REMARKS ON KÄHLER ORBIFOLDS OF NON-NEGATIVE RICCI CURVATURE

YUGUANG ZHANG

Abstract. This note proves orbifold versions of Kobayashi's theorem. The main result asserts that a compact Kähler orbifold with non-negative Ricci curvature, along with certain conditions regarding singularities, is simply connected.

1. Introduction

A theorem due to Kobayashi (cf. [10]) asserts that a compact Kähler manifold with positive Ricci curvature is simply connected. This short note aims to generalise this theorem to Kähler orbifolds. The concept of orbifolds was initially introduced by Satake under the term 'V-manifolds', and was later renamed 'orbifolds' by Thurston (cf. [13, 15]).

We briefly recall the definition of orbifolds and refer to [1, 6, 8, 11, 13] for detailed definitions and basic properties of orbifolds. A complex n-orbifold X is a complex analytic space equipped with a complex orbifold structure (See [9, 1]). More precisely, for any point $x \in X$, there exists a neighbourhood U_x that is the quotient of a finite subgroup G_x of $U(n)$ acting linearly on an open neighbourhood $\tilde{U}_x$ of 0 in $\mathbb{C}^n$, i.e., $\tilde{U}_x/G_x = U_x$. We denote the quotient map as $q_x : \tilde{U}_x \to U_x$, which satisfies $q_x(0) = x$, and call U_x an orbifold chart. Furthermore, if $U_y \subseteq U_x$, then there is a holomorphic open embedding $\iota : \tilde{U}_y \hookrightarrow \tilde{U}_x$ and an injective homomorphism $\kappa : G_y \to G_x$ such that $q_x \circ \iota = q_y$ and $\iota(t \cdot y') = \kappa(t)\iota(y')$ for any $t \in G_y$ and $y' \in \tilde{U}_y$. X admits a regular/singular decomposition $X = X_R \coprod X_S$, where the regular part X_R is a complex manifold of dimension n. The singular set X_S consists of those points with non-trivial orbifold groups, i.e., $x \in X_S$ if and only if $G_x \neq \{1\}$.

An example of complex orbifolds is the quotient space $X = \mathbb{C}P^1/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ through the map $z \mapsto -z$. Note that X is an orbifold with X_S consisting of 2 points, even though X is homeomorphic to $\mathbb{C}P^1$ as topological spaces. A complex projective surface X with only finite ordinary double points as singularities is a complex orbifold of dimension 2, which is not homeomorphic to any manifold.

A Kähler metric g (respectively (p, q)-form β) on a complex orbifold X is a smooth Kähler metric g (resp. (p, q)-form β) on the regular part X_R, and

[1]Supported in part by grant NSFC-12531001.

on any $q_x^{-1}(U_x \cap X_R)$, the pull-back $\tilde{g} = q_x^* g$ (resp. $\tilde{\beta} = q_x^* \beta$) extends to a smooth Kähler metric (resp. (p,q)-form) on $\tilde{U}_x \subseteq \mathbb{C}^n$. We refer to (X, g) as a Kähler orbifold.

Many classical theorems for manifolds have been shown to hold for orbifolds ([1, 2, 5, 6, 8, 9, 13]), for example, the de Rham theorem, the Dolbeault theorem, and the Hirzebruch-Riemann-Roch formula, etc. We aim to generalise Kobayashi's theorem to orbifolds in a manner that takes into account the effects of orbifold singularities on the results.

Note that G_x acts on the $(1,0)$-cotangent space $T_0^{*(1,0)}\tilde{U}_x$ at $0 \in \tilde{U}_x$, and thus also on $\wedge^p T_0^{*(1,0)}\tilde{U}_x$, for all $1 \leqslant p \leqslant n$. Define

$$F_x^{p,0} = \{w \in \wedge^p T_0^{*(1,0)}\tilde{U}_x | t \cdot w = w, \text{ for all } t \in G_x\}, \tag{1.1}$$

i.e., the fixed point set of G_x-action on $\wedge^p T_0^{*(1,0)}\tilde{U}_x$. If x is a regular point, i.e., $x \in X_R$, then $\dim_{\mathbb{C}} F_x^{1,0} = n$ and $\dim_{\mathbb{C}} F_x^{n,0} = 1$. $x \in X_S$ if and only if $\dim_{\mathbb{C}} F_x^{1,0} < n$. If x is an ADE surface singularity, i.e., $\mathbb{C}^2/G_0$ for a finite subgroup G_0 of $SU(2)$, then $\dim_{\mathbb{C}} F_x^{1,0} = 0$ and $\dim_{\mathbb{C}} F_x^{2,0} = 1$.

The main result is the following theorem.

Theorem 1. *Let (X, g) be a compact Kähler n-orbifold with non-negative Ricci curvature, i.e.,* $\mathrm{Ric}(g) \geqslant 0$. *If*

$$\sum_{p=1}^{n} \inf_{x \in X} \dim_{\mathbb{C}} F_x^{p,0} = 0, \tag{1.2}$$

then X is simply connected, i.e., the fundamental group $\pi_1(X)$ is trivial.

The hypothesis (1.2), along with the assumption of non-negative Ricci curvature, plays a similar role to that of positive Ricci curvature in Kobayashi's theorem.

Now we apply Theorem 1 to an example. Let $T_{\mathbb{C}}^2 = \mathbb{C}^2/(\mathbb{Z}^2 + \sqrt{-1}\mathbb{Z}^2)$ be the complex 2-torus. If z_1 and z_2 are the angle coordinates on $T_{\mathbb{C}}^2$ induced from coordinates on $\mathbb{C}^2$, then $\mathbb{Z}_2^2$ acts on $T_{\mathbb{C}}^2$ by $\gamma_1 \cdot (z_1, z_2) = (-z_1, -z_2)$ and $\gamma_2 \cdot (z_1, z_2) = (z_2, z_1)$, where γ_1 and γ_2 are generators of $\mathbb{Z}_2^2$. The quotient space $X = T_{\mathbb{C}}^2/\mathbb{Z}_2^2$ is a complex orbifold that admits a flat orbifold Kähler metric g induced by the Euclidean metric on $\mathbb{C}^2$. Note that $(0,0) \in T_{\mathbb{C}}^2$ is fixed by the $\mathbb{Z}_2^2$-action. We consider the orbifold point x_0 in X which is the image of $(0,0) \in T_{\mathbb{C}}^2$ under the quotient map. The induced $\mathbb{Z}_2^2$-actions on $T_{(0,0)}^{*(1,0)}T_{\mathbb{C}}^2$ and $\wedge^2 T_{(0,0)}^{*(1,0)}T_{\mathbb{C}}^2$ are given by $\gamma_1 \cdot dz_i = -dz_i$, $i = 1, 2$, and $\gamma_2 \cdot dz_1 \wedge dz_2 = -dz_1 \wedge dz_2$. Thus we have

$$\dim_{\mathbb{C}} F_{x_0}^{1,0} + \dim_{\mathbb{C}} F_{x_0}^{2,0} = 0,$$

and X is simply connected by Theorem 1. We remark that although X is simply connected, the orbifold fundamental group $\pi_1^{orb}(X)$ of X is not trivial, and the universal orbifold covering of X is $\mathbb{C}^2$ (See [5, 6] for the definitions of orbifold fundamental group and of universal orbifold covering).

We also have a theorem that includes a broader range of orbifold types than those permitted in Theorem 1, such as the ADE surface singularities.

Theorem 2. *Let (X, g) be a compact Kähler n-orbifold with non-negative Ricci curvature, i.e.,* $\mathrm{Ric}(g) \geqslant 0$*. If $n = 2m$ and*

$$\sum_{p=1}^{n-1} \inf_{x\in X} \dim_{\mathbb{C}} F_x^{p,0} = 0, \tag{1.3}$$

then X is either simply connected or the fundamental group $\pi_1(X) \cong \mathbb{Z}_2$. Furthermore,

i) *if $\pi_1(X) \cong \mathbb{Z}_2$, then $\mathrm{Ric}(g) \equiv 0$, i.e., (X, g) is a Calabi-Yau orbifold,*
ii) *if X admits a non-vanishing holomorphic $2m$-form Ω, then X is a simply connected Calabi-Yau orbifold.*

An implication of Theorem 2 is that the Kummer K3 orbifold $T^2_{\mathbb{C}}/\mathbb{Z}_2$, defined as the quotient of the complex two-dimensional torus $T^2_{\mathbb{C}}$ by the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$, is simply connected. This fact has been proved in [14].

Acknowledgments. The author thanks the referee for helpful suggestions that have improved the present work.

2. Proofs

This section proves Theorem 1 and Theorem 2. We follow the proof of Kobayashi's theorem, and adapt the argument to the case of orbifolds. Let X be a compact complex orbifold of dimension n, g be a Kähler metric on X, and ∇ be the Levi-Civita connection of g.

Denote $A^{p,q}(X)$ (resp. $A^k(X)$) as the space of smooth (p, q)-forms (resp. k-forms) on X. As in the smooth case, the exterior differentiation $d : A^k(X) \to A^{k+1}(X)$, the Cauchy-Riemann operator $\bar\partial : A^{p,q}(X) \to A^{p,q+1}(X)$, and $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$ are well-defined, which satisfy $d = \partial + \bar\partial$ and $d^2 = \partial^2 = \bar\partial^2 = 0$. The De Rham cohomology $H^k(X)$ and the Dolbeault cohomology $H^{p,q}(X)$ are defined in the same way as in the case of smooth manifolds (cf. [1, 2, 9]). Note that $H^{p,0}(X)$ consists of holomorphic p-forms, i.e., $\beta \in A^{p,0}(X)$, $\bar\partial\beta = 0$.

We also have the Laplace operators $\Delta_d = (d + d^*)^2$, $\Delta_{\bar\partial} = (\bar\partial + \bar\partial^*)^2$, and $\Delta_\partial = (\partial + \partial^*)^2$, where d^*, $\bar\partial^*$, and ∂^* are the adjoint operators. The same local calculation as in the manifold case shows $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$ (cf. Theorem 8.6 in [12]). By the Hodge-Kodaira decomposition theorem of [2], $H^{p,q}(X)$ (resp. $H^k(X)$) is isomorphic to the space $\mathcal{H}^{p,q}(X)$ (resp. $\mathcal{H}^k(X)$) of harmonic (p, q)-forms, i.e.,

$$H^{p,q}(X) \cong \mathcal{H}^{p,q}(X) = \{\beta \in A^{p,q}(X) | \Delta_{\bar\partial}\beta = 0\}.$$

Furthermore, $\mathcal{H}^{p,q}(X) = \overline{\mathcal{H}^{q,p}(X)}$ and $\dim_{\mathbb{C}} \mathcal{H}^{0,0}(X) = 1$. The holomorphic Euler characteristic is defined as

$$\chi(X, \mathcal{O}_X) = \sum_{p=0}^{n} (-1)^p \dim_{\mathbb{C}} H^{0,p}(X) = \sum_{p=0}^{n} (-1)^p \dim_{\mathbb{C}} H^{p,0}(X).$$

The following proposition might have some independent interests.

Proposition 3. *If (X, g) is a compact Kähler n-orbifold with non-negative Ricci curvature, i.e.,* $\mathrm{Ric}(g) \geqslant 0$, *then for any $p \geqslant 1$,*

$$\dim_{\mathbb{C}} H^{p,0}(X) \leqslant \inf_{x \in X} \dim_{\mathbb{C}} F_x^{p,0}.$$

Proof. Note that the Bochner technique applies to orbifolds (cf. [7, 5]). The Weitzenböck formula (cf. [4] and Proposition 14.3 in [12]) says that

$$2\bar{\partial}^* \bar{\partial} \beta = 2\Delta_{\bar{\partial}} \beta = \nabla^* \nabla \beta + \mathrm{Ric}^p(\beta).$$

Here $\mathrm{Ric}^p : A^{p,0}(X) \to A^{p,0}(X)$ is a zero order linear operator such that $g(\mathrm{Ric}^p(\beta), \beta) \geqslant 0$, for any $\beta \in A^{p,0}(X)$, when the Ricci curvature of g is non-negative, i.e., $\mathrm{Ric}(g) \geqslant 0$. We recall that for a function f on X, the integration is defined as

$$\int_X f dv_g = \sum_{U_x \in \mathcal{U}} \frac{1}{|G_x|} \int_{\tilde{U}_x} (\eta_{U_x} f) \circ q_x dv_{\tilde{g}},$$

(cf. [1, 9]) where $|G_x|$ is the cardinality of G_x. Here $\mathcal{U}$ is a countable collection of orbifold charts U_x that covers of X, and $\{\eta_{U_x}\}$ is a partition of unity with respect to $\mathcal{U}$, i.e., $\sum\limits_{U_x \in \mathcal{U}} \eta_{U_x} \equiv 1$. Now, for any $\beta \in H^{p,0}(X)$, we integrate the inner product of β with the Weitzenböck formula, and obtain

$$0 = \int_X (|\nabla \beta|_g^2 + g(\mathrm{Ric}^p(\beta), \beta)) dv_g \geqslant \int_X |\nabla \beta|_g^2 dv_g \geqslant 0.$$

Thus $\nabla \beta \equiv 0$, i.e., all holomorphic p-forms β are parallel.

If β is a parallel holomorphic p-form, then for any $x \in X$, $q_x^* \beta$ extends to a parallel G_x-invariant p-form $\tilde{\beta}$ on $\tilde{U}_x$. This implies $\tilde{\beta}(0) \in F_x^{p,0}$. By considering an orthonormal basis of $H^{p,0}(X)$, we obtain the result. □

Similar theorems have been obtained for Riemannian orbifolds (Theorem 2.2 of [7] and Proposition 9.0.1 of [16]). More precisely, the Bochner technique can be deployed to show that the first Betti number of a compact Riemannian orbifold M with non-negative Ricci curvature is zero, i.e., $b_1(M) = 0$, if

$$\inf_{x \in M} \dim_{\mathbb{R}} F_x^1 = 0,$$

where F_x^1 denotes the set of fixed points of the G_x-action on $T_0^* \tilde{U}_x$. Furthermore, the splitting theorem for orbifolds in [6] shows that the fundamental group $\pi_1(M)$ is finite in this case. This result generalises Myers' theorem for Riemannian manifolds. We only state and prove this fact for Kähler orbifolds, as it will be used in the proof of the main theorems.

Lemma 4. *Let (X, g) be a compact Kähler n-orbifold with* $\mathrm{Ric}(g) \geqslant 0$. *If*

$$\inf_{x\in X} \dim_{\mathbb{C}} F_x^{1,0} = 0,$$

then the fundamental group $\pi_1(X)$ is finite.

Proof. Let $\pi : \bar{X} \to X$ be the universal covering of X in the usual topological sense. For an orbifold chart $U_x \subseteq X$, $\pi^{-1}(U_x) = \coprod_{\nu\in\pi_1(X)} \bar{U}_x^\nu$, and any connected component $\bar{U}_x^\nu$ of $\pi^{-1}(U_x)$ is homeomorphic to U_x by shrinking U_x if necessary. Hence $\bar{U}_x^\nu = \tilde{U}_x/G_x$, and $\bar{X}$ is equipped with a natural orbifold structure induced by that of X, more precisely the collection of orbifold charts $\bar{\mathcal{U}} = \{\bar{U}_x^\nu | U_x \in \mathcal{U}, \pi^{-1}(U_x) = \coprod_{\nu\in\pi_1(X)} \bar{U}_x^\nu\}$.

Theorem 1 in [6] shows that

$$\bar{X} = Y \times \mathbb{R}^l$$

where Y is an orbifold containing no geodesic lines. The same arguments as in the proof of Theorem 2 of [6] prove that Y is compact by replacing the orbifold fundamental group in [6] with the usual topological fundamental group.

For any $x \in X$, there is an orbifold chart $\bar{U}_x^\nu \subseteq \bar{X}$ homeomorphic to U_x, and $\bar{U}_x^\nu = \tilde{U}_x/G_x$. Therefore, the fixed point set of the G_x-action on $\mathbb{C}^n$ contains at least a real linear subspace $\mathbb{R}^l$. Since $G_x \subseteq U(n)$ and $T_0^*\tilde{U}_x \otimes \mathbb{C} = T_0^{*(1,0)}\tilde{U}_x \oplus T_0^{*(0,1)}\tilde{U}_x$, there is a non-trivial $w \in T_0^{*(1,0)}\tilde{U}_x$ fixed by the G_x-action if $l > 0$, i.e., $\mathrm{Re}(w) \in T_0^*(\{0\}\times\mathbb{R}^l) \subseteq T_0^*\tilde{U}_x$ and $t\cdot w = w$ for all $t \in G_x$. This contradicts the hypotheses $\inf_{x\in X} \dim_{\mathbb{C}} F_x^{1,0} = 0$. Hence $l = 0$, and the universal covering space $\bar{X} = Y$ is compact, which is equivalent to $\pi_1(X)$ being finite. □

The Hirzebruch-Riemann-Roch formula has been generalised to complex orbifolds by Kawasaki in [9]. More precisely,

$$\chi(X, \mathcal{O}_X) = \sum_{U_x\in\mathcal{U}} \frac{1}{|G_x|} \sum_{t\in G_x} \int_{\tilde{U}_x^t} \eta_{U_x} \circ q_x \mathrm{Todd}^t(\tilde{g}),$$

where $\mathcal{U} = \{U_x\}$ covers X, and $\{\eta_{U_x}\}$ is a partition of unity with respect to $\mathcal{U}$, i.e., $\sum_{U_x\in\mathcal{U}} \eta_{U_x} \equiv 1$. Here $\tilde{g} = q_x^* g$, $\tilde{U}_x^t = \{y \in \tilde{U}_x | t \cdot y = y\}$, for any $t \in G_x$, and $\mathrm{Todd}^t(\tilde{g})$ is the equivariant Todd form on $\tilde{U}_x^t$. Suppose that the universal covering $\pi : \bar{X} \to X$ of X is compact, i.e., the fundamental group $\pi_1(X)$ is finite. Now the collection $\bar{\mathcal{U}} = \{\pi^{-1}(U_x) = \coprod_{\nu\in\pi_1(X)} \bar{U}_x^\nu\}$ covers $\bar{X}$, and $\{\eta_{U_x} \circ \pi\}$ is a partition of unity on $\bar{X}$. We obtain

$$\chi(\bar{X}, \mathcal{O}_{\bar{X}}) = \sum_{\bar{U}_x^\nu\in\bar{\mathcal{U}}} \frac{1}{|G_x|} \sum_{t\in G_x} \int_{\tilde{U}_x^t} \eta_{U_x} \circ q_x \mathrm{Todd}^t(\tilde{g}) = |\pi_1(X)|\chi(X, \mathcal{O}_X).$$

Proof of Theorem 1. Assume that the Ricci curvature $\mathrm{Ric}(g) \geqslant 0$, and the condition (1.2) holds. Hence we have $H^{p,0}(X) = \{0\}$ for any $1 \leqslant p \leqslant n$ by Proposition 3, and

$$\chi(X, \mathcal{O}_X) = 1.$$

Lemma 4 shows that $\pi_1(X)$ is finite. The Ricci curvature of $\pi^* g$ is non-negative, and the condition of (1.2) is also satisfied since $\bar{U}_x^\nu = \tilde{U}_x / G_x$ for any $\nu \in \pi_1(X)$, i.e., for each $p = 1, \cdots, n$, there is a point $y \in \bar{X}$ such that $\pi(y) = x$ and

$$\dim_{\mathbb{C}} F_y^{p,0} = \dim_{\mathbb{C}} F_x^{p,0} = 0.$$

Therefore

$$1 = \chi(\bar{X}, \mathcal{O}_{\bar{X}}) = |\pi_1(X)| \chi(X, \mathcal{O}_X) = |\pi_1(X)|.$$

We conclude that X is simply connected. □

Proof of Theorem 2. The same arguments as above prove that the universal covering $\bar{X}$ is compact, i.e., $\pi_1(X)$ is finite. Since all holomorphic forms are parallel as shown in the proof of Proposition 3, we have

$$\dim_{\mathbb{C}} H^{2m,0}(X) \leqslant 1, \quad \text{and} \quad \dim_{\mathbb{C}} H^{2m,0}(\bar{X}) \leqslant 1.$$

By Proposition 3, the condition (1.3) implies that $H^{p,0}(X) = H^{p,0}(\bar{X}) = 0$ for $1 \leqslant p \leqslant 2m - 1$ as above. Thus $\chi(\bar{X}, \mathcal{O}_{\bar{X}})$ and $\chi(X, \mathcal{O}_X)$ are either 1 or 2. Note that $\chi(\bar{X}, \mathcal{O}_{\bar{X}}) \geqslant \chi(X, \mathcal{O}_X)$. If $\chi(\bar{X}, \mathcal{O}_{\bar{X}}) = \chi(X, \mathcal{O}_X)$, then X is simply connected. If $\chi(\bar{X}, \mathcal{O}_{\bar{X}}) = 2$ and $\chi(X, \mathcal{O}_X) = 1$, then $\pi_1(X) = \mathbb{Z}_2$. In this case, $H^{2m,0}(\bar{X})$ is generated by a non-trivial parallel holomorphic $2m$-form Ω on $\bar{X}$. Hence, the local holonomy group of $\pi^* g$ is reduced to $SU(2m)$, which implies $\mathrm{Ric}(\pi^* g) = \pi^* \mathrm{Ric}(g) \equiv 0$ (cf. Chapter 10 in [3]). Finally, if there is a non-vanishing holomorphic $2m$-form Ω on X, then $\pi^* \Omega$ is also holomorphic on $\bar{X}$ and $\mathrm{Ric}(g) \equiv 0$. Therefore, $\chi(\bar{X}, \mathcal{O}_{\bar{X}}) = \chi(X, \mathcal{O}_X) = 2$, which implies that $\bar{X} = X$ and X is simply connected. □

References

[1] W. Baily Jr., *The decomposition theorem for V-manifolds*, Amer. J. Math., 78 (1956), 862–888.
[2] W. Baily Jr., *On the imbedding of V-manifolds in projective space*, Amer. J. Math., 79 (1957), 403–430.
[3] A. Besse, *Einstein manifolds*, Springer, (1987).
[4] S. Bochner, *Vector fields and Ricci curvature*, Bull. Amer. Math. Soc., 52 (1946), 776–797.
[5] J. Borzellino, *Orbifolds with lower Ricci curvature bounds*, Proc. Amer. Math. Soc., 125 (1997), 3011–3018.
[6] J. Borzellino, S. Zhu, *The splitting theorem for orbifolds*, Illinois J. Math., 38 (1994), 679–691.
[7] F. Caramello, D. Töben, *Equivariant basic cohomology under deformations*, Math. Z., 299 (2021), 2461–2482.
[8] T. Kawasaki, *The signature theorem for V-manifolds*, Topology, 17 (1978), 75–83.
[9] T. Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Math., 16 (1979), 151–159.

[10] R. Kobayashi, A. Todorov, *Polarized period map for genaralized K3 surfaces and the moduli of Einstein metrics*, Tôhoku Math. J., 39 (1987), 341–363.
[11] S. Kobayashi, *On compact Kähler manifolds with positive definite Ricci tensor*, Ann. of Math., 74 (1961), 570–574.
[12] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts 69, Cambridge University Press, (2009).
[13] I. Satake, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan, 9 (1957), 464–492.
[14] E. Spanier, *The homology of Kummer manifolds*, Proc. Amer. Math. Soc., 7 (1956), 155–160.
[15] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton University, (1978).
[16] Y. Zhang, *Convergence of Kähler manifolds and calibrated fibrations*, PhD thesis, Nankai Institute of Mathematics (2006).

Xi'an Jiaotong-Liverpool University, Suzhou, China
Email address: `Yuguang.Zhang@xjtlu.edu.cn`