SELFLESS W*-PROBABILITY SPACES AND CONNES' BICENTRALIZER PROBLEM

CYRIL HOUDAYER AND AMINE MARRAKCHI

ABSTRACT. We introduce the notion of selfless W*-probability space and study its connection with Connes' bicentralizer problem. In particular, we show that if M is a separable type III_1 factor with trivial bicentralizer, then (M, φ) is selfless for every faithful normal state $\varphi \in M_*$.

1. INTRODUCTION

In [Ro23], Robert introduced a new class of C*-probability spaces, which he called *selfless*, characterized by the existence of a copy of themselves in their ultrapower that is freely independent from the diagonal copy (thus being "free from themselves"). This property quickly attracted the attention of numerous researchers as it implies many important regularity properties and is satisfied by a large class of examples (see [AGKEP24, HKER25, RTV25, Vi25, Oz25, FKOCP25, Vi26]).

In this short note, we introduce a parallel notion of selfless W*-probability space and we relate this notion to Connes' bicentralizer problem.

A W*-*probability space* is a pair (M, φ) that consists of a von Neumann algebra M endowed with a faithful normal state $\varphi \in M_*$. For W*-probability spaces (M, φ) and (N, ψ), we say that $(M, \varphi) \subset (N, \psi)$ is an *inclusion* of W*-probability spaces if $M \subset N$ and if there exists a faithful normal conditional expectation $\mathrm{E} : N \to M$ such that $\varphi \circ \mathrm{E} = \psi$. In that case, $\mathrm{E} : N \to M$ is the unique faithful normal conditional expectation such that $\varphi \circ \mathrm{E} = \psi$.

Following [GH21], we say that an inclusion of W*-probability spaces $(M, \varphi) \subset (N, \psi)$ is *existentially closed* if there exists a nonprincipal ultrafilter $\mathcal{U}$ on some directed set I such that $(M, \varphi) \subset (N, \psi) \subset (M, \varphi)^{\mathcal{U}}$, where $(M, \varphi) \subset (M, \varphi)^{\mathcal{U}}$ is the diagonal inclusion. Note that if N is separable (i.e. N has separable predual), then $\mathcal{U}$ can be chosen to be a nonprincipal ultrafilter on $\mathbb{N}$.

Adapting [Ro23, Definition 2.1] to the von Neumann algebraic realm, we say that a W*-probability space (M, φ) is *selfless* if the first factor inclusion $(M, \varphi) \subset (M, \varphi) * (M, \varphi)$ is existentially closed.

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Popa's seminal work [Po95] shows that (M, τ) is selfless for any separable type II_1 factor M endowed with its canonical trace τ. Houdayer–Isono [HI14] extended Popa's result by showing that (M, φ) is selfless for any separable factor M endowed with a faithful normal state $\varphi \in M_*$ for which $(M_\varphi)' \cap M = \mathbb{C}1$.

In this note, we show that a diffuse separable W^*-probability space (M, φ) is selfless if and only if it has a trivial *bicentralizer*. Recall from [Co80, Ha85], that the *bicentralizer* $\mathrm{B}(M, \varphi)$ of a W^*-probability space (M, φ) is the set of all elements $x \in M$ that satisfy the following condition:

> For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $u \in \mathscr{U}(M)$, if $\|u\varphi - \varphi u\| < \delta$, then $\|ux - xu\|_\varphi < \varepsilon$.

We then obtain the following characterization.

Theorem A. *Let (M, φ) be a diffuse separable W^*-probability space. The following assertions are equivalent:*

(i) *(M, φ) is selfless.*
(ii) *The first factor inclusion $(M, \varphi) \subset (M, \varphi) * (N, \psi)$ is existentially closed for some nontrivial W^*-probability space (N, ψ).*
(iii) *The first factor inclusion $(M, \varphi) \subset (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_\mathbb{Z})$ is existentially closed, where $\tau_\mathbb{Z}$ is the canonical trace on $\mathrm{L}(\mathbb{Z})$.*
(iv) *$\mathrm{B}(M, \varphi) = \mathbb{C}1$.*

We point out that the first three conditions above are analogous to the ones appearing in [Ro23, Theorem 2.6] for C^*-probability spaces.

The bicentralizer $\mathrm{B}(M, \varphi) \subset M$ is a von Neumann subalgebra such that $\mathrm{B}(M, \varphi) \subset (M_\varphi)' \cap M$ (see [Ha85, Proposition 1.3]). Thus Theorem A strengthens Houdayer–Isono's result [HI14, Theorem A]. In fact, the proof of Theorem A relies on [HI14, Theorem A] combined with a diagonal argument.

Observe that if $\mathrm{B}(M, \varphi) = \mathbb{C}1$, then M must be a factor. Moreover, according to [Ok21], one and exactly one of the following assertions hold:

- (M, φ) is a tracial factor of type I_n for $n \in \mathbb{N}^*$ or of type II_1.
- There exists $\lambda \in (0, 1)$ such that (M, φ) is a type III_λ factor endowed with its $\frac{2\pi}{|\log(\lambda)|}$-periodic faithful normal state. In that case, we have $(M_\varphi)' \cap M = \mathbb{C}1$.
- M is a type III_1 factor. In that case, using [Ha85, Corollary 1.5], we further have $\mathrm{B}(M, \psi) = \mathbb{C}1$ for *every* faithful normal state $\psi \in M_*$.

In particular, Theorem A implies that no W^*-probability space of type II_∞ or type III_0 can be selfless (this also follows by combining [GH21, Theorem 3.5] and [Ue11, Theorem 4.1]).

On the other hand, a famous conjecture of Connes, known as Connes' bicentralizer problem, claims that $\mathrm{B}(M, \varphi) = \mathbb{C}1$ for *every* type III_1 factor M and every faithful normal state $\varphi \in M_*$. This conjecture has been verified for several families of type III_1 factors such as amenable factors [Ha85], factors with a Cartan subalgebra, free products [HU15], semisolid factors

[HI15], q-deformed Araki–Woods factors [HI20, Bi24] and tensor products of type III_1 factors [Ma25].

Corollary B. *Let M be a separable type III_1 factor satisfying Connes' bicentralizer conjecture. Then for* every *faithful normal state $\varphi \in M_*$, the W^*-probability space (M, φ) is selfless.*

2. Proof of Theorem A

Proof. (i) ⇒ (ii) This is obvious by taking $(N, \psi) = (M, \varphi)$.

(ii) ⇒ (iii) Let (N, ψ) be a nontrivial W^*-probability space such that the first factor inclusion $(M, \varphi) \subset (M, \varphi) * (N, \psi)$ is existentially closed. We may assume that N is separable. Choose a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that we have $(M, \varphi) \subset (M, \varphi) * (N, \psi) \subset (M, \varphi)^{\mathcal{U}}$, where $(M, \varphi) \subset (M, \varphi)^{\mathcal{U}}$ is the diagonal inclusion. There are two cases to consider.

Firstly, assume that $N_\psi = \mathbb{C}1$. Then N is a type III_1 factor (see e.g. [AH12, Lemma 5.3]). Consider the same nonprincipal ultrafilter $\mathcal{V} = \mathcal{U}$ on $\mathbb{N}$ and define $\mathcal{W} = \mathcal{V} \otimes \mathcal{U} = \mathcal{U} \otimes \mathcal{U}$, which is a nonprincipal ultrafilter on $\mathbb{N} \times \mathbb{N}$. By [AHHM18, Proposition 2.5], we naturally have $(M^{\mathcal{U}}, \varphi^{\mathcal{U}})^{\mathcal{V}} = (M^{\mathcal{U}}, \varphi^{\mathcal{U}})^{\mathcal{U}} = (M, \varphi)^{\mathcal{W}}$. Then we have

$$(M, \varphi) \subset (M, \varphi) * (N, \psi)^{\mathcal{U}} \subset (M, \varphi)^{\mathcal{U}} * (N, \psi)^{\mathcal{U}} \subset (M^{\mathcal{U}}, \varphi^{\mathcal{U}})^{\mathcal{U}} = (M, \varphi)^{\mathcal{W}},$$

where $(M, \varphi) \subset (M, \varphi)^{\mathcal{W}}$ is the diagonal inclusion. By [AH12, Proposition 4.24], the centralizer $(N^{\mathcal{U}})_{\psi^{\mathcal{U}}}$ is a type II_1 factor and so $(\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) \subset ((N^{\mathcal{U}})_{\psi^{\mathcal{U}}}, \psi^{\mathcal{U}})$. This futher implies that

$$(M, \varphi) \subset (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) \subset (M, \varphi) * ((N^{\mathcal{U}})_{\psi^{\mathcal{U}}}, \psi^{\mathcal{U}}) \subset (M, \varphi)^{\mathcal{W}},$$

where $(M, \varphi) \subset (M, \varphi)^{\mathcal{W}}$ is the diagonal inclusion. Therefore, the first factor inclusion $(M, \varphi) \subset (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}})$ is existentially closed.

Secondly, assume that $N_\psi \neq \mathbb{C}1$. Upon replacing (N, ψ) by (N_ψ, ψ), we may assume that (N, ψ) is tracial. Reasoning as is the first case, we obtain

$$\begin{aligned}(M, \varphi) &\subset (M, \varphi) * (N, \psi)^{*2} \\ &= ((M, \varphi) * (N, \psi)) * (N, \psi) \\ &\subset (M, \varphi)^{\mathcal{U}} * (N, \psi) \\ &\subset (M, \varphi)^{\mathcal{U}} * (N, \psi)^{\mathcal{U}} \\ &\subset (M^{\mathcal{U}}, \varphi^{\mathcal{U}})^{\mathcal{U}} = (M, \varphi)^{\mathcal{W}}.\end{aligned}$$

By [Ro23, Lemma 2.5], there exists $n \in \mathbb{N}$ large enough so that the iterated free product $(N, \psi)^{*n}$ is diffuse and so $(\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) \subset (N, \psi)^{*n}$. Upon iterating n times the ultraproduct construction and replacing $\mathcal{U}$ by the appropriate ultrafilter $\mathcal{W} = \mathcal{U}^{\otimes n}$, we obtain

$$(M, \varphi) \subset (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) \subset (M, \varphi) * (N, \psi)^{*n} \subset (M, \varphi)^{\mathcal{W}},$$

where $(M, \varphi) \subset (M, \varphi)^{\mathcal{W}}$ is the diagonal inclusion. Therefore, the first factor inclusion $(M, \varphi) \subset (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}})$ is existentially closed.

(iii) $\Rightarrow$ (iv) Choose a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $(M, \varphi) \subset (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) \subset (M, \varphi)^{\mathcal{U}}$, where $(M, \varphi) \subset (M, \varphi)^{\mathcal{U}}$ is the diagonal inclusion. By [HI15, Proposition 3.3], we know that $\mathrm{B}(M, \varphi) = ((M^{\mathcal{U}})_{\varphi^{\mathcal{U}}})' \cap M$. Since $(\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) \subset ((M^{\mathcal{U}})_{\varphi^{\mathcal{U}}}, \varphi^{\mathcal{U}})$, [Ue11, Proposition 3.1] implies that

$$\mathrm{B}(M, \varphi) \subset \mathrm{L}(\mathbb{Z})' \cap M \cap (M, \varphi) * (\mathrm{L}(\mathbb{Z}), \tau_{\mathbb{Z}}) = \mathrm{L}(\mathbb{Z}) \cap M = \mathbb{C}1.$$

(iv) $\Rightarrow$ (i) When (M, φ) is a tracial factor, the implication follows by Popa's result [Po95]. When $\lambda \in (0, 1)$ and (M, φ) is a type III_λ factor endowed with its $\frac{2\pi}{|\log(\lambda)|}$-periodic faithful normal state, the implication follows by [HI14, Theorem A]. Therefore, we may assume that M is a type III_1 factor. By [Ha85, Theorem 3.1], there exists a faithful normal state $\theta \in M_*$ such that $(M_\theta)' \cap M = \mathbb{C}1$. Then by [CS76, Theorem 4], we can choose a sequence of faithful normal states $\varphi_k \in M_*$ such that $\lim_k \|\varphi_k - \varphi\| = 0$ and $(M_{\varphi_k})' \cap M = \mathbb{C}1$ for every $k \in \mathbb{N}$. Choose a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. By [HI14, Theorem A], for every $k \in \mathbb{N}$, there exists a unitary $u_k \in \mathscr{U}((M^{\mathcal{U}})_{\varphi_k^{\mathcal{U}}})$ such that M and $u_k M u_k^*$ are $*$-free inside $M^{\mathcal{U}}$ with respect to $\varphi_k^{\mathcal{U}}$. Choose another nonprincipal ultrafilter $\mathcal{V}$ on $\mathbb{N}$ and define $\mathcal{W} = \mathcal{V} \otimes \mathcal{U}$, which is a nonprincipal ultrafilter on $\mathbb{N} \times \mathbb{N}$. We have $(\varphi_k^{\mathcal{U}})^{\mathcal{V}} = \varphi^{\mathcal{W}}$. Set $u = (u_k)^{\mathcal{V}} \in (M^{\mathcal{U}})^{\mathcal{V}}$ and observe that $u \in \mathscr{U}((M^{\mathcal{W}})_{\varphi^{\mathcal{W}}})$. Then M and uMu^* are $*$-free inside $M^{\mathcal{W}}$ with respect to $\varphi^{\mathcal{W}}$. This further implies that the first factor inclusion $(M, \varphi) \subset (M, \varphi) * (M, \varphi)$ is existentially closed and so (M, φ) is selfless. $\square$

Remark. It is well known that all type III_1 free Araki–Woods factors have trivial bicentralizer [Ho08]. Another proof of this fact was recently obtained in [HI20]. More precisely, in the particular case $q = 0$, the proof of [HI20, Main Theorem] shows that for any type III_1 free Araki–Woods factor $\Gamma(H_{\mathbb{R}}, U)''$ for which the orthogonal representation $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$ has a nonzero weakly mixing part, the W*-probability space $(\Gamma(H_{\mathbb{R}}, U)'', \varphi_U)$ is selfless. It would be interesting to obtain new classes of type III_1 factors with trivial bicentralizer by showing that they are selfless.

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École normale supérieure, Département de mathématiques et applications, Université Paris-Saclay, 45 rue d'Ulm, 75230 Paris Cedex 05, France

E-mail address: cyril.houdayer@ens.psl.eu

CNRS, École normale supérieure de Lyon, Unité de mathématiques pures et appliques, 46 allée dItalie, 69364 Lyon, France

E-mail address: amine.marrakchi@ens-lyon.fr