ANISOTROPIC ESTIMATES FOR A SOLUTION TO THE HEAT EQUATION ON AN INFINITE LAYER AND ITS APPLICATION TO THE PRIMITIVE EQUATIONS

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Abstract. In this paper, we consider the heat equation on the infinite layer $\Omega = \mathbb{R}^d \times (-1, 0)$. By an explicit expression of the solution, we establish an $L^1(\mathbb{R}_+)$-$\dot{B}^s_{p,1}(\mathbb{R}^d; L^r(-1, 0))$ estimate. As an application, we solve the primitive equations in the framework of scaling invariant spaces globally in time. Although our result on the existence of the global solutions with small initial data is not altogether new, our method relies on the new anisotropic estimate of the heat equations, which might have future application to other equations.

1. Introduction

We consider the heat equation on the infinite layer $\Omega = \mathbb{R}^d \times (-1, 0)$ $(d \geq 1)$:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \times J, \\ \partial_z u = 0 & \text{on } \mathbb{R}^d \times \{-1, 0\} \times J, \\ u(0) = a & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $J = (0, T)$ or $J = \mathbb{R}_+ = (0, \infty)$ and u is a unknown function, while u_0 and f are given initial data and external data, respectively.

The first aim of this paper is to establish an L^1 in time estimate for a solution to (1.1). Generally speaking, estimates of solutions not only for (1.1), but also for the one to the X-valued abstract evolution equation

$$\begin{cases} \partial_t u + Au = f & \text{in } J, \\ u(0) = a \end{cases} \tag{1.2}$$

play a crucial role to construct a local solution and a global small solution to various nonlinear partial differential equations of parabolic type, where X is a proper Banach space and A is a closed operator on X, such as $-\Delta$, Stokes operator, or elliptic operator of second order. Typical examples are L^p-L^q estimates for the semigroup with its generator $-A$, and maximal L^q-regularity of (1.2). The method of solving nonlinear partial differential equations (more precisely, their transformed equations of integral form) with L^p-L^q estimates of the corresponding semigroup is often called the Kato iteration method due to his pioneer work [30] in 1984. This method is quite effective to prove the local well-posedness and the global well-posedness for small initial data of various semilinear equations, such as the Navier–Stokes equations (cf. Kato [30]), semilinear heat equations (cf. Giga [19]), the Keller-Segel equation (cf. Nagai [41]), and drift-diffusion systems (cf. Kurokiba–Ogawa [36]) etc.

However, this method cannot be applied for quasilinear equations, such as free-boundary problems for fluids in Lagrangian coordinates. One of the reasons is that it would be necessary to estimate the second derivative of the solution, which yields the time singularity t^{-1}. The approach of the maximal regularity is effective to avoid this difficulty. Here, we say that a proper closed operator A has maximal

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L^q-regularity if for given $a \in X_\gamma = (X, D(A))_{1-1/q,q}$ $(1 < q < \infty)$ or $a \in X_\gamma = X$ $(q = 1)$, there exists a solution $u \in C(\overline{J}; X_\gamma)$ of (1.2) with the estimate

$$\|u\|_{L^\infty(J;X_\gamma)} + \left\|\frac{du}{dt}\right\|_{L^q(J;X)} + \|Au\|_{L^q(J;X)} \le C\left(\|a\|_{X_\gamma} + \|f\|_{L^q(J;X)}\right), \tag{1.3}$$

where C is a positive constant independent of the data u_0 and f. For $1 < q < \infty$, general theories of the maximal L^q-regularity were developed by many authors (cf. Ladyzěnskaya–Solonnikov–Ural'ceva [37], Dore–Venni [17], Giga–Sohr [24], Amann [2], Kalton–Weis [29], Weis [49], Denk–Hieber–Prüss [16]) within the framework on UMD Banach spaces X. On the other hand, such general theories cannot be applied to show the maximal L^1-regularity. One of the advantages of the maximal L^1-regularity is that initial data does not require a higher regularity. Results of the maximal L^1-regularity were established by Da Prato–Grisvard [10], Chemin [9], Danchin [11, 12], Danchin–Mucha [14, 15], Danchin–Hieber–Mucha–Tolksdorf [13], Ogawa–Shimizu [42, 43, 44, 45] in the setting of homogeneous Besov spaces $\dot{B}^\alpha_{p,1}$, and by Giga–Saal [23] and Iwabuchi [28], in the settings of non-UMD spaces. See also Shibata–Watanabe [47] and Kuo–Shibata [35] to overview this direction. In this paper, we define the homogeneous Besov spaces $\dot{B}^\alpha_{p,q}(\mathbb{R}^d)$ on the whole space $\mathbb{R}^d$ as follows:

Definition 1.1. Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be a Littlewood–Paley dyadic partition of unity satisfying

$$\begin{aligned}
&\mathcal{F}\varphi_j \in C_0^\infty(\mathbb{R}^d) \text{ with } 0 \le [\mathcal{F}\varphi_j](\xi) \le 1 \ (\xi \in \mathbb{R}^d),\\
&\operatorname{supp} \mathcal{F}\varphi_j \subset \{\xi \in \mathbb{R}^d\,;\, 2^{j-1} \le |\xi| \le 2^{j+1}\},\\
&\sum_{j\in\mathbb{Z}} [\mathcal{F}\varphi_j](\xi) = 1 \qquad (\xi \in \mathbb{R}^d \setminus \{0\}),
\end{aligned}$$

where $[\mathcal{F}\varphi_j](\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi_j(x) e^{-ix\cdot\xi}\, dx$ denotes the Fourier transformation of φ_j. We define the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ $(s \in \mathbb{R},\ 1 \le p, q \le \infty)$ by

$$\dot{B}^s_{p,q}(\mathbb{R}^d) \equiv \left\{ f \in \mathcal{S}'/\mathcal{P}\,;\, \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty \right\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} = \left\| \left\{ 2^{sj} \|\varphi_j * f\|_{L^p(\mathbb{R}^d)} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q_j},$$

where $\mathcal{S}'$ and $\mathcal{P}$ denote the set of tempered distributions and the set of polynomials, respectively.

Remark 1.2. *It is well-known that if $s < d/p$ or $(s,q) = (d/p, 1)$, then the above Besov space $\dot{B}^s_{p,q}(\mathbb{R}^d)$ can be identified with*

$$\left\{ f \in \mathcal{S}'\,;\, \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^d)} < \infty,\ f = \sum_{j\in\mathbb{Z}} \varphi_j * f \text{ in } \mathcal{S}' \right\}.$$

Under the setting of anisotropic domains, such as the infinite layer $\Omega = \mathbb{R}^d \times (-1, 0)$, it might be natural to set X to be also anisotropic, for instance, $X = \dot{B}^\alpha_{p,1}(\mathbb{R}^d; L^r(-1,0))$ with the norm

$$\|f\|_{\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0))} = \sum_{j\in\mathbb{Z}} 2^{\alpha j} \left(\int_{\mathbb{R}^d} \left(\int_{-1}^0 |(\varphi_j *_{\mathrm{H}} f)(x,z)|^r \, dz \right)^{p/r} dx \right)^{1/p},$$

where $*_{\mathrm{H}}$ denotes the convolution on $\mathbb{R}^d$. Before stating our main result, for given function a on Ω with

$$\overline{a}(x) \equiv \int_{-1}^0 a(x,z)\, dz = 0, \quad x \in \mathbb{R}^d,$$

we define the solution operator $S(t)a$ of (1.1) with $f = 0$ by

$$
\begin{aligned}
[S(t)a](x,z) &= \sum_{\ell=1}^{\infty} e^{-\pi^2\ell^2 t}\left(\int_{-1}^{0}(G_t *_{\mathrm{H}} a)(x,\zeta)\cos(\pi\ell\zeta)\,d\zeta\right)\cos(\ell\pi z) \\
&= \frac{1}{2}\sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t}\left(\int_{-1}^{1}(G_t *_{\mathrm{H}} (Ea))(x,\zeta)e^{-i\pi\ell\zeta}\,d\zeta\right)e^{i\pi\ell z},
\end{aligned}
\tag{1.4}
$$

where $G_t(x) = \dfrac{1}{(4\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}}$ $(x\in\mathbb{R}^d,\ t>0)$ denotes the d-dimensional heat kernel, and Ea denotes the even extension of a in z-component with respect to $z=0$, respectively. It is not difficult to show that $S(t)a$ formally solves (1.1) with $f=0$. Our first main result is the following smoothing estimate of $S(t)a$ in the space L^1 in time.

Theorem 1.3. *Let $1\le p,\, r\le\infty$ and $\alpha\in\mathbb{R}$.*

(1) *For every*
$$a\in\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)) \quad \text{with } \overline{a}=0,$$
it holds that
$$\Delta_{\mathrm{H}}S(\cdot)a\in L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))$$
with the estimate
$$\|\Delta_{\mathrm{H}}S(\cdot)a\|_{L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\le C\|a\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))},$$
where $\Delta_{\mathrm{H}}=\sum_{j=1}^{d}\partial_j^2$;

(2) *For every a with*
$$\partial_z a\in\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0)) \quad \text{and } \overline{a}=0,$$
it holds that
$$\partial_z^2 S(\cdot)a\in L^1(\mathbb{R}_+;\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0)))$$
with the estimate
$$\left\|\partial_z^2 S(\cdot)a\right\|_{L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\le C\|\partial_z a\|_{\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0))};$$

(3) *For every a with*
$$a\in\dot{B}^{\alpha}_{p,q}(\mathbb{R}^d;L^r(-1,0)),\ \partial_z a\in\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0)), \quad \text{and } \overline{a}=0,$$
$S(t)a$ solves the heat equation (1.1) with $f=0$ and has the regularity properties
$$S(\cdot)a\in C\left(\overline{\mathbb{R}_+};\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))\right)$$
and
$$\partial_t S(\cdot)a,\ \Delta S(\cdot)a\in L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))$$
with the estimate
$$
\begin{aligned}
&\|\partial_t S(\cdot)a\|_{L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}+\|\Delta S(\cdot)a\|_{L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))} \\
&\qquad\le C\left(\|a\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))}+\|\partial_z a\|_{\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0))}\right),
\end{aligned}
$$
where C is a positive constant.

Remark 1.4. (i) *In the case $\Omega=\mathbb{R}^{d+1}$ instead of $\Omega=\mathbb{R}^d\times(-1,0)$, it is well-known that from the theory of the maximal L^1-regularity, the solution u to (1.1) with $f=0$ satisfies the estimate*
$$\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^{d+1}))}+\|\Delta u\|_{L^1(\mathbb{R}_+;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^{d+1}))}\le C\|a\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^{d+1})}$$
for all $\alpha\in\mathbb{R}$ and $1\le p\le\infty$. The additional assumption of the regularity $\partial_z a\in\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0))$ in (2) and (3) may be regarded as a "cost" of choosing the larger space $L^r(-1,0)$ than $\dot{B}^{0}_{r,1}(-1,0)$ in the vertical component.

(ii) *From this theorem, it seems to be difficult to show that* $\{S(t)\}_{t\geq 0}$ *is a bounded* C_0*-semigroup. Indeed, (3) shows us that the strong continuity requires the additional regularity* $\partial_z a \in \dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d; L^r(-1,0))$.

(iii) *The Neumann boundary condition is not essential for this theorem, and it might not be difficult to establish the same assertion for the Dirichlet boundary condition, or the mixed Dirichlet–Neumann boundary condition. In such cases, the mean-free condition* $\overline{a} = 0$ *seems to be redundant.*

(iv) *It seems to be possible to replace the assumption* $\partial_z a \in \dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d; L^r(-1,0))$ *by* $\partial_z^\gamma Ea \in \dot{B}^{\alpha-\gamma}_{p,1}(\mathbb{R}^d; L^r(-1,1))$ $(0 < \gamma < 2)$*, where* E *denotes the even extension operator, and* ∂_z^γ *denotes the fractional derivative in* z *of order* γ*, which is defined via Fourier coefficients. However, in such cases, we cannot take* $\gamma = 0$.

As a corollary, we also establish the following type of estimate for the Duhamel term with the given external data f.

Corollary 1.5. *Let* $1 \leq p, r \leq \infty$ *and* $\alpha \in \mathbb{R}$. *For* $J = (0,T)$ *with* $0 < T \leq \infty$*, define*

$$(Ff)(t) = \int_0^t S(t-s)f(s)\,ds, \qquad t \in J.$$

(1) *For every*

$$f \in L^1(J; \dot{B}^\alpha_{p,1}(\mathbb{R}^d; L^r(-1,0))) \quad \text{with } \overline{f} = 0,$$

it holds that

$$\Delta_{\mathrm{H}} Ff \in L^1(J; \dot{B}^\alpha_{p,1}(\mathbb{R}^d; L^r(-1,0)))$$

with the estimate

$$\|\Delta_{\mathrm{H}} Ff\|_{L^1(J;\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0)))} \leq C \|f\|_{L^1(J;\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0)))},$$

where $C > 0$ *is independent of* T*;*

(2) *For every* f *with*

$$\partial_z f \in L^1(J; \dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d; L^r(-1,0))) \quad \text{and } \overline{f} = 0,$$

it holds that

$$\partial_z^2 Ff \in L^1(J; \dot{B}^\alpha_{p,1}(\mathbb{R}^d; L^r(-1,0)))$$

with the estimate

$$\left\|\partial_z^2 Ff\right\|_{L^1(J;\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0)))} \leq C \|\partial_z f\|_{L^1(J;\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0)))},$$

where $C > 0$ *is independent of* T*;*

(3) *For every*

$$f \in L^1(J; \dot{B}^\alpha_{p,1}(\mathbb{R}^d; L^r(-1,0)))$$

with

$$\partial_z f \in L^1(J; \dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d; L^r(-1,0))) \quad \text{and } \overline{f} = 0,$$

it holds that

$$\partial_t Ff \in L^1(J; \dot{B}^\alpha_{p,1}(\mathbb{R}^d; L^r(-1,0)))$$

with the estimate

$$\begin{aligned}&\|\partial_t Ff\|_{L^1(J;\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\\ &\quad\leq C\left(\|f\|_{L^1(J;\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0)))} + \|\partial_z f\|_{L^1(J;\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\right),\end{aligned}$$

where $C > 0$ *is independent of* T.

Remark 1.6. *In view of* [4, Theorem 4.5]*, it seems impossible to prove maximal* L^1*-regularity with* $f \in \dot{B}^\alpha_{p,q}(\mathbb{R}^d; L^r(-1,0))$*, especially in the case* $r = 1$*. However, by an additional regularity of* f *such as* $\partial_z f \in \dot{B}^{\alpha-1}_{p,q}(\mathbb{R}^2; L^r(-1,0))$*, we may recover the corresponding estimate to the maximal* L^1*-regularity.*

As an application of the linear theory in Theorem 1.3 and Corollary 1.5, we shall solve the three-dimensional primitive equations on the flat layer $\Omega = \mathbb{R}^2 \times (-1, 0)$:

$$\text{(PE)} \qquad \begin{cases} \partial_t v - \Delta v + (v \cdot \nabla_{\mathrm{H}}) v + w(\partial_z v) + \nabla_{\mathrm{H}} \Pi = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_z \Pi = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div}_{\mathrm{H}} v + \partial_z w = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_z v = 0, \quad w = 0 & \text{in } \mathbb{R}^2 \times \{-1, 0\} \times \mathbb{R}_+, \\ v(0) = v_0 & \text{in } \Omega. \end{cases}$$

The unknown functions $v = {}^{\top}(v^1, v^2)$, w and Π denote the horizontal velocity, the vertical velocity, and the scalar pressure of the fluid, respectively. The full velocity is given by ${}^{\top}(v, w) = (v^1, v^2, w)$. The operator ∇_{H} stands for the horizontal gradient, namely, $\nabla_{\mathrm{H}} = {}^{\top}(\partial_{x_1}, \partial_{x_2})$. The operator $\operatorname{div}_{\mathrm{H}}$ denotes the horizontal divergence, namely, $\operatorname{div}_{\mathrm{H}} v = \partial_{x_1} v^1 + \partial_{x_2} v^2$. The primitive equations (PE) are derived from the incompressible Navier–Stokes equations with the hydrostatic approximation. We note that, under the boundary condition $(\mathrm{PE})_4$, the vertical velocity w is determined from the horizontal velocity v by

$$w(x, z) = -\int_{-1}^{z} \operatorname{div}_{\mathrm{H}} v(x, \zeta)\, d\zeta =: [w(v)](x, z).$$

The boundary condition $w = 0$ on $\mathbb{R}^2 \times \{0\}$ yields

$$\operatorname{div}_{\mathrm{H}} \overline{v} = \operatorname{div}_{\mathrm{H}} \left(\int_{-1}^{0} v(x, \zeta)\, d\zeta \right) = 0, \qquad x \in \mathbb{R}^2.$$

Accordingly, we may rewrite (PE) to the following system of v:

$$\text{(1.5)} \qquad \begin{cases} \partial_t v + (v \cdot \nabla_{\mathrm{H}}) v + [w(v)] \partial_z v - \Delta v + \nabla_{\mathrm{H}} \Pi = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_z \Pi = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div}_{\mathrm{H}} \overline{v} = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_z v = 0 & \text{on } \mathbb{R}^2 \times \{-1, 0\} \times \mathbb{R}_+, \\ v(0) = v_0 & \text{in } \Omega. \end{cases}$$

In 1990s, Lions–Temam–Wang [38, 39, 40] started the mathematical studies for the primitive equations. Cao–Titi [8] and Kukavica–Ziane [34] proved the global well-posedness for large initial data in H^1 in the framework of L^2. Later, by applying the Fujita–Kato method, Hieber–Kashiwabara [27] (and also Hieber–Hussein–Kashiwabara [26]) proved the global well-posedness for large data in the critical Sobolev space $H^{2/p,p}$ $(1 < p < \infty)$ in the framework of L^p-spaces. Giga–Gries–Hieber–Hussein–Kashiwabara [20] applied the maximal L^q-regularity of the linearized system to construct a global solution for large data. They also proved the analyticity of the solution.

On the other hand, from the view point of the Fujita–Kato principle (cf. [18]), it is natural to choose a scaling invariant space as the solution space. To see this, let us review the well-posedness results of the Navier–Stokes equations. The Sobolev space $\dot{H}^{-1+d/2,2}(\mathbb{R}^d)$, which is found by [18] is very well-known as the space of initial data to guarantee well-posedness of the d-dimensional Navier–Stokes equations on the whole space $\mathbb{R}^d$ $(d \geq 2)$. Kato [30] and Giga–Miyakawa [22] extend the initial data space to $L^d(\mathbb{R}^d)$. Later on, Kozono–Yamazaki [33], Cannone–Planchon [7], Cannone [6] and Planchon [46] also extended it to the homogeneous Besov spaces $\dot{B}^{-1+d/p}_{p,q}(\mathbb{R}^d)$ $(d < p < \infty,\ 1 \leq q \leq \infty)$. See also Kozono–Shimizu [31, 32]. Whereas, Bourgain–Pavlović [5], Yoneda [50] and Wang [48] proved ill-posedness of the Navier–Stokes equations for initial data in $\dot{B}^{-1}_{\infty,q}(\mathbb{R}^d)$ $(1 \leq q \leq \infty)$ in the sense of break down of continuity of solutions with respect to initial data, which means that the spaces of initial data satisfying well-posedness and ill-posedness are characterized within the framework of homogeneous Besov spaces.

Getting back to the primitive equations, in this direction, Giga–Gries–Hieber–Hussein–Kashiwabara [21] constructed a local solution for initial data with the regularity $L^\infty(\mathbb{R}^2; L^1(-1, 0))$ and also showed that the solution still has the regularity $L^q(\delta, T; H^{2,p}(\Omega)) \cap H^{1,q}(\delta, T; L^p(\Omega))$ $(1/p + 1/q \leq 1)$ for any

$T>0$ with some $\delta>0$. The anisotropic space $L^\infty(\mathbb{R}^2;L^1(-1,0))$ comes from the scaling invariance

$$\|v_\lambda\|_{C([0,T/\lambda^2);L^\infty_{\mathrm{H}}(\mathbb{R}^2;L^1(-1/\lambda,0)))}=\|v\|_{C([0,T);L^\infty_{\mathrm{H}}(\mathbb{R}^2;L^1(-1,0)))},$$

where $v_\lambda(x_1,x_2,z,t)=\lambda v(\lambda x_1,\lambda x_2,\lambda z,\lambda^2 t)$ $(\lambda>0)$. It should be noticed that v_λ satisfies (1.5) if v is a solution to (1.5). It is an interesting problem that to what extent it is possible to extend the space of initial data to guarantee well-posedness of the primitive equations. To consider this problem, we prove the well-posedness of the primitive equations in the framework of homogeneous Besov spaces.

Our second main result is the well-posedness of (1.5) in other scaling invariant spaces which come from the structure of the primitive equations. To state the result, we introduce the function spaces X_{H} and X_{V} by

$$\begin{aligned}
X_{\mathrm{H}}&\equiv\left\{v_{\mathrm{H}}\in C([0,\infty);\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2))\,;\,\partial_t v_{\mathrm{H}},\,Av_{\mathrm{H}}\in L^1(\mathbb{R}_+;\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2))\right\},\\
X_{\mathrm{V}}&\equiv\Big\{v_{\mathrm{V}}\in C([0,\infty);\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))\\
&\qquad;\,\partial_t v_{\mathrm{V}},\,\Delta_{\mathrm{H}}v_{\mathrm{V}},\,\partial_z^2 v_{\mathrm{V}},\,\nabla_{\mathrm{H}}\partial_z v_{\mathrm{V}}\in L^1(\mathbb{R}_+;\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))),\,\int_{-1}^0 v_{\mathrm{V}}\,d\zeta=0\Big\},
\end{aligned}$$

which are equipped with the norms

$$\begin{aligned}
\|v_{\mathrm{H}}\|_{X_{\mathrm{H}}}&=\|v_{\mathrm{H}}\|_{L^\infty(\mathbb{R}_+;\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2))}+\|\partial_t v_{\mathrm{H}},\,Av_{\mathrm{H}}\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2))},\\
\|v_{\mathrm{V}}\|_{X_{\mathrm{V}}}&=\|v_{\mathrm{V}}\|_{L^\infty(\mathbb{R}_+;\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}+\left\|\partial_t v_{\mathrm{V}},\,\Delta_{\mathrm{H}}v_{\mathrm{V}},\,\partial_z^2 v_{\mathrm{V}},\,\nabla_{\mathrm{H}}\partial_z v_{\mathrm{V}}\right\|_{L^1(\mathbb{R}_+;\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))},
\end{aligned}$$

respectively. Here, $\mathbb{P}=I+\nabla_{\mathrm{H}}(-\Delta_{\mathrm{H}})^{-1}\mathrm{div}_{\mathrm{H}}$ and $A=-\mathbb{P}\Delta_{\mathrm{H}}$ denote the two-dimensional Helmholtz projection and the Stokes operator, respectively. We note that for all $v_{\mathrm{H}}\in X_{\mathrm{H}}$ and $v_{\mathrm{V}}\in X_{\mathrm{V}}$, it holds that

$$\begin{aligned}
\|v_{\mathrm{H}}\|_{X_{\mathrm{H}}}&=\|v_{\mathrm{H},\lambda}\|_{L^\infty(\mathbb{R}_+;\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2))}+\|\partial_t v_{\mathrm{H},\lambda},\,Av_{\mathrm{H},\lambda}\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2))},\\
\|v_{\mathrm{V}}\|_{X_{\mathrm{V}}}&=\|v_{\mathrm{V},\lambda}\|_{L^\infty(\mathbb{R}_+;\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1/\lambda,0)))}\\
&\qquad+\left\|\partial_t v_{\mathrm{V},\lambda},\,\Delta_{\mathrm{H}}v_{\mathrm{V},\lambda},\,\partial_z^2 v_{\mathrm{V},\lambda},\,\nabla_{\mathrm{H}}\partial_z v_{\mathrm{V},\lambda}\right\|_{L^1(\mathbb{R}_+;\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1/\lambda,0)))}
\end{aligned}$$

for all $\lambda>0$, where $v_{\mathrm{H},\lambda}(x_1,x_2,t)=\lambda v_{\mathrm{H}}(\lambda x_1,\lambda x_2,\lambda^2 t)$ and $v_{\mathrm{V},\lambda}(x_1,x_2,z,t)=\lambda v_{\mathrm{V}}(\lambda x_1,\lambda x_2,\lambda z,\lambda^2 t)$.

Theorem 1.7. *Let $1\le p<4$. Let $v_0=\overline{v_0}+\widetilde{v_0}$ satisfy $\overline{v_0}=\int_{-1}^0 v_0\,dz\in\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2)$ and $\widetilde{v_0}=v_0-\overline{v_0}\in\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))$ and $\partial_z\widetilde{v_0}\in\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))$. There exists $\delta>0$ such that if $\|\overline{v_0}\|_{\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2)}+\|\widetilde{v_0}\|_{\dot B^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\le\delta$, then there exists a unique solution $v=\overline{v}+\widetilde{v}$ to (1.5) with $(\overline{v},\widetilde{v})\in X_{\mathrm{H}}\times X_{\mathrm{V}}$.*

Remark 1.8. (i) *In fact, $\overline{v}$ is governed by a two-dimensional incompressible flow interacted with the remaining part $\widetilde{v}=v-\overline{v}$ (we will give the detail in (3.1)). This is the reason why we chose X_{H} defined above as the solution space for $\overline{v}$. Although [21] required the full initial velocity has the regularity $v_0\in L^\infty(\mathbb{R}^2;L^1(-1,0))$, this theorem states that the weaker regularity assumption $\overline{v_0}\in\mathbb{P}\dot B^{-1+2/p}_{p,1}(\mathbb{R}^2)$ for $\overline{v_0}$ is sufficient to solve (1.5).*

(ii) *The restriction on p as $1\le p<4$ comes from the following type of bilinear estimate with $d=2$*

$$\|fg\|_{\dot B^{-1+d/p}_{p,1}(\mathbb{R}^d)}\le C\,\|f\|_{\dot B^{-1+d/p}_{p,1}(\mathbb{R}^d)}\,\|g\|_{\dot B^{d/p}_{p,1}(\mathbb{R}^d)}$$

for $1\le p<2d$. See, Proposition 3.3 (2).

2. Linear theory

This section is devoted to the proof of Theorem 1.3 and Corollary 1.5. We first introduce the following lemma about characterization of Besov norms, which plays an essential role to prove Theorem 1.3.

Proposition 2.1 (Bahouri–Chemin–Danchin [3, Theorem 2.34]). *For $\alpha < 0$, there exist $C_1, C_2 > 0$ such that*

$$C_1 \|u\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d)} \le \int_0^\infty t^{-\alpha/2} \left\|e^{t\Delta}u\right\|_{L^p(\mathbb{R}^d)} \frac{dt}{t} \le C_2 \|u\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d)}$$

holds, where C_1, $C_2 > 0$ are independent of u.

Remark 2.2. *The proof of Proposition 2.1 is based on direct estimates of the norm via dyadic decomposition. Thus, we also see that*

$$C_1 \|u\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^1(-1,0))} \le \int_0^\infty t^{-\alpha/2} \|G_t *_{\mathrm{H}} u\|_{L^p(\mathbb{R}^d;L^1(-1,0))} \frac{dt}{t} \le C_2 \|u\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^1(-1,0))}$$

holds.

Proof of Theorem 1.3. It suffices to show the assertion for $\alpha = -2s < 0$ since $(-\Delta)^\alpha$ is an isomorphism from $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^d)$ to $\dot{B}^{-\alpha}_{p,q}(\mathbb{R}^d)$. The approach is closed to the proof of [13, Proposition 1.18].
(1) By (1.4) and Proposition 2.1, we have

$$\begin{aligned}
&\|\Delta_{\mathrm{H}} S(\cdot)a\|_{L^1(0,T;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))} \\
&\le C \int_0^T \int_0^\infty \tau^s \left\| \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t} \left(\int_{-1}^1 (\Delta_{\mathrm{H}} G_{\tau+t}) *_{\mathrm{H}} (Ea))(\cdot,\zeta) e^{-i\pi\ell\zeta}\, d\zeta \right) e^{i\pi\ell z} \right\|_{L^p(\mathbb{R}^d;L^r_z(-1,0))} \frac{d\tau}{\tau}\, dt
\end{aligned}$$

Here using the periodicity of the integrand function and Ea, the Minkowski inequality for integration, and the Hausdorff–Young inequality for convolution, we have

$$\begin{aligned}
&\left\| \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t} \left(\int_{-1}^1 (\Delta_{\mathrm{H}} G_{\tau+t}) *_{\mathrm{H}} (Ea))(x,\zeta) e^{-i\pi\ell\zeta}\, d\zeta \right) e^{i\pi\ell z} \right\|_{L^r_z(-1,0)} \\
&= \frac{1}{2} \left\| \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t} \left(\int_{-1}^1 (\Delta_{\mathrm{H}} G_{\tau+t}) *_{\mathrm{H}} (Ea))(x,\zeta) e^{-i\pi\ell\zeta}\, d\zeta \right) e^{i\pi\ell z} \right\|_{L^r_z(-1,1)} \\
&= \frac{1}{2} \left\| \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t} \left(\int_{-1}^1 (\Delta_{\mathrm{H}} G_{\tau+t}) *_{\mathrm{H}} (Ea))(x,z-\zeta) e^{i\pi\ell\zeta}\, d\zeta \right) \right\|_{L^r_z(-1,1)} \\
&\le \left(\int_{-1}^1 \left| \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t} e^{i\pi\ell\zeta} \right| d\zeta \right) \cdot \|((\Delta_{\mathrm{H}} G_{\tau+t}) *_{\mathrm{H}} a)(x,\cdot)\|_{L^r_z(-1,0)}\,.
\end{aligned} \tag{2.1}$$

Since $e^{-\pi^2\ell^2 t} = \widehat{H_{t/\pi^2}}(\ell)$ holds, where $H_t(z) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}$ is the one-dimensional heat kernel on $(-1,1)$, we have

$$\begin{aligned}
\int_{-1}^1 \left| \sum_{\ell\in\mathbb{Z}\setminus\{0\}} e^{-\pi^2\ell^2 t} e^{i\pi\ell\zeta} \right| d\zeta &= \int_{-1}^1 \left| \sqrt{\frac{2}{\pi}} \left(\sum_{\ell\in\mathbb{Z}} \widehat{H_{t/\pi^2}}(\ell) e^{i\pi\ell\zeta} - 1 \right) \right| d\zeta \\
&\le 2\sqrt{\frac{2}{\pi}} + \frac{2}{\sqrt{\pi}} \int_{-1}^1 \sum_{k\in\mathbb{Z}} H_{t/\pi^2}(\zeta + 2k)\, d\zeta \\
&= 2\sqrt{\frac{2}{\pi}} + \frac{2}{\sqrt{\pi}} \int_{\mathbb{R}} H_{t/\pi^2}(\zeta)\, d\zeta \\
&= 2\sqrt{\frac{2}{\pi}} + \frac{2}{\sqrt{\pi}}.
\end{aligned} \tag{2.2}$$

Here we used the Poisson summation formula (cf. Grafakos [25, Theorem 3.2.8])

$$\frac{1}{\sqrt{2}}\sum_{\ell\in\mathbb{Z}}\widehat{f}(\ell)e^{i\pi\ell z}=\sum_{k\in\mathbb{Z}}f(z+2k),\qquad \widehat{f}(\xi)=\frac{1}{\sqrt{2}}\int_{\mathbb{R}}f(\zeta)e^{-i\pi\zeta\xi}\,d\zeta$$

for any $-1\le z\le 1$. Hence, again by the property of the heat kernel and the Minkowski inequality for integration, we have

$$\begin{aligned}
&\left\|\sum_{\ell\in\mathbb{Z}\setminus\{0\}}e^{-\pi^2\ell^2t}\left(\int_{-1}^{1}((\Delta_{\mathrm{H}}G_{\tau+t})*_{\mathrm{H}}(Ea))(\cdot,\zeta)e^{-i\pi\ell\zeta}\,d\zeta\right)e^{i\pi\ell z}\right\|_{L^p(\mathbb{R}^d;L^r_z(-1,0))}\\
&\le C\left\|\|((\Delta_{\mathrm{H}}G_{\tau+t})*_{\mathrm{H}}a)(x,z)\|_{L^r_z(-1,0)}\right\|_{L^p_x(\mathbb{R}^d)}\\
&\le\frac{C}{\tau+t}\left\|G_{(\tau+t)/2}*_{\mathrm{H}}a\right\|_{L^p(\mathbb{R}^d;L^r(-1,0))}
\end{aligned}$$

for $\tau,t>0$. Here we used $\Delta_{\mathrm{H}}G_{\tau+t}=(\Delta_{\mathrm{H}}G_{(\tau+t)/2})*_{\mathrm{H}}G_{\tau+t}$ and $\left\|\Delta_{\mathrm{H}}G_{(\tau+t)/2}\right\|_{L^1(\mathbb{R}^d)}\le C(\tau+t)^{-1}$. Therefore, by the Fubini theorem and Proposition 2.1, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty\tau^s\left\|\sum_{\ell\in\mathbb{Z}\setminus\{0\}}e^{-\pi^2\ell^2t}\left(\int_{-1}^{0}(G_\tau*_{\mathrm{H}}(\Delta_{\mathrm{H}}G_t)*_{\mathrm{H}}a)(\cdot,\zeta)e^{-i\pi\ell\zeta}\,d\zeta\right)e^{i\pi\ell z}\right\|_{L^p(\mathbb{R}^d;L^r_z(-1,0))}\frac{d\tau}{\tau}\,dt\\
&\le C\int_0^\infty\int_0^\infty\frac{\tau^s}{\tau+t}\left\|G_{(\tau+t)/2}*_{\mathrm{H}}a\right\|_{L^p(\mathbb{R}^d;L^r(-1,0))}\frac{d\tau}{\tau}\,dt\\
&\le C\int_0^\infty\int_t^\infty(\tau-t)^{s-1}\frac{1}{\tau}\left\|G_{\tau/2}*_{\mathrm{H}}a\right\|_{L^p(\mathbb{R}^d;L^r(-1,0))}\,d\tau dt\\
&=C\int_0^\infty\left(\int_0^\tau(\tau-t)^{s-1}\,dt\right)\left\|G_{\tau/2}*_{\mathrm{H}}a\right\|_{L^p(\mathbb{R}^d;L^r(-1,0))}\frac{d\tau}{\tau}\\
&\le C\int_0^\infty\tau^s\left\|G_{\tau/2}*_{\mathrm{H}}a\right\|_{L^p(\mathbb{R}^d;L^r(-1,0))}\frac{d\tau}{\tau}\\
&\le C\|a\|_{B^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0))}.
\end{aligned}$$

Consequently, we have already obtained

$$\|\Delta_{\mathrm{H}}S(\cdot)a\|_{L^1(\mathbb{R}_+;\dot{B}^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\le C\|a\|_{B^\alpha_{p,1}(\mathbb{R}^d;L^r(-1,0))},\tag{2.3}$$

for some positive constant C.

(2) We shall estimate $\partial_z^2S(\cdot)a$. For $\alpha=-s/2<0$, in the same way as in (2.1), we have

$$\begin{aligned}
&\left\|\partial_z^2S(\cdot)a\right\|_{L^1(0,T;\dot{B}^\alpha_{p,q}(\mathbb{R}^d;L^r(-1,0)))}\\
&\le C\int_0^T\int_0^\infty\tau^s\left\|\sum_{\ell\in\mathbb{Z}\setminus\{0\}}\ell e^{-\pi^2\ell^2t}\left(\int_{-1}^{1}(G_{\tau+t}*_{\mathrm{H}}(E(\partial_za)))(\cdot,\zeta)e^{-i\pi\ell\zeta}\,d\zeta\right)e^{i\pi\ell z}\right\|_{L^p(\mathbb{R}^d;L^r_z(-1,1))}\frac{d\tau}{\tau}\,dt\\
&\le C\int_0^T\int_0^\infty\tau^s\left(\int_{-1}^{1}\left|\sum_{\ell\in\mathbb{Z}\setminus\{0\}}\ell e^{-\pi^2\ell^2t}e^{i\pi\ell\zeta}\right|d\zeta\right)\cdot\|G_{\tau+t}*_{\mathrm{H}}(\partial_za)\|_{L^p(\mathbb{R}^d;L^r_z(-1,0))}\frac{d\tau}{\tau}dt.
\end{aligned}$$

In the same way as in (2.2), one can see that

$$\int_{-1}^{1}\left|\sum_{\ell\in\mathbb{Z}\setminus\{0\}}\ell e^{-\pi^2\ell^2t}e^{i\pi\ell\zeta}\right|d\zeta\le\frac{C}{\sqrt{t}}.$$

Thus by Proposition 2.1 and the Fubini theorem, we have

$$
\begin{aligned}
&\left\|\partial_z^2 S(\cdot)a\right\|_{L^1(0,\infty;\dot{B}^{\alpha}_{p,q}(\mathbb{R}^2;L^r(-1,0)))}\\
\leq& C\int_0^\infty t^{-1/2}\int_0^\infty \tau^s \left\|G_{\tau+t} *_{\mathrm{H}} (\partial_z a)\right\|_{L^p(\mathbb{R}^2;L^r_z(-1,0))}\,\frac{d\tau}{\tau}dt\\
\leq& C\int_0^\infty t^{-1/2}\int_t^\infty (\tau-t)^{s-1}\left\|G_{\tau} *_{\mathrm{H}} (\partial_z a)\right\|_{L^p(\mathbb{R}^2;L^r_z(-1,0))}\,d\tau dt\\
=& C\int_0^\infty \left(\int_0^\tau (\tau-t)^{s-1}t^{-1/2}\,dt\right)\cdot\left\|G_{\tau} *_{\mathrm{H}} (\partial_z a)\right\|_{L^p(\mathbb{R}^2;L^r_z(-1,0))}\,d\tau\\
=& C\int_0^\infty B(s,1/2)\cdot\tau^{s+1/2}\cdot\left\|G_{\tau} *_{\mathrm{H}} (\partial_z a)\right\|_{L^p(\mathbb{R}^2;L^r_z(-1,0))}\,\frac{d\tau}{\tau}\\
\leq& C\left\|\partial_z a\right\|_{\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r_z(-1,0))},
\end{aligned}
\tag{2.4}
$$

where $B(\cdot,\cdot)$ denotes the beta function.
(3) Since $\partial_t(S(t)a)=\Delta(S(t)a)=\Delta_{\mathrm{H}}(S(t)a)+\partial_z^2(S(t)a)$, it follows from (1) and (2) that

$$
\|\partial_t(S(\cdot)a)\|_{L^1(0,\infty;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\leq C\left(\|a\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r_z(-1,0))}+\|\partial_z a\|_{\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^d;L^r_z(-1,0))}\right).
$$

The time continuity at $t=0$ follows by

$$
\|S(t)a-a\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^2;L^r(-1,0))}\leq C\int_0^t\|\partial_s S(s)a\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^2;L^r(-1,0))}\,ds.
$$

□

Proof of Corollary 1.5. (1) Combining (2.3) with the Fubini theorem, we can also establish the estimate of the Duhamel term as

$$
\begin{aligned}
&\|\Delta_{\mathrm{H}}(Ff)\|_{L^1(0,T;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\\
\leq&\int_0^T\int_0^t\|\Delta_{\mathrm{H}}S(t-s)f(s)\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))}\,dsdt\\
=&\int_0^T\left(\int_s^T\|\Delta_{\mathrm{H}}S(t-s)f(s)\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))}\,dt\right)ds\\
\leq& C\|f\|_{L^1(0,T;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}.
\end{aligned}
\tag{2.5}
$$

The above positive constant C can be chosen to be independent of T.
(2) In the same way as in (2.5), combining (2.4) with the Fubini theorem, we have

$$
\begin{aligned}
&\left\|\partial_z^2(Ff)\right\|_{L^1_t(0,T;\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0)))}\\
\leq&\int_0^T\int_0^t\left\|\partial_z^2 S(t-s)f(s)\right\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))}\,dsdt\\
=&\int_0^T\left(\int_s^T\left\|\partial_z^2 S(t-s)f(s)\right\|_{\dot{B}^{\alpha}_{p,1}(\mathbb{R}^2;L^r(-1,0))}\,dt\right)ds\\
\leq& C\|\partial_z f\|_{L^1(0,T;\dot{B}^{\alpha-1}_{p,1}(\mathbb{R}^2;L^r(-1,0)))}.
\end{aligned}
\tag{2.6}
$$

(3) Since $(Ff)(t)=\displaystyle\int_0^t S(t-s)f(s)\,ds$ satisfies

$$
\partial_t(Ff)-\Delta(Ff)=f,
$$

the assertion is an immediate consequence of (1) and (2). □

3. Well-posedness of the primitive equations

This section is devoted to the proof of the well-posedness of the primitive equations, Theorem 1.7.

3.1. **Setup of the contraction mapping.** Decomposing v into $\overline{v}$ and $\widetilde{v}$, we note that (1.5) is equivalent to

$$\begin{cases} \partial_t \overline{v} + A\overline{v} = -\mathbb{P}F_1(\overline{v},\widetilde{v}), & \text{in } \mathbb{R}^2\times\mathbb{R}_+,\\ \partial_t \widetilde{v} - \Delta\widetilde{v} = -F_2(\overline{v},\widetilde{v}), & \text{in } \Omega\times\mathbb{R}_+,\\ \partial_z \widetilde{v} = 0 & \text{on } \mathbb{R}^2\times\{-1,0\}\times\mathbb{R}_+,\\ \overline{v}(0) = \overline{v_0}, & \text{in } \mathbb{R}^2,\\ \widetilde{v}(0) = \widetilde{v_0}, & \text{in } \Omega, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} F_1(\overline{v},\widetilde{v}) &\equiv \overline{\operatorname{div}(u\otimes v)}\\ &= \operatorname{div}_{\mathrm{H}}(\overline{v}\otimes\overline{v}) + \overline{\operatorname{div}_{\mathrm{H}}(\widetilde{v}\otimes\widetilde{v})},\\ F_2(\overline{v},\widetilde{v}) &\equiv \widetilde{\operatorname{div}(u\otimes v)}\\ &= (\widetilde{v}\cdot\nabla_{\mathrm{H}})\overline{v} + (\overline{v}\cdot\nabla_{\mathrm{H}})\widetilde{v} - \widetilde{v}\operatorname{div}_{\mathrm{H}}\widetilde{v} + \widetilde{\operatorname{div}_{\mathrm{H}}(\widetilde{v}\otimes\widetilde{v})} + w(\partial_z\widetilde{v}). \end{aligned}$$

We shall apply the Banach fixed point argument for (3.1) in the space $X_{\mathrm{H}}\times X_{\mathrm{V}}$. For $(\overline{v},\widetilde{v})\in X_{\mathrm{H}}\times X_{\mathrm{V}}$, let $\Phi(\overline{v},\widetilde{v}) = (\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}),\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}))$, where

$$\begin{aligned} [\Phi_{\mathrm{H}}(\overline{v},\widetilde{v})](t) &= e^{-tA}\overline{v_0} - \int_0^t e^{-(t-s)A}\mathbb{P}F_1(\overline{v}(s),\widetilde{v}(s))\,ds,\\ [\Phi_{\mathrm{V}}(\overline{v},\widetilde{v})](t) &= S(t)\widetilde{v_0} - \int_0^t S(t-s)F_2(\overline{v}(s),\widetilde{v}(s))\,ds. \end{aligned}$$

In the following, we shall prove that Φ is a contraction mapping in the ball

$$\mathbb{B}_r \equiv \{(\overline{v},\widetilde{v})\in X_{\mathrm{H}}\times X_{\mathrm{V}}\,;\, \|\overline{v} - e^{-tA}\overline{v_0}\|_{X_{\mathrm{H}}} + \|\widetilde{v} - S(t)\widetilde{v_0}\|_{X_{\mathrm{V}}} \le r,\ \overline{v}(0) = \overline{v_0},\ \widetilde{v}(0) = \widetilde{v_0}\},$$

for small $r>0$.

3.2. **Linear theory.** We first introduce the maximal L^1-regularity for the Stokes equations.

Proposition 3.1 (Maximal L^1-regularity of the Stokes equations, cf. [9, 14, 15, 42, 44])**.** *Let $s\in\mathbb{R}$, $1\le p\le\infty$ and $0<T\le\infty$. Then, for $a_{\mathrm{H}}\in\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2)$ and $f_{\mathrm{H}}\in L^1(0,T;\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))$, there exists a unique solution v_{H} to*

$$\begin{cases} \partial_t v_{\mathrm{H}} + Av_{\mathrm{H}} = f_{\mathrm{H}}, & \text{in } \mathbb{R}^2\times(0,T),\\ v_{\mathrm{H}}(0) = a_{\mathrm{H}}, & \text{in } \mathbb{R}^2, \end{cases}$$

with $v_{\mathrm{H}}\in C([0,T);\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))$ and $\partial_t v_{\mathrm{H}}$, $Av_{\mathrm{H}}\in L^1(0,T;\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))$ with the estimate

$$\begin{aligned} &\|v_{\mathrm{H}}\|_{L^\infty(0,T;\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))} + \|\partial_t v_{\mathrm{H}}\|_{L^1(0,T;\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))} + \|Av_{\mathrm{H}}\|_{L^1(0,T;\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))}\\ &\qquad\le C\left(\|a_{\mathrm{H}}\|_{\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2)} + \|f_{\mathrm{H}}\|_{L^1(0,T;\dot{B}^s_{p,1}(\mathbb{R}^2))}\right). \end{aligned}$$

Applying this proposition with $T=\infty$, we can see that $\Phi_{\mathrm{H}}(\overline{v},\widetilde{v})$ solves

$$\begin{cases} \partial_t(\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}) - e^{-tA}\overline{v_0}) + A(\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}) - e^{-tA}\overline{v_0}) = -\mathbb{P}F_1(\overline{v},\widetilde{v}) & \text{in } \mathbb{R}^2\times\mathbb{R}_+,\\ [\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}) - e^{-tA}\overline{v_0}]\big|_{t=0} = 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{3.2}$$

with the estimate

$$\|\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}) - e^{-tA}\overline{v_0}\|_{X_{\mathrm{H}}} \le C_1\|\mathbb{P}F_1(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}. \tag{3.3}$$

Here we used the inequality

$$\|\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}) - e^{-tA}\overline{v_0}\|_{L^\infty(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))} \le C\|\partial_t(\Phi_{\mathrm{H}}(\overline{v},\widetilde{v}) - e^{-tA}\overline{v_0})\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}$$

by virtue of the zero initial value. Regarding $\Phi_{\mathrm{V}}(\overline{v},\widetilde{v})$, applying Corollary 1.5 with $d=2$, we can see that it solves

$$\begin{cases} \partial_t(\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}) - S(t)\widetilde{v_0}) - \Delta(\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}) - S(t)\widetilde{v_0}) = -F_2(\overline{v},\widetilde{v}) & \text{in } \Omega\times\mathbb{R}_+, \\ \partial_z(\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}) - S(t)\widetilde{v}) = 0 & \text{on } \mathbb{R}^2\times\{-1,0\}\times\mathbb{R}_+, \\ [\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}) - S(t)\widetilde{v}]|_{t=0} = 0 & \text{in } \Omega \end{cases} \tag{3.4}$$

with the estimate

$$\begin{aligned} &\|\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}) - S(\cdot)\widetilde{v_0}\|_{X_{\mathrm{V}}} \\ &\le C_2\left(\|F_2(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} + \|\partial_z F_2(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\right). \end{aligned} \tag{3.5}$$

3.3. **Self mapping property of Φ.** We show $\Phi:\mathbb{B}_r\to\mathbb{B}_r$ for small $r>0$. Since

$$\mathbb{P}f = f + \sum_{\ell=1,2}\begin{pmatrix} R_1R_\ell f^\ell \\ R_2R_\ell f^\ell \end{pmatrix},$$

where R_k and R_ℓ denote the Riesz operators, which are bounded in $\dot{B}^s_{p,q}(\mathbb{R}^2)$, it holds that

$$\|\mathbb{P}F_1(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))} \le C\|F_1(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}.$$

To estimate the right hand side, we use the following type of the bilinear estimate in homogeneous Besov spaces.

Proposition 3.2 (cf. [1, 15, 42, 44]). (1) *Let $1\le p<\infty$. If $f\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)$ and $g\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)$, then $fg\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)$ with the estimate*

$$\|fg\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)} \le C\|f\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)}\|g\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)},$$

where C is independent of f and g.

(2) *Let $1\le p<2d$. If $f\in\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d)$ and $g\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)$, then $fg\in\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d)$ with the estimate*

$$\|fg\|_{\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d)} \le C\|f\|_{\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d)}\|g\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d)},$$

where C is independent of f and g.

The following bilinear estimates for the anisotropic spaces $\dot{B}^{\alpha}_{p,1}(\mathbb{R}^d;L^r(-1,0))$ hold as well, and the proof is straight forward.

Proposition 3.3. (1) *Let $1\le p<\infty$. Let $1\le r,r_1,r_2\le\infty$ satisfy $1/r_1+1/r_2=1/r$. If $f\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^{r_1}(-1,0))$ and $g\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^{r_2}(-1,0))$, then $fg\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^r(-1,0))$ with the estimate*

$$\|fg\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^r(-1,0))} \le C\|f\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^{r_1}(-1,0))}\|g\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^{r_2}(-1,0))},$$

where C is independent of f and g.

(2) *Let $1\le p<2d$. Let $1\le r_1,r_2\le\infty$ satisfy $1/r_1+1/r_2=1$. If $f\in\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d;L^{r_1}(-1,0))$ and $g\in\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^{r_2}(-1,0))$, then $fg\in\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d;L^r(-1,0))$ with the estimate*

$$\|fg\|_{\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d;L^r(-1,0))} \le C\|f\|_{\dot{B}^{-1+d/p}_{p,1}(\mathbb{R}^d;L^{r_1}(-1,0))}\|g\|_{\dot{B}^{d/p}_{p,1}(\mathbb{R}^d;L^{r_2}(-1,0))},$$

where C is independent of f and g.

By this Proposition 3.3 and the inequality

$$\|f\|_{L^\infty(-1,0)} \le \int_{-1}^0 |\partial_z f(z)|\,dz$$

for $f\in L^1(-1,0)$ with $\overline{f}=0$, we have

$$\|F_1(\overline{v},\widetilde{v})\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}$$

$$
\begin{aligned}
&\le C\left(\|\mathrm{div}_{\mathrm{H}}(\overline{v}\otimes\overline{v})\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}+\left\|\overline{\mathrm{div}_{\mathrm{H}}(\widetilde{v}\otimes\widetilde{v})}\right\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}\right)\\
&\le C\left(\left\|\overline{v}-e^{-tA}\overline{v_0}\right\|^2_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}+\left\|e^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\left\|\overline{v}-e^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}+\left\|e^{-tA}\overline{v_0}\right\|^2_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\right)\\
&\quad+C\left(\|\partial_z\widetilde{v}-\partial_z S(t)\widetilde{v_0}\|^2_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\right.\\
&\qquad\left.+\|\partial_z S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\|\partial_z\overline{v}-\partial_z S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}+\|\partial_z S(t)\widetilde{v_0}\|^2_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\right).
\end{aligned}
$$

Thus, by virtue of the interpolation inequalities

$$
\|f\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\le C\|f\|^{1/2}_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}\|Af\|^{1/2}_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)} \tag{3.6}
$$

for $f\in\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)$ with $Af\in\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)$ and

$$
\|\partial_z g\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\le C\|g\|^{1/2}_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\left\|\partial_z^2 g\right\|^{1/2}_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \tag{3.7}
$$

for $g\in\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))$ with $\partial_z^2 g\in\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))$ and $\overline{g}=0$, we have

$$
\begin{aligned}
&\|\mathbb{P}F_1(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\\
&\le C\left\|\overline{v}-e^{-tA}\overline{v_0}\right\|^2_{X_{\mathrm{H}}}\\
&\quad+C\left\|e^{-tA}\overline{v_0}\right\|^{1/2}_{L^\infty(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\left\|Ae^{-tA}\overline{v_0}\right\|^{1/2}_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\left\|\overline{v}-e^{-tA}\overline{v_0}\right\|_{X_{\mathrm{H}}}\\
&\quad+C\left\|e^{-tA}\overline{v_0}\right\|_{L^\infty(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\left\|Ae^{-tA}\overline{v_0}\right\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\\
&\quad+C\|\widetilde{v}-S(t)\widetilde{v_0}\|^2_{X_{\mathrm{V}}}\\
&\quad+C\|S(t)\widetilde{v_0}\|^{1/2}_{L^\infty(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\left\|\partial_z^2 S(t)\widetilde{v_0}\right\|^{1/2}_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\widetilde{v}-S(t)\widetilde{v_0}\|_{X_{\mathrm{V}}}\\
&\quad+C\|\widetilde{v_0}\|_{L^\infty(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\left\|\partial_z^2 S(t)\widetilde{v_0}\right\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}.
\end{aligned}
$$

Thus, for $(\overline{v},\widetilde{v})\in\mathbb{B}_r$, choosing $r>0$ to be sufficiently small, we have

$$
\|\mathbb{P}F_1(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\le\frac{r}{2C_1},
$$

where C_1 is a constant appearing in (3.3), provided $(\overline{v_0},\widetilde{v_0})$ is small enough. By (3.3), we eventually obtain

$$
\left\|\Phi_{\mathrm{H}}(\overline{v},\widetilde{v})-e^{-tA}\overline{v_0}\right\|_{X_{\mathrm{H}}}\le\frac{r}{2}.
$$

At the next step, we estimate $F_2(\overline{v},\widetilde{v})$. We note that the following estimate holds:

$$
\begin{aligned}
&\|F_2(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\le\|(\widetilde{v}\cdot\nabla_{\mathrm{H}})\overline{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\quad+\|(\overline{v}\cdot\nabla_{\mathrm{H}})\widetilde{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}+\|\widetilde{v}\,(\mathrm{div}_{\mathrm{H}}\,\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\quad+\left\|\mathrm{div}_{\mathrm{H}}\widetilde{(\widetilde{v}\otimes\widetilde{v})}\right\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}+\left\|\left(\int_{-1}^{z}\mathrm{div}_{\mathrm{H}}\,\widetilde{v}\,d\zeta\right)(\partial_z\widetilde{v})\right\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&=:I_{1,1}+\cdots+I_{1,5}.
\end{aligned}
$$

Let $(\overline{v},\widetilde{v})\in\mathbb{B}_r$. Since it holds by Proposition 3.3 that

$$
\begin{aligned}
&\|(\widetilde{v}\cdot\nabla_{\mathrm{H}})\overline{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\
&\le C\|\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\|A\overline{v}\|_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}\\
&\le C\left(\|\widetilde{v}-S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\left\|A(\overline{v}-e^{-tA}\overline{v_0})\right\|_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \|S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \left\|A(\overline{v} - e^{-tA}\overline{v_0})\right\|_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)} \\
&+ \|\widetilde{v} - S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \left\|Ae^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)} \\
&+ \|S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \left\|Ae^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}\Big),
\end{aligned}
$$

we see that

$$
I_{1,1} \le \frac{r}{20C_2},
$$

provided $r > 0$ and $(\overline{v_0}, \widetilde{v_0})$ are sufficiently small. Here C_2 is specified in (3.5). In the same way as above, we have

$$
\begin{aligned}
&\|(\overline{v}\cdot\nabla_{\mathrm{H}})\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\le C\|\overline{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)} \|\widetilde{v}\|_{\dot{B}^{1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\le C\Big(\left\|\overline{v} - e^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)} \|\widetilde{v} - S(t)\widetilde{v_0}\|_{\dot{B}^{1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\qquad + \left\|e^{-tA}\overline{v_0}\right\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)} \|\widetilde{v} - S(t)\widetilde{v_0}\|_{\dot{B}^{1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\qquad + \left\|\overline{v} - e^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)} \|S(t)\widetilde{v_0}\|_{\dot{B}^{1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\qquad + \left\|e^{-tA}\overline{v_0}\right\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)} \|S(t)\widetilde{v_0}\|_{\dot{B}^{1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\Big).
\end{aligned}
$$

Thus, by using the interpolation inequalities (3.6)–(3.7) and

$$
\|g\|_{\dot{B}^{1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \le C\|g\|^{1/2}_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\Delta_{\mathrm{H}} g\|^{1/2}_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}, \tag{3.8}
$$

we have

$$
I_{1,2} \le \frac{r}{20C_2}.
$$

Since

$$
\begin{aligned}
&\|\widetilde{v}\,(\mathrm{div}_{\mathrm{H}}\,\widetilde{v})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} + \left\|\widetilde{\mathrm{div}_{\mathrm{H}}\,(\widetilde{v}\otimes\widetilde{v})}\right\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\le C\|\nabla_{\mathrm{H}}\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\partial_z\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\le C\Big(\|\nabla_{\mathrm{H}}(\widetilde{v} - S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\partial_z(\widetilde{v} - S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\qquad + \|\nabla_{\mathrm{H}}S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\partial_z(\widetilde{v} - S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\qquad + \|\nabla_{\mathrm{H}}(\widetilde{v} - S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\partial_z S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \\
&\qquad + \|\nabla_{\mathrm{H}}S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\partial_z S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\Big),
\end{aligned}
$$

we have

$$
I_{1,3} + I_{1,4} \le \frac{r}{10C_2},
$$

where we used (3.7) and (3.8). Since

$$
\left\|\left(\int_{-1}^{z} \mathrm{div}_{\mathrm{H}}\,\widetilde{v}\,d\zeta\right)(\partial_z\widetilde{v})\right\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \le C\|\nabla_{\mathrm{H}}\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} \|\partial_z\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))},
$$

we have by the same way as $I_{1,3} + I_{1,4}$ that

$$
I_{1,5} \le \frac{r}{20C_2}.
$$

Consequently,

$$
\|F_2(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} \le \frac{r}{4C_2}.
$$

We also have to consider

$$\begin{aligned}\partial_z F_2(\overline{v},\widetilde{v}) =& ((\partial_z\widetilde{v})\cdot\nabla_{\mathrm{H}})\overline{v} + (\overline{v}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v})\\ &+ ((\partial_z\widetilde{v})\cdot\nabla_{\mathrm{H}})\widetilde{v} + (\widetilde{v}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v})\\ &- (\mathrm{div}_{\mathrm{H}}\,\widetilde{v})(\partial_z\widetilde{v}) + w\cdot\partial_z^2\widetilde{v},\end{aligned}$$

which is bounded from above as

$$\begin{aligned}&\|\partial_z F_2(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\ \le& \Big(\|((\partial_z\widetilde{v})\cdot\nabla_{\mathrm{H}})\overline{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} + \|(\overline{v}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\ &+ \|((\partial_z\widetilde{v})\cdot\nabla_{\mathrm{H}})\widetilde{v}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} + \|(\widetilde{v}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\ &+ \|(\mathrm{div}_{\mathrm{H}}\,\widetilde{v})(\partial_z\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} + \left\|\left(\int_{-1}^{z}\mathrm{div}_{\mathrm{H}}\,\widetilde{v}\,d\zeta\right)\partial_z^2\widetilde{v}\right\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\Big)\\ =& I_{2,1}+\cdots+I_{2,6}.\end{aligned}$$

Since it follows from Proposition 3.3 (here the restriction $1\le p<4$ is required) that

$$\begin{aligned}&\|((\partial_z\widetilde{v})\cdot\nabla_{\mathrm{H}})\overline{v}\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))} + \|(\overline{v}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v})\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ \le& C\,\|\partial_z\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\overline{v}\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\\ \le& C\Big(\|\partial_z(\widetilde{v}-S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\overline{v}-e^{-tA}\overline{v_0}\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\\ &+ \|\partial_z S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\overline{v}-e^{-tA}\overline{v_0}\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\\ &+ \|\partial_z(\widetilde{v}-S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|e^{-tA}\overline{v_0}\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\\ &+ \|\partial_z S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|e^{-tA}\overline{v_0}\|_{\mathbb{P}\dot{B}^{2/p}_{p,1}(\mathbb{R}^2)}\Big),\end{aligned}$$

we have by (3.6) and (3.7) that

$$I_{2,1}+I_{2,2}\le\frac{r}{12C_2}.$$

Furthermore, since it holds

$$\begin{aligned}&\|((\partial_z\widetilde{v})\cdot\nabla_{\mathrm{H}})\widetilde{v},\,(\widetilde{v}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v}),\,(\mathrm{div}_{\mathrm{H}}\,\widetilde{v})(\partial_z\widetilde{v})\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ \le& C\,\|\partial_z\widetilde{v}\|^2_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))},\end{aligned}$$

we have

$$I_{2,3}+I_{2,4}+I_{2,5}\le\frac{r}{8C_2}.$$

For $I_{2,6}$, since

$$\begin{aligned}&\left\|\left(\int_{-1}^{z}\mathrm{div}_{\mathrm{H}}\,\widetilde{v}\,d\zeta\right)\partial_z^2\widetilde{v}\right\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ \le& C\,\|\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\partial_z^2\widetilde{v}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ \le& C\Big(\|\widetilde{v}-S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\partial_z^2(\widetilde{v}-S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ &+ \|S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\partial_z^2(\widetilde{v}-S(t)\widetilde{v_0})\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ &+ \|\widetilde{v}-S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\partial_z^2 S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\ &+ \|S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\,\|\partial_z^2 S(t)\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\Big),\end{aligned}$$

we have

$$I_{2,6} \le \frac{r}{24C_2},$$

provided $\|\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}$ is sufficiently small. Consequently,

$$\|\partial_z F_2(\overline{v},\widetilde{v})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} \le \frac{r}{4C_2}.$$

Hence

$$\|\Phi_{\mathrm{V}}(\overline{v},\widetilde{v}) - S(t)\widetilde{v_0}\|_{X_{\mathrm{V}}} \le \frac{r}{2},$$

which implies $\Phi(\overline{v},\widetilde{v}) \in \mathbb{B}_r$.

3.4. **Contraction property of Φ and the existence.** For $(\overline{v_1},\widetilde{v_1})$, $(\overline{v_2},\widetilde{v_2}) \in \mathbb{B}_r$, where $r>0$ is chosen in the previous subsection, we show

$$\begin{aligned} &\|\Phi_{\mathrm{H}}(\overline{v_1},\widetilde{v_1}) - \Phi_{\mathrm{H}}(\overline{v_2},\widetilde{v_2})\|_{X_{\mathrm{H}}} + \|\Phi_{\mathrm{V}}(\overline{v_1},\widetilde{v_1}) - \Phi_{\mathrm{V}}(\overline{v_2},\widetilde{v_2})\|_{X_{\mathrm{V}}} \\ &\le \frac{1}{2}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H}}} + \|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V}}}\right). \end{aligned} \tag{3.9}$$

Firstly, we notice that, from Proposition 3.1 and Corollary 1.5, $(\delta v)_{\mathrm{H}} \equiv \Phi_{\mathrm{H}}(\overline{v_1},\widetilde{v_1}) - \Phi_{\mathrm{H}}(\overline{v_2},\widetilde{v_2})$ solves

$$\begin{cases} \partial_t(\delta v)_{\mathrm{H}} + A(\delta v)_{\mathrm{H}} = \mathbb{P}\left[F_1(\overline{v_1},\widetilde{v_1}) - F_1(\overline{v_2},\widetilde{v_2})\right] & \text{in } \mathbb{R}^2\times\mathbb{R}_+, \\ (\delta v)_{\mathrm{H}}|_{t=0} = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

with the estimate

$$\|(\delta v)_{\mathrm{H}}\|_{X_{\mathrm{H}}} \le C_1 \left\|\mathbb{P}\left[F_1(\overline{v_1},\widetilde{v_1}) - F_1(\overline{v_2},\widetilde{v_2})\right]\right\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}$$

and $(\delta v)_{\mathrm{V}} \equiv \Phi_{\mathrm{V}}(\overline{v_1},\widetilde{v_1}) - \Phi_{\mathrm{V}}(\overline{v_2},\widetilde{v_2})$ solves

$$\begin{cases} \partial_t(\delta v)_{\mathrm{V}} - \Delta(\delta v)_{\mathrm{V}} = F_2(\overline{v_1},\widetilde{v_1}) - F_2(\overline{v_2},\widetilde{v_2}) & \text{in } \Omega\times\mathbb{R}_+, \\ \partial_z(\delta v)_{\mathrm{V}} = 0 & \text{in } \mathbb{R}^2\times\{-1,0\}\times\mathbb{R}_+, \\ (\delta v)_{\mathrm{V}}|_{t=0} = 0 & \text{in } \Omega \end{cases}$$

with the estimate

$$\begin{aligned} \|(\delta v)_{\mathrm{V}}\|_{X_{\mathrm{V}}} \le & C_2\Big(\|F_2(\overline{v_1},\widetilde{v_1}) - F_2(\overline{v_2},\widetilde{v_2})\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))} \\ &+ \|\partial_z F_2(\overline{v_1},\widetilde{v_1}) - \partial_z F_2(\overline{v_2},\widetilde{v_2})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\Big), \end{aligned}$$

respectively. Here,

$$F_1(\overline{v_1},\widetilde{v_1}) - F_1(\overline{v_2},\widetilde{v_2}) = \mathrm{div}_{\mathrm{H}}\left(\overline{v_1}\otimes\overline{v_1}\right) - \mathrm{div}_{\mathrm{H}}\left(\overline{v_2}\otimes\overline{v_2}\right) + \overline{\mathrm{div}_{\mathrm{H}}\left(\widetilde{v_1}\otimes\widetilde{v_1}\right)} - \overline{\mathrm{div}_{\mathrm{H}}\left(\widetilde{v_2}\otimes\widetilde{v_2}\right)},$$

$$\begin{aligned} &F_2(\overline{v_1},\widetilde{v_1}) - F_2(\overline{v_2},\widetilde{v_2}) \\ &= (\widetilde{v_1}\cdot\nabla_{\mathrm{H}})\overline{v_1} - (\widetilde{v_2}\cdot\nabla_{\mathrm{H}})\overline{v_2} + (\overline{v_1}\cdot\nabla_{\mathrm{H}})\widetilde{v_1} - (\overline{v_2}\cdot\nabla_{\mathrm{H}})\widetilde{v_2} \\ &\quad - [\widetilde{v_1}\mathrm{div}_{\mathrm{H}}\,\widetilde{v_1} - \widetilde{v_2}\mathrm{div}_{\mathrm{H}}\widetilde{v_2}] + \mathrm{div}_{\mathrm{H}}\,[\widetilde{v_1}\otimes\widetilde{\widetilde{v_1}-\widetilde{v_2}}\otimes\widetilde{v_2}] \\ &\quad - \left[\left(\int_{-1}^{z}\mathrm{div}_{\mathrm{H}}\,\widetilde{v_1}\,d\zeta\right)(\partial_z\widetilde{v_1}) - \left(\int_{-1}^{z}\mathrm{div}_{\mathrm{H}}\,\widetilde{v_2}\,d\zeta\right)(\partial_z\widetilde{v_2})\right] \end{aligned}$$

and

$$\begin{aligned} \partial_z F_2(\overline{v_1},\widetilde{v_1}) - \partial_z F_2(\overline{v_2},\widetilde{v_2}) = &((\partial_z\widetilde{v_1})\cdot\nabla_{\mathrm{H}})\overline{v_1} - ((\partial_z\widetilde{v_2})\cdot\nabla_{\mathrm{H}})\overline{v_2} \\ &+ (\overline{v_1}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v_1}) - (\overline{v_2}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v_2}) \\ &+ ((\partial_z\widetilde{v_1})\cdot\nabla_{\mathrm{H}})\widetilde{v_1} - ((\partial_z\widetilde{v_2})\cdot\nabla_{\mathrm{H}})\widetilde{v_2} \\ &+ (\widetilde{v_1}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v_1}) - (\widetilde{v_2}\cdot\nabla_{\mathrm{H}})(\partial_z\widetilde{v_2}) \\ &- [(\mathrm{div}_{\mathrm{H}}\,\widetilde{v_1})(\partial_z\widetilde{v_1}) - (\mathrm{div}_{\mathrm{H}}\,\widetilde{v_2})(\partial_z\widetilde{v_2})] \\ &+ w(v_1)\cdot\partial_z^2\widetilde{v_1} - w(v_2)\cdot\partial_z^2\widetilde{v_2}. \end{aligned}$$

In the same manner as for the estimate of $F_1(\overline{v},\widetilde{v})$ in the previous subsection, we see that

$$\begin{aligned}
&\|\mathbb{P}\left[F_1(\overline{v_1},\widetilde{v_1})-F_1(\overline{v_2},\widetilde{v_2})\right]\|_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^s_{p,1}(\mathbb{R}^2))}\\
&\le\left(r+\|\overline{v_0}\|^{1/2}_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}\left\|Ae^{-tA}\overline{v_0}\right\|^{1/2}_{L^1(\mathbb{R}_+;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\right)\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H}}}\\
&\quad+C\left(r+\|\widetilde{v_0}\|^{1/2}_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\|\Delta S(\cdot)\widetilde{v_0}\|^{1/2}_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\right)\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V}}}\\
&\le\frac{1}{4C_1}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H}}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V}}}\right).
\end{aligned}$$

Furthermore, we also see that

$$\begin{aligned}
&\|F_2(\overline{v_1},\widetilde{v_1})-F_2(\overline{v_2},\widetilde{v_2})\|_{L^1(\mathbb{R}_+;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\le\frac{1}{8C_2}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H}}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V}}}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_zF_2(\overline{v_1},\widetilde{v_1})-\partial_zF_2(\overline{v_2},\widetilde{v_2})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\le\frac{1}{8C_2}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H}}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V}}}\right),
\end{aligned}$$

provided $(\overline{v_0},\widetilde{v_0})$ is small. Consequently (3.9) holds, which implies that Φ is a contraction mapping on $\mathbb{B}_r$. Applying the Banach fixed point theorem, we see that (3.1) admits a solution $(\overline{v},\widetilde{v})\in\mathbb{B}_r$ on $\mathbb{R}_+$, provided $\|\overline{v_0}\|_{\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2)}$ and $\|\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}+\|\partial_z\widetilde{v_0}\|_{\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}$ are sufficiently small, which implies that the existence part of Theorem 1.7 holds.

3.5. **Uniqueness.** Let v_1,v_2 be the solutions in the class $X_{\mathrm{H}}\times X_{\mathrm{V}}$ sharing the same initial data. As already seen before, it also holds that

$$\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}\le C_1\|\mathbb{P}\left[F_1(\overline{v_1},\widetilde{v_1})-F_1(\overline{v_2},\widetilde{v_2})\right]\|_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}$$

and

$$\begin{aligned}
\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\le&C_2\Big(\|F_2(\overline{v_1},\widetilde{v_1})-F_2(\overline{v_2},\widetilde{v_2})\|_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&+\|\partial_zF_2(\overline{v_1},\widetilde{v_1})-\partial_zF_2(\overline{v_2},\widetilde{v_2})\|_{L^1(0,T;\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\Big),
\end{aligned}$$

where $X_{\mathrm{H},T}$ and $X_{\mathrm{V},T}$ are defined by replacing the time interval $\mathbb{R}_+$ with $(0,T)$ in the definition of X_{H} and X_{V}, respectively. From the above discussion, we can see that there exists $0<T=T(v_1,v_2)\le\infty$ such that

$$\begin{aligned}
&\|\mathbb{P}\left[F_1(\overline{v_1},\widetilde{v_1})-F_1(\overline{v_2},\widetilde{v_2})\right]\|_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\\
&\le C\left(\|\overline{v_1}\|^{1/2}_{L^\infty(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\|A\overline{v_1}\|^{1/2}_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\right.\\
&\qquad\left.+\|\overline{v_2}\|^{1/2}_{L^\infty(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\|A\overline{v_2}\|^{1/2}_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\right)\cdot\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}\\
&\quad+C\left(\|\widetilde{v_1}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\left\|\partial_z^2\widetilde{v_1}\right\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\right.\\
&\qquad\left.+\|\widetilde{v_2}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\left\|\partial_z^2\widetilde{v_2}\right\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\right)\cdot\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\\
&\le\frac{1}{4C_1}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\right).
\end{aligned}$$

For $F_2(\overline{v_j},\widetilde{v_j})$, we obtain

$$
\begin{aligned}
&\|F_2(\overline{v_1},\widetilde{v_1})-F_2(\overline{v_2},\widetilde{v_2})\|_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
\le& C\left(\|\widetilde{v_2}\|_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}+\|\widetilde{v_2}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\Delta_{\mathrm{H}}\widetilde{v_2}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\right)\cdot\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}\\
&+C\Big(\|A\overline{v_1}\|_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\\
&\qquad+\|\overline{v_1}\|^{1/2}_{L^\infty(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\|A\overline{v_1}\|^{1/2}_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\\
&\qquad+\|\widetilde{v_1}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\partial_z^2\widetilde{v_1}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\qquad+\|\widetilde{v_2}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\Delta_{\mathrm{H}}\widetilde{v_2}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\Big)\cdot\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\\
\le&\frac{1}{8C_2}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\right)
\end{aligned}
$$

for small $T=T(v_1,v_2)$, provided $\|\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}$ is small. Here, we used

$$
\begin{aligned}
&\|\widetilde{v_2}(t)\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\
\le&\|\widetilde{v_2}(t)-\widetilde{v_2}(0)\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}+\|\widetilde{v_2}(0)\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))}\\
\le&\|\partial_t\widetilde{v_2}(t)\|_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}+\|\widetilde{v_0}\|_{\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0))},\qquad 0\le t\le T,
\end{aligned}
$$

which implies the smallness of $\|\widetilde{v_2}\|_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}$ with small T. Similarly, we also have

$$
\begin{aligned}
&\|\partial_zF_2(\overline{v_1},\widetilde{v_1})-\partial_zF_2(\overline{v_2},\widetilde{v_2})\|_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
\le&C\|\widetilde{v_1}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\partial_z^2\widetilde{v_1}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}\\
\le&C\Big(\|\overline{v_2}\|^{1/2}_{L^\infty(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\|A\overline{v_2}\|^{1/2}_{L^1(0,T;\mathbb{P}\dot{B}^{-1+2/p}_{p,1}(\mathbb{R}^2))}\\
&\qquad+\|\widetilde{v_1}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\partial_z^2\widetilde{v_1}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\qquad+\|\widetilde{v_2}\|^{1/2}_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\|\partial_z^2\widetilde{v_2}\|^{1/2}_{L^1(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\\
&\qquad+\|\widetilde{v_1}\|_{L^\infty(0,T;\dot{B}^{2/p}_{p,1}(\mathbb{R}^2;L^1(-1,0)))}\Big)\cdot\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\\
\le&\frac{1}{8C_2}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\right).
\end{aligned}
$$

Consequently we have

$$
\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\le\frac{1}{2}\left(\|\overline{v_1}-\overline{v_2}\|_{X_{\mathrm{H},T}}+\|\widetilde{v_1}-\widetilde{v_2}\|_{X_{\mathrm{V},T}}\right),
$$

which implies $\overline{v_1}=\overline{v_2}$ and $\widetilde{v_1}=\widetilde{v_2}$ on $[0,T]$. Repeating the same procedure, we may see that $\overline{v_1}=\overline{v_2}$ and $\widetilde{v_1}=\widetilde{v_2}$ on $[T,2T]$. Thus, repeating this finite times, we obtain uniqueness on every compact time interval. This completes the proof of Theorem 1.7. □

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