, s) = (12t - 4, -8t + 3)$ for an integer t, then the ideal $\overline{I}^1_{r,s}$ is generated by $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$ and $a_{0,1}y + a_{1,1}x^2$. Therefore, the ideal $I_{r,s}$ is generated by $xy - yx$, $b_{0,1}^2 + e_{0,1}$, $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$, and $a_{0,1}y + a_{1,1}x^2$.

If we have $(r, s) = (12t + 5, -8t - 3)$ for an integer t, then the ideal $\overline{I}^1_{r,s}$ is generated by $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$ and $a_{0,1}x + a_{1,1}y$. Therefore, the ideal $I_{r,s}$ is generated by $xy - yx$, $b_{0,1}^2 + e_{0,1}$, $b_{0,1}a_{1,1}y^2 - a_{1,1}x^3$, and $a_{0,1}x + a_{1,1}y$.

The surjective morphism $\varphi_{r,s}\colon kQ \to \Gamma$ of $\mathbb{Q}$-algebras is given as follows. The images of the idempotents corresponding to the vertices i (respectively, $(0, 1)$) under $\varphi_{r,s}$ are the idempotents of Γ corresponding to the indecomposable direct summands V^i (respectively, V^{8,f_1}). The images of the elements x_i, y_i, $a_{i,1}$, and $b_{0,1}$ under $\varphi_{r,s}$ are given as follows. The elements $\varphi_{r,s}(x_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i + 2)_{\geq 0}$$

given by multiplication by x. The elements $\varphi_{r,s}(y_i)$ are the morphisms

$$R(i)_{\geq 0} \longrightarrow R(i + 3)_{\geq 0}$$

given by multiplication by y. The element $\varphi_{r,s}(a_{0,1})$ is the morphism

$$R(6)_{\geq 0} \longrightarrow (\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]/(x^6 + y^4))(8)_{\geq 0}$$

given by multiplication by $(x^r y^s)^2$. The element $\varphi_{r,s}(a_{1,1})$ is the morphism

$$R(7)_{\geq 0} \longrightarrow (\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]/(x^6 + y^4))(8)_{\geq 0}$$

given by multiplication by $x^r y^s$. The element $\varphi_{r,s}(b_{0,1})$ is the morphism

$$(\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]/(x^6 + y^4))(8)_{\geq 0} \longrightarrow (\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]/(x^6 + y^4))(8)_{\geq 0}$$

given by multiplication by $\frac{x^3}{y^2}$. Then the kernel of $\varphi_{r,s}$ coincides with $I_{r,s}$.

4.4. **The structure theorem for negative a-invariants.** Recall that we write

$$f = x^{n_x} y^{n_y} \prod_{j=1}^{q} f_j^{n_j} ,$$

where f_j are monic irreducible polynomials. We assume that $m = \deg x$ and $n = \deg y$ are coprime with $m \leq n$.

Lemma 4.12. *The a-invariant of R is negative if and only if either the $\mathbb{N}$-graded commutative ring R is regular, or f equals x^{n_x} with $n_x > 1$ and $(n_x - 1)m < n$.*

Proof. We first prove the sufficiency. If R is regular, then the polynomial f does not lie in the ideal $(x,y)^2$ of $k[x,y]$. Therefore, the degree of f is m or n and hence the a-invariant $a = \deg f - m - n$ is negative. If f equals x^{n_x} with $n_x > 1$ and $(n_x - 1)m < n$, then a is also negative.

We now prove the necessity. Since the a-invariant $a = \deg f - m - n$ of R is negative, we have $\deg f < m + n$. If n_y is positive, then we must have $n_x = n_j = 0$ for all $1 \leq j \leq q$ and $f = y$. If n_j is positive for some $1 \leq j \leq q$, then we must have $n_x = n_y = n_{j'} = 0$ for all $j' \neq j$ and $f = \alpha x^n + \beta y$ for α, $\beta \neq 0$ since m must equal 1. If we have $n_y = n_j = 0$ for all $1 \leq j \leq q$ and $n_x = 1$, then we have $f = x$. In these cases, the $\mathbb{N}$-graded commutative ring R is regular. Otherwise, we have $f = x^{n_x}$ with $n_x > 1$ and $(n_x - 1)m < n$. $\surd$

We now suppose that the a-invariant of R is negative and the stable category $\mathcal{T} = \underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ does not vanish. For any positive integer i, we define graded R-modules

$$V^i = (k[x, y^{\pm 1}]/(x^{n_x}))(i)_{\geq 0}\,.$$

By Theorem 1.6 of [5], the category $\mathcal{T}$ contains a standard silting object $V = \bigoplus_{i=1}^{a+n} V^i$. Since the triangulated category $\underline{\mathrm{CM}}^{\mathbb{Z}} R$ is enhanced by the dg category $\mathcal{C}_{ac,dg}(\mathrm{proj}^{\mathbb{Z}} R)$ of acyclic complexes of finitely generated graded projective R-modules, we can define the dg endomorphism algebra of any object $X \in \mathcal{T} \subset \underline{\mathrm{CM}}^{\mathbb{Z}} R$. To describe the dg endomorphism algebra of V, we introduce the following dg path algebra (kQ', d).

Definition 4.13. The set Q_0 of the vertices of Q is $\mathbb{Z}/n\mathbb{Z}$. For a positive even integer p and $i \in \mathbb{Z}/n\mathbb{Z}$, we define $\nu_p(i) = i + \frac{p}{2} n_x m$. For a positive odd integer p and $i \in \mathbb{Z}/n\mathbb{Z}$, we define $\nu_p(i) = i + (\frac{p-1}{2} n_x + 1)m$. The set Q_1 of the arrows of Q is

$$\{\beta_{p,i} \colon i \to \nu_p(i) \mid p > 0, i \in \mathbb{Z}/n\mathbb{Z}\}\,,$$

where $\beta_{p,i}$ is of degree $1 - p$. Let (kQ, d) be the dg path algebra kQ with the differential determined by

$$\begin{aligned} d(\beta_{p,i}) = &\sum_{p_1+p_2=p, 2 \mid p_1 p_2} (-1)^{p_2+1} \beta_{p_2, \nu_{p_1}(i)} \beta_{p_1, i} \\ &+ \sum_{p_1+\cdots+p_{n_x}=p+n_x-2, 2 \nmid p_1 \ldots p_{n_x}} (-1)^{\frac{n_x(n_x-1)}{2}+1} \beta_{p_{n_x}, (\nu_{p_{n_x-1}} \circ \cdots \circ \nu_{p_1})(i)} \cdots \beta_{p_1, i}\,. \end{aligned}$$

Let Q' be the graded quiver obtained from Q by removing the vertices $0, \ldots, -a-1$ and the arrows incident with them. The differential d induced by that of (kQ, d) makes kQ' into a dg algebra.

Theorem 4.14. *Let R and (kQ', d) be as above. Denote the dg endomorphism algebra of the standard silting object V in $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ by Γ.*

a) The dg algebra Γ is proper and Gorenstein.
b) The dg algebra Γ is quasi-isomorphic to (kQ', d). In particular, the stable category $\underline{\mathrm{CM}}_0^{\mathbb{Z}} R$ is triangle equivalent to $\mathrm{per}\,(kQ', d)$.

4.5. **An example of Theorem 4.14.** The following example serves to illustrate the construction of dg path algebras (kQ', d). Let k be an arbitrary field. For $R = k[x,y]/(x^3)$ with $\deg x = 2$ and $\deg y = 9$, we have $a = -5$. The graded quiver Q is shown in Figure 6. In Figure 6, each displayed arrow represents a family of arrows parameterized by $\mathbb{N} \setminus \{0\}$. The six families are distinguished by the line styles and loops shown in the legend. The

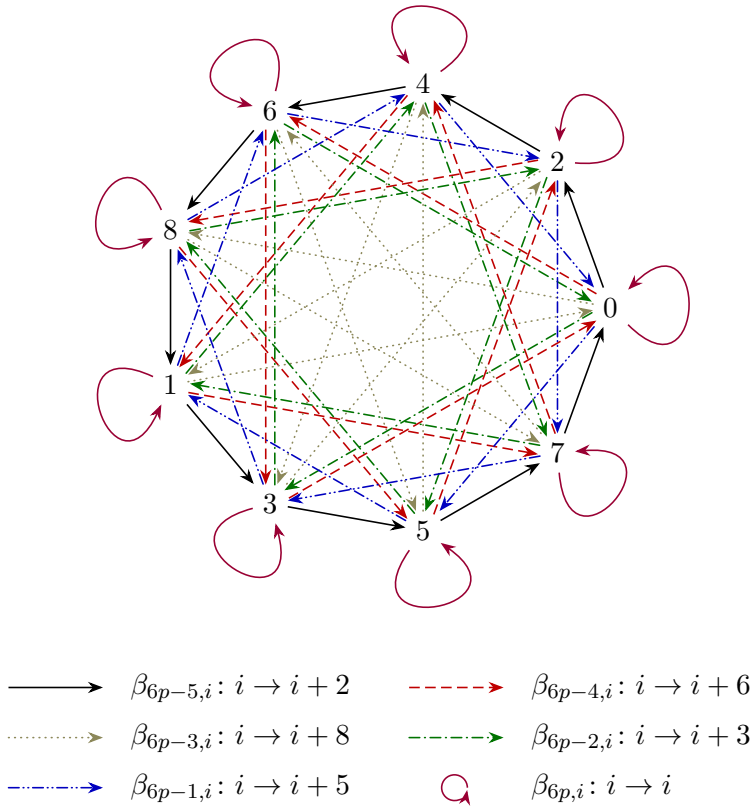


FIGURE 6. The graded quiver Q for $R = k[x,y]/(x^3)$ with $\deg x = 2$ and $\deg y = 9$. In the legend, $p \geq 1$, $i \in Q_0$, and vertex indices are taken modulo 9.

differentials of $\beta_{p,i}$ for small p are given by

$$\begin{aligned}
d(\beta_{1,i}) =& 0\,,\\
d(\beta_{2,i}) =& \beta_{1,i+4}\beta_{1,i+2}\beta_{1,i}\,,\\
d(\beta_{3,i}) =& -\beta_{2,i+2}\beta_{1,i} + \beta_{1,i+6}\beta_{2,i}\,,\\
d(\beta_{4,i}) =& -\beta_{2,i+6}\beta_{2,i} + \beta_{3,i+4}\beta_{1,i+2}\beta_{1,i} + \beta_{1,i+10}\beta_{3,i+2}\beta_{1,i} + \beta_{1,i+10}\beta_{1,i+8}\beta_{3,i}\,,\\
d(\beta_{5,i}) =& -\beta_{4,i+2}\beta_{1,i} + \beta_{3,i+6}\beta_{2,i} - \beta_{2,i+8}\beta_{3,i} + \beta_{1,i+12}\beta_{4,i}\,,\\
d(\beta_{6,i}) =& -\beta_{4,i+6}\beta_{2,i} - \beta_{2,i+12}\beta_{4,i} + \beta_{5,i+4}\beta_{1,i+2}\beta_{1,i} + \beta_{1,i+16}\beta_{5,i+2}\beta_{1,i} + \beta_{1,i+16}\beta_{1,i+14}\beta_{5,i}\\
&+ \beta_{3,i+10}\beta_{3,i+2}\beta_{1,i} + \beta_{3,i+10}\beta_{1,i+8}\beta_{3,i} + \beta_{1,i+16}\beta_{3,i+8}\beta_{3,i}\,.
\end{aligned}$$

Let Q' be the graded quiver obtained from Q by removing the vertices 0, 1, 2, 3, and 4 and the arrows which are adjacent to them. The differential d induced by that of (kQ,d) makes kQ' into a dg algebra. The dg endomorphism algebra Γ of the standard silting object V in $\underline{\mathrm{CM}}_0^{\mathbb{Z}}R$ is quasi-isomorphic to (kQ',d).

5. Proof of our results

5.1. **Structures of the morphism spaces.** In this section, we suppose that the a-invariant of R is nonnegative. In the following lemmas, we analyze the morphism spaces between direct summands of V, together with the corresponding components of the radical

and its square. They will be used to prove Theorem 4.4. We denote the morphism spaces and the radical morphism spaces in the full subcategory $\operatorname{add} V \subseteq \mathrm{CM}_0^{\mathbb{Z}} R$ by $\operatorname{Hom}_R(?,-)_0$ and $\operatorname{rad}_V(?,-)_0$, respectively. For $1 \leq u \leq a$, we also set

$$V^{u,x} = K^x(u)_{\geq 0}\ ,\ V^{u,y} = K^y(u)_{\geq 0}\ ,\ V^{u,f_j} = K^j(u)_{\geq 0}\ .$$

Lemma 5.1. *For all sufficiently large integers N, the subcategory $\operatorname{add}(\bigoplus_{i=a+1}^{N} V^i) \subseteq \mathrm{CM}_0^{\mathbb{Z}} R$ coincides with $\operatorname{add}(V^x \oplus V^y \oplus \bigoplus_{j=1}^{q} V^{f_j})$. In particular, we have $\operatorname{add}(\bigoplus_{i=1}^{N} V^i) = \operatorname{add} V$.*

Proof. By part (a) of Lemma 4.11 of [5] and part (c) of Theorem 2.1 of [5], the object V^i is isomorphic to $V^{i,x} \oplus V^{i,y} \oplus \bigoplus_{j=1}^{q} V^{i,f_j}$ for all $i \geq a+1$. Since N is large enough, by part (b) of Lemma 4.11 of [5], we have

$$\operatorname{add}(\bigoplus_{i=a+1}^{N}(V^{i,x} \oplus V^{i,y} \oplus \bigoplus_{j=1}^{q} V^{i,f_j})) = \operatorname{add}(V^x \oplus V^y \oplus \bigoplus_{j=1}^{q} V^{f_j})\ .$$

Then the statements follow. √

Lemma 5.2. *We have the isomorphism $\operatorname{End}_R(V)_0 \xrightarrow{\sim} \operatorname{End}_{\mathcal{T}}(V)$ of algebras.*

Proof. The statement follows from Lemma 5.1 and part (b) of Lemma 4.13 of [5]. √

Consider the full subquivers Q^x, Q^y, and $Q^j \subseteq Q$ with vertex sets $\{(i,x) \mid 0 \leq i < n\}$, $\{(i,y) \mid 0 \leq i < m\}$, and $\{(0,j)\}$, respectively.

Lemma 5.3.

a) For any $1 \leq i, i' \leq a$, we have a bijection $\operatorname{Hom}_R(V^i, V^{i'})_0 \xrightarrow{\sim} R_{i'-i}$.

b) We have an isomorphism

$$\operatorname{End}_R(V^x)_0 \xrightarrow{\sim} kQ^x/(b_x^{n_x})$$

of algebras. For any $a+1 \leq i \leq a+n$ and positive integer t satisfying $i - tn \geq 1$, the morphism $\cdot y^t \colon V^{i-tn} \to V^{i,x}$ induces the bijection

$$\operatorname{Hom}_R(V^{i,x}, V^x)_0 \xrightarrow{\sim} \operatorname{Hom}_R(V^{i-tn}, V^x)_0\ .$$

c) We have an isomorphism

$$\operatorname{End}_R(V^y)_0 \xrightarrow{\sim} kQ^y/(b_y^{n_y})$$

of algebras. For any $a+1 \leq i \leq a+m$ and positive integer t satisfying $i - tm \geq 1$, the morphism $\cdot x^t \colon V^{i-tm} \to V^{i,y}$ induces the bijection

$$\operatorname{Hom}_R(V^{i,y}, V^y)_0 \xrightarrow{\sim} \operatorname{Hom}_R(V^{i-tm}, V^y)_0\ .$$

d) For any $1 \leq j \leq q$, we have an isomorphism

$$\operatorname{End}_R(V^{f_j})_0 \xrightarrow{\sim} kQ^j/(g_j(b_j)^{n_j})$$

of algebras. For any $1 \leq j \leq q$ and $1 \leq t \leq a$, the morphism $\cdot(x^r y^s)^t \colon V^{a+1-t} \to V^{f_j}$ induces the bijection

$$\operatorname{End}_R(V^{f_j})_0 \xrightarrow{\sim} \operatorname{Hom}_R(V^{a+1-t}, V^{f_j})_0\ .$$

e) We have

$$\mathrm{Hom}_R(V^x, V^{[1,a]} \oplus V^y \oplus \bigoplus_{j=1}^{q} V^{f_j})_0 = 0 \text{ and}$$

$$\mathrm{Hom}_R(V^y, V^{[1,a]} \oplus V^x \oplus \bigoplus_{j=1}^{q} V^{f_j})_0 = 0\,.$$

For any $1 \le j \le q$, we have

$$\mathrm{Hom}_R(V^{f_j}, V^{[1,a]} \oplus V^x \oplus V^y \oplus \bigoplus_{j' \neq j} V^{f_{j'}})_0 = 0\,.$$

f) For any $1 \le i \le a$, all morphisms in $\mathrm{rad}_V(V^i, V)_0$ factor through the morphism $\cdot x \colon V^i \to V^{i+m}$ or the morphism $\cdot y \colon V^i \to V^{i+n}$.

Proof. a) This follows from part (c) of Theorem 1.4 of [5].

b) In the definition

$$V^x = \bigoplus_{i=a+1}^{a+n} V^{i,x} = \bigoplus_{i=a+1}^{a+n} (k[x, y^{\pm 1}]/(x^{n_x}))(i)_{\ge 0}\,,$$

the index i runs from $a+1$ to $a+n$. Since the invertible element y is of degree n, the morphism space $\mathrm{Hom}_R(V^{i,x}, V^x)_0$ is spanned by the morphisms given by multiplication by nonnegative powers of x times the uniquely determined nonnegative powers of y^{-1}. So the algebra $\mathrm{End}_R(V^x)_0$ is isomorphic to $kQ^x/(b_x^{n_x})$. We now prove the second statement. By the universal properties of localizations and quotients, we have the bijection

$$\mathrm{Hom}_R(V^{i-tn,x}, V^x)_0 \xrightarrow{\sim} \mathrm{Hom}_R(V^{i-tn}, V^x)_0\,.$$

Since the morphism

$$\cdot y^t \colon V^{i-tn,x} \xrightarrow{\sim} V^{i,x}$$

is an isomorphism, we have the bijection

$$\mathrm{Hom}_R(V^{i,x}, V^x)_0 \xrightarrow{\sim} \mathrm{Hom}_R(V^{i-tn,x}, V^x)_0\,.$$

Then the statement follows from the above bijections.

c) Similar to part b).

d) Since the morphism space $\mathrm{End}_R(V^{f_j})_0$ is spanned by the morphisms given by multiplication by nonnegative powers of $\frac{x^n}{y^m}$, the algebra $\mathrm{End}_R(V^{f_j})_0$ is isomorphic to $kQ^j/(g_j(b_j)^{n_j})$. We now prove the second statement. By the universal properties of localizations and quotients, we have the bijection

$$\mathrm{Hom}_R(V^{a+1-t,f_j}, V^{f_j})_0 \xrightarrow{\sim} \mathrm{Hom}_R(V^{a+1-t}, V^{f_j})_0\,.$$

Since the morphism

$$\cdot (x^r y^s)^t \colon V^{a+1-t,f_j} \xrightarrow{\sim} V^{a+1,f_j}$$

is an isomorphism, we have the bijection

$$\mathrm{End}_R(V^{f_j})_0 \xrightarrow{\sim} \mathrm{Hom}_R(V^{a+1-t,f_j}, V^{f_j})_0\,.$$

Then the statement follows from the above bijections.

e) Since V^x is annihilated by x^{n_x} but V^y and V^{f_j} do not contain a nonzero element annihilated by x^{n_x}, we have $\mathrm{Hom}_R(V^x, V^y \oplus \bigoplus_{j=1}^{q} V^{f_j})_0 = 0$. By Lemma 5.1 and part (a)

of Lemma 4.13 of [5], we have $\mathrm{Hom}_R(V^x, V^{[1,a]})_0 = 0$. So the first statement follows. The proofs of the other statements are similar.

f) This is a consequence of parts a), b), c), and d). √

Lemma 5.4.

a) The radical of $\mathrm{End}_R(V^{[1,a]})_0$ *is generated by the morphisms in* $\mathrm{Hom}_R(V^i, V^{[1,a]})_0$ *given by multiplication by* x *or* y.

b) The radical of $\mathrm{End}_R(V^x)_0$ *is generated by the morphisms in* $\mathrm{Hom}_R(V^{i,x}, V^x)_0$ *given by multiplication by* x *times the uniquely determined nonnegative powers of* y^{-1}.

c) The radical of $\mathrm{End}_R(V^y)_0$ *is generated by the morphisms in* $\mathrm{Hom}_R(V^{i,y}, V^y)_0$ *given by multiplication by* y *times the uniquely determined nonnegative powers of* x^{-1}.

d) For any $1 \le j \le q$, *the radical of* $\mathrm{End}_R(V^{f_j})_0$ *is generated by the morphism given by multiplication by* f_j *times the uniquely determined positive power of* y^{-1}.

e) The object V *is basic. In particular, all morphisms between different indecomposable direct summands of* V *are radical morphisms.*

Proof. a) This follows from part a) of Lemma 5.3.

b) The ideal of $\mathrm{End}_R(V^x)_0$, generated by the morphisms in $\mathrm{Hom}_R(V^{i,x}, V^x)_0$ given by multiplication by x times the uniquely determined nonnegative powers of y^{-1}, is nilpotent. The quotient of $\mathrm{End}_R(V^x)_0$ by this ideal is isomorphic to the direct product of n copies of k. Therefore, this ideal equals the radical of $\mathrm{End}_R(V^x)_0$.

c) Similar to part b).

d) Similar to part b).

e) The statement follows from parts a), b), c), d) and e) of Lemma 5.3. √

Lemma 5.5.

a) The square of the radical of $\mathrm{End}_R(V^{[1,a]})_0$ *is generated by the morphisms in* $\mathrm{Hom}_R(V^i, V^{[1,a]})_0$ *given by multiplication by* x^2, xy, *or* y^2.

b) The square of the radical of $\mathrm{End}_R(V^x)_0$ *is generated by the morphisms in* $\mathrm{Hom}_R(V^{i,x}, V^x)_0$ *given by multiplication by* x^2 *times the uniquely determined nonnegative powers of* y^{-1}.

c) The square of the radical of $\mathrm{End}_R(V^y)_0$ *is generated by the morphisms in* $\mathrm{Hom}_R(V^{i,y}, V^y)_0$ *given by multiplication by* y^2 *times the uniquely determined nonnegative powers of* x^{-1}.

d) For any $1 \le j \le q$, *the square of the radical of* $\mathrm{End}_R(V^{f_j})_0$ *is generated by the morphism given by multiplication by* f_j^2 *times the uniquely determined positive power of* y^{-1}.

e) For any $a+1+\max\{n-a, 0\} \le i \le a+n$, *the morphism* $\cdot y\colon V^{i-n} \to V^{i,x}$ *induces the bijection*

$$\mathrm{rad}_V(V^{i,x}, V^x)_0 \xrightarrow{\sim} \mathrm{rad}_V^2(V^{i-n}, V^x)_0 .$$

For any $a+1 \le i \le a+n$ *and integer* t *satisfying* $t \ge 2$ *and* $i - tn \ge 1$, *the morphism* $\cdot y^t\colon V^{i-tn} \to V^{i,x}$ *induces the bijection*

$$\mathrm{Hom}_R(V^{i,x}, V^x)_0 \xrightarrow{\sim} \mathrm{rad}_V^2(V^{i-tn}, V^x)_0 .$$

f) For any $a+1+\max\{m-a, 0\} \le i \le a+m$, *the morphism* $\cdot x\colon V^{i-m} \to V^{i,y}$ *induces the bijection*

$$\mathrm{rad}_V(V^{i,y}, V^y)_0 \xrightarrow{\sim} \mathrm{rad}_V^2(V^{i-m}, V^y)_0 .$$

For any $a+1 \leq i \leq a+m$ and integer t satisfying $t \geq 2$ and $i-tm \geq 1$, the morphism $\cdot x^t \colon V^{i-tm} \to V^{i,y}$ induces the bijection

$$\mathrm{Hom}_R(V^{i,y}, V^y)_0 \xrightarrow{\sim} \mathrm{rad}_V^2(V^{i-tm}, V^y)_0 \,.$$

g) Fix j with $1 \leq j \leq q$. For any $a+1+\max\{m-a, 0\} \leq i \leq a+m$, the morphism

$$\cdot (x^r y^s)^{a+m+1-i} \colon V^{i-m} \to V^{f_j}$$

induces the bijection

$$\mathrm{rad}_V(V^{f_j}, V^{f_j})_0 \xrightarrow{\sim} \mathrm{rad}_V^2(V^{i-m}, V^{f_j})_0 \,.$$

For any $a+1 \leq i \leq a+m$ and integer t satisfying $t \geq 2$ and $i-tm \geq 1$, the morphism

$$\cdot x^{t-1}(x^r y^s)^{a+m+1-i} \colon V^{i-tm} \to V^{f_j}$$

induces the bijection

$$\mathrm{End}_R(V^{f_j})_0 \xrightarrow{\sim} \mathrm{rad}_V^2(V^{i-tm}, V^{f_j})_0 \,.$$

Proof. a) This follows from part a) of Lemma 5.4.

b) This follows from part b) of Lemma 5.4.

c) This follows from part c) of Lemma 5.4.

d) This follows from part d) of Lemma 5.4.

e) We first prove the first bijection. Since we have $a-(i-n) < n$, for any $i-n \leq i' \leq a$, the morphism space $\mathrm{Hom}_R(V^{i-n}, V^{i'})_0$ is spanned by the morphisms given by multiplication by nonnegative powers of x. For any $a+1 \leq i'' \leq a+n$, we have the commutative diagram

$$\begin{array}{ccc} V^{i-n} & \longrightarrow & V^{i'} \\ \downarrow & & \downarrow \\ V^{i-n+i''-i',x} & \longrightarrow & V^{i'',x} \,, \end{array}$$

where the two horizontal morphisms are given by multiplication by the same element of R, and the two vertical morphisms are given by multiplication by the same element of K^x. This implies that

$$\mathrm{Hom}_R(V^{i'}, V^{i'',x})_0 \cdot \mathrm{rad}_V(V^{i-n}, V^{i'})_0$$

is contained in

$$\mathrm{rad}_V(V^{i-n+i''-i',x}, V^{i'',x})_0 \cdot \mathrm{Hom}_R(V^{i-n}, V^{i-n+i''-i',x})_0 \,.$$

Therefore, by part e) of Lemma 5.3, we have

$$\begin{aligned} \mathrm{rad}_V^2(V^{i-n}, V^x)_0 =& \mathrm{rad}_V(V^{[1,a]}, V^x)_0 \cdot \mathrm{rad}_V(V^{i-n}, V^{[1,a]})_0 \\ &+ \mathrm{rad}_V(V^x, V^x)_0 \cdot \mathrm{rad}_V(V^{i-n}, V^x)_0 \\ =& \mathrm{rad}_V(V^x, V^x)_0 \cdot \mathrm{rad}_V(V^{i-n}, V^x)_0 \,. \end{aligned}$$

Then the statement follows from part e) of Lemma 5.4 and part b) of Lemma 5.3.

We now prove the second bijection. By the universal properties of localizations and quotients, any morphism $\varphi \colon V^{i-tn} \to V^x$ factors uniquely through the canonical morphism $V^{i-tn} \to V^{i-tn,x}$. The induced morphism $V^{i-tn,x} \to V^x$ factors uniquely through the isomorphism

$$\cdot y^{t-1} \colon V^{i-tn,x} \xrightarrow{\sim} V^{i-n,x} \,.$$

Then by the commutative diagram

$$\begin{array}{ccc} V^{i-tn} & \xrightarrow{\cdot y^{t-1}} & V^{i-n} \\ \downarrow & & \downarrow \\ V^{i-tn,x} & \xrightarrow{\cdot y^{t-1}} & V^{i-n,x} \end{array},$$

in which the vertical morphisms are canonical, we deduce that φ factors through V^{i-n}. By part e) of Lemma 5.4, this implies that φ lies in $\mathrm{rad}_V^2(V^{i-tn}, V^x)_0$. So we have

$$\mathrm{rad}_V^2(V^{i-tn}, V^x)_0 = \mathrm{Hom}_R(V^{i-tn}, V^x)_0 .$$

Then the statement follows from part b) of Lemma 5.3.

f) Similar to part e).

g) We first prove the first bijection. Since we have $a-(i-m) < m$, for any $i-m < i' \leq a$, the morphism space $\mathrm{Hom}_R(V^{i-m}, V^{i'})_0$ vanishes. Therefore, by part e) of Lemma 5.3, we have

$$\begin{aligned} \mathrm{rad}_V^2(V^{i-m}, V^{f_j})_0 =& \mathrm{rad}_V(V^{i-m}, V^{f_j})_0 \cdot \mathrm{rad}_V(V^{i-m}, V^{i-m})_0 \\ &+ \mathrm{rad}_V(V^{f_j}, V^{f_j})_0 \cdot \mathrm{rad}_V(V^{i-m}, V^{f_j})_0 . \end{aligned}$$

Then the statement follows from parts a) and e) of Lemma 5.4 and part d) of Lemma 5.3.

We now prove the second bijection. By the universal properties of localizations and quotients, any morphism $\varphi\colon V^{i-tm} \to V^{f_j}$ factors uniquely through the canonical morphism $V^{i-tm} \to V^{i-tm,f_j}$. The induced morphism $V^{i-tm,f_j} \to V^{f_j}$ factors uniquely through the isomorphism

$$\cdot x^{t-1}\colon V^{i-tm,f_j} \xrightarrow{\sim} V^{i-m,f_j} .$$

Then by the commutative diagram

$$\begin{array}{ccc} V^{i-tm} & \xrightarrow{\cdot x^{t-1}} & V^{i-m} \\ \downarrow & & \downarrow \\ V^{i-tm,f_j} & \xrightarrow{\cdot x^{t-1}} & V^{i-m,f_j} \end{array},$$

in which the vertical morphisms are canonical, we deduce that φ factors through V^{i-m}. By part e) of Lemma 5.4, this implies that φ lies in $\mathrm{rad}_V^2(V^{i-tm}, V^{f_j})_0$. So we have

$$\mathrm{rad}_V^2(V^{i-tm}, V^{f_j})_0 = \mathrm{Hom}_R(V^{i-tm}, V^{f_j})_0 .$$

Then the statement follows from part d) of Lemma 5.3.

√

5.2. **Proof of Theorem 4.4.** By Lemma 5.2, we may and will assume that we have $\mathrm{End}_R(V)_0 = \Gamma$.

a) Since the category $\underline{\mathrm{CM}}_0^{\mathbb{Z}}R$ is Hom-finite, the algebra Γ is finite-dimensional. To prove it is Iwanaga–Gorenstein, we first claim that the projective dimension of the simple Γ^{op}-module S_i is less than or equal to 2 for all $1 \leq i \leq a$. In fact, we have the short exact sequence

$$0 \longrightarrow R(i)_{\geq 0} \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} R(i+m)_{\geq 0} \oplus R(i+n)_{\geq 0} \xrightarrow{\begin{bmatrix} y & -x \end{bmatrix}} R(i+m+n)_{\geq 0} \longrightarrow 0$$

in $\mathrm{CM}_0^{\mathbb{Z}}R$. If we apply the functor $\mathrm{Hom}_R(?, V)_0$ to it, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(V^{i+m+n}, V)_0 \xrightarrow{\begin{bmatrix} y \\ -x \end{bmatrix}} \mathrm{Hom}_R(V^{i+m} \oplus V^{i+n}, V)_0 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} \mathrm{Hom}_R(V^i, V)_0$$

of Γ^{op}-modules. By Lemma 5.1, the Γ^{op}-module $\mathrm{Hom}_R(V^{i+m+n}, V)_0$ is projective. By part f) of Lemma 5.3, the image of the last morphism is $\mathrm{rad}_V(V^i, V)_0$. So the cokernel of the last morphism is the simple Γ^{op}-module S_i. This implies that its projective dimension is less than or equal to 2.

We now prove that the projective dimension of any indecomposable injective Γ^{op}-module is less than or equal to 2. For any $1 \leq i \leq a$, by part e) of Lemma 5.3, we have

$$\mathrm{Hom}_R(V^x, V^i)_0 = \mathrm{Hom}_R(V^y, V^i)_0 = \mathrm{Hom}_R(V^{f_j}, V^i)_0 = 0\,.$$

So the indecomposable injective Γ^{op}-module I_i is filtered by the simple Γ^{op}-modules $S_{i'}$, $1 \leq i' \leq a$. So the projective dimension of I_i is less than or equal to 2. For any $0 \leq i < n$, since we have

$$\mathrm{Hom}_R(V^y, V^{i,x})_0 = \mathrm{Hom}_R(V^{f_j}, V^{i,x})_0 = \mathrm{Hom}_R(V^{i,x}, V^{[1,a]})_0 = 0\,,$$

we have the short exact sequence

$$0 \longrightarrow M \longrightarrow I_{(i,x)} \longrightarrow N \longrightarrow 0$$

of Γ^{op}-modules, where M (respectively, N) is filtered by the simple Γ^{op}-modules $S_{(i',x)}$, $0 \leq i' < n$ (respectively, $S_{i'}$, $1 \leq i' \leq a$). By the previous argument, the projective dimension of N is less than or equal to 2. The Γ^{op}-module M is the image of the indecomposable injective $\mathrm{End}_R(V^x)_0^{op}$-module $J_{(i,x)}$ under the fully faithful functor

$$F\colon \mathrm{mod}\,(\mathrm{End}_R(V^x)_0^{op}) \longrightarrow \mathrm{mod}\,\Gamma^{op}\,.$$

By part b) of Lemma 5.3, the algebra $\mathrm{End}_R(V^x)_0$ is self-injective. This implies that the $\mathrm{End}_R(V^x)_0^{op}$-module $J_{(i,x)}$ is also projective and hence it is isomorphic to the indecomposable projective $\mathrm{End}_R(V^x)_0^{op}$-module $Q_{(i-(n_x-1)m,x)}$. Since we have

$$\mathrm{Hom}_R(V^{i',x}, V^{[1,a]})_0 = \mathrm{Hom}_R(V^{i',x}, V^y)_0 = \mathrm{Hom}_R(V^{i',x}, V^{f_j})_0 = 0\,,$$

the Γ^{op}-module M, as the image $\mathrm{Hom}_R(V^{i-(n_x-1)m,x}, V)_0$ of $\mathrm{Hom}_R(V^{i-(n_x-1)m,x}, V^x)_0$ under the functor F, is also projective. Since the projective dimensions of the Γ^{op}-modules M and N are less than or equal to 2, so is the projective dimension of their extension $I_{(i,x)}$. Similarly, the projective dimensions of the indecomposable injective Γ^{op}-modules $I_{(i,y)}$ and $I_{(0,j)}$ are less than or equal to 2. Therefore, the injective dimension of Γ as a left module over itself is less than or equal to 2. By part (d) of Theorem 1.4 of [5], we conclude that the algebra Γ is Iwanaga–Gorenstein of self-injective dimension less than or equal to 2. Its global dimension is finite if and only if it is less than or equal to 2 if and only if the projective dimensions of the simple Γ^{op}-modules $S_{(i,x)}$, $S_{(i,y)}$, and $S_{(0,j)}$ are less than or equal to 2. Since the algebras $\mathrm{End}_R(V^x)_0$, $\mathrm{End}_R(V^y)_0$, and $\mathrm{End}_R(V^{f_j})_0$ are isomorphic to self-injective Nakayama algebras, the above conditions are equivalent to the condition that n_x, n_y, and n_j are less than or equal to 1 for all $1 \leq j \leq q$. This means that R is reduced.

b) By part e) of Lemma 5.4, the algebra Γ is basic. By Lemmas 5.3, 5.4 and 5.5, we know the structures of $\Gamma/\mathrm{rad}\,\Gamma$ and $\mathrm{rad}\,\Gamma/\mathrm{rad}^2\Gamma$. So we have the surjective morphism $\varphi_{r,s}\colon kQ \to \Gamma$ of algebras defined in Section 4.1. By the descriptions of the images of the arrows under $\varphi_{r,s}$, we see that the kernel of $\varphi_{r,s}$ coincides with $I_{r,s}$.

5.3. **Proof of Theorem 4.14.** a) The statement follows from Proposition 4.1.

b) Set $\Lambda = \mathrm{End}_{\mathcal{T}}(\bigoplus_{i=1}^{n} V^i)$ and $P = \bigoplus_{i=a+n+1}^{n} V^i$. By part (b) of Theorem 1.6 of [5], the dg endomorphism algebra Γ of the standard silting object V in $\underline{\mathrm{CM}}_0^{\mathbb{Z}}R$ is quasi-isomorphic to the dg endomorphism algebra of Λ in the Verdier quotient of $\mathrm{per}\,\Lambda$ by its thick subcategory generated by P. By part (a) of Corollary 2.12 of [22], it is the dg quotient of Λ by $\mathrm{End}_\Lambda(P)$. On the other hand, by part b) of Lemma 5.3 and Lemma 5.2, the algebra

Λ is isomorphic to the self-injective Nakayama algebra N_{n,n_x} defined in Proposition A.1. By Proposition A.1, it is quasi-isomorphic to the cofibrant dg algebra (kQ, d) defined in Definition 4.13. Therefore, by Theorem 7.1 of [21], the above dg quotient is quasi-isomorphic to (kQ', d).

6. Auslander–Reiten quivers for finite and countable Cohen–Macaulay types

Buchweitz–Greuel–Schreyer gave a classification of commutative complete local Gorenstein rings of finite (respectively, countable) Cohen–Macaulay representation type: Under mild assumptions, these are precisely the ADE singularities (respectively, the A_∞ and D_∞ singularities) of the form

$$k[\![x, y, z_2, \dots, z_d]\!]/(f(x, y) + z_2^2 + \cdots + z_d^2)\,,$$

where $f(x, y)$ is one of the polynomials listed in Table 1, cf. Theorem 8.8 of [35], Corollary 10.19 and Theorem 14.16 of [28].

In this section, we consider the corresponding $\mathbb{N}$-graded hypersurface singularities $R = k[x, y]/(f)$, which are of finite (respectively, countable) Cohen–Macaulay representation type in the sense that there are only finitely (respectively, countably) many indecomposable objects in $\mathrm{CM}^{\mathbb{Z}}R$ up to isomorphism and degree shifts.

R	A_n (n is odd)	A_n (n is even)	D_n (n is even)	D_n (n is odd)
f	$x^{n+1} - y^2$	$x^{n+1} - y^2$	$x^{n-1} - xy^2$	$x^{n-1} - xy^2$
$(\deg x, \deg y)$	$(1, \frac{n+1}{2})$	$(2, n+1)$	$(1, \frac{n}{2} - 1)$	$(2, n-2)$
irreducible factors of f	$f_1 = x^{\frac{n+1}{2}} - y\,,$ $f_2 = x^{\frac{n+1}{2}} + y$	f	$x\,,$ $f_1 = x^{\frac{n}{2}-1} - y\,,$ $f_2 = x^{\frac{n}{2}-1} + y$	$x\,,$ $f_1 = x^{n-2} - y^2$

R	E_6	E_7	E_8	A_∞	D_∞
f	$x^4 - y^3$	$x^3y - y^3$	$x^5 - y^3$	y^2	xy^2
$(\deg x, \deg y)$	$(3, 4)$	$(2, 3)$	$(3, 5)$	(m, l)	(m, l)
irreducible factors of f	f	$y,$ $f_1 = x^3 - y^2$	f	y	$x,$ y

Table 1. Defining polynomials, gradings, and irreducible factors for the finite and countable Cohen–Macaulay types.

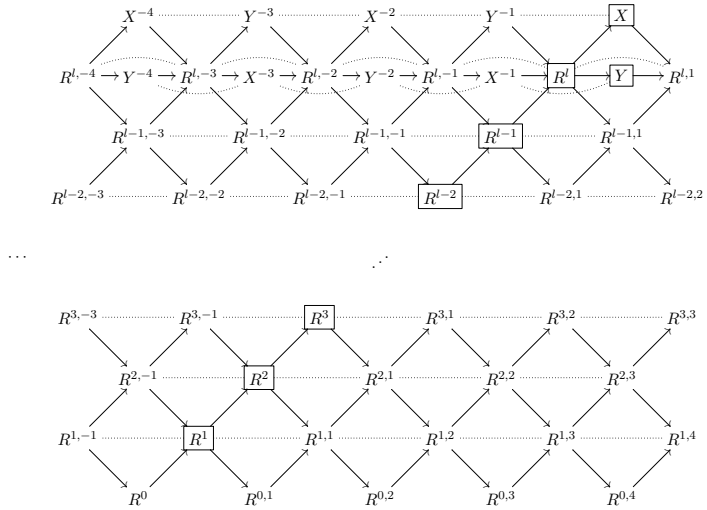
In the remainder of this section, we draw the Auslander–Reiten quiver of the category $\mathrm{CM}^{\mathbb{Z}}_0 R$ for the rings R listed in Table 1. Our Auslander–Reiten quivers are displayed in a standard way following the theory of translation quivers [32, 2], cf. Section 3 of [1], see also Section 3 of [7], Chapter 9 of [35], Section 4.3 of [14], and Section 13.2 of [28] for the ungraded case.

Moreover, we give an explicit description of each indecomposable object $M \in \mathrm{CM}^{\mathbb{Z}}_0 R$ by giving a homogeneous k-basis. For the reduced cases, write $f = \prod_{j=1}^{p} f_j$ with distinct irreducible factors f_j. We view R as an $\mathbb{N}$-graded subalgebra of $\prod_{j=1}^{p} R/(f_j)$. In each basis diagram in this section, a column labeled by f_{j_i} represents a $\mathbb{Z}$-graded R-submodule M^i of the graded normalization of $R/(f_{j_i})$ whose homogeneous k-basis is given by black circles $\bullet$. A picture with r columns labeled by $f_{j_1}, \dots, f_{j_r}$ represents a $\mathbb{Z}$-graded R-submodule of $\bigoplus_{i=1}^{r} M^i$ spanned by the vectors indicated in the picture, where $\bullet{=}\bullet$ and $\bullet{=}\bullet{=}\bullet$ each

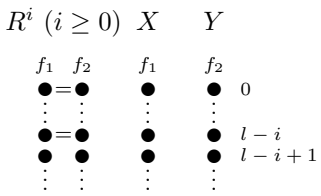
represent the sum of the basis vectors indicated by the connected black circles. If we have $p = 1$, then we omit the label f for simplicity.

6.1. **Finite Cohen–Macaulay representation type.** In the Auslander–Reiten quivers for the finite Cohen–Macaulay types, we illustrate the standard tilting object V (cf. Section 4.1) as the direct sum of the framed objects.

The Auslander–Reiten quivers of $\mathrm{CM}_0^{\mathbb{Z}} R$ and homogeneous k-bases for type A_n with n odd and even are shown in Figures 7 and 8, respectively. For type D_n with n even, the quiver and bases are shown in Figures 9 and 10; for n odd, they are shown in Figures 11 and 12. For types E_6, E_7, and E_8, the quivers and bases are shown in Figures 13 and 14, Figures 15 and 16, and Figures 17 and 18, respectively.



(a) Auslander–Reiten quiver



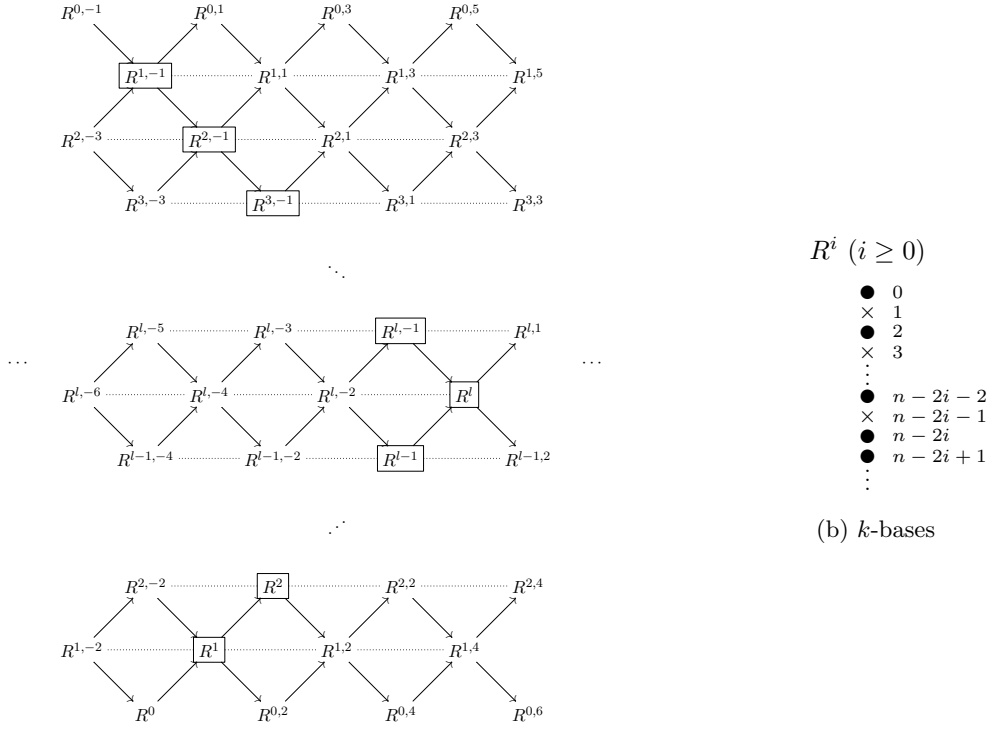
(b) k-bases

FIGURE 7. Type A_n with n odd: (a) the Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}} R$; (b) homogeneous k-bases of the $\mathbb{Z}$-graded modules. Here $l = \frac{n-1}{2}$ and $*^j = *(j)$ for $* = R^i$, X, or Y.

In Figure 9, the last framed object in the bottom row is $\tau Z^{1,l}$.

In Figure 10, the symbol •–•–• means the subspace $\{(a, b, c) \in k^3 \mid 2a - b + c = 0\}$, where the natural basis of k^3 is given by the black circles in the columns labeled by x, f_1, and f_2, respectively.

In Figure 11, the last framed object in the bottom (respectively, top) row is $\tau X^{1,n-3}$ (respectively, $\tau X^{1,n-2}$).



(a) Auslander–Reiten quiver

(b) k-bases

FIGURE 8. Type A_n with n even: (a) the Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$; (b) homogeneous k-bases of the $\mathbb{Z}$-graded modules. Here $l = \frac{n}{2}$ and $*^j = *(j)$ for $* = R^i$.

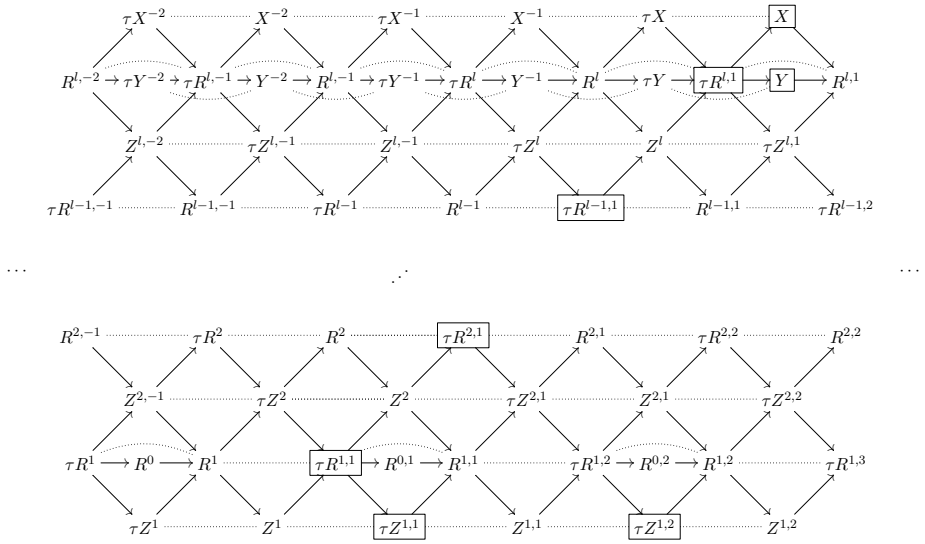


FIGURE 9. The Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$ for type D_n (n is even), where $l = \frac{n}{2} - 1$ and $*^j = *(j)$ for $* = R^i$, X, Y, Z^i, τR^i, τX, τY, or τZ^i.

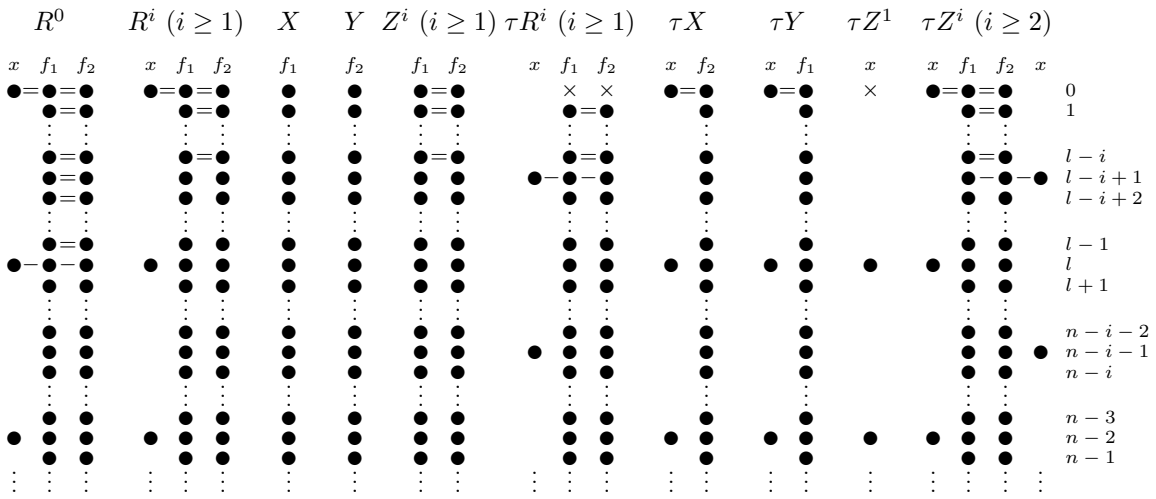


FIGURE 10. Homogeneous k-bases of $\mathbb{Z}$-graded modules for type D_n (n is even).

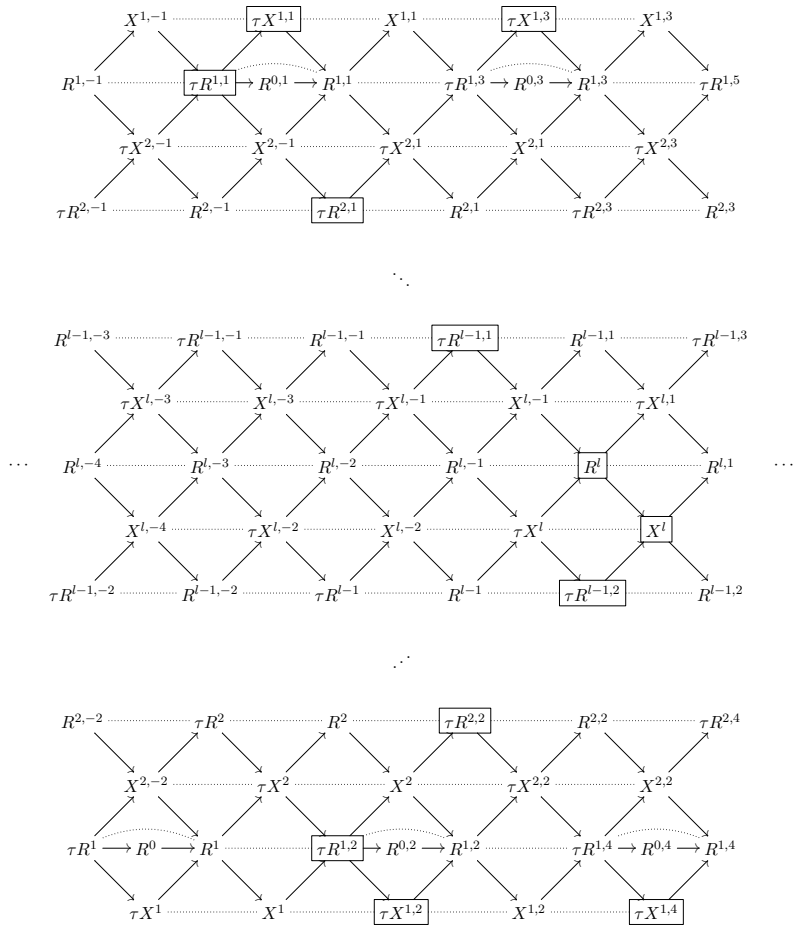


FIGURE 11. The Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$ for type D_n (n is odd), where $l = \frac{n-1}{2}$ and $*^j = *(j)$ for $* = R^i$, X^i, τR^i, or τX^i.

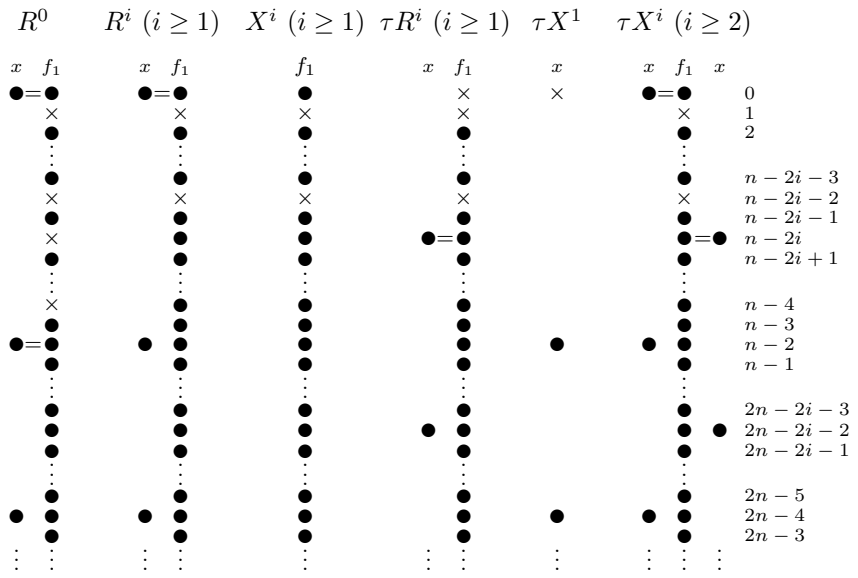


FIGURE 12. Homogeneous k-bases of $\mathbb{Z}$-graded modules for type D_n (n is odd).

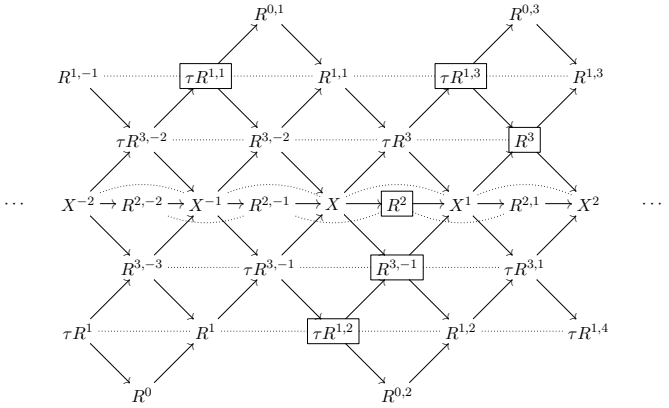


FIGURE 13. The Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$ for type E_6, where $*^j = *(j)$ for $* = R^0$, R^1, R^2, R^3, τR^1, τR^3, or X.

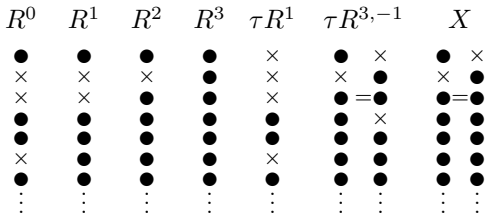


FIGURE 14. Homogeneous k-bases of $\mathbb{Z}$-graded modules for type E_6. The top row represents the component of degree 0.

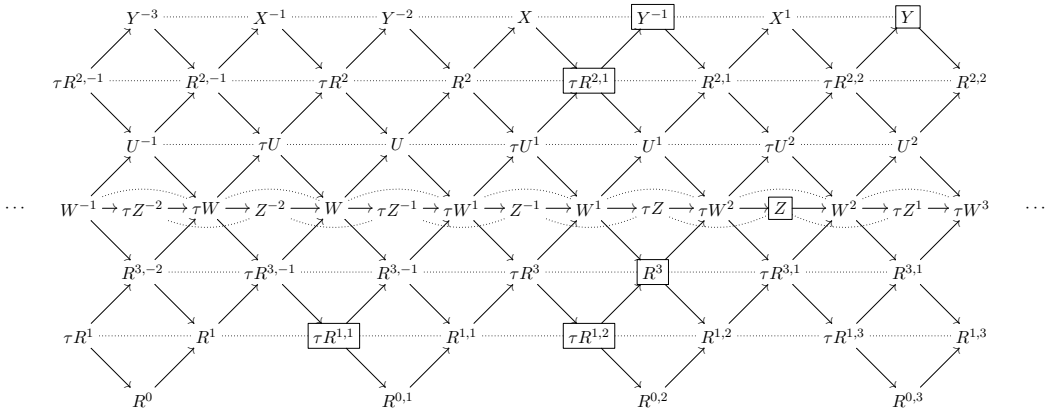


FIGURE 15. The Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$ for type E_7, where $*^j = *(j)$ for $* = R^0$, R^1, R^2, R^3, X, Y, Z, W, U, τR^1, τR^2, τR^3, τZ, τW, or τU.

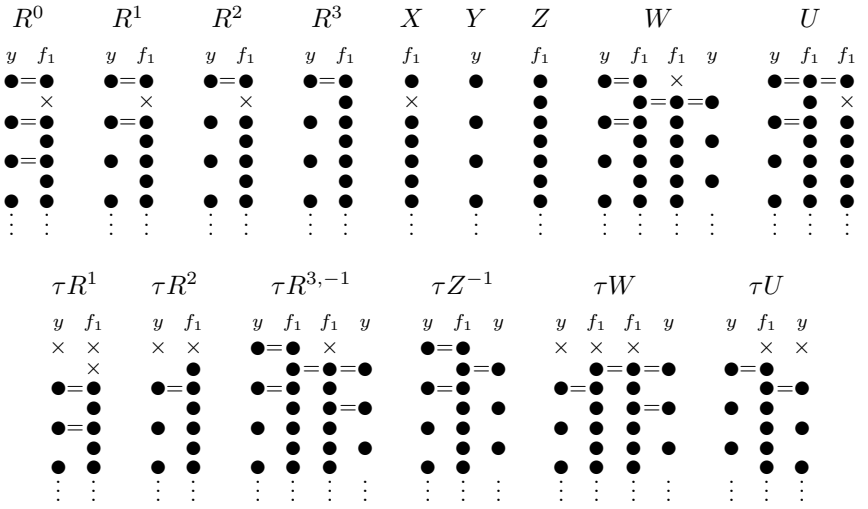


FIGURE 16. Homogeneous k-bases of $\mathbb{Z}$-graded modules for type E_7. The top row represents the component of degree 0.

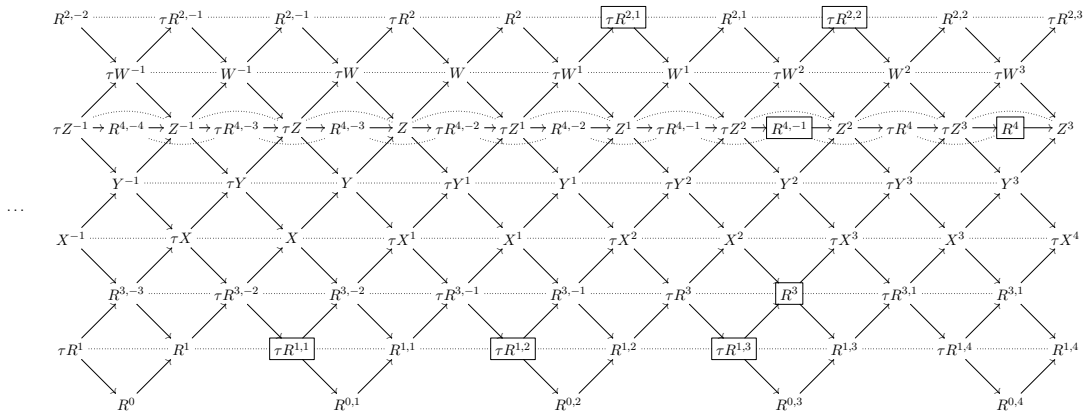


FIGURE 17. The Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$ for type E_8, where $*^j = *(j)$ for $* = R^0$, R^1, R^2, R^3, R^4, X, Y, Z, W, τR^1, τR^2, τR^3, τR^4, τX, τY, τZ, or τW.

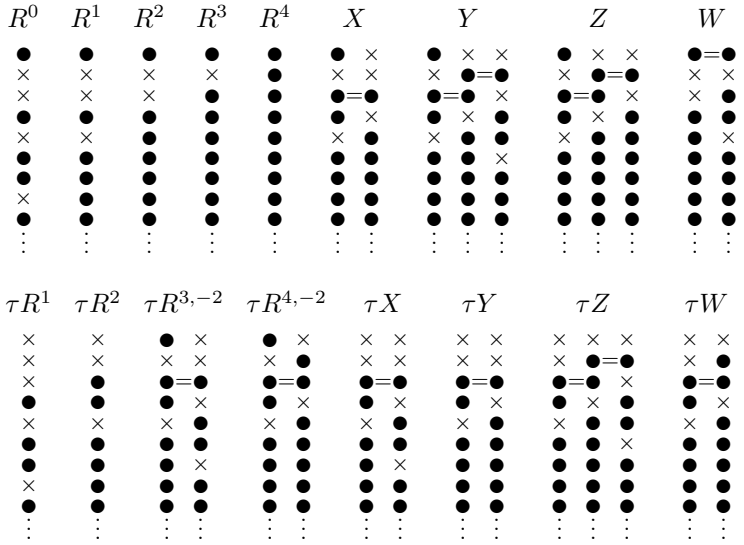
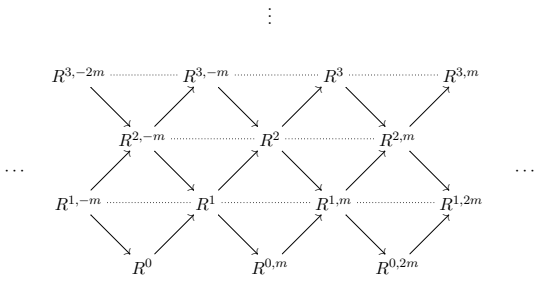
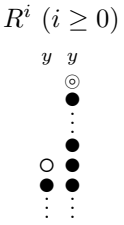


FIGURE 18. Homogeneous k-bases of $\mathbb{Z}$-graded modules for type E_8. The top row represents the component of degree 0.

6.2. **Countable Cohen–Macaulay representation type.** For each of the types A_∞ and D_∞, the Auslander–Reiten quiver of the category $\mathrm{CM}_0^{\mathbb{Z}}R$ has precisely m connected components C^j, $j \in \mathbb{Z}/m\mathbb{Z}$. The components C^0 for types A_∞ and D_∞ are shown in Figures 19(a) and 20, respectively; the corresponding bases are shown in Figures 19(b) and 21. The component C^j is the image of C^0 under the grading shift (j).



(a) Auslander–Reiten quiver



(b) k-bases

FIGURE 19. Type A_∞: (a) the component C^0 of the Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$; (b) homogeneous k-bases of the $\mathbb{Z}$-graded modules. Here $*^j = *(j)$ for $* = R^i$.

In Figure 19(b), the symbols $\circ$ and $\circledcirc$ represent 1 and $x^{-i}y$, respectively. Each black circle represents the preceding basis element in the same column multiplied by x. For type A_∞, the standard silting object is

$$V = \Big(\bigoplus_{i=1}^{\lceil \frac{l}{m} \rceil - 1} \bigoplus_{j=1}^{m} R^{i,j-m}\Big) \oplus \Big(\bigoplus_{j=1}^{l+m-\lceil \frac{l}{m} \rceil m} R^{\lceil \frac{l}{m} \rceil, j-m}\Big).$$

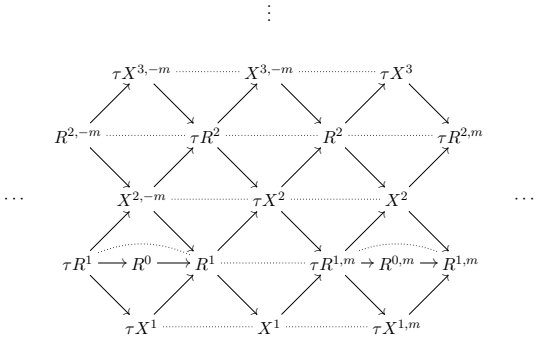


FIGURE 20. The component C^0 of the Auslander–Reiten quiver of $\mathrm{CM}_0^{\mathbb{Z}}R$ for type D_∞, where $*^j = *(j)$ for $* = R^i$, X^i, τR^i, or τX^i.

In Figure 21, the symbols $\circ$, $\circledast$, $\odot$, and $\circledcirc$ mean 1, x, y, and $x^{1-i}y$ in the bases, respectively. In a column labeled by x (respectively, y), each black circle represents the preceding basis element in the same column multiplied by y (respectively, x). For type D_∞, the standard silting object is

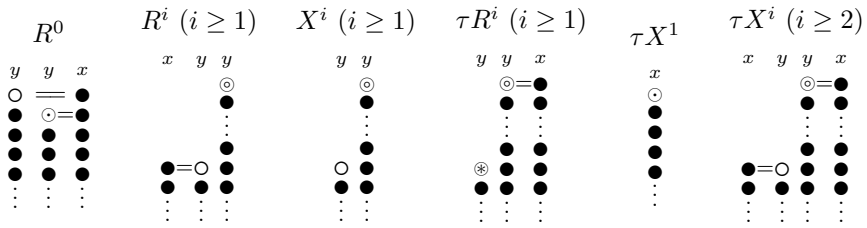


FIGURE 21. Homogeneous k-bases of $\mathbb{Z}$-graded modules for type D_∞.

$$V = (\bigoplus_{j=1}^{m} \tau R^{1,j}) \oplus (\bigoplus_{i=2}^{\lceil \frac{l}{m} \rceil - 1} \bigoplus_{j=1}^{m} \tau R^{i,j}) \oplus (\bigoplus_{j=1}^{l+m-\lceil \frac{l}{m} \rceil m} \tau R^{\lceil \frac{l}{m} \rceil, j})$$
$$\oplus (\bigoplus_{j=1}^{l} \tau X^{1,j}) \oplus (\bigoplus_{j=1}^{m} X^{\lceil \frac{l+j}{m} \rceil + 1, l-m+j-\lceil \frac{l+j}{m} \rceil m}) .$$

Appendix A. Dg resolutions of self-injective Nakayama algebras

A.1. **A_∞-algebras.** Following Section 3.1 of [24], an *A_∞-algebra* is a $\mathbb{Z}$-graded k-vector space A endowed with k-linear maps $m_n\colon A^{\otimes_k n} \to A$ of degree $2-n$, $n \geq 1$, satisfying

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t} \circ (\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$

for all positive integers n, where r and t run through nonnegative integers and s through positive integers. If m_n vanishes for all $n > 2$, then $d = m_1$ and m_2 make A into a dg algebra.

A.2. **Cofibrant dg replacements of self-injective Nakayama algebras.**

Proposition A.1. *Let $n \geq 1$ and $n_x > 1$ be integers. Let C_n be an n-cycle whose arrows are labeled by x and $N_{n,n_x} = kC_n/(x^{n_x})$ the associated self-injective Nakayama algebra. Then N_{n,n_x} is quasi-isomorphic to the cofibrant dg algebra (kQ, d) defined in Definition 4.13.*

Proof. Assume that the vertex set of C_n is $\mathbb{Z}/n\mathbb{Z}$ and that, for each $i \in \mathbb{Z}/n\mathbb{Z}$, there is an arrow from i to $i+m$, where $m \leq n$ is a positive integer coprime to n. Consider the algebra N_{n,n_x}. For a positive even integer p, the set of p-chains (in the sense of [34]) is $\{x^{\frac{p}{2}n_x+1}\}$. For a positive odd integer p, the set of p-chains is $\{x^{\frac{p+1}{2}n_x}\}$. If we have a decomposition of a p-chain as a p_1-chain, ..., and a p_l-chain, then, using the equality $\sum_{i=1}^{l} p_i = p-1$, one checks that the only possibilities are $l = 2$ or $l = n_x$. If we identify the p-chains with the arrows $\beta_{p+1,i}$ in Definition 4.13, the statement follows from Theorem 4.8 of [34]. √

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