Geometrical Proof of Generalized Mirror Transformation for Multi-Point Virtual Structure Constants of Projective Hypersurfaces

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Abstract

In this paper, we propose a geometric proof of the generalized mirror transformation for multi-point virtual structure constants of degree k hypersurfaces in CP^{N-1}.

1 Introduction

1.1 Notation and Main Theorem

Let N and k be positive integers, and let M_N^k denote a degree k hypersurface in CP^{N-1}. We consider $\overline{M}_{0,n}(CP^{N-1}, d)$, the moduli space of stable maps of degree d from genus 0 semi-stable curves with n marked points to CP^{N-1} [8]. The genus 0 n-pointed Gromov-Witten invariant of M_N^k is defined as [8, 4]:

$$\langle \prod_{j=1}^{n} \mathcal{O}_{h^{c_j}} \rangle_{0,d} = \int_{\overline{M}_{0,n}(CP^{N-1},d)} c_{top}(\overline{\mathcal{E}}_d^k) \wedge \left(\bigwedge_{j=1}^{n} ev_j^*(h^{c_j}) \right), \tag{1.1}$$

where h is the hyperplane class in $H^*(CP^{N-1}, \mathbf{C})$, and $ev_j : \overline{M}_{0,n}(CP^{N-1}, d) \to CP^{N-1}$ $(j = 1, 2, \cdots, n)$ are the evaluation maps at the j-th marked point. Here, $\overline{\mathcal{E}}_d^k$ is the vector bundle on $\overline{M}_{0,n}(CP^{N-1}, d)$ that imposes the condition that the image of the stable map is contained in the hypersurface. This invariant is non-vanishing only if the following condition holds:

$$\sum_{j=1}^{n} c_j = N - 5 + (N-k)d + n. \tag{1.2}$$

We also introduce the compactified moduli space of quasimaps from CP^1 with $2+n$ marked points (0, ∞, and $z_j \in CP^1 - \{0, \infty\}$ for $j = 1, 2, \cdots, n$) to CP^{N-1} of degree d, denoted by $\widetilde{Mp}_{0,2|n}(N, d)$. We have to note here that the latter n marked points do not have to be distinct. For details on the construction, see [3] for the case of two marked points, and [7] for the case of $2 + n$ marked points. We define the intersection number $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b} | \prod_{j=1}^{n} \mathcal{O}_{h^{c_j}})_{0,d}$ on $\widetilde{Mp}_{0,2|n}(N, d)$, analogous to $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \prod_{j=1}^{n} \mathcal{O}_{h^{c_j}} \rangle_{0,d}$ on

$\overline{M}_{0,2+n}(\mathbb{CP}^{N-1},d)$, as:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^{n}\mathcal{O}_{h^{c_j}})_{0,d}=\int_{\widetilde{Mp}_{0,2|n}(N,d)}c_{top}(\widetilde{\mathcal{E}}_d^k)\wedge ev_0^*(h^a)\wedge ev_\infty^*(h^b)\wedge\left(\bigwedge_{j=1}^{n}ev_j^*(h^{c_j})\right). \tag{1.3}$$

Here, ev_0 and ev_∞ are the evaluation maps at 0 and ∞, respectively, ev_j is the evaluation map at z_j, and $\widetilde{\mathcal{E}}_d^k$ is the vector bundle on $\widetilde{Mp}_{0,2|n}(N,d)$ with the same geometric meaning as $\overline{\mathcal{E}}_d^k$. This intersection number is non-vanishing only if the following condition is satisfied:

$$a+b+\sum_{j=1}^{n}(c_j-1)=N-3+(N-k)d\quad\left(\Longleftrightarrow\quad a+b+\sum_{j=1}^{n}c_j=N-3+(N-k)d+n\right). \tag{1.4}$$

We refer to this intersection number as the "multi-point virtual structure constant" [7].

Let P_g^l be the set of partitions of the positive integer g into l parts:

$$P_g^l=\{\sigma_g=(g_1,\cdots,g_l)\mid 1\le g_1\le g_2\le\cdots\le g_l,\ \sum_{j=1}^{l}g_j=g\ \}. \tag{1.5}$$

For a partition $\sigma_g\in P_g^l$, we define the multiplicity $\mathrm{mul}(i,\sigma_g)$ of σ_g as:

$$\mathrm{mul}(i,\sigma_g)=(\text{number of subscripts } j \text{ such that } g_j=i). \tag{1.6}$$

We define the combinatorial factor $S(\sigma_g)$ as:

$$S(\sigma_g)=\prod_{i=1}^{g}\frac{1}{(\mathrm{mul}(i,\sigma_g))!}. \tag{1.7}$$

In this paper, we prove the following theorem, which describes the relationship between the intersection numbers $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^{n}\mathcal{O}_{h^{c_j}})_{0,d}$ and the Gromov-Witten invariants $\langle\prod_{j=1}^{n}\mathcal{O}_{h^{c_j}}\rangle_{0,d}$.

Theorem 1.1. The following equality holds:

$$\begin{aligned}
&w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{n_j})_{0,d}\\
&=\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{n_j}\rangle_{0,d}+w(\mathcal{O}_{h^{a+b}}\mathcal{O}_1|\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{n_j})_{0,d}+\\
&+\sum_{g=1}^{d-1}\sum_{l=1}^{g}\sum_{\substack{\sigma_g\in P_g^l\\(\sigma_g=(g_1,\cdots,g_l))}}S(\sigma_g)\Bigg(\sum_{\substack{m_0^j,m_1^j,m_2^j,\cdots,m_l^j\ge 0\\ m_0^j+\sum_{i=1}^{l}m_i^j=n_j}}\sum_{e_1,\cdots,e_l=0}^{N-2}\left(\prod_{j=2}^{N-2}\frac{n_j!}{m_0^j!\prod_{p=1}^{l}m_p^j!}\right)\\
&\times\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{m_0^j}\prod_{p=1}^{l}\mathcal{O}_{h^{e_p}}\rangle_{0,d-g}\left(\prod_{p=1}^{l}\frac{w(\mathcal{O}_{h^{N-2-e_p}}\mathcal{O}_1|\prod_{j=2}^{l}(\mathcal{O}_{h^j})^{m_p^j})_{0,g_p}}{k}\right)\Bigg),
\end{aligned} \tag{1.8}$$

where

$$a+b+\sum_{j=2}^{N-2}(j-1)n_j=N-3+(N-k)d.$$

1.2 Usage and Examples of the Main Theorem

In [7], we established the following properties for multi-point virtual structure constants:

(i) $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\mathcal{O}_{h^c})_{0,0} = k \cdot \delta_{N-2,a+b+c}$.

(ii) If $n \neq 1$, then $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^{n}\mathcal{O}_{h^{c_j}})_{0,0} = 0$.

(iii) $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{p=0}^{N-2}(\mathcal{O}_{h^p})^{m_p})_{0,d} = \delta_{m_0,0} \cdot d^{m_1} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{p=2}^{N-2}(\mathcal{O}_{h^p})^{m_p})_{0,d}$ for $d \geq 1$.

The characteristics (i) and (ii) follow from the fact that the degrees of freedom of $n(\geq 2)$ marked points cannot be fixed by insertion of operators in the $d = 0$ case. The property (iii) follows from Theorem 1 in [7]. Consequently, we will only consider $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{p=2}^{N-2}(\mathcal{O}_{h^p})^{m_p})_{0,d}$ for $d \geq 1$ henceforth. Let us introduce the generating function for the multi-point virtual structure constants:

$$\begin{aligned} & w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_0(x^0, x^1, \cdots, x^{N-2}) \\ = \quad & kx^{N-2-a-b} + \sum_{m_2=0}^{\infty} \cdots \sum_{m_{N-2}=0}^{\infty} \sum_{d=1}^{\infty} e^{dx^1} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{p=2}^{N-2}(\mathcal{O}_{h^p})^{m_p})_{0,d} \prod_{q=2}^{N-2} \frac{(x^q)^{m_q}}{m_q!}. \end{aligned} \tag{1.9}$$

In [7], we defined the "mirror map", a coordinate change of deformation variables, as:

$$t^p(x^0, x^1, \cdots, x^{N-2}) := \frac{1}{k} w(\mathcal{O}_{h^{N-2-p}}\mathcal{O}_1)_0 \quad (p = 0, 1, \cdots, N-2). \tag{1.10}$$

Equation (1.9) reveals the structure of this mirror map:

$$t^p(x^0, x^1, \cdots, x^{N-2}) = x^p + \frac{1}{k} \sum_{m_2=0}^{\infty} \cdots \sum_{m_{N-2}=0}^{\infty} \sum_{d=1}^{\infty} e^{dx^1} w(\mathcal{O}_{h^{N-2-p}}\mathcal{O}_1|\prod_{p=2}^{N-2}(\mathcal{O}_{h^p})^{m_p})_{0,d} \prod_{q=2}^{N-2} \frac{(x^q)^{m_q}}{m_q!} \tag{1.11}$$

This structure allows us to invert the mirror map, yielding:

$$x^p = x^p(t^0, t^1, \cdots, t^{N-2}) \quad (p = 0, 1, \cdots, N-2). \tag{1.12}$$

On the other hand, Gromov-Witten invariants satisfy the following properties [9]:

(i) $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\mathcal{O}_{h^c}\rangle_{0,0} = k \cdot \delta_{N-2,a+b+c}$.

(ii) If $d = 0$ and $n \neq 3$, $\langle \prod_{j=1}^{n}\mathcal{O}_{h^{c_j}}\rangle_{0,0}$ vanishes.

(iii) $\langle \prod_{p=0}^{N-2}(\mathcal{O}_{h^p})^{m_p}\rangle_{0,d} = \delta_{m_0,0} \cdot d^{m_1}\langle \prod_{p=2}^{N-2}(\mathcal{O}_{h^p})^{m_p}\rangle_{0,d} \quad (d \geq 1)$.

Analogous to the generating function in (1.9), we introduce the perturbed two-point genus 0 Gromov-Witten invariant:

$$\begin{aligned} & \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_0(t^0, t^1, \cdots, t^{N-2}) \\ := \quad & \sum_{m_0=0}^{\infty} \cdots \sum_{m_{N-2}=0}^{\infty} \sum_{d=0}^{\infty} \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \prod_{p=0}^{N-2}(\mathcal{O}_{h^p})^{m_p}\rangle_{0,d} \prod_{q=0}^{N-2} \frac{(t^q)^{m_q}}{m_q!} \\ = \quad & kt^{N-2-a-b} + \sum_{m_2=0}^{\infty} \cdots \sum_{m_{N-2}=0}^{\infty} \sum_{d=1}^{\infty} e^{dt^1} \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \prod_{p=2}^{N-2}(\mathcal{O}_{h^p})^{m_p}\rangle_{0,d} \prod_{q=2}^{N-2} \frac{(t^q)^{m_q}}{m_q!}. \end{aligned} \tag{1.13}$$

Then, the conjecture proposed in [7] is:

Conjecture 1.1. [7]

$$\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_0(t^0,t^1,\cdots,t^{N-2}) = w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_0(x^0(t^*),x^1(t^*),\cdots,x^{N-2}(t^*))$$
$$(\Longleftrightarrow \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_0(t^0(x^*),t^1(x^*),\cdots,t^{N-2}(x^*)) = w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_0(x^0,x^1,\cdots,x^{N-2})). \quad (1.14)$$

Here, $x^p(t^*)$ (resp. $t^p(x^*)$) abbreviates $x^p(t^0,t^1,\cdots,t^{N-2})$ (resp. $t^p(x^0,x^1,\cdots,x^{N-2})$) as given in (1.12) (resp. (1.10)).

By expanding the right-hand side of the second line of (1.14) after substituting (1.11), we observe that Theorem 1.1 is equivalent to this conjecture. Therefore, this paper provides a proof of Conjecture 1.1.

Example 1.1. Consider the case $N = 4, k = 1$, corresponding to the CP^2 model. As shown in [5], the generalized mirror transformation for the two-point virtual structure constant is trivial:

$$w(\mathcal{O}_{h^2}\mathcal{O}_{h^2})_{0,1} = \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1}. \quad (1.15)$$

However, for multi-point virtual structure constants, the transformation can be non-trivial even for $d = 1$. For $w(\mathcal{O}_h\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,1}$, Theorem 1.1 yields:

$$w(\mathcal{O}_h\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,1} = \langle \mathcal{O}_h\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1}(=1).. \quad (1.16)$$

While this specific case is trivial, the theorem also indicates:

$$\begin{aligned}(2=)w(\mathcal{O}_h\mathcal{O}_h|\mathcal{O}_{h^2}\mathcal{O}_{h^2})_{0,1} &= \langle \mathcal{O}_h\mathcal{O}_h\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1} + \langle \mathcal{O}_h\mathcal{O}_h\mathcal{O}_1\rangle_{0,0} w(\mathcal{O}_{h^2}\mathcal{O}_1|\mathcal{O}_{h^2}\mathcal{O}_{h^2})_{0,1} \\ &= \langle \mathcal{O}_h\mathcal{O}_h\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1} + w(\mathcal{O}_{h^2}\mathcal{O}_1|\mathcal{O}_{h^2}\mathcal{O}_{h^2})_{0,1}(=1+1), \quad (1.17)\end{aligned}$$

which is non-trivial. The reason for this difference will be discussed in the next section.

Example 1.2. Next, we consider the $N = k = 8$ case, representing a Calabi-Yau hypersurface. In this scenario, the generalized mirror transformation for the two-point virtual structure constants is non-trivial. For $d = 1$, Theorem 1.1 provides the following equality:

$$\begin{aligned}(83871744=)w(\mathcal{O}_{h^2}\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,1} &= \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1} + \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,0} w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,1}/k \\ &= \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1} + w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,1} \\ &(= 59021312 + 24850432\ ) \quad (1.18)\end{aligned}$$

This equality shares a fundamental structure with the two-point case. For $d = 2$, Theorem 1.1 yields:

$$\begin{aligned}(1238948617930752 &=) \\ w(\mathcal{O}_{h^2}\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,2} &= \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,2} + \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,0} w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,2}/k \\ &\quad + \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_h\rangle_{0,1} w(\mathcal{O}_{h^5}\mathcal{O}_1)_{0,1}/k + \langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1} w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,1}/k \\ &(= 821654084851712 + 201251978293248 \\ &\quad + 59021312 \cdot 4432896/8 + 59021312 \cdot 24850432/8). \quad (1.19)\end{aligned}$$

In the right-hand side of the above equality, the first three terms exhibit characteristics similar to the two-point case, while the fourth term introduces a new effect specific to the multi-point scenario. We will discuss this example further in the next section.

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2 New Excess Intersections (Erratum of page 463, lines 6–10 of the paper: Period integrals (Givental's I-function) of Calabi-Yau hypersurfaces in CP^{N-1} as generating functions of intersection numbers on the moduli space of quasimaps [6])

Let us begin by defining $Mp_{0,2|n}(N,d)$, which represents the highest dimensional stratum of $\widetilde{Mp}_{0,2|n}(N,d)$, the compactified moduli space of quasimaps from CP^1 with $2+n$ marked points to CP^{N-1}. For a detailed construction of $\widetilde{Mp}_{0,2|n}(N,d)$, please refer to our previous articles [3, 7].

To describe a point in $Mp_{0,2|n}(N,d)$, we consider a vector-valued complex polynomial:

$$\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j} \quad \left(\mathbf{a}_j \in \mathbf{C}^N (j=0,1,\cdots,d).\ \mathbf{a}_0, \mathbf{a}_d \neq \mathbf{0}\right).$$

Additionally, we have $z_j \in \mathbf{C}^\times$ for $j = 1,2,\ldots,n$. Each $z_j \in \mathbf{C}^\times$ represents the point $(1:z_j) \in CP^1 - \{0,\infty\}$, where $0 = (1:0) \in CP^1$ and $\infty = (0:1) \in CP^1$. These data are represented by the tuple:

$$((\mathbf{a}_0,\mathbf{a}_1,\ldots,\mathbf{a}_d),(z_1,z_2,\ldots,z_n)) \in (\mathbf{C}^N - \{\mathbf{0}\}) \times (\mathbf{C}^N)^{d-1} \times (\mathbf{C}^N - \{\mathbf{0}\}) \times (\mathbf{C}^\times)^n.$$

Next, we define a $(\mathbf{C}^\times)^2$ action on $(\mathbf{C}^N - \{\mathbf{0}\}) \times (\mathbf{C}^N)^{d-1} \times (\mathbf{C}^N - \{\mathbf{0}\}) \times (\mathbf{C}^\times)^n$ as follows:

$$G((\mu,\nu),((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))) = \left((\mu\mathbf{a}_0,\mu\nu\mathbf{a}_1,\cdots,\mu\nu^d\mathbf{a}_d),(\nu z_1,\nu z_2,\cdots,\nu z_n)\right) \quad \left((\mu,\nu) \in (\mathbf{C}^\times)^2\right). \tag{2.20}$$

This action defines equivalence classes of elements in $(\mathbf{C}^N - \{\mathbf{0}\}) \times (\mathbf{C}^N)^{d-1} \times (\mathbf{C}^N - \{\mathbf{0}\}) \times (\mathbf{C}^\times)^n$ based on the $(\mathbf{C}^\times)^2$ orbits. We denote the equivalence class containing $((\mathbf{a}_0,\mathbf{a}_1,\ldots,\mathbf{a}_d),(z_1,z_2,\ldots,z_n))$ in the $(\mathbf{C}^\times)^2$ orbit by $[(\mathbf{a}_0,\mathbf{a}_1,\ldots,\mathbf{a}_d),(z_1,z_2,\ldots,z_n)]$. This equivalence class defines a point in $Mp_{0,2|n}(N,d)$. Note that z_i and z_j can coincide for $i \neq j$ in our definition.

In our previous work [7], we defined the evaluation map $ev_m : \widetilde{Mp}_{0,2|n}(N,d) \to CP^{N-1}$ at the m-th marked point $(1:z_m) \in CP^1 - \{0,\infty\}$. Let us take a point $[(\mathbf{a}_0,\mathbf{a}_1,\ldots,\mathbf{a}_d),(z_1,z_2,\ldots,z_n)] \in Mp_{0,2|n}(N,d)$. Then, we consider the vector-valued polynomial $\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j}$. It can be uniquely factored into the following form:

$$\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j} = \left(\sum_{i=0}^{g} c_i s^i t^{g-i}\right) \cdot \left(\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j}\right) \quad (c_0,\cdots,c_g \in \mathbf{C},\ \mathbf{b}_0,\cdots,\mathbf{b}_{d-g} \in \mathbf{C}^N), \tag{2.21}$$

where $\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j}$ satisfies the condition:

$$\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j} = \mathbf{0} \Longrightarrow (s,t) = (0,0). \tag{2.22}$$

With this setup, we "may" define ev_m on $Mp_{0,2|n}(N,d)$ as follows:

$$ev_m([((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))]) := \pi_N\left(\sum_{j=0}^{d-g} \mathbf{b}_j (z_m)^{d-g-j}\right), \tag{2.23}$$

where $\pi_N : (\mathbf{C}^N - \{\mathbf{0}\}) \to CP^{N-1}$ is the projective equivalence map.

At first glance, the initial definition of the evaluation map ev_m appears satisfactory, as it uniquely determines the image for any point in $Mp_{0,2|n}(N,d)$. However, it suffers from a significant discontinuity problem, as pointed out in [2]. This discontinuity is the error in the definition of the evaluation map in [6]. Specifically, ev_m **can be discontinuous at points** satisfying:

$$[((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))] \in Mp_{0,2|n}(N,d) \text{ which satisfies } \sum_{j=0}^{d}\mathbf{a}_j(z_m)^{d-j}=\mathbf{0}. \tag{2.24}$$

This discontinuity arises when the non-negative integer g in the factorization (2.21) is a positive integer and

$$\sum_{i=0}^{g} c_i(z_m)^{g-i}=0. \tag{2.25}$$

To illustrate this, let us consider a vector-valued polynomial $\sum_{j=0}^{d}\mathbf{e}_j s^j t^{d-j}$ that satisfies the condition:

$$\sum_{j=0}^{d}\mathbf{e}_j s^j t^{d-j}=\mathbf{0} \implies (s,t)=(0,0). \tag{2.26}$$

Now, consider the following deformation of the original polynomial:

$$\begin{aligned}\varphi_\varepsilon(s,t) &:= \sum_{j=0}^{d}\mathbf{a}_j s^j t^{d-j}+\varepsilon\left(\sum_{j=0}^{d}\mathbf{e}_j s^j t^{d-j}\right)\\ &= \left(\sum_{i=0}^{g} c_i s^i t^{g-i}\right)\cdot\left(\sum_{j=0}^{d-g}\mathbf{b}_j s^j t^{d-g-j}\right)+\varepsilon\left(\sum_{j=0}^{d}\mathbf{e}_j s^j t^{d-j}\right).\end{aligned} \tag{2.27}$$

When $\varepsilon=0$, $ev_m([((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))])$ is given by $[\sum_{j=0}^{d-g}\mathbf{b}_j(z_m)^{d-g-j}]$ as defined in (2.23). However, for any non-zero ε, the image abruptly changes to $\left[\varepsilon\left(\sum_{j=0}^{d}\mathbf{e}_j(z_m)^{d-j}\right)\right]=[\sum_{j=0}^{d}\mathbf{e}_j(z_m)^{d-j}]$ because (2.25) holds. Consequently, the image jumps from $[\sum_{j=0}^{d-g}\mathbf{b}_j(z_m)^{d-g-j}]$ to $[\sum_{j=0}^{d}\mathbf{e}_j(z_m)^{d-j}]$, indicating that ev_m is discontinuous at points satisfying the factorization condition (2.26) with positive g.

To resolve this discontinuity issue, we adopt the approach proposed by Ciocan-Fontanine and Kim [1], which involves:

(i) Changing the target domain of ev_m from CP^{N-1} to the algebraic stack $\mathbf{C}^N/\mathbf{C}^\times$.

This stack may be viewed as $CP^{N-1}\cup\{\mathbf{0}\}$ as a quotient set. However, with the quotient topology, it is not a Hausdorff space, due to the following property:

(ii) Every point $p\in CP^{N-1}$ is infinitesimally close to $\mathbf{0}$, meaning any open set containing $\mathbf{0}$ also contains the entire CP^{N-1}.

We therefore modify the definition of ev_m as follows:

Definition 2.1.

$$ev_m([((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))]) := \tilde{\pi}_N\left(\sum_{j=0}^{d}\mathbf{a}_j(z_m)^{d-j}\right),$$

where $\tilde{\pi}_N:\mathbf{C}^N\to\mathbf{C}^N/\mathbf{C}^\times$ is the quotient map.

Now, let us reconsider the image of the point corresponding to $\varphi_\varepsilon(s,t)$ in (2.27) with $\sum_{i=0}^{g} c_i(z_m)^{g-i} = 0$. When $\varepsilon = 0$, the image is $\mathbf{0}$. When $\varepsilon \neq 0$, it becomes $[\sum_{j=0}^{d} \mathbf{e}_j (z_m)^{d-j}]$. Due to property (ii), $\mathbf{0}$ is infinitesimally close to $[\sum_{j=0}^{d} \mathbf{e}_j (z_m)^{d-j}]$, ensuring that ev_m is continuous at this point.

In sum, we write down the Erratum of [6] in order to resolve the discontinuity problem of the evaluation map.

Erratum 2.1. ([6]) The definition of $ev_1 : Mp_{0,2|1}(N,d) \to CP^{N-1}$ on page 463, lines 6–10 of [6] (stated as (2.23) in the present paper) should be replaced with Definition 2.1 for $m = 1$.

With these preparations, we now discuss the novel excess intersections that become relevant in the proof of Theorem 1.1. As demonstrated in [5], the generalized mirror transformation for virtual structure constants with two operator insertions arises from excess intersections. In essence, we can interpret the mirror map as **the generating function of these excess intersections**. These intersections primarily manifest because the dimension of the moduli space either decreases or remains invariant as the degree d increases, particularly when the degree of the hypersurface k is no less than the positive integer N of CP^{N-1}. However, Theorem 1.1 reveals that the generalized mirror transformation is non-trivial even when N exceeds k. Consequently, we must identify new excess intersections that account for the non-trivial terms in the theorem. These intersections can indeed be found by considering the revised definition of ev_m and the characteristic property (ii) of $\mathbf{C}^N/\mathbf{C}^\times$.

Let us fix g to a positive integer no greater than d and examine vector-valued polynomials factorized as:

$$\begin{aligned} \varphi(s,t) &= \sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j} \\ &= \left(\sum_{i=0}^{g} c_i s^i t^{g-i} \right) \cdot \left(\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j} \right), \end{aligned} \tag{2.28}$$

where $\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g}$ satisfies the condition (2.22). We further factorize $\sum_{i=0}^{g} c_i s^i t^{g-i}$ into the form:

$$\sum_{i=0}^{g} c_i s^i t^{g-i} = c_0 \prod_{i=1}^{l} (t - w_i s)^{g_i} \qquad \left((g_1, g_2, \cdots, g_l) = \sigma_g \in P_g^l \right), \tag{2.29}$$

where the w_i's are l distinct elements of $\mathbf{C}^\times$. With this setup, we consider the locus $\mathcal{L}_{(m,i)} \subset \widetilde{Mp}_{0,2|n}(N,d)$ where the m-th marked point z_m coincides with w_i. We then analyze the contribution of the locus under consideration to the multi-point virtual structure constant:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^{n}\mathcal{O}_{h^{c_j}})_{0,d} = \int_{\widetilde{Mp}_{0,2|n}(N,d)} c_{top}(\widetilde{\mathcal{E}}_d^k) \wedge ev_0^*(h^a) \wedge ev_\infty^*(h^b) \wedge \left(\bigwedge_{j=1}^{n} ev_j^*(h^{c_j}) \right). \tag{2.30}$$

Given the factorization condition (2.28), the number of degrees of freedom arising from $\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j}$ and $\sum_{i=0}^{g} c_i s^i t^{g-i}$ is $N - 3 + (N-k)(d-g) + l$[1]. Incorporating the degrees of freedom from the marked points other than z_m, the total dimension of the locus $\mathcal{L}_{(m,i)}$ is:

$$N - 3 + (N-k)(d-g) + l + n - 1.$$

[1]We must account for the constraint that the image of the polynomial map lies within the hypersurface M_N^k, reflected by the insertion of $c_{top}(\widetilde{\mathcal{E}}_d^k)$ in (1.3). This constraint decreases the degrees of freedom of $\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j}$, $N(d-g+1)-2 = N-2+N(d-g)$ (-2 comes from the $(\mathbf{C}^\times)^2$ action in (2.20)) into $N-3+(N-k)(d-g)$.

Applying the condition (1.4), the number of degrees of freedom of $\mathcal{L}_{(m,i)}$ minus the constraints imposed by operator insertions is:

$$\begin{aligned} & N-3+(N-k)(d-g)+l+n-1-\left(a+b+\sum_{j=1}^{n} c_j\right) \\ = \;& N-3+(N-k)(d-g)+l+n-1-(N-3+(N-k)d+n) \\ = \;& l-1-(N-k)g \\ = \;& -(g-l)-1-(N-k-1)g. \end{aligned} \tag{2.31}$$

Since $g \geq l$, the integer in the last line is consistently negative if $N-k-1 \geq 0$. Assuming $ev_m^*(h^{c_m})$ is an ordinary cohomology element of (complex) degree c_m, we conclude that the locus under consideration does not contribute to the intersection number (2.30) when $N-k-1 \geq 0$.

At this juncture, we must emphasize the condition where the marked point z_m coincides with w_i, expressed as:

$$\tilde{\pi}_N\left(\varphi(1,z_m)\right) = \mathbf{0} \in \mathbf{C}^N/\mathbf{C}^\times \Longleftrightarrow ev_m([((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))]) = \mathbf{0} \in \mathbf{C}^N/\mathbf{C}^\times. \tag{2.32}$$

The right-hand side of (2.30) indicates that $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^n \mathcal{O}_{h^{c_j}})_{0,d}$ generically counts the number of quasimaps within the following subset of $\widetilde{Mp}_{0,2|n}(N,d)$:

$$PD(c_{top}(\mathcal{E}_d^k)) \cap ev_0^{-1}(PD(h^a)) \cap ev_\infty^{-1}(PD(h^b)) \cap \left(\bigcap_{j=1}^{m} ev_j^{-1}(PD(h^{c_j}))\right),$$

where $PD(*)$ denotes the subvariety Poincaré dual to the cohomology element $*$. Let us examine $ev_m^{-1}(PD(h^{c_m}))$ under the conditions of (2.32). Recall that $PD(h^{c_m})$ is a linear space of codim c_m in CP^{N-1}, and the fact that $\mathbf{0} \in \mathbf{C}^N/\mathbf{C}^\times$ is infinitesimally close to (or indistinguishable in the quotient topology from) any point in CP^{N-1}. The condition (2.32) implies

$$ev_m^{-1}(PD(h^{c_m})) = ev_m^{-1}(\mathbf{C}^N/\mathbf{C}^\times) \quad \text{on } \mathcal{L}_{(m,i)}, \tag{2.33}$$

meaning $ev_m^*(h^{c_m})$ **imposes no constraints on** $\mathcal{L}_{(m,i)}$. This conclusion is also derived from elementary consideration. Since $PD(h^{c_m})$ is a linear space, restriction to $ev_m^{-1}(PD(h^{c_m}))$ is given by linear equations of homogeneous coordinates of the image $ev_m([((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_1,z_2,\cdots,z_n))])$. But the condition (2.32) turns these linear equations into $0=0$, i.e., trivial equations. Consequently, the dimensional count of (2.31) must be adjusted as follows:

$$\begin{aligned} & N-3+(N-k)(d-g)+l+n-1-\left(a+b+\sum_{j\neq m} c_j\right) \\ = \;& N-3+(N-k)(d-g)+l+n-1-(N-3+(N-k)d+n-c_m) \\ = \;& c_m+l-1-(N-k)g \\ = \;& (c_m-1)-(g-l)-(N-k-1)g. \end{aligned} \tag{2.34}$$

This adjustment allows the last line to yield a non-negative integer. This is a case where the locus $\mathcal{L}_{(m,i)}$ contributes to the intersection number (2.30) as new excess intersections even when $N-k-1 \geq 0$. More generally, we can consider scenarios where p marked points $z_{m_1}, z_{m_2}, \cdots, z_{m_p}$ $(1 \leq m_1 < m_2 < \cdots < m_p \leq n)$ coincide with elements of $\{w_1, w_2, \cdots, w_l\}$. In such cases, the dimensional count for the new excess intersection is modified to:

$$\begin{aligned} & N-3+(N-k)(d-g)+l+n-p-\left(a+b+\sum_{j\notin\{m_1,\cdots,m_p\}} c_j\right) \\ = \;& \sum_{j=1}^{p}(c_{m_j}-1)-(g-l)-(N-k-1)g. \end{aligned} \tag{2.35}$$

Example 2.1. The generalized mirror transformation for $w(\mathcal{O}_h\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,1}$ in (1.16) is trivial, as demonstrated by setting $c_1 = 2, g = l = 1, N = 4, k = 1$ in (2.34):

$$2 - 1 - (1 - 1) - (4 - 1 - 1) \cdot 1 = -1 < 0.$$

However, for $w(\mathcal{O}_h\mathcal{O}_h|\mathcal{O}_{h^2}\mathcal{O}_{h^2})_{0,1}$ in (1.17), we obtain from (2.35) with $c_1 = c_2 = 2, g = l = 1, N = 4, k = 1$:

$$(2 - 1) + (2 - 1) - (1 - 1) - (4 - 1 - 1) \cdot 1 = 0.$$

Thus, the term $w(\mathcal{O}_{h^2}\mathcal{O}_1|\mathcal{O}_{h^2}\mathcal{O}_{h^2})_{0,1}$ arises from the new excess intersection.

Example 2.2. For $w(\mathcal{O}_{h^2}\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,1}$ in (1.18), the term $w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,1}$ emerges from (2.34) with $c_1 = 2, g = l = 1, N = k = 8$:

$$(2 - 1) - (1 - 1) - (8 - 8 - 1) \cdot 1 = 2 \geq 0.$$

In the case of $w(\mathcal{O}_{h^2}\mathcal{O}_{h^2}|\mathcal{O}_{h^2})_{0,2}$ in (1.19), the term $\langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,0} w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,2}/k$ appears from (2.34) with $c_1 = 2, g = 2, l = 1, N = k = 8$:

$$2 - 1 - (2 - 1) - (8 - 8 - 1) \cdot 2 = 2 \geq 0.$$

$\langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_h\rangle_{0,1} w(\mathcal{O}_{h^5}\mathcal{O}_1)_{0,1}/k$ results from the excess intersection already discussed in [5], with the following count:

$$-(g - l) - (N - k - 1)g = -(1 - 1) - (8 - 8 - 1) \cdot 1 = 1 \geq 0.$$

The term $\langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}\mathcal{O}_{h^2}\rangle_{0,1} w(\mathcal{O}_{h^4}\mathcal{O}_1|\mathcal{O}_{h^2})_{0,1}/k$ arises from the new excess intersection (2.34) with $c_1 = 2, g = l = 1, N = k = 8$:

$$2 - 1 - (1 - 1) - (8 - 8 - 1) \cdot 1 = 2 \geq 0.$$

3 Proof of Theorem 1.1

The term $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{n_j})_{0,d}$ on the left-hand side of (1.8) is defined as an intersection number on the moduli space $\widetilde{Mp}_{0,2|\sum_{j=2}^{N-2} n_j}(N,d)$. In this proof, we focus on the highest dimensional stratum $Mp_{0,2|\sum_{j=2}^{N-2} n_j}(N,d)$.[2] We introduce the notational setting for $Mp_{0,2|\sum_{j=2}^{N-2} n_j}(N,d)$:

$$\begin{aligned} &Mp_{0,2|\sum_{j=2}^{N-2} n_j}(N,d) = \\ &\{\big((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_{(2,1)},\cdots,z_{(2,n_2)},\cdots,z_{(N-2,1)},\cdots,z_{(N-2,n_{N-2})})\big) \mid \mathbf{a}_i \in \mathbf{C}^N, \mathbf{a}_0,\mathbf{a}_d \neq \mathbf{0}, z_{(j,i)} \in \mathbf{C}^\times\}/(\mathbf{C}^\times)^2, \end{aligned} \tag{3.36}$$

where the two $\mathbf{C}^\times$ actions are given by:

$$\begin{aligned} &\lambda \cdot \big((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_{(2,1)},\cdots,z_{(2,n_2)},\cdots,z_{(N-2,1)},\cdots,z_{(N-2,n_{N-2})})\big) = \\ &\big((\lambda\mathbf{a}_0,\lambda\mathbf{a}_1,\cdots,\lambda\mathbf{a}_d),(z_{(2,1)},\cdots,z_{(2,n_2)},\cdots,z_{(N-2,1)},\cdots,z_{(N-2,n_{N-2})})\big), \\ &\nu \cdot \big((\mathbf{a}_0,\mathbf{a}_1,\cdots,\mathbf{a}_d),(z_{(2,1)},\cdots,z_{(2,n_2)},\cdots,z_{(N-2,1)},\cdots,z_{(N-2,n_{N-2})})\big) = \\ &\big((\mathbf{a}_0,\nu\mathbf{a}_1,\nu^2\mathbf{a}_2,\cdots,\nu^d\mathbf{a}_d),(\nu z_{(2,1)},\cdots,\nu z_{(2,n_2)},\cdots,\nu z_{(N-2,1)},\cdots,\nu z_{(N-2,n_{N-2})})\big), \\ &\qquad\qquad (\lambda,\nu \in \mathbf{C}^\times). \end{aligned} \tag{3.37}$$

[2] See the last part of the proof of the two-point version of Theorem 1.1 [5]. We will also explain the reason in the last part of this paper.

The points $(1 : z_{(j,i)}) \in (CP^1 - \{0, \infty\})$ for $(i = 1, \cdots, n_j)$ are the marked points where the operator $\mathcal{O}_{h^j}$ is inserted. Note that the $z_{(j,i)}$'s can move freely within $\mathbf{C}^\times$, and they may coincide. The tuple $(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_{d-1}, \mathbf{a}_d)$ represents a polynomial map $\varphi(s : t) = \tilde{\pi}_N \left(\sum_{j=0}^d \mathbf{a}_j s^j t^{d-j} \right)$ from CP^1 to $\mathbf{C}^N/\mathbf{C}^\times$, modulo the ν-action, which is introduced from the $\mathbf{C}^\times$ action on CP^1 that fixes $0 = (0 : 1), \infty = (1 : 0) \in CP^1$:

$$(s : t) \to (\nu s : t). \tag{3.38}$$

As was discussed in the previous section, $\sum_{j=0}^d \mathbf{a}_j s^j t^{d-j}$ can be uniquely factorized into the following form:

$$\sum_{j=0}^{d} \mathbf{a}_j s^j t^{d-j} = c_0 \Big(\prod_{j=1}^{l} (t - w_j s)^{g_j} \Big) \cdot \Big(\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j} \Big), \tag{3.39}$$

where the following condition holds:

$$\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j} = \mathbf{0} \Longrightarrow (s, t) = (0, 0),$$

or, equivalently, sending (s, t) to $\pi_N(\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j})$ defines a well-defined map from CP^1 to CP^{N-1}.

Since $\mathbf{a}_0, \mathbf{a}_d \neq \mathbf{0}$, it follows that w_j belongs to $\mathbf{C}^\times$, which means $(1 : w_j)$ never coincides with 0 or ∞ in CP^1. These distinct points $(1 : w_j)$ $(j = 1, \cdots, l(\sigma_g))$ also represent points where $\varphi(s : t)$ evaluates to $\mathbf{0}$. Given that $[(\mathbf{a}_0, \mathbf{a}_1, \cdots, \mathbf{a}_{d-1}, \mathbf{a}_d)]$ represents $\varphi(s : t)$ modulo the $\mathbf{C}^\times$ action defined in (3.38), the configuration of the $(1 : w_j)$'s must be considered modulo this $\mathbf{C}^\times$ action. For brevity, we will henceforth not distinguish between $w_j \in \mathbf{C}^\times$ and $(1 : w_j) \in (CP^1 - \{0, \infty\})$, nor between $z_{(j,i)} \in \mathbf{C}^\times$ and $(1 : z_{(j,i)}) \in (CP^1 - \{0, \infty\})$.

It is evident that the factorization in (3.39) remains invariant under permutations of w_j's with subscripts j that share the same value of g_j. Based on these observations, we obtain the following decomposition of $Mp_{0,2|\sum_{j=2}^{N-2} n_j}(N, d)$:

$$Mp_{0,2|\sum_{j=2}^{N-2} n_j}(N, d) = \coprod_{g=0}^{d} \coprod_{l=1}^{g} \coprod_{\sigma_g \in P_g^l} M_{0,2+l|\sum_{j=2}^{N-2} n_j}(CP^{N-1}, d-g, \sigma_g). \tag{3.40}$$

This decomposition includes $M_{0,2+l|\sum_{j=2}^{N-2} n_j}(CP^{N-1}, d-g, \sigma_g)$, which denotes the uncompactified moduli space of degree $d-g$ holomorphic maps from CP^1 to CP^{N-1}. This space features $2+l$ distinct marked points and $\sum_{j=2}^{N-2} n_j$ marked points free to move within $CP^1 - \{0, \infty\}$, modulo the action of $\prod_{i=1}^{g} \mathrm{Sym}(\mathrm{mul}(i, \sigma_g))$ on the set $\{w_1, w_2, \cdots, w_l\}$.

$$\begin{aligned} &M_{0,2+l|\sum_{j=2}^{N-2} n_j}(CP^{N-1}, d-g, \sigma_g) \\ &:= \{[(\tilde{\pi}_N(\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j}), (0, \infty, w_1, w_2, \cdots, w_l), (z_{(2,1)}, \cdots, z_{(N-2,n_{N-2})}))]\} / \Big(\prod_{i=1}^{g} \mathrm{Sym}(\mathrm{mul}(i, \sigma_g)) \Big). \end{aligned} \tag{3.41}$$

In this definition, the tuple $\big(\pi_N(\sum_{j=0}^{d-g} \mathbf{b}_j s^j t^{d-g-j}), (0, \infty, w_1, w_2, \cdots, w_l), (z_{(2,1)}, \cdots, z_{(N-2,n_{N-2})})\big)$ is considered modulo the $\mathbf{C}^\times$ action (3.38) on CP^1 , and the symmetric group $\mathrm{Sym}(\mathrm{mul}(i, \sigma_g))$ permutes the w_j's for which $g_j = i$.

As observed in the discussion given in first half of Section 3 of [5], ill-definedness of the image of the polynomial map φ at the points w_j leads to the excess intersections in the two-pointed case. These

remain relevant in our current analysis. However, we must also account for the new excess intersections introduced in the preceding section. To this end, we define a decomposition of the subscript set for the $z_{(*,*)}$'s as follows:

$$\{(j,1),(j,2),\cdots,(j,n_j)\} = \coprod_{p=0}^{l}\{(j,h_1^p),(j,h_2^p),\cdots,(j,h^p_{m_p^j})\} =: \coprod_{p=0}^{l} W_{(j,p)}$$

$$(j=2,3,\cdots,N-2,\ \ m_p^j\geq 0,\quad m_0^j+\sum_{p=1}^{l} m_p^j = n_j \quad 1\leq h_1^p<h_2^p<\cdots<h^p_{m_p^j}\leq n_j,\ \ ). \tag{3.42}$$

Under this decomposition, we impose the following restrictions on the positions of the $z_{(j,i)}$'s:

$$\begin{aligned} &z_{(j,h_i^0)}\notin\{w_1,w_2,\cdots,w_l\}\ \ (i=1,2,\cdots,m_0^j),\\ &z_{(j,h_i^p)}=w_p\ \ (p=1,2,\cdots,l,\ \ i=1,2,\cdots,m_p^j). \end{aligned} \tag{3.43}$$

For fixed positive integers m_a^j's ($a=0,1,\cdots,l, j=2,3,\cdots,N-2$), the number of distinct decompositions is given by:

$$\prod_{j=2}^{N-2}\frac{n_j!}{m_0^j!\left(\prod_{p=1}^{l} m_p^j!\right)}. \tag{3.44}$$

Let us fix a decomposition type $\coprod_{j=2}^{N-2}\coprod_{p=0}^{l} W_{(j,p)}$ and denote by $M_{0,2+l|\coprod_{j=2}^{N-2}\coprod_{p=0}^{l} W_{(j,p)}}(CP^{N-1},d-g,\sigma_g)$ the locus within $M_{0,2+l|\sum_{j=2}^{N-2} n_j}(CP^{N-1},d-g,\sigma_g)$ where the $z_{(*,*)}$'s satisfy the restrictions of (3.43). To evaluate the contribution from the excess intersection arising from this locus, we introduce a "perturbation space" defined as follows [3]:

$$\begin{aligned} &M^{pert.}_{0,2+l|\coprod_{j=2}^{N-2}\coprod_{p=0}^{l} W_{(j,p)}}(CP^{N-1},d-g,\sigma_g)\\ &:=\left(M_{0,2+l|\coprod_{j=2}^{N-2} W_{(j,0)}}(CP^{N-1},d-g)\left(\prod_{p=1}^{l}(\underset{CP^{N-1}}{\times}\widetilde{Mp}_{0,2|\sum_{j=2}^{N-2} m_p^j}(N,g_p))\right)\right)/\left(\prod_{i=1}^{g}\mathrm{Sym}(\mathrm{mul}(i,\sigma_g))\right). \end{aligned} \tag{3.45}$$

Here, $M_{0,2+l\coprod_{j=2}^{N-2} W_{(j,0)}}(CP^{N-1},d-g)$ represents the uncompactified moduli space of holomorphic maps from CP^1 with $2+l$ distinct marked points and the marked points $z_{(*,*)}$'s whose subscripts belong to $\coprod_{j=2}^{N-2} W_{(j,0)}$. The p-th fiber product in (3.45) is defined through the following diagram:

$$M_{0,2+l(\sigma_g)|\coprod_{j=2}^{N-2} W_{(j,0)}}(CP^{N-1},d-g)\overset{ev_{w_p}}{\longrightarrow} CP^{N-1}\overset{ev_0}{\longleftarrow}\widetilde{Mp}_{0,2|\sum_{j=2}^{N-2} m_p^j}(N,g_p))\quad (p=1,2,\cdots,l). \tag{3.46}$$

In the above diagram, ev_{w_p} is the evaluation map of $\tilde{\pi}_N(\sum_{j=0}^{d-g}\mathbf{b}_j s^j t^{d-g-j})$ at w_p. The symmetric group $\prod_{i=1}^{g}\mathrm{Sym}(\mathrm{mul}(i,\sigma_g))$ permutes the w_i's along with $\times_{CP^{N-1}}\widetilde{Mp}_{0,2|\sum_{j=2}^{N-2} m_i^j}(N,g_i)$ in a manner consistent with the definition of $M_{0,2+l|\sum_{j=2}^{N-2} n_j}(CP^{N-1},d-g,\sigma_g)$.

[3] The logic behind the introduction of this perturbation space mirrors that used in [5]. See Appendix A of [5].

The corresponding contribution is then given by:

$$\int_{M^{pert.}_{0,2+l(\sigma_g)|\coprod_{j=2}^{N-2}\coprod_{p=0}^{l}W_{(j,p)}}(CP^{N-1},d-g,\sigma_g))} c_{top}(\widetilde{\mathcal{E}}_d^k)\wedge ev_0^*(h^a)\wedge ev_\infty^*(h^b)\wedge\left(\bigwedge_{\substack{j=2,\cdots,N-2\\ i=1,2,\cdots,n_j}} ev_{(j,i)}^*(h^j)\right).$$

$$= \sum_{e_1,\cdots,e_l=0}^{N-2}\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{m_0^j}\prod_{p=1}^{l}\mathcal{O}_{h^{e_p}}\rangle_{0,d-g}\Big(\prod_{p=1}^{l}\frac{w(\mathcal{O}_{h^{N-2-e_p}}\mathcal{O}_1|\prod_{j=2}^{l}(\mathcal{O}_{h^j})^{m_p^j})_{0,g_p}}{k}\Big)\Big(\prod_{i=1}^{g}\frac{1}{(\mathrm{mul}(i,\sigma_g))!}\Big), \tag{3.47}$$

This follows because $w(\mathcal{O}_{h^{N-2-e_i}}\mathcal{O}_1|\prod_{j=2}^{l}(\mathcal{O}_{h^j})^{m_p^j})_{0,g_p}$ (which includes the insertion of "1", or no insertion) is non-vanishing. In deriving this formula, we have used the splitting axiom [9] (2.2.6. page 529) for the fiber product, i.e., the fiber product $\times_{CP^{N-1}}$ under the diagram:

$$M_{0,2+l|\coprod_{j=2}^{N-2}W_{(j,0)}}(CP^{N-1},d-g)\xrightarrow{ev_p}CP^{N-1}\xleftarrow{ev_0}\widetilde{Mp}_{0,2}(N,g_p)$$

corresponds to,

$$\frac{1}{k}\sum_{e_p=0}^{N-2}ev_{w_p}^*(h^{e_p})\wedge ev_0^*(h^{N-2-e_p}).$$

This correspondence arises from the description of the Poincaré dual of the diagonal Δ in $CP^{N-1}\times CP^{N-1}$ (specifically, $M_N^k\times M_N^k$) using cohomology elements.

The term $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{m_0^j}\prod_{i=1}^{l}\mathcal{O}_{h^{e_i}}\rangle_{0,d-g}$ in (3.47) represents the Gromov-Witten invariant of the hypersurface, obtained by integration over $M_{0,2+l|\coprod_{j=2}^{N-2}W_{(j,0)}}(CP^{N-1},d-g)$. While the moduli space $M_{0,2+l|\coprod_{j=2}^{N-2}W_{(j,0)}}(CP^{N-1},d-g)$ is uncompactified and the marked points $z_{(j,i)}$'s can coincide, the boundary components added during compactification by stable maps, and the coincidence loci of the $z_{(j,i)}$'s, do not contribute to the intersection numbers due to their positive complex codimensions. The factor $\prod_{i=1}^{g}\frac{1}{(\mathrm{mul}(i,\sigma_g))!}$ on the right-hand side of (3.47) accounts for the division by the group $\prod_{i=1}^{g}\mathrm{Sym}(\mathrm{mul}(i,\sigma_g))$.

In the $g=0$ case, the right-hand side of (3.47) simplifies to $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{n_j}\rangle_{0,d}$. In the $g=d$ case, $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{m_0^j}\prod_{i=1}^{l}\mathcal{O}_{h^{e_i}}\rangle_{0,0}$ vanishes if $l>1$, or if there exists a $j\in\{2,3,\cdots,N-2\}$ such that $m_0^j>0$ [9] (2.2.5. page 529). If $l=1$ and $m_0^2=m_0^3=\cdots=m_0^{N-2}=0$, we have:

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\mathcal{O}_{h^{e_1}}\rangle_{0,0}=k\cdot\delta_{e_1,N-2-a-b}. \tag{3.48}$$

Consequently, the right-hand side of (3.47) becomes $w(\mathcal{O}_{h^{a+b}}\mathcal{O}_1|\prod_{j=2}^{N-2}(\mathcal{O}_{h^j})^{n_j})_{0,d}$. We can also introduce perturbation spaces for the lower-dimensional strata of $\widetilde{Mp}_{0,2|\sum_{j=2}^{N-2}n_j}(N,d)$, but these are irrelevant to the intersection number due to their positive codimensions.

By multiplying the contribution (3.47) by the combinatorial factor given in (3.44), and summing over all possible choices of g's, l's, σ_g's and m_p^j's, we obtain the right-hand side of Theorem 1.1. □

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References

[1] I. Ciocan-Fontanine, B. Kim. *Big I-functions.* Development of moduli theory-Kyoto 2013. Adv. Stud. Pure Math., 69, (2016), 323–347,

[2] K. Hori. *Private communication.* Feb. 2025.

[3] M. Jinzenji. *Mirror map as generating function of intersection numbers: toric manifolds with two Kähler forms.* Comm. Math. Phys. 323 (2013), no. 2, 747–811.

[4] M. Jinzenji. *On the quantum cohomology rings of general type projective hypersurfaces and generalized mirror transformation.* Internat. J. Modern Phys. A 15 (2000), no. 11, 1557–1595.

[5] M. Jinzenji. *Geometrical proof of generalized mirror transformation of projective hypersurfaces.* Internat. J. Math. 34 (2023), no. 2, Paper No. 2350006, 17 pp.

[6] M. Jinzenji, K. Matsuzaka. *Period integrals (Givental's I-function) of Calabi-Yau hypersurface in CP^{N-1} as generating functions of intersection numbers on the moduli space of quasimaps.* J. Math. Soc. Japan 77 (2025), no. 2, 449–482.

[7] M. Jinzenji, M. Shimizu. *Multi-Point Virtual Structure Constants and Mirror Computation of CP^2-model.* Commun.Num.Theor Phys. 07 (2013) 411–468.

[8] M. Kontsevich. *Enumeration of rational curves via torus actions. The moduli space of curves* (Texel Island, 1994), 335–368, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[9] M. Kontsevich, Yu. I. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry.* Comm. Math. Phys. 164 (1994), no. 3, 525–562.