

1                   **Harish-Chandra Theorem for the Multi-Parameter**  
2                   **Quantum Groups of Okado-Yamane Type**

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5                   **Abstract.** This paper is devoted to studying the centre of the multi-  
6                   parameter quantum group  $U_{q,G}(\mathfrak{g})$  introduced by Okado and Yamane, where  
7                    $\mathfrak{g}$  is a complex simple Lie algebra, and all parameters lie in general position.  
8                   We mainly establish the Harish-Chandra theorem, proving that the Harish-  
9                   Chandra homomorphism is an isomorphism; in particular, we determine the  
10                  centre  $Z(U_{q,G}) \cong (U_{\mathfrak{b}}^0)^W$  is isomorphic to a polynomial algebra or a quotient  
11                  algebra of a polynomial algebra. The same result holds for the  $(U_{\mathfrak{b}}^0)^W$  of the  
12                  two-parameter quantum group  $U_{r,s}(\mathfrak{g})$ .

13                  **1. Introduction.**

14                  In mathematics and theoretical physics, “quantum group”  $U_q(\mathfrak{g})$  refers to a class of  
15                  Hopf algebras that are neither commutative nor cocommutative. One of the most well-  
16                  known and widely studied classes of quantum groups is the Drinfeld-Jimbo presentation,  
17                  which arises as a  $q$ -deformation of the universal enveloping algebra  $U(\mathfrak{g})$  of a complex  
18                  simple Lie algebra  $\mathfrak{g}$ . This construction was independently introduced by Drinfeld [7] and  
19                  Jimbo [13] in the mid of 1980s, and has played a fundamental role in the development of  
20                  quantum integrable systems, representation theory, and the theory of braid group actions  
21                  etc. (see [5, 7, 12, 15, 18]). In 1990, Okado and Yamane [19] introduced a class of  
22                  multi-parameter quantum groups  $U_{q,G}(\mathfrak{g})$ , defined via the parameter matrix  $G = (q_{ij})$   
23                  and modified commutation relations, and investigated the basic  $R$ -matrix of  $U_{q,G}(\widehat{\mathfrak{sl}}_n)$ .  
24                  Subsequently, Hayashi [10] showed that it admits a unique bilinear pairing and similar  
25                  highest weight representations. In fact,  $U_{q,G}(\mathfrak{g})$  can be realized as a specialization of the  
26                  general multi-parameter quantum groups constructed in the second author’s joint work  
27                  with Pei and Rosso [20] under the appropriate parameter constraints.

28                  Several works on the centre of quantum groups have been developed over the last  
29                  three decades. In 1990, Rosso [21] defined a significant  $ad$ -invariant bilinear form on  
30                   $U_q(\mathfrak{g})$  with a generic  $q$ , where  $\mathfrak{g}$  is a finite-dimensional simple Lie algebra. This form,  
31                  often referred to as the Rosso form or quantum Killing form, has paved the way for the  
32                  theorem of quantum Harish-Chandra isomorphism (later completed by Tanisaki [22, 23]  
33                  and Joseph-Letzter [14] etc. through different approaches). The centre  $Z(U_q)$  is a poly-  
34                  nomial algebra for the most of types and a quotient thereof in the remaining types, as

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1 studied by Li-Xia-Zhang [16, 17]. For quantum groups of weight lattice types  $\check{U}_q(\mathfrak{g})$ ,  
 2 their centres are all isomorphic to polynomial algebras. (First proposed by Etingof [8];  
 3 a complete proof with explicit central element construction was later provided by Dai  
 4 [6], building upon the work of R. B. Zhang et al. [9, 25]). In the broader context of  
 5 quantum doubles of the bosonization of Nichols algebras of (finite) diagonal type, in-  
 6 cluding multiparameter quantum groups, quantum superalgebras, and Lusztig's small  
 7 quantum groups with free Cartan part, Batra and Yamane [1, 2, 3] established the gen-  
 8 eral Harish-Chandra-type theorem for (skew) centres and proposed a conjectural basis  
 9 for these algebras. Their work provides a powerful, unified framework: [2, Thm. 10.4]  
 10 characterizes the image of the Harish-Chandra homomorphism as an abstract subalgebra  
 11  $\mathfrak{B}_{\check{\omega}, \pi}^{\chi, \pi}$  defined by a system of equations, while [3, Conj. 3.13] proposes that the centre  
 12 should admit a natural basis parameterized by finite-dimensional weight modules. How-  
 13 ever, for concrete families of quantum groups, the explicit structure of the centre and  
 14 the verification of this conjectural basis often require further investigation.

15 Recently, Hu and Wang [11] established the Harish-Chandra theorem for two-  
 16 parameter quantum groups  $U_{r,s}(\mathfrak{g})$  with detailed centre structure in the case when two-  
 17 parameter  $r, s$  are generic and central elements, and the open question in the odd-rank  
 18 case solved by Xia [24] (with the modified relations in his definition of two-parameter  
 19 quantum groups):

20 PROPOSITION 1.1. [11, 24] *Let  $n = \text{rank}(\mathfrak{g})$ , parameters  $r$  and  $s$  be in general*  
 21 *position,  $U = U_{r,s}(\mathfrak{g})$ , and  $\check{U} = \check{U}_{r,s}(\mathfrak{g})$  be the weight lattice type of  $U$ , then we have the*  
 22 *following Harish-Chandra theorem:*

23 1. *The Harish-Chandra homomorphism  $\xi : Z(U) \rightarrow U^0$  is injective. The image*  
 24  *$\xi(Z(U)) = (U_b^0)^W$  in the case when  $n$  is even, and  $\xi(Z(U)) \supseteq (U_b^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}]$*   
 25 *in the case when  $n$  is odd.*

26 2. *The Harish-Chandra homomorphism  $\check{\xi} : Z(\check{U}) \rightarrow \check{U}^0$  is injective. The image*

$$27 \quad \xi(Z(\check{U})) = \begin{cases} (\check{U}_b^0)^W & = \mathbb{K}[z_{\varpi_1}, \dots, z_{\varpi_n}], & \text{if } n \text{ is even,} \\ (\check{U}_b^0)^W \otimes \mathbb{K}[z_*^{\frac{1}{2}}, z_*^{-\frac{1}{2}}] & = \mathbb{K}[z_{\varpi_1}, \dots, z_{\varpi_n}] \otimes \mathbb{K}[z_*^{\frac{1}{2}}, z_*^{-\frac{1}{2}}], & \text{if } n \text{ is odd,} \end{cases}$$

28 *where  $\varpi_i$  is the  $i$ -th fundamental weight (shown in Appendix A), central element*  
 29  *$z_\lambda$  is obtained by the Rosso form realization of quantum trace on weight module*  
 30  *$L(\lambda)$ ;  $z_*$  is an additional invertible central generator (degenerating to the 1 in the*  
 31 *one-parameter specialization), and  $\ell = 2$ , except  $\ell = 4$  for  $D_{2k+1}$ .*

32 Inspired by the above-mentioned works, we investigate the centre of the quantum  
 33 group  $U = U_{q,G}(\mathfrak{g})$  ( $n \geq 2$ , otherwise it is  $U_{q,q^{-1}}(\mathfrak{sl}_2)$ ), under the simplest assumption  
 34 that all parameters in  $\{q\} \cup \{q_{ij} \mid i < j\}$  are algebraically independent, and prove  
 35 that the Harish-Chandra homomorphism  $\xi : Z(U) \rightarrow U^0$  is injective, and the image  
 36  $\text{Im}(\xi) = (U_b^0)^W$  for all types. The key difference in our approach arises from the fact  
 37 that  $U_{q,G}(\mathfrak{g})$  involves more algebraically independent parameters than  $U_{r,s}(\mathfrak{g})$ . These  
 38 parameters induce stronger conditions (see Lemma 2), which are essential in guaranteeing  
 39  $\text{Im}(\xi) \subseteq (U_b^0)^W$  for all types. The isomorphism  $\xi : Z(U) \rightarrow (U_b^0)^W$  indicates  $Z(U) =$

1  $\text{span}_{\mathbb{K}}\{z_\lambda \mid \lambda \in \Psi := \Lambda^+ \cap Q\}$ , which confirms that the Batra-Yamane conjecture [3,  
2 Conj. 3.13] holds for  $Z(U_{q,G})$  under the given conditions (see Corollary 1).

3 Although the set  $\{z_\lambda \mid \lambda \in \Psi_{\min}\}$  generates  $Z(U)$ , where  $\Psi_{\min}$  is the minimal  
4 generating set of the monoid  $\Psi$ , the algebraic relations among them depend on Steinberg's  
5 formula. To obtain a more direct formulation of  $Z(U)$ , motivated by Li-Xia-Zhang's work  
6 [16] on  $U_q(\mathfrak{g})$  (where their  $\Psi' := \Lambda^+ \cap \frac{Q}{2}$ ), we shall construct an algebra monomorphism  
7  $\theta : (U_b^0)^W \rightarrow \mathbb{K}[\mathfrak{h}^*]^W$  and find another generating set of  $\text{Im}(\theta)$  whose multiplicative  
8 relations are completely determined by the additive relations among  $\Psi_{\min}$  in  $\Psi$ . So we  
9 can describe the  $Z(U) \cong (U_b^0)^W \cong \text{Im}(\theta)$  by (a quotient of) a polynomial algebra with  
10 the explicit generators (and generating relations). Our main results can be summarized  
11 as follows:

12 **THEOREM 1.2.** *Let  $U = U_{q,G}(\mathfrak{g})$  ( $n \geq 2$ ), we have the following Harish-Chandra*  
13 *theorem:*

- 14 1. *The Harish-Chandra homomorphism  $\xi : Z(U) \rightarrow U^0$  is injective. The image*  
15  *$\xi(Z(U)) = (U_b^0)^W$ .*  
16 2. *The diagram*

$$\begin{array}{ccc}
 R(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{K} & \dashrightarrow & Z(U_{q,G}(\mathfrak{g})) \\
 \text{Ch} \downarrow \wr & & \xi \downarrow \wr \\
 \mathbb{K}[\mathfrak{h}^*]^W & \xleftarrow{\theta} & (U_b^0)^W
 \end{array}
 \quad
 \begin{array}{ccc}
 [L(\lambda)] & \overset{\lambda \in \Lambda^+ \cap Q}{\dashrightarrow} & z_\lambda \\
 \downarrow & & \downarrow \\
 \sum_{\mu \leq \lambda} \dim(L(\lambda)_\mu) e^\mu & \longleftarrow & \sum_{\mu \leq \lambda} \dim(L(\lambda)_\mu) \omega'_\mu \omega_{-\mu}
 \end{array}$$

18 *commutes, where  $R(\mathfrak{g})$  denotes the Green ring of  $\mathfrak{g}$ , and the dashed arrow represents*  
19 *a partial map only defined on  $[L(\lambda)]$ ,  $\lambda \in \Psi = \Lambda^+ \cap Q$ .*

- 20 3. *The centre  $Z(U) \cong (U_b^0)^W$  is isomorphic to a polynomial algebra  $S = \mathbb{K}[t_1, \dots, t_n]$*   
21 *when  $\mathfrak{g}$  is of type  $B_n, E_8, F_4$  or  $G_2$ , otherwise it is isomorphic to a quotient  $S/I$ ,*  
22 *where  $S$  is a polynomial algebra of rank  $|\Psi_{\min}|$ , and  $I$  is the relation ideal given in*  
23 *Theorem 5.1.*  
24 4. *With the same monoid  $\Psi = \Lambda^+ \cap Q$ , the structure of  $(U_b^0)^W$  in Proposition 1.1.1*  
25 *for the two-parameter quantum group  $U_{r,s}(\mathfrak{g})$  matches Theorem 1.2.3 identically.*  
26 *That is,*

$$\xi(Z(U_{r,s})) \begin{cases} = (U_b^0)^W & \cong S/I, & \text{if } n \text{ is even,} \\ \supseteq (U_b^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}] \cong S/I \otimes \mathbb{K}[z_*, z_*^{-1}], & \text{if } n \text{ is odd.} \end{cases}$$

28 The paper is organized as follows. In Section 2, we present the definition of the  
29 quantum groups  $U_{q,G}(\mathfrak{g})$ , the Rosso form, and the weight module. In Section 3, we  
30 introduce the Harish-Chandra homomorphism  $\xi$ , prove its injectivity, and show that  
31  $\text{Im}(\xi) \subseteq (U_b^0)^W$ . In Section 4, we construct central elements  $z_\lambda$  by realizing the quantum  
32 trace  $t_\lambda$  on the weight module  $L(\lambda)$  via the Rosso form. We prove that the images  $\xi(z_\lambda)$   
33 span the subalgebra  $(U_b^0)^W$ , thereby establishing  $\text{Im}(\xi) \supseteq (U_b^0)^W$ . Consequently, the

1 proof of Theorem 1.2.1 is finished, and then  $Z(U)$  has the basis  $\{z_\lambda \mid \lambda \in \Psi\}$ , which  
 2 supports the Batra-Yamane conjecture in our case. In Section 5, we construct an algebra  
 3 monomorphism  $\theta : (U_b^0)^W \rightarrow \mathbb{K}[\mathfrak{h}^*]^W$ , making the diagram in Theorem 1.2.2 commute.  
 4 Then we provide minimal generating sets for both  $\Psi$  and  $\text{Im}(\theta)$ . Finally, we construct  
 5 an epimorphism  $\phi$  from a rank- $|\Psi_{\min}|$  polynomial algebra  $S$  onto  $\text{Im}(\theta)$ , and determine  
 6 a generating set for the ideal  $I = \ker(\phi)$ . Then it turns out that  $(U_b^0)^W \cong \text{Im}(\theta) \cong S/I$ ,  
 7 which completes the proofs of Theorems 1.2.3 and 1.2.4.

## 8 2. Preliminaries.

### 9 2.1. Specialized multi-parameter quantum groups and the Rosso form

10 Let  $I$  be a set of integers and  $A = [a_{ij}]_{i,j \in I}$  be a symmetrizable Cartan matrix cor-  
 11 responding to a complex simple Lie algebra  $\mathfrak{g}$ . Let  $\{d_i\}_{i \in I}$  be positive integers such that  
 12  $d_i a_{ij} = d_j a_{ji}$  and  $\gcd\{d_i\}_{i \in I} = 1$ . Let  $\mathbb{K}$  be an algebraically closed field of characteristic  
 13 zero,  $q \in \mathbb{K}^\times$ , and  $G = [q_{ij}]_{i,j \in I}$  be a matrix such that  $q_{ij} \in \mathbb{K}^\times$ ,  $q_{ii} = 1$  and  $q_{ij}q_{ji} = 1$ .  
 14 We assume that the parameters in the set  $\{q\} \cup \{q_{ij} \mid i < j\}$  are algebraically independent  
 15 (certainly this excludes the root of unity cases).

16 **DEFINITION 1.** [10, 19] *Let  $U = U_{q,G}(\mathfrak{g})$  be the unital associative algebra over  $\mathbb{K}$*   
 17 *generated by elements  $e_i, f_i, \omega_i^{\pm 1}, \omega_i^{\prime \pm 1}$  satisfying:*

$$(A1) \quad \omega_i \omega_i^{-1} = \omega_i' \omega_i'^{-1} = 1,$$

$$\omega_i \omega_j = \omega_j \omega_i, \quad \omega_i' \omega_j' = \omega_j' \omega_i', \quad \omega_i' \omega_j = \omega_j \omega_i',$$

$$(A2) \quad \omega_i e_j \omega_i^{-1} = q_i^{a_{ij}} q_{ij} e_j, \quad \omega_i' e_j \omega_i'^{-1} = q_j^{-a_{ji}} q_{ji}^{-1} e_j,$$

$$\omega_i f_j \omega_i^{-1} = q_i^{-a_{ij}} q_{ij}^{-1} f_j, \quad \omega_i' f_j \omega_i'^{-1} = q_j^{a_{ji}} q_{ji} f_j,$$

$$18 \quad (A3) \quad e_i f_j - f_j e_i = \delta_{ij} \cdot \frac{\omega_i - \omega_i'}{q_i - q_i^{-1}},$$

$$(A4) \quad \sum_{n=0}^{1-a_{ij}} (-q_{ij})^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_i e_i^{1-a_{ij}-n} e_j e_i^n = 0, \quad (i \neq j),$$

$$\sum_{n=0}^{1-a_{ij}} (-q_{ij})^n \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_i f_i^{1-a_{ij}-n} f_j f_i^n = 0, \quad (i \neq j),$$

19 where  $q_i = q^{d_i}$ ,  $i \in I$ , and  $\begin{bmatrix} m \\ n \end{bmatrix}_i$  is defined by

$$20 \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i [m-1]_i \cdots [m-n+1]_i}{[n]_i [n-1]_i \cdots [1]_i}, \quad [m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}.$$

21 The algebra  $U = U_{q,G}$  constitutes a multi-parameter quantum enveloping algebra  
 22 (see [20] for general definition), equipped with the Hopf algebra structure  $(U, \Delta, \varepsilon, S)$   
 23 defined by

$$\begin{aligned}
\text{(C1)} \quad & \Delta(\omega_i) = \omega_i \otimes \omega_i, & \Delta(\omega'_i) &= \omega'_i \otimes \omega'_i, \\
& \Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i, & \Delta(f_i) &= f_i \otimes \omega'_i{}^{-1} + 1 \otimes f_i, \\
\text{(C2)} \quad & \varepsilon(\omega_i) = \varepsilon(\omega'_i) = 1, & \varepsilon(e_i) &= \varepsilon(f_i) = 0, \\
\text{(S1)} \quad & S(\omega_i) = \omega_i^{-1}, & S(\omega'_i) &= \omega'_i{}^{-1}, & S(e_i) &= -\omega_i^{-1}e_i, & S(f_i) &= -f_i\omega'_i.
\end{aligned}$$

2 Refer to [20, Thm. 20], there exists a unique bilinear pairing  $\langle -, - \rangle: U^{\leq 0} \times U^{\geq 0} \rightarrow \mathbb{K}$   
3 such that for all  $x, x' \in U^{\leq 0}$ ,  $y, y' \in U^{\geq 0}$ ,  $\mu, \nu \in Q$ , and  $i, j \in I$

$$\begin{aligned}
& \langle y, xx' \rangle = \langle \Delta(y), x' \otimes x \rangle, & \langle yy', x \rangle &= \langle y \otimes y', \Delta(x) \rangle, \\
& \langle f_i, e_j \rangle = \delta_{ij} \cdot \frac{1}{q_i^{-1} - q_i}, \\
& \langle \omega'_i, \omega_j \rangle = q_j^{a_{ji}} q_{ji}, & \langle \omega'_i{}^{\pm 1}, \omega_j^{-1} \rangle &= \langle \omega'_i{}^{\pm 1}, \omega_j \rangle^{-1} = \langle \omega'_i, \omega_j \rangle^{\mp 1},
\end{aligned}$$

5 for  $1 \leq i, j \leq n$ , and all other pairs of generators are 0.

6 **THEOREM 2.1.** [20, Cor. 22] *The algebra  $U = U_{q,G}(\mathfrak{g})$  has the triangular decompo-*  
7 *sition  $U \cong U_{q,G}(\mathfrak{n}^-) \otimes U^0 \otimes U_{q,G}(\mathfrak{n})$ , abbreviated as  $U = U^- U^0 U^+$ , where  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$*   
8 *is the semisimple Lie algebra corresponding to the Cartan matrix  $A$ .*

9 The algebra  $U$  is  $Q$ -graded given by

$$10 \quad \deg e_i = \alpha_i, \quad \deg f_i = -\alpha_i, \quad \deg \omega_i = \deg \omega'_i = 0,$$

11 it has  $Q^+$ -graded subalgebras

$$12 \quad U^\pm = \bigoplus_{\mu \in Q^+} U_{\pm\mu}^\pm = \bigoplus_{\mu \in Q^+} \left\{ x \in U^\pm \mid \begin{array}{l} \omega_\eta x \omega_\eta^{-1} = \langle \omega'_{\pm\mu}, \omega_\eta \rangle x \\ \omega'_\eta x \omega_\eta'^{-1} = \langle \omega'_\eta, \omega_{\pm\mu} \rangle^{-1} x, \eta \in Q \end{array} \right\}.$$

13 One can also define a homomorphism  $\varrho^\lambda: U^0 \rightarrow \mathbb{K}$  for each  $\lambda = \sum_{i=1}^n \lambda_i \alpha_i \in \Lambda$  satisfying

$$\begin{aligned}
14 \quad & \varrho^\lambda(\omega_j) = \prod_{i=1}^n \langle \omega'_i, \omega_j \rangle^{\lambda_i} = \prod_{i=1}^n (q_j^{a_{ji}} q_{ji})^{\lambda_i}, \\
& \varrho^\lambda(\omega'_j) = \prod_{i=1}^n \langle \omega'_j, \omega_i \rangle^{-\lambda_i} = \prod_{i=1}^n (q_i^{a_{ij}} q_{ij})^{-\lambda_i}.
\end{aligned}$$

15 The following lemma is the key in proving the injectivity of the Harish-Chandra  
16 homomorphism.

17 **LEMMA 1.** *Let  $\rho$  be the half sum of positive roots. Given any  $\eta, \phi \in Q$  and*  
18 *any integer  $N$ , we have  $\varrho^\lambda(\omega'_\eta \omega_\phi) = 1$  for any  $\lambda \in \Lambda^+$  with  $\lambda \geq N\rho$ , if and only if*  
19  *$(\eta, \phi) = (0, 0)$ .*

20 **PROOF.** It suffices to prove the necessity. Fix elements  $\eta, \phi \in Q$  and  $\lambda \in \Lambda^+$ ,  
21 viewing all three as column vectors in the basis  $\{\alpha_i\}_{i=1}^n$ , one has

$$\begin{aligned}
\varrho^\lambda(\omega_\phi) &= \prod_{j=1}^n (\varrho^\lambda(\omega_j))^{\phi_j} = \prod_{i,j=1}^n (q_j^{a_{ji}} q_{ji})^{\lambda_i \phi_j} \\
&= q^{\sum_{i,j} \phi_j d_j a_{ji} \lambda_i} \left( \prod_{i,j=1}^n q_{ji}^{\lambda_i \phi_j} \right) = q^{\phi^T D A \lambda} \left( \prod_{i,j=1}^n q_{ij}^{-\lambda_i \phi_j} \right).
\end{aligned}$$

Similarly, one has  $\varrho^\lambda(\omega'_\eta) = q^{-\eta^T D A \lambda} \left( \prod_{i,j} q_{ij}^{-\lambda_i \eta_j} \right)$ . Hence, the condition  $\varrho^\lambda(\omega'_\eta \omega_\phi) = 1$  is equivalent to  $(\phi - \eta)^T D A \lambda = 0$ , and  $-\lambda_i (\eta + \phi)_j + \lambda_j (\eta + \phi)_i = 0$  for all  $\lambda \in \Lambda^+$ ,  $\lambda \geq N\rho$ , which easily leads to  $(\eta, \phi) = (0, 0)$ .  $\square$

The proof of  $\xi(Z(U)) \supseteq (U_b^0)^W$  for the Harish-Chandra theorem requires central elements constructed via the following quantum Killing form.

**DEFINITION 2.** *The bilinear form  $\langle - | - \rangle: U \times U \rightarrow \mathbb{K}$  defined by*

$$\begin{aligned}
\langle (y_1 \omega'_{\nu_1}) \omega'_{\eta_1} \omega_{\phi_1} x_1 | (y_2 \omega'_{\nu_2}{}^{-1}) \omega'_{\eta_2} \omega_{\phi_2} x_2 \rangle &= \langle y_2, x_1 \rangle \langle \omega'_{\eta_2}, \omega_{\phi_1} \rangle \langle \omega'_{\eta_1}, \omega_{\phi_2} \rangle \langle S^2(y_1), x_2 \rangle \\
&= q^{2(\rho, \nu_1)} \langle y_2, x_1 \rangle \langle \omega'_{\eta_2}, \omega_{\phi_1} \rangle \langle \omega'_{\eta_1}, \omega_{\phi_2} \rangle \langle y_1, x_2 \rangle
\end{aligned}$$

is called the Rosso form of  $U$ , where  $x_i \in U_{\mu_i}^+$ ,  $y_i \in U_{-\nu_i}^-$ ,  $\mu_i, \nu_i, \phi_i, \eta_i \in Q^+$ .

**PROPOSITION 2.2.** [20] *As a specialized multi-parameter quantization, the algebra  $U = U_{q,G}$  admits:*

1. *The Rosso form is  $\text{ad}_l$ -invariant, i.e., for all  $a, b, c \in U$ , one has*

$$\langle \text{ad}_l(a) b | c \rangle = \langle b | \text{ad}_l(S(a)) c \rangle,$$

2. *The Rosso form satisfies the orthogonality condition: if  $\mu_i, \nu_i \in Q^+$ , then  $\langle U_{-\nu_1}^- U^0 U_{\mu_1}^+ | U_{-\nu_2}^- U^0 U_{\mu_2}^+ \rangle \neq 0$  if and only if  $\mu_1 = \nu_2, \nu_1 = \mu_2$ .*

**THEOREM 2.3.** *The Rosso form  $\langle - | - \rangle$  of  $U$  is nondegenerate.*

**PROOF.** Since the pair  $\langle -, - \rangle$  has orthogonality for the grading, it suffices to check the case when  $u \in U_{-\nu}^- U^0 U_{\mu}^+$ . If  $\langle u | v \rangle = 0$  holds for all  $v \in U_{-\mu}^- U^0 U_{\nu}^+$ , then  $u = 0$ .

Denote  $d_\mu = \dim U_\mu^+$ . Let  $\{u_1^\mu, \dots, u_{d_\mu}^\mu\}$  be a basis of  $U_\mu^+$  and  $\{v_1^\mu, \dots, v_{d_\mu}^\mu\}$  be its dual basis in  $U_{-\mu}^-$  with respect to the pair, that is,  $\langle v_i^\mu, u_j^\mu \rangle = \delta_{ij}$ . Hence

$$U_{-\nu}^- U^0 U_\mu^+ = \text{span}_{\mathbb{K}} \left\{ (v_i^\nu \omega_\nu'^{-1}) \omega'_\eta \omega_\phi u_j^\mu \mid 1 \leq i \leq d_\nu, 1 \leq j \leq d_\mu \right\}.$$

Let  $u = \sum_{i,j,\eta,\phi} k_{i,j,\eta,\phi} (v_i^\nu \omega_\nu'^{-1}) \omega'_\eta \omega_\phi u_j^\mu$ , and  $v = (v_k^\mu \omega_\mu'^{-1}) \omega'_\eta \omega_{\phi'} u_l^\nu$ , where  $1 \leq k \leq d_\mu, 1 \leq l \leq d_\nu, \eta', \phi' \in Q$ . Suppose  $\langle u | v \rangle = 0$ , we have

$$\begin{aligned}
0 &= \sum_{\eta, \phi} k_{k,l,\eta,\phi} q^{2(\rho,\nu)} \langle \omega'_\eta, \omega_{\phi'} \rangle \langle \omega'_{\eta'}, \omega_\phi \rangle \\
&= \sum_{\eta, \phi} k_{k,l,\eta,\phi} q^{2(\rho,\nu)} \chi_{\eta,\phi}(\eta', \phi'),
\end{aligned} \tag{2.1}$$

where  $\chi_{\eta,\phi} : Q \times Q \rightarrow \mathbb{K}^\times$  denotes the group homomorphism by  $\chi_{\eta,\phi}(\eta', \phi') = \langle \omega'_\eta, \omega_{\phi'} \rangle \langle \omega'_{\eta'}, \omega_\phi \rangle$ .

We claim that these  $\chi_{\eta,\phi}$  in Eq.(2.1) are different characters, i.e., if  $\chi_{\eta,\phi} = \chi_{\eta',\phi'}$ , then  $(\eta, \phi) = (\eta', \phi')$ . This is verified by

$$1 = \frac{\chi_{\eta,\phi}(0, \alpha_j)}{\chi_{\eta',\phi'}(0, \alpha_j)} = \prod_{i=1}^n \langle \omega'_i, \omega_j \rangle^{\eta_i - \eta'_i} = \prod_{i=1}^n a_{ji}^{\eta_i - \eta'_i} = \prod_{i=1}^n (q_j^{\alpha_{ji}} q_{ji})^{\eta_i - \eta'_i},$$

which in turn gives  $DA(\eta - \eta') = 0$ , then  $\eta = \eta'$ , since  $DA$  is invertible; a similar argument leads to the conclusion  $\phi = \phi'$ .

Finally, we have  $k_{l,k,\eta,\phi} = 0$  from Eq.(2.1) by Dedekind theorem, which leads to  $u = 0$ .  $\square$

## 2.2. Weight modules of $U_{q,G}$

Let  $\mathcal{B} = U_{q,G}(\mathfrak{b})$ , and  $V^\psi$  be the one-dimensional  $\mathcal{B}$ -module, on which each  $e_i$  acts as multiplication by 0, and  $U^0$  acts via an algebra homomorphism  $\psi : U^0 \rightarrow \mathbb{K}$ . By Theorem 2.1, the Verma module  $M(\psi)$  with highest weight  $\psi$  is induced from  $V^\psi$  as  $M(\psi) = U_{q,G} \otimes_{\mathcal{B}} V^\psi$ .

PROPOSITION 2.4. [20] For  $U = U_{q,G}$ , we have

1. Let  $v_\lambda$  be a highest weight vector of  $M(\lambda)$  for  $\lambda \in \Lambda^+$ , then

$$L(\lambda) = M(\lambda) / \left( \sum_{i=1}^n U f_i^{(\lambda, \alpha_i^\vee) + 1} \cdot v_\lambda \right)$$

is simple, with the decomposition of the weight space  $L(\lambda) = \bigoplus_{\eta \leq \lambda} L(\lambda)_\eta$ , where

$$L(\lambda)_\eta = \{ x \in L(\lambda) \mid \omega_i \cdot x = \varrho^\eta(\omega_i)x, \omega'_i \cdot x = \varrho^\eta(\omega'_i)x, \forall i \in I \}.$$

2. The elements  $e_i, f_i, \forall i \in I$  act locally nilpotently on  $L(\lambda)$ .

3.  $\dim L(\lambda)_\eta = \dim L(\lambda)_{\sigma(\eta)}, \forall \eta \in \Lambda, \sigma \in W$ .

4. For any  $\beta = \sum_{i=1}^n m_i \alpha_i \in Q^+$  such that all  $m_i \leq (\lambda, \alpha_i^\vee)$ , the linear map  $x \mapsto x.v_\lambda$  from  $U_{-\beta}^-$  to  $M(\lambda)$  is injective.

## 3. Injectivity and the image of Harish-Chandra homomorphism $\xi$

### 3.1. Injective properties

Let  $Z(U)$  be the centre of  $U$ , then  $Z(U) \subset U_0$ . Defining an algebra homomorphism  $\gamma^{-\rho} : U^0 \rightarrow U^0$  as  $\gamma^{-\rho}(\omega'_\eta \omega_\phi) = \varrho^{-\rho}(\omega'_\eta \omega_\phi) \omega'_\eta \omega_\phi$ , and the canonical projection  $\pi : U_0 \rightarrow U^0$ .

1 DEFINITION 3. The Harish-Chandra homomorphism  $\xi : Z(U) \rightarrow U^0$  is defined by

$$2 \quad \xi = \gamma^{-\rho} \circ \pi|_{Z(U)},$$

3 and the Weyl group  $W$  has a natural action on  $U_{\mathfrak{b}}^0 := \bigoplus_{\eta \in Q} \mathbb{K} \omega'_{\eta} \omega_{-\eta}$  that

$$4 \quad \sigma(\omega'_{\eta} \omega_{-\eta}) := \omega'_{\sigma(\eta)} \omega_{-\sigma(\eta)}, \quad \forall \sigma \in W, \eta \in Q.$$

5 THEOREM 3.1. The Harish-Chandra homomorphism  $\xi : Z(U) \rightarrow U^0$  is injective.

6 PROOF. We aim to show that if  $z \in Z(U)$  satisfies  $\xi(z) = 0$ , then  $z = 0$ .

7 Since  $Z(U) \subset U_0 = U^0 \oplus K$ , where  $K$  is the two-sided ideal  $\bigoplus_{\nu > 0} U_{-\nu}^- U^0 U_{\nu}^+$  and  
8 is the kernel of  $\xi$ , we write  $z = \sum_{\nu \in Q^+} z_{\nu}$  with each  $z_{\nu} \in U_{-\nu}^- U^0 U_{\nu}^+$ . The condition  
9  $\xi(z) = 0$  implies  $z_0 = 0$ . Now assume that  $z \neq 0$ , then there exists a minimal  $\nu \in Q^+$   
10 such that  $z_{\nu} \neq 0$ .

11 Choose bases  $\{x_l\}$  and  $\{y_k\}$  for  $U_{\nu}^+$  and  $U_{-\nu}^-$ , respectively. Writing  $z_{\nu} =$   
12  $\sum_{k,l} y_k t_{k,l} x_l$ ,  $t_{k,l} \in U^0$ , we have for each  $i$ :

$$13 \quad 0 = e_i z - z e_i = \sum_{\gamma \neq \nu} (e_i z_{\gamma} - z_{\gamma} e_i) + (e_i z_{\nu} - z_{\nu} e_i).$$

14 For the last term

$$15 \quad e_i z_{\nu} - z_{\nu} e_i = \sum_{k,l} (e_i y_k - y_k e_i) t_{k,l} x_l + \sum_{k,l} y_k (e_i t_{k,l} x_l - t_{k,l} x_l e_i),$$

16 we have  $\sum_{k,l} (e_i y_k - y_k e_i) t_{k,l} x_l = 0$  since this term alone lies in  $U_{-(\nu - \alpha_i)}^- U^0 U_{\nu}^+$ . Hence  
17  $\sum_k e_i y_k t_{k,l} = \sum_k y_k e_i t_{k,l}$  for each  $l$  since  $\{x_l\}$  forms a basis of  $U_{\nu}^+$ . Fixing an index  $l$ ,  
18 for each  $\lambda \in \Lambda^+$ , we claim that

$$19 \quad m := \left( \sum_k y_k t_{k,l} \right) \cdot v_{\lambda} = 0$$

20 in the irreducible weight module  $L(\lambda)$  with the highest weight vector  $v_{\lambda}$ . Indeed, the  
21 identities

$$22 \quad e_i m = \sum_k e_i y_k t_{k,l} \cdot v_{\lambda} = \sum_k y_k e_i t_{k,l} \cdot v_{\lambda} = 0, \quad \forall i,$$

23 imply that  $m$  is a weight vector annihilated by all  $e_i$  and lies in a lower weight space,  
24 forcing  $m = 0$  to prevent generating a proper submodule of  $L(\lambda)$ . It follows that

$$25 \quad \sum_k y_k \varrho^{\lambda}(t_{k,l}) v_{\lambda} = 0.$$

26 Taking  $N = \max \{k_i | \nu = k_1 \alpha_1 + \dots + k_n \alpha_n\}$  and for any  $\lambda \geq N\rho$ , we have  $(\lambda, \alpha_i^{\vee}) \geq k_i$   
27 and the injective map  $\sum_k y_k \varrho^{\lambda}(t_{k,l}) \mapsto (\sum_k y_k \varrho^{\lambda}(t_{k,l})) \cdot v_{\lambda}$  through Lemma 2.4, then  
28 conclude that  $\sum_k \varrho^{\lambda}(t_{k,l}) y_k = 0$ , which leads to  $\varrho^{\lambda}(t_{k,l}) = 0$  for each  $k$ .

1 Next, write each  $t_{k,l}$  as a finite sum:  $t_{k,l} = \sum_{\eta,\phi} k_{\eta,\phi} \omega'_\eta \omega_\phi$ ,  $k_{\eta,\phi} \in \mathbb{K}$ . Then for all  
 2  $m \in \mathbb{Z}_{>0}$ , we have

$$3 \quad 0 = \varrho^{m\lambda}(t_{k,l}) = \sum_{\eta,\phi} (\varrho^\lambda(\omega'_\eta \omega_\phi))^m k_{\eta,\phi}.$$

4 This gives a homogeneous linear system with a Vandermonde matrix since the values  
 5  $\varrho^\lambda(\omega'_\eta \omega_\phi)$  are pairwise distinct by Proposition 1. Therefore, the only solution is  $k_{\eta,\phi} = 0$   
 6 for all  $\eta, \phi$ , implying  $t_{k,l} = 0$ . Hence  $z_\nu = 0$ , contradicting the assumption  $z_\nu \neq 0$ . This  
 7 contradiction implies that  $z = 0$ .  $\square$

### 8 3.2. The image $\text{Im}(\xi)$

9 Let  $\lambda, \mu \in \Lambda$ . Define an algebra map  $\varrho^{\lambda,\mu} : U^0 \rightarrow \mathbb{K}$  by  $\varrho^{\lambda,\mu} = \varrho^{0,\lambda} \varrho^{\mu,0}$ , where

$$10 \quad \varrho^{0,\mu} : \omega'_\eta \omega_\phi \mapsto q^{(\eta+\phi,\mu)}, \quad \varrho^{\lambda,0} : \omega'_\eta \omega_\phi \mapsto \varrho^\lambda(\omega'_\eta \omega_\phi).$$

11 LEMMA 2. *We have the following properties:*

12 1. *Let  $u = \omega'_\eta \omega_\phi$ ,  $\eta, \phi \in Q$ . If  $\varrho^{\lambda,\mu}(u) = 1$  for all  $\lambda, \mu \in \Lambda$ , then  $u = 1$ ;*

13 2. *If  $u \in U^0$  satisfies  $\varrho^{\lambda,\mu}(u) = 0$  for all  $\lambda, \mu \in \Lambda$ , then  $u = 0$ .*

14 PROOF. (1) Since  $1 = \varrho^{\lambda,\mu}(\omega'_\eta \omega_\phi) = q^{(\phi-\eta)^T DA \lambda} \left( \prod_{i,j} q_{ij}^{-\lambda_i(\phi+\eta)_j} \right) \cdot q^{(\eta+\phi,\mu)}$ , the  
 15 algebraic independence of the parameters  $q$  and  $\{q_{ij}\}$  implies the identity

$$16 \quad \begin{cases} (\phi - \eta)^T DA \lambda + (\eta + \phi, \mu) = 0, \\ -\lambda_i (\eta + \phi)_j + \lambda_j (\eta + \phi)_i = 0. \end{cases} \quad (3.1)$$

17 When  $\lambda = 0$ , the equality  $\eta + \phi = 0$  holds. Setting  $\mu = 0$ , the invertibility of  $DA$  implies  
 18  $\eta - \phi = 0$ . Consequently, we conclude  $\eta = \phi = 0$  and  $u = 1$ .

19 (2) Fixing a pair  $(\eta, \phi) \in Q \times Q$ , one can define a character  $\kappa_{\eta,\phi}$  on the group  $\Lambda \times \Lambda$   
 20 to be  $\kappa_{\eta,\phi} : (\lambda, \mu) \mapsto \varrho^{\lambda,\mu}(\omega'_\eta \omega_\phi)$ . Let  $u = \sum_{(\eta,\phi)} k_{\eta,\phi} \omega'_\eta \omega_\phi$ ,  $k_{\eta,\phi} \in \mathbb{K}$ . Then

$$21 \quad 0 = \varrho^{\lambda,\mu}(u) = \sum_{(\eta,\phi)} k_{\eta,\phi} \varrho^{\lambda,\mu}(\omega'_\eta \omega_\phi) = \sum_{(\eta,\phi)} k_{\eta,\phi} \kappa_{\eta,\phi}(\omega'_\eta \omega_\phi).$$

22 The property (1) implies that all characters  $\kappa_{\eta,\phi}$  are distinct, which forces all  $k_{\eta,\phi} = 0$   
 23 and  $u = 0$ .  $\square$

24 REMARK 1. *This result differs from [11, Lem. 23] in the two-parameter quantum*  
 25 *groups  $U_{r,s}(\mathfrak{g})$ , where  $\mathfrak{g}$  is of even rank. The reason is that  $U_{q,G}(\mathfrak{g})$  involves more*  
 26 *algebraically independent parameters  $q$  and  $\{q_{ij}\}$ , and provides more restrictive conditions*  
 27 *(3.1) on weights  $\phi$  and  $\eta$ , and further guarantees  $\xi(Z(U)) \subseteq (U_b^0)^W$  for the Harish-*  
 28 *Chandra homomorphism for any rank in the following process.*

29 LEMMA 3.  $\varrho^{\sigma(\lambda),\mu}(u) = \varrho^{\lambda,\mu}(\sigma^{-1}(u))$ , for  $u \in U_b^0$ ,  $\sigma \in W$ ,  $\lambda, \mu \in \Lambda$ .

1 PROOF. It suffices to check when  $u = \omega'_\eta \omega_{-\eta}$  and  $\sigma$  is a simple reflection. Let  $\Sigma$   
2 denote the matrix of  $\sigma$ , we have

$$\begin{aligned} \varrho^{\lambda,0}(\sigma_i^{-1}(u)) &= \varrho^\lambda(\omega'_{\sigma(\eta)} \omega_{-\eta}) = q^{-2(\sigma(\eta))^T DA\lambda} = q^{-2\eta^T \Sigma DA\lambda} \\ &= q^{-2\eta^T DA\Sigma\lambda} = q^{-2\eta DA\sigma(\lambda)} = \varrho^{\sigma(\lambda),0}(\omega'_\eta \omega_{-\eta}), \\ \varrho^{0,\mu}(\sigma^{-1}(u)) &= q^{(\sigma^{-1}(\eta) + \sigma^{-1}(-\eta), \mu)} = q^{(0,\mu)} = \varrho^{0,\mu}(\omega'_\eta \omega_{-\eta}). \end{aligned}$$

4 This completes the proof.  $\square$

5 LEMMA 4. For  $\sigma \in W$ ,  $\lambda, \mu \in \Lambda$ . If  $u \in U^0$  satisfies  $\varrho^{\sigma(\lambda),\mu}(u) = \varrho^{\lambda,\mu}(u)$ , then  
6  $u \in (U_b^0)^W$ .

7 PROOF. Let  $u = \sum_{(\eta,\phi)} k_{\eta,\phi} \omega'_\eta \omega_\phi \in U^0$ , then the assumption implies  
8  $\sum_{(\eta,\phi)} k_{\eta,\phi} \varrho^{\lambda,\mu}(\omega'_\eta \omega_\phi) = \sum_{(\zeta,\psi)} k_{\zeta,\psi} \varrho^{\sigma(\lambda),\mu}(\omega'_\zeta \omega_\psi)$ , hence we have an equation for char-  
9 acters:

$$10 \quad \sum_{(\eta,\phi)} k_{\eta,\phi} \kappa_{\eta,\phi} = \sum_{(\zeta,\psi)} k_{\zeta,\psi} \kappa_{\zeta,\psi}^i.$$

11 For each  $k_{\eta,\phi} \neq 0$ , there exists a pair  $(\zeta, \psi) \in Q \times Q$ , such that  $\kappa_{\eta,\phi} = \kappa_{\zeta,\psi}^i$  and  
12  $k_{\zeta,\psi} = k_{\eta,\phi}$ . Then

$$13 \quad \kappa_{\eta,\phi}(0, \varpi_j) = \varrho^{0,\varpi_j}(\omega'_\eta \omega_\phi) = q^{(\eta+\phi, \varpi_j)} = \kappa_{\zeta,\psi}^i(0, \varpi_j) = \varrho^{0,\varpi_j}(\omega'_\zeta \omega_\psi) = q^{(\zeta+\psi, \varpi_j)},$$

14 yields  $\eta + \phi = \zeta + \psi$ , and  $\varrho^{\varpi_i}(\omega'_\eta \omega_\phi) = \varrho^{\sigma_i(\varpi_i)}(\omega'_\zeta \omega_\psi) = \varrho^{\varpi_i - \alpha_i}(\omega'_\zeta \omega_\psi)$ . Substituting  
15  $\phi = \zeta + \psi - \eta$ , one has  $\varrho^{\varpi_i}(\omega'_{\eta-\zeta} \omega_{\zeta-\eta}) \varrho^{\alpha_i}(\omega'_\zeta \omega_{-\zeta}) = \varrho^{-\alpha_i}(\omega_{\psi+\zeta})$ , that is,

$$16 \quad q^{2(\zeta-\eta)^T DA\varpi_i - 2\zeta^T DA\alpha_i} = q^{(\psi+\zeta)^T DA(-\alpha_i)} \left( \prod_{j=1}^n q_{ij}^{(\psi+\zeta)_j} \right).$$

17 For  $i = 1, \dots, n$ , the algebraic independence of the parameters implies

$$18 \quad \begin{cases} 2(\zeta - \eta)^T DA\varpi_i + (\psi - \zeta)^T DA\alpha_i = 0, \\ (\psi + \zeta)_j = 0, \quad j \neq i. \end{cases}$$

19 So we have  $\psi + \zeta = 0$ , which forces

$$20 \quad u = \sum_{(\eta,-\eta)} k_{\eta,-\eta} \omega'_\eta \omega_{-\eta} \in U_b^0.$$

21 Finally, by Lemma 3, we get  $\varrho^{\lambda,\mu}(\sigma^{-1}(u)) = \varrho^{\sigma(\lambda),\mu}(u) = \varrho^{\lambda,\mu}(u)$  for all  $\lambda, \mu \in \Lambda$  and  
22  $\sigma \in W$ . It implies  $\sigma^{-1}(u) = u$  for all  $\sigma \in W$ . So  $u \in (U_b^0)^W$ .  $\square$

23 THEOREM 3.2.  $\varrho^{\lambda+\rho,\mu}(\xi(z)) = \varrho^{\sigma(\lambda+\rho),\mu}(\xi(z))$ , for all  $z \in Z(U)$ ,  $\sigma \in W$ , which  
24 implies  $Im(\xi) \subseteq (U_b^0)^W$ .

1 PROOF. Let  $z \in Z(U)$  and  $\mu \in \Lambda$ , take a  $\lambda \in \Lambda$  such that  $(\lambda, \alpha_i^\vee) \geq 0$  for some  
2 fixed  $i$ . Let  $v_{\lambda, \mu}$  be the highest weight vector of the Verma module  $M(\varrho^{\lambda, \mu})$ . Then

$$3 \quad z.v_{\lambda, \mu} = \varrho^{\lambda, \mu}(\pi(z))v_{\lambda, \mu} = \varrho^{\lambda+\rho, \mu}(\xi(z))v_{\lambda, \mu}.$$

4 On the other hand, according to [20, Lem. 33], we have

$$5 \quad e_i f_i^{(\lambda, \alpha_i^\vee)+1}.v_{\lambda, \mu} = [((\lambda, \alpha_i^\vee) + 1)]_i f_i^{(\lambda, \alpha_i^\vee)} \frac{q_i^{-(\lambda, \alpha_i^\vee)} \omega_i - q_i^{(\lambda, \alpha_i^\vee)} \omega_i'}{q_i - q_i^{-1}}.v_{\lambda, \mu} = 0,$$

6 since  $q_i^{-(\lambda, \alpha_i^\vee)} \varrho^{\lambda, \mu}(\omega_i) = q_i^{(\lambda, \alpha_i^\vee)} \varrho^{\lambda, \mu}(\omega_i')$ . It follows that  $e_j f_i^{(\lambda, \alpha_i^\vee)+1}.v_{\lambda, \mu} = 0$ ,  $j =$   
7  $1, \dots, n$ , and

$$8 \quad \begin{aligned} z f_i^{(\lambda, \alpha_i^\vee)+1}.v_{\lambda, \mu} &= \pi(z) f_i^{(\lambda, \alpha_i^\vee)+1}.v_{\lambda, \mu} = \varrho^{\sigma_i(\lambda+\rho)-\rho, \mu}(\pi(z)) f_i^{(\lambda, \alpha_i^\vee)+1}.v_{\lambda, \mu} \\ &= \varrho^{\sigma_i(\lambda+\rho), \mu}(\xi(z)) f_i^{(\lambda, \alpha_i^\vee)+1}.v_{\lambda, \mu}. \end{aligned}$$

9 Hence,  $z$  acts on  $M(\varrho^{\lambda, \mu})$  by scalar  $\varrho^{\sigma_i(\lambda+\rho), \mu}(\xi(z))$ , so we have  $\varrho^{\lambda+\rho, \mu}(\xi(z)) =$   
10  $\varrho^{\sigma_i(\lambda+\rho), \mu}(\xi(z))$ .

11 Moreover, this result holds for any  $\lambda \in \Lambda$ . This is because if  $(\lambda, \alpha_i^\vee) = -1$ , then  
12  $\lambda + \rho = \sigma_i(\lambda' + \rho)$  such that this equation holds. If  $(\lambda, \alpha_i^\vee) < -1$ , let  $\lambda' = \sigma_i(\lambda + \rho) - \rho$ ,  
13 then  $(\lambda', \alpha_i^\vee) \geq 0$  such that the equation holds for  $\lambda'$ . Replace  $\lambda'$  with  $\sigma_i(\lambda + \rho) - \rho$  into  
14 the result such that the equation holds for  $\lambda$  in this case. Finally, since the equation  
15 holds for each  $\sigma_i$ , so it holds for all  $\sigma \in W$ . It implies  $Im(\xi) \subseteq (U_b^0)^W$  by Lemma 6.  $\square$

## 16 4. Central elements and the Harish-Chandra theorem

### 17 4.1. Rosso form realization and central elements

18 In what follows, we construct central elements by realizing the quantum trace on  
19 weight modules using the Rosso form.

20 Since the Rosso form is nondegenerate and ad-invariant, there is an injective mor-  
21 phism of  $U$ -module  $\beta : U \rightarrow U^*$ ,  $u \mapsto \langle u | - \rangle$ , where the  $U$ -module structure on  $U^*$  is  
22 defined by  $(x.f)(v) = f(\text{ad}_l(S(x))v)$ . for all  $x, v \in U$ ,  $f \in U^*$ .

23 DEFINITION 4. For  $\lambda \in \Lambda^+$ , define the quantum trace  $t_\lambda \in U^*$  by

$$24 \quad t_\lambda(v) := \text{tr}_{L(\lambda)}(v \circ \Theta),$$

25 where  $\Theta \in \text{End}_{\mathbb{K}}(L(\lambda))$  is a linear map defined by  $m \mapsto q^{-2(\rho, \mu)}m$ ,  $\forall m \in L(\lambda)_\mu$ ,  $\mu \in \Lambda$ .

26 Let  $d = \dim L(\lambda)$ , and choose a basis  $\{m_i\}_{i=1}^d$  of  $L(\lambda)$  with the corresponding dual  
27 basis  $\{f_i\}_{i=1}^d \subset L(\lambda)^*$ . Then, for any  $v \in U$ ,

$$28 \quad v \cdot \Theta(m_i) = \sum_{j=1}^d f_j(v \cdot \Theta(m_i))m_j = \sum_{j=1}^d C_{f_j, \Theta(m_i)}(v)m_j,$$

1 where  $C_{f,m} \in U^*$  denotes the matrix coefficient satisfying  $C_{f,m}(v) = f(v.m)$ , for all  
 2  $v \in U$ . The quantum trace can then be expressed as

$$3 \quad t_\lambda(v) = \text{tr}_{L(\lambda)}(v \circ \Theta) = \sum_{i=1}^d C_{f_i, \Theta(m_i)}(v)$$

4 and the matrix coefficient can be realized by the Rosso form.

5 **PROPOSITION 4.1.** *If the weight set of a finite-dimensional weight module  $M$  of*  
 6  *$U$  satisfies  $\text{wt}(M) \subseteq Q$ , then for each  $m \in M$ ,  $f \in M^*$ , there exists a unique element*  
 7  *$u \in U$  such that  $C_{f,m}(v) = \langle u | v \rangle$ ,  $\forall v \in U$ .*

8 **PROOF.** It suffices to verify the case when  $m \in M_\lambda$ , since  $C_{f,m}$  is linear in  $m \in M$ .  
 9 Suppose  $v$  is a monomial of the form  $v = (y \omega'_\mu{}^{-1}) \omega'_\eta \omega_\phi x$ , where  $y \in U_{-\mu}^-$ ,  $x \in U_\nu^+$ . Then  
 10 for any  $f \in M^*$ , we have

$$11 \quad \begin{aligned} C_{f,m}(v) &= C_{f,m}((y \omega'_\mu{}^{-1}) \omega'_\eta \omega_\phi x) \\ &= \langle \omega'_\eta, \omega_{-(\lambda+\nu)} \rangle \langle \omega'_{\lambda+\nu}, \omega_\phi \rangle \cdot f((y \omega'_\mu{}^{-1}) x \cdot m). \end{aligned}$$

12 Noting the bilinearity of  $\Psi : U_{-\mu}^- \times U_\nu^+ \rightarrow \mathbb{K}$ ,  $(y, x) \mapsto f((y \omega'_\mu{}^{-1}) x \cdot m)$ , we claim that  
 13 for a given  $\Psi$  and a pair  $(\eta, \phi) \in Q \times Q$ , there exists an element  $u \in U_{-\mu}^- U^0 U_\nu^+$  such that

$$14 \quad \langle u | (y \omega'_\mu{}^{-1}) \omega'_\zeta \omega_\psi x \rangle = \langle \omega'_\zeta, \omega_\phi \rangle \langle \omega'_\eta, \omega_\psi \rangle \Psi(y, x)$$

15 for any  $x \in U_\nu^+$ ,  $y \in U_{-\mu}^-$  and  $(\zeta, \psi) \in Q \times Q$ . One can easily check that

$$16 \quad u = \sum_{i,j} q^{-2(\rho,\nu)} \Psi(v_j^\mu, u_i^\nu) (v_i^\nu \omega'_\nu{}^{-1}) \omega'_\eta \omega_\phi u_j^\mu,$$

17 where  $\{u_1^\mu, \dots, u_{d_\mu}^\mu\}$  is a basis of  $U_\mu^+$  with its dual basis  $\{v_1^\mu, \dots, v_{d_\mu}^\mu\}$  in  $U_{-\mu}^-$  with respect  
 18 to  $\langle -, - \rangle$ . Therefore, there exists a unique  $u_{\nu\mu} \in U$  such that  $C_{f,m}(v) = \langle u_{\nu\mu} | v \rangle$ ,  $\forall v \in$   
 19  $U_{-\mu}^- U^0 U_\nu^+$ .

20 Now consider a general element  $v \in U$  and write it as  $v = \sum_{(\mu,\nu)} v_{\mu\nu}$ , where  $v_{\mu\nu} \in$   
 21  $U_{-\mu}^- U^0 U_\nu^+$ . Since  $M$  is finite-dimensional, there exists a finite subset  $\Omega \subset Q \times Q$  such  
 22 that  $C_{f,m}(v) = C_{f,m}(\sum_{(\mu,\nu) \in \Omega} v_{\mu\nu})$  for all  $v \in U$ . Define  $u := \sum_{(\mu,\nu) \in \Omega} u_{\nu\mu}$ , then we  
 23 have

$$24 \quad \langle u | v \rangle = \sum_{(\mu,\nu) \in \Omega} \langle u_{\nu\mu} | v \rangle = \sum_{(\mu,\nu) \in \Omega} \langle u_{\nu\mu} | v_{\mu\nu} \rangle = C_{f,m}(v).$$

25 This proves the existence of  $u$  and provides an explicit construction.  $\square$

26 **THEOREM 4.2.** *For each  $\lambda \in \Lambda^+ \cap Q$ , there exists a unique element  $z_\lambda$  such that*  
 27  *$\beta(z_\lambda) = t_\lambda$ . The explicit expression of  $z_\lambda$  is*

$$z_\lambda = \sum_{\tau \leq \lambda} \sum_{\mu \in Q^+} \sum_{i,j} q^{-2(\rho, \tau + \mu)} \langle \omega'_\mu, \omega_{\tau + \mu} \rangle \text{tr}(v_j^\mu u_i^\mu \circ P_\tau) v_i^\mu \omega'_\tau \omega_{\tau + \mu}^{-1} u_j^\mu.$$

where  $\{u_j^\mu\}_{j=1}^{d_\mu}$  is a basis of  $U_\mu^+$  and  $\{v_i^\mu\}_{i=1}^{d_\mu}$  is the dual basis of  $U_{-\mu}^-$  with respect to the restriction of  $\langle -, - \rangle$  to  $U_{-\mu}^- \times U_\mu^+$ , and  $P_\tau$  is the projector from  $L(\lambda)$  to  $L(\lambda)_\tau$ .

REMARK 2. The explicit expression of central elements  $z_\lambda$  in the one-parameter quantum groups  $U_q(\mathfrak{g})$  (see, e.g. [6, 12]) corresponds to the specialization of  $U_{q,G}$  where all  $q_{ij} = 1$ ,  $\omega_i = K_i$ ,  $\omega'_i = K_i^{-1}$ .

We proceed to show that these elements  $\{z_\lambda\}_{\lambda \in \Lambda^+ \cap Q}$  lie in the centre  $Z(U)$ .

LEMMA 5. Let  $z \in U$ , then  $z \in Z(U)$  if and only if  $\text{ad}_l(x)z = \varepsilon(x)z$ , for all  $x \in U$ .

PROOF. Suppose  $z \in Z(U)$ . Then for all  $x \in U$ , we have

$$\text{ad}_l(x)z = \sum_{(x)} x_{(1)} z S(x_{(2)}) = z \sum_{(x)} x_{(1)} S(x_{(2)}) = \varepsilon(x)z.$$

Conversely, if  $\text{ad}_l(x)z = \varepsilon(x)z$  holds for all  $x \in U$ , then  $\omega_i z \omega_i^{-1} = \text{ad}_l(\omega_i)z = \varepsilon(\omega_i)z = z$ . For each generator  $e_i$  and  $f_i$  of  $U$ , we have

$$\begin{aligned} e_i z - z e_i &= e_i z + \omega_i z S(e_i) = \text{ad}_l(e_i)z = \varepsilon(e_i)z = 0, \\ (f_i z - z f_i) \omega_i'^{-1} &= z S(f_i) + f_i z S(\omega_i') = \text{ad}_l(f_i)z = \varepsilon(f_i)z = 0. \end{aligned}$$

This completes the proof.  $\square$

PROPOSITION 4.3. The following statements hold for the weight modules  $L(\lambda)$  with  $\lambda \in \Lambda^+$ .

1. For all  $u \in U$ , we have  $\Theta(u.m) = S^2(u) \cdot \Theta(m)$ ,  $\forall m \in L(\lambda)$ , or  $\Theta \circ u = S^2(u) \circ \Theta$  for short.
2. For all  $x \in U$ , we have  $x \cdot t_\lambda = \varepsilon(x) t_\lambda$ .
3. If  $\lambda \in \Lambda^+ \cap Q$ , then  $z_\lambda = \beta^{-1}(t_\lambda) \in Z(U)$ .

PROOF. (1) It is sufficient to verify the statement on the generators  $e_i$  and  $f_i$ . Notice that

$$\langle \omega'_i, \omega_i \rangle = a_{ii} = q^{2d_i} = q^{2(\varpi_i, \alpha_i)} = q^{2(\rho, \alpha_i)}.$$

Now, for each  $m \in L(\lambda)_\mu$ , we compute

$$\begin{aligned} S^2(e_i) \cdot \Theta(m) &= q^{-2(\rho, \mu)} \cdot S^2(e_i) \cdot m = q^{-2(\rho, \mu)} \cdot \langle \omega'_i, \omega_i \rangle^{-1} e_i \cdot m \\ &= q^{-2(\rho, \mu + \alpha_i)} e_i \cdot m = \Theta(e_i \cdot m), \\ S^2(f_i) \cdot \Theta(m) &= S^2(f_i) \cdot q^{-2(\rho, \mu)} m = \langle \omega'_i, \omega_i \rangle \cdot q^{-2(\rho, \mu)} f_i \cdot m \\ &= q^{-2(\rho, \mu - \alpha_i)} f_i \cdot m = \Theta(f_i \cdot m). \end{aligned}$$

1 (2) For any  $x, u \in U$ , we compute

$$\begin{aligned}
& (S^{-1}(x) \cdot t_\lambda)(u) = t_\lambda(\text{ad}_l(x)u) = \text{tr}_{L(\lambda)} \left( \sum_{(x)} x_{(1)} u S(x_{(2)}) \circ \Theta \right) \\
& = \text{tr}_{L(\lambda)} \left( u \cdot \sum_{(x)} S(x_{(2)}) \Theta x_{(1)} \right) = \text{tr}_{L(\lambda)} \left( u \cdot \sum_{(x)} S(x_{(2)}) S^2(x_{(1)}) \circ \Theta \right) \\
& = \text{tr}_{L(\lambda)} \left( u \cdot S \left( \sum_{(x)} S(x_{(1)}) x_{(2)} \right) \circ \Theta \right) = \varepsilon(x) \text{tr}_{L(\lambda)}(u \circ \Theta) = \varepsilon(x) t_\lambda(u).
\end{aligned}$$

3 Replacing  $x$  by  $S(x)$ , we get  $x \cdot t_\lambda = \varepsilon(x) t_\lambda$  for all  $x \in U$ .

4 (3) We compute that

$$\begin{aligned}
& x \cdot t_\lambda = x \cdot \beta(z_\lambda) = (x \cdot \beta)(z_\lambda) = \beta(\text{ad}_l(x)z_\lambda), \\
& \varepsilon(x) t_\lambda = \varepsilon(x) \beta(z_\lambda) = \beta(\varepsilon(x)z_\lambda).
\end{aligned}$$

6 By the injectivity of  $\beta$ , it follows that  $\text{ad}_l(x)(z_\lambda) = \varepsilon(x)z_\lambda$  for all  $x \in U$ , which implies  
7  $z_\lambda \in Z(U)$  by Lemma 5.  $\square$

#### 8 4.2. The Harish-Chandra theorem

9 Set  $\Psi = \Lambda^+ \cap Q$ . For each  $\eta \in Q$ , there exists a unique  $\sigma \in W$  such that  $\sigma(\eta) \in \Psi$ .

10 Therefore, the elements

$$11 \quad \text{av}(\lambda) = \frac{1}{|W|} \sum_{\sigma \in W} \omega'_{\sigma(\lambda)} \omega_{-\sigma(\lambda)} \quad (\lambda \in \Psi)$$

12 form a basis of the  $W$ -invariant subalgebra  $(U_b^0)^W$ .

13 **THEOREM 4.4.** *The Harish-Chandra homomorphism  $\xi : Z(U) \rightarrow (U_b^0)^W$  is an*  
14 *algebra isomorphism.*

15 **PROOF.** By Theorem 4.2, for each  $\lambda \in \Lambda^+ \cap Q$ , the central element  $z_\lambda$  satisfies

$$16 \quad z_\lambda^0 = \sum_{\mu \leq \lambda} q^{-2(\rho, \mu)} \dim(L(\lambda)_\mu) \omega'_\mu \omega_{-\mu}.$$

17 Applying the definition of the map  $\xi$  in Section 3.1, we obtain

$$\begin{aligned}
& \xi(z_\lambda) = \gamma^{-\rho}(z_\lambda^0) = \sum_{\mu \leq \lambda} q^{-2(\rho, \mu)} \dim(L(\lambda)_\mu) \varrho^{-\rho}(\omega'_\mu \omega_{-\mu}) \omega'_\mu \omega_{-\mu} \\
& = \sum_{\mu \leq \lambda} \dim(L(\lambda)_\mu) \omega'_\mu \omega_{-\mu} \in (U_b^0)^W.
\end{aligned}$$

19 It remains to prove that each  $\text{av}(\lambda)$  lies in  $\text{Im}(\xi)$ . We proceed by induction on the  
20 height of  $\lambda$ .

21 If  $\lambda = 0$ , then  $\text{av}(0) = 1 = \xi(z_0) \in \text{Im}(\xi)$ . Suppose  $\lambda > 0$ . The properties  
22  $\dim(L(\lambda)_\mu) = \dim(L(\lambda)_{\sigma(\mu)})$  and  $\dim(L(\lambda)_\lambda) = 1$  imply that

$$1 \quad \xi(z_\lambda) = \sum_{\mu \leq \lambda} \dim(L(\lambda)_\mu) \omega'_\mu \omega_{-\mu} = |W| \cdot \text{av}(\lambda) + |W| \sum_{\mu < \lambda, \mu \in \Psi} \dim(L(\lambda)_\mu) \text{av}(\mu).$$

2 By the induction hypothesis, each  $\text{av}(\mu)$  for  $\mu < \lambda$  lies in  $\text{Im}(\xi)$ , then

$$3 \quad \text{av}(\lambda) = \frac{1}{|W|} \xi(z_\lambda) - \sum_{\mu < \lambda, \mu \in \Psi} \dim(L(\lambda)_\mu) \text{av}(\mu) \in \text{Im}(\xi).$$

4 Now we conclude that

$$5 \quad (U_b^0)^W = \text{span}_{\mathbb{K}}\{\text{av}(\lambda) \mid \lambda \in \Psi\} \subseteq \text{Im}(\xi),$$

6 which completes the proof together with Theorem 3.2.  $\square$

7 **COROLLARY 1.** *The centre  $Z(U)$  admits the basis  $\{z_\lambda \mid \lambda \in \Psi\}$  over  $\mathbb{K}$ , thereby*  
 8 *supporting the Batra-Yamane conjecture [3, Conj. 3.13].*

9 **PROOF.** The previous proof has shown that  $(U_b^0)^W = \text{span}_{\mathbb{K}}\{\text{av}(\lambda) \mid \lambda \in \Psi\} =$   
 10  $\text{span}_{\mathbb{K}}\{\xi(z_\lambda) \mid \lambda \in \Psi\}$ . Moreover, the set  $\{\xi(z_\lambda) \mid \lambda \in \Psi\}$  is a  $\mathbb{K}$ -linearly independent,  
 11 otherwise  $\exists \lambda^1, \dots, \lambda^s \in \Psi$ , such that

$$12 \quad 0 = \sum_{i=1}^s a_i \xi(z_{\lambda^i}) = |W| \sum_{i=1}^s a_i \left( \text{av}(\lambda^i) + \sum_{\mu < \lambda^i, \mu \in \Psi} \dim(L(\lambda^i)_\mu) \text{av}(\mu) \right),$$

13 and all  $a_i \neq 0$ , we thus arrive at a contradiction: there exists a weight  $\lambda^{i_0}$  such that  
 14  $\lambda^i \not\asymp \lambda^{i_0}$  for all other  $i$ , which necessarily implies  $a_{i_0} \text{av}(\lambda^{i_0}) = 0$  in the summand. Hence  
 15 the set  $\{\xi(z_\lambda) \mid \lambda \in \Psi\}$  forms a basis of  $(U_b^0)^W$ . By applying the Harish-Chandra  
 16 isomorphism, we conclude that the centre  $Z(U)$  admits the basis  $\{z_\lambda \mid \lambda \in \Psi\}$ .

17 Following the notation established in [3] and under the assumptions of the present  
 18 work, we set  $\pi(i) = \alpha_i$ ,  $\mathfrak{A} = Q$ , and the map  $\chi(\alpha_i, \alpha_j) = \langle w'_j, w_i \rangle$  for all  $i, j$ . Then the  
 19 set  $\mathfrak{Z}_1(\chi, \pi)$  is just the centre  $Z(U)$ , and  $\mathfrak{B}_1^{\chi, \pi} = (U_b^0)^W$  — an explicit realization of the  
 20 abstract subalgebra in [2, Thm. 10.4]. The set

$$21 \quad \text{Fin}_1^\chi := \{(\lambda, \mu) \in \mathfrak{A} \times \mathfrak{A} \mid \dim \mathcal{L}(\Lambda_{\lambda, \mu; 1}^\chi) < \infty\}$$

22 where the map  $\Lambda_{\lambda, \mu; 1}^\chi(w'_{\mu'} w_{\lambda'}) := \chi(\lambda, \mu') \chi(\lambda', \mu) = \langle w'_{\mu'}, w_{\lambda'} \rangle \langle w'_{\mu'}, w_\lambda \rangle$  (also denoted by  
 23  $\varrho^{(\lambda, \mu)}$ ). Our  $\varrho^\lambda = \Lambda_{-\lambda, \lambda; 1}^\chi = \varrho^{(-\lambda, \lambda)}$ , then  $(-\lambda, \lambda) \in \text{Fin}_1^\chi$  if and only if  $\lambda \in \Lambda^+ \cap \mathfrak{A} = \Psi$ .  
 24 On the other hand, there is no finite-dimensional weight module with the highest weight  
 25  $\varrho^{(\lambda, \mu)}$  such that  $\lambda + \mu \neq 0$ , this fact is guaranteed by [20, Prop.36] through quick  
 26 verification. Above all, we have  $\Psi = \text{Fin}_1^\chi$  and confirm that the Batra-Yamane conjecture  
 27  $\mathfrak{Z}_1(\chi, \pi) = \text{span}_{\mathbb{K}}\{z_\lambda \mid \lambda \in \text{Fin}_1^\chi\}$  holds for  $Z(U_{q, G}(\mathfrak{g}))$ .  $\square$

## 28 5. The Centre of $U_{q, G}(\mathfrak{g})$

29 Now we determine generators of the subalgebra  $(U_b^0)^W$ . There is a natural monomor-  
 30 phism  $\theta : (U_b^0)^W \rightarrow \mathbb{K}[\mathfrak{h}^*]^W$  defined by

$$\theta \left( \sum_{\sigma \in W} \omega'_{\sigma(\lambda)} \omega_{-\sigma(\lambda)} \right) = \sum_{\sigma \in W} e^{\sigma(\lambda)}, \quad \forall \lambda \in \Psi.$$

2 Meanwhile, the homomorphism  $\text{Ch} : R(\mathfrak{g}) \rightarrow \mathbb{Z}[\Lambda]$  induces an isomorphism  $R(\mathfrak{g}) \cong \mathbb{Z}[\Lambda]^W$   
 3 (see, e.g., [4]) and  $R(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{K} \cong \mathbb{K}[\mathfrak{h}^*]^W$ . Since the element  $z_\lambda := \text{Ch}(L(\lambda))$  satisfies  
 4  $z_\lambda = \theta\xi(z_\lambda)$  when  $\lambda \in \Psi$ , we have  $\text{Im}(\theta) = \text{span}_{\mathbb{K}}\{z_\lambda \mid \lambda \in \Psi\}$ . For simplicity of  
 5 notation, we write  $z_i := z_{\varpi_i}$ ,  $i = 1, \dots, n$ .

6 **LEMMA 6.** *The elements  $z_1, z_2, \dots, z_n$  are algebraically independent over  $\mathbb{K}$ .*

7 **PROOF.** If there exists a polynomial  $f$  in  $\mathbb{K}[v_1, \dots, v_n]$  such that  
 8  $f(z_1, z_2, \dots, z_n) = 0$ , writing

$$f = c_{k_1, \dots, k_n} v_1^{k_1} v_2^{k_2} \dots v_n^{k_n} + \sum_{(a_1, \dots, a_n) < (k_1, \dots, k_n)} c_{a_1, \dots, a_n} v_1^{a_1} v_2^{a_2} \dots v_n^{a_n},$$

10 where “ $<$ ” denotes the lexicographical order, and  $v_1^{k_1} v_2^{k_2} \dots v_n^{k_n}$  is the maximal one with  
 11  $c_{k_1, \dots, k_n} \neq 0$ . Through the isomorphism  $\text{Ch} : R(\mathfrak{g}) \otimes_{\mathbb{Z}} \mathbb{K} \cong \mathbb{K}[\mathfrak{h}^*]^W$ , the equation  
 12  $f(z_1, z_2, \dots, z_n) = 0$  corresponds to

$$c_{k_1, \dots, k_n} \left[ \bigotimes_{i=1}^n L(\varpi_i)^{\otimes k_i} \right] + \sum_{(a_1, \dots, a_n) < (k_1, \dots, k_n)} c_{a_1, \dots, a_n} \left[ \bigotimes_{i=1}^n L(\varpi_i)^{\otimes a_i} \right] = 0.$$

14 If we rewrite the equation above as a linear combination of the basis elements in  $\{[L(\lambda)] \mid$   
 15  $\lambda \in \Lambda^+\}$ , then  $[L(k_1 \varpi_1 + \dots + k_n \varpi_n)]$  only appears in  $[\bigotimes_{i=1}^n L(\varpi_i)^{\otimes k_i}]$ , then  $c_{k_1, \dots, k_n}$   
 16 has to be 0, contradiction.  $\square$

### 17 **5.1. Minimal generating system $\Psi_{\min}$**

18 An element  $\lambda \in \Psi \setminus \{0\}$  is said to be indecomposable if  $\nexists \lambda', \lambda'' \in \Psi \setminus \{0\}$ , such that  
 19  $\lambda = \lambda' + \lambda''$ . Let  $\Psi_{\min}$  be the set containing all indecomposable weights in  $\Psi \setminus \{0\}$ ,  
 20 which is the minimal generating system of  $\Psi$ . We adopt an approach analogous to that  
 21 in [16] and adapt it to our multi-parameter setting. We will show that the following  
 22 lemmas, together with our main result (Theorem 5.1), hold in this setting, where the  
 23 constructions of  $\Psi$  and  $\Psi_{\min}$  are considerably more refined.

24 **LEMMA 7.** *The set  $\Psi_{\min}$  is finite.*

25 **PROOF.** According to the table in Appendix A, for each  $1 \leq k \leq n$ , there exist  
 26 elements  $c \in \mathbb{Z}_+$  such that  $c\varpi_k \in \Psi \setminus \{0\}$ . The smallest  $c$  is denoted by  $r_k$  (e.g.,  
 27  $r_k = \frac{n+1}{(n+1, k)}$  for type  $A_n$ ), and the weight  $\mathfrak{s}(k) := r_k \varpi_k \in \Psi_{\min}$  is called a single  
 28 weight. Hence if  $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in \Psi_{\min}$ , then each  $\lambda_i \leq r_i$  (otherwise  $\exists i_0$ , such that  
 29  $\lambda - \mathfrak{s}(i_0) \in \Psi \setminus \{0\}$ , contradiction). The finiteness is clear.  $\square$

30 Except for type  $A_n$ , one can directly check that the given set forms a minimal  
 31 generating system for the additive monoid  $\Psi$  by straightforward enumeration and com-  
 32 binatorial arguments.

1 LEMMA 8. For  $\text{rank}(\mathfrak{g}) \geq 2$ , the elements of  $\Psi_{\min}$  are shown explicitly as follows

Type	Minimal generating system $\Psi_{\min}$
$A_n$	$\{\mathbf{s}(k) := r_k \varpi_k \mid r_k = \frac{n+1}{(n+1, k)}, 1 \leq k \leq n\}_{(\text{single})}$ $\cup \{\mathbf{e}(k) := d_k \varpi_1 + \varpi_k \mid d_k = n+1-k, 2 \leq k \leq n\}_{(\text{special})}$ $\cup \{\tau \mid \text{not single and not special}\}_{=:T}$
$B_n$	$\{\varpi_1, \dots, \varpi_{n-2}, \varpi_{n-1}, 2\varpi_n\}$
$C_{2k+1}$	$\{2\varpi_1, \varpi_2, 2\varpi_3, \dots, \varpi_{2k}, 2\varpi_{2k+1},$ $\varpi_u + \varpi_v \ (u, v \equiv 1 \pmod{2}, u < v)\}$
$C_{2k+2}$	$\{2\varpi_1, \varpi_2, 2\varpi_3, \dots, \varpi_{2k}, 2\varpi_{2k+1}, \varpi_{2k+2},$ $\varpi_u + \varpi_v \ (u, v \equiv 1 \pmod{2}, u < v)\}$
$D_{2k+2}$	$\{2\varpi_1, \varpi_2, 2\varpi_3, \dots, 2\varpi_{2k-1}, \varpi_{2k}, 2\varpi_{2k+1}, 2\varpi_{2k+2},$ $\varpi_u + \varpi_v \ (u, v \equiv 1 \pmod{2}, u < v), \varpi_{2k+1} + \varpi_{2k+2}\}$
$D_{2k+3}$	$\{2\varpi_1, \varpi_2, 2\varpi_3, \dots, 2\varpi_{2k-1}, \varpi_{2k}, 2\varpi_{2k+1}, 4\varpi_{2k+2}, 4\varpi_{2k+3},$ $\varpi_u + \varpi_v \ (u, v \equiv 1 \pmod{2}, u < v \leq 2k+1),$ $\varpi_u + 2\varpi_{2k+2} \ (u \equiv 1 \pmod{2}, u \leq 2k+1),$ $\varpi_u + 2\varpi_{2k+3} \ (u \equiv 1 \pmod{2}, u \leq 2k+1),$ $2\varpi_{2k+2} + 2\varpi_{2k+3}, \varpi_{2k+2} + 3\varpi_{2k+3}, 3\varpi_{2k+2} + \varpi_{2k+3}\}$
$E_6$	$\{3\varpi_1, \varpi_2, 3\varpi_3, \varpi_4, 3\varpi_5, 3\varpi_6,$ $\varpi_1 + \varpi_3, \varpi_1 + \varpi_6, \varpi_3 + \varpi_5, \varpi_5 + \varpi_6,$ $\varpi_1 + 2\varpi_5, 2\varpi_1 + \varpi_5, \varpi_3 + 2\varpi_6, 2\varpi_3 + \varpi_6\}$
$E_7$	$\{\varpi_1, 2\varpi_2, \varpi_3, \varpi_4, 2\varpi_5, \varpi_6, 2\varpi_7,$ $\varpi_2 + \varpi_5, \varpi_2 + \varpi_7, \varpi_5 + \varpi_7\}$
$E_8, F_4, G_2$	$\{\varpi_1, \varpi_2, \dots, \varpi_n\}$

3 Here we give an explicit description of  $\Psi_{\min}$  for type  $A_n$ . Let  $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in \Lambda$ .  
 4 Set  $|\lambda| := \sum_{k=1}^n k \lambda_k \in \mathbb{Z}_+$ . Then we write

$$5 \quad \begin{aligned} \lambda &= \lambda_1 \varpi_1 - \sum_{i=2}^n \lambda_i (\alpha_{i-1} + 2\alpha_{i-2} + \dots + (i-1)\alpha_1 - i\varpi_1) \\ &= (\sum_{i=1}^n i \lambda_i) \varpi_1 - (\lambda_n \alpha_{n-1} + (2\lambda_n + \lambda_{n-1}) \alpha_{n-2} + \dots + ((n-1)\lambda_n + \dots + \lambda_2) \alpha_1). \end{aligned}$$

6 Hence,  $\lambda \in Q$  if and only if  $(\sum_{i=1}^n i \lambda_i) \varpi_1 \in Q$ , if and only if  $(n+1) \mid |\lambda|$ . This fact shows  
 7 that

$$8 \quad \Psi = \{\lambda \in \Lambda^+ \mid (n+1) \mid |\lambda|\}.$$

9 Hence, single weights  $\mathbf{s}(k) = r_k \varpi_k$  ( $1 \leq k \leq n$ ) and special weights  $\mathbf{e}(k) = d_k \varpi_1 + \varpi_k$  ( $2 \leq$   
 10  $k \leq n$ ) are contained in  $\Psi_{\min}$ . We end this subsection by proving a property of the  
 11 elements in  $T$ .

12 LEMMA 9. For each  $\tau = \sum_{k=1}^n \tau_k \varpi_k \in T$ , set  $\|\tau\| = -\tau_1 + \sum_{k \neq 1} \tau_k d_k$ , then we  
 13 have

$$1 \quad \tau + \|\tau\|\varpi_1 = \sum_{k \neq 1} \tau_k \mathbf{e}(k),$$

2 *and  $\|\tau\|$  is a positive integer such that  $(n+1) \mid \|\tau\|$ .*

3 **PROOF.** One can directly verify that

$$4 \quad \sum_{k \neq 1} \tau_k \mathbf{e}(k) = \sum_{k \neq 1} \tau_k (d_k \varpi_1 + \varpi_k) = \left( \sum_{k \neq 1} \tau_k d_k \varpi_1 \right) + \tau - \tau_1 \varpi_1 = \tau + \|\tau\|\varpi_1.$$

5 Since  $\sum_{k \neq 1} \tau_k \mathbf{e}(k) \in Q$  and  $\tau \in Q$ , we have  $\|\tau\|\varpi_1 \in Q$ , that is,  $(n+1) \mid \|\tau\|$ .

6 Assume that  $\|\tau\| \leq 0$ , then  $\tau_1 \geq \sum_{k \neq 1} \tau_k d_k$ . Since  $\tau \neq \tau_1 \varpi_1$  (otherwise  $(n+1) \mid \tau_1$   
7 and  $\tau = \frac{\tau_1}{n+1} \mathbf{s}(1)$ , contradiction), there exists  $k_0 \neq 1$  such that  $\tau_{k_0} \geq 1$ . Then

$$8 \quad Q \ni \tau - \mathbf{e}(k_0) = (\tau_1 - d_{k_0})\varpi_1 + (\tau_{k_0} - 1)\varpi_{k_0} + \sum_{k \neq 1, k_0} \tau_k \varpi_k.$$

9 However,  $\tau_1 - d_{k_0} \geq \tau_1 - \sum_{k \neq 1} \tau_k d_k \geq 0$  implies that the right-hand side of the above  
10 expression belongs to  $\Lambda^+$ . This leads to  $\tau - \mathbf{e}(k_0) \in \Lambda^+ \cap Q = \Psi$ , and then  $\tau$  has to be  
11  $\mathbf{e}(k_0)$  since  $\tau$  is indecomposable, which contradicts to  $\tau \in T$ . Thus, we have  $\|\tau\| > 0$ , the  
12 Lemma holds.  $\square$

### 13 **5.2. Presentation of $Z(U_{q, \mathcal{G}})$**

14 Now we write  $m = |\Psi_{\min}|$  and  $\Psi_{\min} = \{\mu_1, \dots, \mu_m\}$  with the order presented in the  
15 Table of Lemma 8.

16 **LEMMA 10.** *The set  $\{z_{\mu_i}\}_{i=1}^m$  forms a minimal generating set of  $\text{Im}(\theta)$ . For each  
17  $\mu_i = \sum_{j=1}^n \mu_{ij} \varpi_j \in \Psi_{\min}$ , define another element  $x_{\mu_i} = \prod_{j=1}^n z_j^{\mu_{ij}}$ . Then the set  $\{x_{\mu_i}\}_{i=1}^m$   
18 provides another minimal generating set of  $\text{Im}(\theta)$ .*

19 **PROOF.** Notice that  $\text{Im}(\theta) = \text{span}_{\mathbb{K}} \{z_\lambda \mid \lambda \in \Psi\}$ , if  $\lambda \in \Psi \setminus \Psi_{\min}$ , then  $\exists \mu \in$   
20  $\Psi_{\min}$ ,  $\lambda - \mu \in \Psi$ , and

$$21 \quad z_{\lambda - \mu} z_\mu = z_\lambda + \sum_{\nu < \lambda, \nu \in \Lambda^+} [L(\lambda - \mu) \otimes L(\mu) : L(\nu)] z_\nu,$$

22 which leads to  $z_\lambda \in \langle z_\mu, z_{\lambda - \mu}, z_\nu \mid 0 < \nu < \lambda, \nu \in \Psi \rangle$ . Hence by induction on the  
23 square length  $(\lambda, \lambda)$  of  $\lambda$ , one obtains  $z_\lambda \in \langle z_{\mu_i} \rangle_{i=1}^m$ . The first claim is clear.

24 Similarly, if  $\lambda = \sum_{i=1}^n \lambda_i \varpi_i \in \Psi$ , then

$$25 \quad \prod_{i=1}^n z_i^{\lambda_i} = z_\lambda + \sum_{\gamma < \lambda, \gamma \in \Lambda^+} d_\gamma^\lambda(\lambda_1, \dots, \lambda_n) z_\gamma. \quad (5.1)$$

26 where  $d_\gamma^\lambda(\lambda_1, \dots, \lambda_n) := [L(\varpi_1)^{\otimes \lambda_1} \otimes \dots \otimes L(\varpi_n)^{\otimes \lambda_n} : L(\gamma)]$ . If  $d_\gamma^\lambda(\lambda_1, \dots, \lambda_n) \neq 0$ , then  
27  $\lambda - \gamma \in Q$ , which forces  $\gamma \in \Psi$ , that is,  $z_\gamma \in \text{Im}(\theta)$ . Therefore  $\prod_{i=1}^n z_i^{\lambda_i} \in \text{Im}(\theta)$ , and in  
28 particular, each  $x_{\mu_i} = \prod_{j=1}^n z_j^{\mu_{ij}} \in \text{Im}(\theta) = \langle z_{\mu_i} \rangle_{i=1}^m$ .

29 To prove each  $z_{\mu_i}$  can be generated by  $\{x_{\mu_i} \mid \mu_i \in \Psi_{\min}\}$ , we temporarily reorder  
30  $\Psi_{\min}$  such that if  $\mu_i < \mu_j$  then  $i > j$ . Then Eq.(5.1) leads to  $x_{\mu_m} \in z_{\mu_m} + \mathbb{Z}$  and  
31  $x_{\mu_i} \in z_{\mu_i} + \mathbb{Z}[z_{\mu_{i+1}}, \dots, z_{\mu_m}]$  ( $1 \leq i < m$ ). Then one has  $z_{\mu_m} \in x_{\mu_m} + \mathbb{Z}$ , and  $z_{\mu_i} \in$

1  $x_{\mu_i} + \mathbb{Z}[x_{\mu_{i+1}}, \dots, x_{\mu_m}]$  by descending induction on  $1 \leq i < m$ . That is, each  $Z_{\mu_j}$  belongs  
 2 to  $\langle x_{\mu_i} \rangle_{i=1}^m$ , for  $j = 1, \dots, m$ . The second claim  $\text{Im}(\theta) = \langle x_{\mu_i} \rangle_{i=1}^m$  is clear.  $\square$

3 In the following theorem, we present a polynomial algebra  $S$  together with a relation  
 4 ideal  $I$  such that  $S/I$  gives a presentation of  $\langle x_{\mu_i} \rangle_{i=1}^m$ , from which  $Z(U)$  is obtained.

5 THEOREM 5.1. 1. The centre  $Z(U)$  is isomorphic to a polynomial algebra  
 6  $\mathbb{K}[t_1, \dots, t_n]$  when  $\mathfrak{g}$  is of type  $B_n, E_8, F_4$  or  $G_2$ .

7 2. For the other types, the centre  $Z(U)$  is isomorphic to a quotient  $\mathfrak{R} = S/I$ , where  
 8  $S$  is a polynomial algebra over  $\mathbb{K}$  of rank  $|\Psi_{\min}|$ , and  $I$  is the relation ideal whose  
 9 generators are specified below

10

Type	Generators $x_{\mu_i}$ of $\text{Im} \theta$	Generators of $S$	Generating set of $I$
$A_{n \geq 2}$	$Z_1^1, Z_2^2, \dots, Z_n^n,$ $Z_1^2 Z_2, \dots, Z_1^n Z_n,$ $x_\tau = \prod_{i=1}^n Z_i^{\tau_i} (\tau \in T)$	$t_1, t_2, \dots, t_n,$ $p_2, \dots, p_n,$ $w_\tau (\tau \in T)$	$t_1^{\frac{d_k}{(n+1, k)}} t_k - p_k^{r_k} (k \neq 1),$ $\prod_{k=2}^n p_k^{\tau_k} - t_1^{\frac{\ \tau\ }{n+1}} w_\tau$
$B_n$	$Z_1, Z_2, Z_3, \dots, Z_{n-1}, Z_n^2$	$t_1, t_2, t_3, \dots, t_{n-1}, t_n$	0
$C_{2k+1}$	$Z_1^2, Z_2, Z_3^2, \dots, Z_{2k}, Z_{2k+1}^2,$ $Z_u Z_v (u, v \text{ odd}, u < v)$	$t_1, t_2, t_3, \dots, t_{2k}, t_{2k+1},$ $t_{u,v} (u, v \text{ odd}, u < v)$	$t_1 t_u - t_{1,u}^2 (u > 1),$ $t_1 t_u t_v - t_{1,u} t_{1,v} t_{u,v} (u > 1)$
$C_{2k+2}$	$Z_1^2, Z_2, \dots, Z_{2k}, Z_{2k+1}^2, Z_{2k+2}^2$ $Z_u Z_v (u, v \text{ odd}, u < v)$	$t_1, t_2, \dots, t_{2k}, t_{2k+1}, t_{2k+2}$ $t_{u,v} (u, v \text{ odd}, u < v)$	$t_1 t_u - t_{1,u}^2 (u > 1),$ $t_1 t_u t_v - t_{1,u} t_{1,v} t_{u,v} (u > 1)$
$D_{2k+2}$	$Z_1^2, Z_2, \dots, Z_{2k-2}, Z_{2k-1}^2, Z_{2k},$ $Z_{2k+1}^2, Z_{2k+2}^2$ $Z_u Z_v (u, v \text{ odd}, u < v \leq 2k),$ $Z_{2k+1} Z_{2k+2}$	$t_1, t_2, \dots, t_{2k-2}, t_{2k-1}, t_{2k},$ $t_{2k+1}, t_{2k+2},$ $t_{u,v} (u, v \text{ odd}, u < v \leq 2k),$ $t_{2k+1, 2k+2}$	$t_1 t_u - t_{1,u}^2 (u > 1),$ $t_1 t_u t_v - t_{1,u} t_{1,v} t_{u,v} (u > 1),$ $t_{2k+1} t_{2k+2} - t_{2k+1, 2k+2}^2$
$D_{2k+3}$	$Z_1^2, Z_2, Z_3^2, \dots, Z_{2k}, Z_{2k+1}^2,$ $Z_{2k+2}^4, Z_{2k+3}^4,$ $Z_u Z_v (u, v \text{ odd}, u < v \leq 2k+1),$ $Z_u Z_{2k+2}^2 (u \text{ odd}, u \leq 2k+1)$ $Z_u Z_{2k+3}^2 (u \text{ odd}, u \leq 2k+1)$ $Z_{2k+2} Z_{2k+3}^3,$ $Z_{2k+2}^3 Z_{2k+3},$ $Z_{2k+2}^2 Z_{2k+3}^2$	$t_1, t_2, t_3, \dots, t_{2k}, t_{2k+1},$ $t_{2k+2}, t_{2k+3},$ $t_{u,v} (u, v \text{ odd}, u < v \leq 2k+1),$ $p_u (u \text{ odd}, u \leq 2k+1),$ $q_u (u \text{ odd}, u \leq 2k+1),$ $w_1,$ $w_2,$ $w_3$	$t_1 t_u - t_{1,u}^2 (u > 1),$ $t_1 t_u t_v - t_{1,u} t_{1,v} t_{u,v} (u > 1),$ $t_1 t_{2k+2} - p_1^2, t_1 t_{2k+3} - q_1^2,$ $t_{2k+2} t_{1,u} - p_{1,u} (u > 1),$ $t_{2k+3} t_{1,u} - q_{1,u} (u > 1),$ $t_{2k+2} t_{2k+3}^3 - w_1^4,$ $t_{2k+2}^3 t_{2k+3} - w_2^4,$ $t_{2k+2} t_{2k+3}^2 - w_3^2$
$E_6$	$Z_1^3, Z_2, Z_3^3, Z_4, Z_5^3, Z_6^3,$ $Z_1 Z_3, Z_1 Z_6, Z_3 Z_5, Z_5 Z_6$ $Z_1 Z_5^2, Z_1^2 Z_5, Z_3 Z_6^2, Z_3^2 Z_6$	$t_1, t_2, t_3, t_4, t_5, t_6,$ $t_7, t_8, t_9, t_{10},$ $t_{11}, t_{12}, t_{13}, t_{14}$	$t_1 t_3 - t_7^3, t_1 t_6 - t_8^3,$ $t_3 t_5 - t_9^3, t_8 t_9 - t_7 t_{10},$ $t_7 t_9^2 - t_3 t_{11}, t_7^2 t_9 - t_3 t_{12},$ $t_7 t_8^2 - t_1 t_{13}, t_7^2 t_8 - t_1 t_{14}$
$E_7$	$Z_1, Z_2^2, Z_3, Z_4, Z_5^2, Z_6, Z_7^2,$ $Z_2 Z_5, Z_2 Z_7, Z_5 Z_7$	$t_1, t_2, t_3, t_4, t_5, t_6, t_7,$ $t_8, t_9, t_{10}$	$t_2 t_5 - t_8^2, t_2 t_7 - t_9^2,$ $t_2 t_5 t_7 - t_8 t_9 t_{10}$
$E_8, F_4, G_2$	$Z_1, Z_2, \dots, Z_n$	$t_1, t_2, \dots, t_n$	0

12

PROOF. (1) When  $\mathfrak{g}$  is of type  $B_n, E_8, F_4$ , or  $G_2$ , Lemmas 6 and 10 yield

13

$$Z(U) \cong \text{Im}(\theta) = \begin{cases} \langle Z_1, \dots, Z_{n-1}, Z_n \rangle, & \text{for types } E_8, F_4, \text{ and } G_2, \\ \langle Z_1, \dots, Z_{n-1}, Z_n^2 \rangle, & \text{for type } B_n, \end{cases}$$

1 and  $\{z_i\}_{i=1}^n$  is an algebraically independent set. Therefore, the centre  $Z(U) \cong \text{Im}(\theta)$  is  
 2 isomorphic to a polynomial algebra of rank  $n$ .

3 (2.1) From type  $C_n$  to  $E_7$ , we only show the proof of type  $C_{2k+1}$ , as the other types  
 4 follow analogously.

5 Define an algebra epimorphism  $\phi : S \rightarrow \text{Im}(\theta)$  by

$$6 \quad \phi(t_{2i-1}) = z_{2i-1}^2, \quad \phi(t_{2i}) = z_{2i}, \quad (1 \leq i \leq k+1), \quad \phi(t_{u,v}) = z_u z_v, \quad \text{for odd } u < v \leq 2k+1,$$

7 and one can directly check that  $I \subseteq \ker \phi$ .

8 Let  $S_1 = \mathbb{K}[t_1, \dots, t_{2k+1}]$  and  $S_2 = \mathbb{K}[z_1^2, z_2, \dots, z_{2k}, z_{2k+1}^2]$ . Since  $z_1^2, z_2, \dots, z_{2k},$   
 9  $z_{2k+1}^2$  are algebraically independent, the map  $\phi|_{S_1}$  is an algebra isomorphism from  $S_1$  to  
 10  $S_2$ , and  $S_1 \cap \ker \phi = \{0\}$ . Now let  $T_i := S_i \setminus \{0\}$  for  $i = 1, 2$ . It is obvious that  $\phi$  can be  
 11 extended to an algebra epimorphism  $\bar{\phi} : T_1^{-1}S \rightarrow T_2^{-1}\text{Im}(\theta)$  as follows

$$12 \quad \bar{\phi}(a^{-1}b) = \phi(a)^{-1}\phi(b), \quad \forall a \in T_1, b \in S.$$

13 There is a canonical embedding  $T_1^{-1} : S \hookrightarrow T_1^{-1}S$  via  $s \mapsto \frac{s}{1}$ , and we define  $J := T_1^{-1}I \subseteq$   
 14  $\ker \bar{\phi}$ .

15 We next prove  $I \supseteq \ker \phi$  by showing that the quotient ring  $T_1^{-1}S/J$  is a field.

16 Let  $F_0 = \mathbb{K}(t_1, \dots, t_{2k+1}) = T_1^{-1}S_1$ , the function field of  $S_1$ . Then  $F_1 := F_0[t_{2k+2}] \cong$   
 17  $F_0[t]/(t^2 - t_1 t_3)$ .  $F_1$  is a field since  $t^2 - t_1 t_3$  is irreducible in ring  $F_0[t]$ . Similarly, we  
 18 have  $F_2 = F_1[t_{1,5}] = F_1[t]/(t^2 - t_1 t_5), \dots, F_k = F_{k-1}[t_{1,2k+1}] = F_{k-1}[t]/(t^2 - t_1 t_{2k+1})$   
 19 and  $F_2, \dots, F_k$  all are field. Moreover, by  $t_{u,v} = t_{1,u}^{-1} t_{1,v}^{-1} t_1 t_u t_v$ , we have all  $t_{u,v} \in F_k$ , that  
 20 is,  $T_1^{-1}S/J = F_k$ . Now  $T_1^{-1}S/J$  is a field, it implies that  $J$  is a maximal ideal, then (i)  
 21  $\ker(\bar{\phi})$  has to be the maximal ideal  $J$ , and  $J$  a prime ideal in  $T_1^{-1}S$ , which implies (ii)  $I$   
 22 is prime in  $S$  since it is the contraction of a prime ideal  $J$ . Finally, notice that  $I \cap T_1 = \emptyset$ ,  
 23 we have  $J \cap S = T_1^{-1}I \cap S \stackrel{(ii)}{=} I$ . Then one can obtain that

$$24 \quad \ker \phi \subseteq \ker \bar{\phi} \cap S \stackrel{(i)}{=} J \cap S \stackrel{(ii)}{=} I.$$

25 Above all, we have  $\ker \phi = I$ , and then  $\mathfrak{R} = S/I \cong \text{Im}(\theta) \cong Z(U)$ .

26 (2.2) For type  $A_n$ , define an algebra epimorphism  $\phi : S \rightarrow \text{Im}(\theta)$  via

$$27 \quad \phi(t_i) = z_i^{r_i}, \quad \phi(p_k) = z_1^{d_k} z_k, \quad \phi(w_\tau) = x_\tau = \prod_{i=1}^n z_i^{\tau_i}$$

28 for  $1 \leq i, k \leq n$  ( $k \neq 1$ ) and all  $\tau = \sum_{i=1}^n \tau_i \varpi_i \in T$ . Then by Lemma 9, we have  
 29  $\frac{\|\tau\|}{n+1} \in \mathbb{Z}_{>0}$  and

$$30 \quad \phi(p_k^{r_k}) = z_1^{r_k d_k} z_k^{r_k} = z_1^{\frac{(n+1)d_k}{(n+1, k)}} z_k^{r_k} = \phi\left(t_1^{\frac{d_k}{(n+1, k)}} t_k\right),$$

$$\phi\left(\prod_{k \neq 1} p_k^{\tau_k}\right) = \prod_{k \neq 1} z_1^{d_k \tau_k} z_k^{\tau_k} = z_1^{\|\tau\|} \prod_{k=1}^n z_k^{\tau_k} = \phi\left(t_1^{\frac{\|\tau\|}{n+1}} w_\tau\right),$$

31 that is,  $I \subseteq \ker \phi$ .

32 In order to prove  $\phi$  is injective, i.e.,  $I \supseteq \ker \phi$ , similarly we let  $S_1 = \mathbb{K}[t_1, \dots, t_n]$ ,  
 33  $S_2 = \mathbb{K}[z_1^{r_1}, \dots, z_n^{r_n}]$ ,  $T_i := S_i \setminus \{0\}$  for  $i = 1, 2$ , and repeat the process in (2.1). It suffices

1 to prove that  $T_1^{-1}S/J$  is a field, where  $J := T_1^{-1}I \subseteq \ker \bar{\phi}$ .

2 Let  $F_0 = \mathbb{K}(t_1, \dots, t_n)$  be the fraction field of  $S_1$ . Then  $F_1 := F_0[p_2]$  is a field since  
 3  $t^{r_2} - t_1^{\frac{d_2}{(n+1, 2)}} t_2$  is irreducible in  $F_0[t]$ , and  $F_2 := F_1[p_2] = F_0[p_2, p_3]$  is also a field since  
 4  $t^{r_3} - t_1^{\frac{d_3}{(n+1, 3)}} t_3$  is irreducible in  $F_1[t]$ . In the same way,  $F_{n-1} := F_0[p_2, \dots, p_n]$  is a field.  
 5 Meanwhile, we have  $w_\tau = t_1^{-\frac{\|\tau\|}{n+1}} \prod_{k=2}^n p_k^{\tau_k} \in F_{n-1}$  for all  $\tau \in T$ . Thus,  $T_1^{-1}S/J = F_{n-1}$ ,  
 6 which is indeed a field.

7 This completes the proof of  $S/I \cong \text{Im}(\theta)$ .  $\square$

8 **THEOREM 5.2.** *The result above provides the same descriptions of  $\Psi = \Lambda^+ \cap Q$*   
 9 *and  $(U_b^0)^W$  for the two-parameter quantum group  $U_{r,s}(\mathfrak{g})$  in [11]. That is,*

$$10 \quad \xi(Z(U_{r,s})) \begin{cases} = (U_b^0)^W & \cong S/I, & \text{if } n \text{ is even,} \\ \supseteq (U_b^0)^W \otimes \mathbb{K}[z_*, z_*^{-1}] & \cong S/I \otimes \mathbb{K}[z_*, z_*^{-1}], & \text{if } n \text{ is odd,} \end{cases}$$

11 where the polynomial algebra  $S$  and the ideal  $I$  is shown in the table of Theorem 5.1.

12 **EXAMPLE 1.** *We provide an explicit description of  $Z(U_{q,G})$  for  $\mathfrak{g}$  of types  $A_2$ ,  $A_3$ ,*  
 13 *and  $A_4$ .*

14

Type	Description of $Z(U_{q,G})$
$A_2$	$\Psi_{\min} = \{ \text{single} : 3\varpi_1, 3\varpi_2, \text{special} : \varpi_1 + \varpi_2 \}$ $Z(U_{q,G}) \cong \mathbb{K}[t_1, t_2, p_2] / (t_1 t_2 - p_2^3)$
$A_3$	$\Psi_{\min} = \{ \text{single} : 4\varpi_1, 2\varpi_2, 4\varpi_3, \text{special} : 2\varpi_1 + \varpi_2, \varpi_1 + \varpi_3, T : \varpi_2 + 2\varpi_3 \}$ $Z(U_{q,G}) \cong \mathbb{K}[t_1, t_2, t_3, p_2, p_3, w_1] / (p_2^2 - t_1 t_2, p_3^4 - t_1 t_3, t_1 w_1 - p_2 p_3^2)$
$A_4$	$\Psi_{\min} = \{ \text{single} : 5\varpi_1, 5\varpi_2, 5\varpi_3, 5\varpi_4,$ $\text{special} : 3\varpi_1 + \varpi_2, 2\varpi_1 + \varpi_3, \varpi_1 + \varpi_4,$ $T : \varpi_2 + \varpi_3, \varpi_1 + 2\varpi_2, \varpi_1 + 3\varpi_3, \varpi_2 + 2\varpi_4,$ $3\varpi_2 + \varpi_4, 2\varpi_3 + \varpi_4, \varpi_3 + 3\varpi_4 \}$ $Z(U_{q,G}) \cong \mathbb{K}[t_1, t_2, t_3, t_4, p_2, p_3, p_4, w_1, w_2, w_3, w_4, w_5, w_6, w_7] / I,$ $I = \langle t_1^3 t_2 - p_2^5, t_1^2 t_3 - p_3^5, t_1 t_4 - p_4^5, t_1 w_1 - p_2 p_3, t_1 w_2 - y_2^2,$ $t_1 w_3 - p_3^3, t_1 w_4 - p_2 p_4^2, t_1 w_5 - p_2^3 p_4, t_1 w_6 - p_3^2 p_4, t_1 w_7 - p_3 p_4^3 \rangle$

16 We compare these results with those in [16] and write  $\Psi' = \Lambda^+ \cap \frac{Q}{2}$ . When  $n$  is even,  
 17 we have  $Z(U_{q,G}(\mathfrak{sl}_{n+1})) \cong Z(U_q(\mathfrak{sl}_{n+1}))$  since monoid  $\Psi = \Psi'$ . When  $n$  is odd, we prove  
 18 that  $\Psi \not\cong \Psi'$  as follows.

19 Assume there exists  $\varphi : \Psi \cong \Psi'$  as monoids, then  $\varphi$  bijectively maps their minimal  
 20 generating systems. Let  $r = \frac{n+1}{2}$ , notice that  $\varpi_r \in \Psi'_{\min}$ , then  $\exists \alpha \in \Psi_{\min}$ ,  $\varphi(\alpha) =$   
 21  $\varpi_r$ . This forces that  $\varphi(\beta) = \sum_{i \neq r} k_i \varpi_i$  for all  $\beta \in \Psi_{\min}$ ,  $\beta \neq \alpha$  (otherwise,  $\varphi(\beta)$  is  
 22 decomposable). Therefore,  $\varpi_r$  cannot appear as an additive component in any  $\mathbb{Z}$ -multiple  
 23 of  $\varphi(\beta)$ .

24 If  $\alpha = \mathfrak{s}(1) = (n+1)\varpi_1$ , then

$$1 \quad r_2\varphi(\mathbf{e}(2)) = r_2\varphi(d_2\varpi_1 + \varpi_2) = \frac{d_2}{(n+1, 2)}\varphi((n+1)\varpi_1) + \varphi(r_2\varpi_2) = \frac{d_2}{(n+1, 2)}\varpi_r + \varphi(\mathbf{s}(2)).$$

2 *If  $\alpha = \mathbf{s}(k) = r_k\varpi_k (k > 1)$ , then*

$$3 \quad r_k\varphi(\mathbf{e}(k)) = r_k\varphi(d_k\varpi_1 + \varpi_k) = \frac{d_k}{(n+1, k)}\varphi((n+1)\varpi_1) + \varphi(r_k\varpi_k) = \frac{d_k}{(n+1, k)}\varphi(\mathbf{s}(1)) + \varpi_r.$$

4 *If  $\alpha = \mathbf{e}(k) = d_k\varpi_1 + \varpi_k (k > 1)$ , then*

$$5 \quad r_k\varpi_r = r_k\varphi(\mathbf{e}(k)) = \frac{d_k}{(n+1, k)}\varphi(\mathbf{s}(1)) + \varphi(\mathbf{s}(k)).$$

6 *If  $\alpha = \sum_{i=1}^n \alpha_i\varpi_i \in T$ , applying  $\varphi$  to Lemma 9 yields*

$$7 \quad \sum_{k \neq 1} \alpha_k\varphi(\mathbf{e}(k)) = \varphi(\alpha) + \frac{\|\alpha\|}{n+1}\varphi((n+1)\varpi_1) = \varpi_r + \frac{\|\alpha\|}{n+1}\varphi(\mathbf{s}(1)).$$

8 *In all cases, contradictions arise in which  $\varpi_r$  appears in some  $\mathbb{Z}$ -multiple of  $\varphi(\beta)$  for*  
 9  *$\beta \in \Psi_{\min}$  with  $\beta \neq \alpha$ . It turns out that  $\Psi \not\cong \Psi'$  and hence  $Z(U_{q,G}(\mathfrak{sl}_{n+1})) \not\cong Z(U_q(\mathfrak{sl}_{n+1}))$*   
 10 *when  $n > 1$  is odd.*

## 11 Appendix A

12 The fundamental weights  $\varpi_i (i = 1, \dots, n)$  have the following  $\mathbb{Q}$ -linear expressions  
 13 in terms of the simple roots  $\{\alpha_i\}_{i=1}^n$ :

14

$A_n$	$\varpi_i = \left( \frac{(n-i+1)}{n+1}, \frac{2(n-i+1)}{n+1}, \dots, \frac{(i-1)(n-i+1)}{n+1}, \frac{i(n-i+1)}{n+1}, \frac{i(n-i)}{n+1}, \dots, \frac{i}{n+1} \right)$		
$B_n$	$\varpi_i = (1, 2, \dots, i-1, i, i, \dots, i)$ $\varpi_n = \frac{1}{2}(1, 2, \dots, n)$	$D_n$	$\varpi_i = (1, 2, \dots, i-1, i, i, \dots, i, \frac{i}{2}, \frac{i}{2})$ $\varpi_{n-1} = \frac{1}{2}(1, 2, \dots, n-2, \frac{n}{2}, \frac{n-2}{2})$ $\varpi_n = \frac{1}{2}(1, 2, \dots, n-2, \frac{n-2}{2}, \frac{n}{2})$
$C_n$	$\varpi_i = (1, 2, \dots, i-1, i, i \dots i, \frac{i}{2})$		
$E_6$	$\varpi_1 = \frac{1}{3}(4, 3, 5, 6, 4, 2)$ $\varpi_2 = (1, 2, 2, 3, 2, 1)$ $\varpi_3 = \frac{1}{3}(5, 6, 10, 12, 8, 4)$ $\varpi_4 = (2, 3, 4, 6, 4, 2)$ $\varpi_5 = \frac{1}{3}(4, 6, 8, 12, 10, 5)$ $\varpi_6 = \frac{1}{3}(2, 3, 4, 6, 5, 4)$	$E_7$	$\varpi_1 = (2, 2, 3, 4, 3, 2, 1)$ $\varpi_2 = \frac{1}{3}(4, 7, 8, 12, 9, 6, 3)$ $\varpi_3 = (3, 4, 6, 8, 6, 4, 2)$ $\varpi_4 = (4, 6, 8, 12, 9, 6, 3)$ $\varpi_5 = \frac{1}{3}(6, 9, 12, 18, 15, 10, 5)$ $\varpi_6 = (2, 3, 4, 6, 5, 4, 2)$ $\varpi_7 = \frac{1}{2}(2, 3, 4, 6, 5, 4, 3)$
$E_8$	$\varpi_1 = (4, 5, 7, 10, 8, 6, 4, 2)$ $\varpi_2 = (5, 8, 10, 15, 12, 9, 6, 3)$ $\varpi_3 = (7, 10, 14, 20, 16, 12, 8, 4)$ $\varpi_4 = (10, 15, 20, 30, 24, 18, 12, 6)$ $\varpi_5 = (8, 12, 16, 24, 20, 15, 10, 5)$ $\varpi_6 = (6, 9, 12, 18, 15, 12, 8, 4)$ $\varpi_7 = (4, 6, 8, 12, 10, 8, 6, 3)$ $\varpi_8 = (2, 3, 4, 6, 5, 4, 3, 2)$	$F_4$	$\varpi_1 = (2, 3, 4, 2)$ $\varpi_2 = (3, 6, 8, 4)$ $\varpi_3 = (2, 4, 6, 3)$ $\varpi_4 = (1, 2, 3, 2)$
		$G_2$	$\varpi_1 = (2, 1)$ $\varpi_2 = (3, 2)$

15

1

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