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On the L_1-maximal regularity in the study of free boundary problem for the compressible fluid flows

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Abstract. In this paper, we consider the Stokes equations with non-homogeneous free boundary conditions, which is obtained by the linearization procedure of the free boundary problem of the Navier-Stokes equations describing the viscous compressible fluid flows. We prove the L_1 maximal regularity of solutions to this Stokes equations. This is an extension result of L_p-L_q maximal regularity result obtained by D. Götz and Y. Shibata [**16**] to the L_1 in time maximal regularity case.

1. Introduction.

1.1. Problem and Result

Let Ω be a domain in the N dimensional Euclidean space $\mathbb{R}^N$, whose boundary $\partial\Omega$ is a C^3 compact hypersurface. Let $\mathbf{n}$ denote the unit outer normal to $\partial\Omega$. In this paper, we consider Stokes equations with free boundary conditions, which read as

$$\begin{cases} \partial_t\rho + \eta_0 \mathrm{div}\mathbf{u} = F & \text{in } \Omega\times(0,\infty), \\ \eta_0\partial_t\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I}) = \mathbf{G} & \text{in } \Omega\times(0,\infty), \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I})\mathbf{n} = \mathbf{H} & \text{on } \partial\Omega\times(0,\infty), \\ (\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{1}$$

Here, ρ and $\mathbf{u}$ denote unknown density field and velocity field, while F, $\mathbf{G}$, $\mathbf{H}$, ρ_0 and $\mathbf{u}_0$ are given right members and initial data. The α and β denote viscosity constants such that $\alpha > 0$ and $\alpha+\beta > 0$. The $P(\rho)$ is a C^∞ function of $\rho \in (0,\infty)$ such that $P'(\rho) > 0$, which denotes the pressure in the Navier-Stokes equations describing the compressible viscous fluid flow. Let ρ_* be a positive constant describing the mass density of the reference body and $\eta_0(x) = \rho_* + \tilde\eta_0(x)$, where $\tilde\eta_0(x)$ belongs to some Besov space. Throughout the paper, we assume that there exist two positive constants $\rho_1 < \rho_2$ such that

$$\rho_1 < \rho_* < \rho_2, \quad \rho_1 < \eta_0(x) < \rho_2, \quad \rho_1 < P'(\rho_*) < \rho_2, \quad \rho_1 < P'(\eta_0(x)) < \rho_2. \tag{2}$$

Moreover, $\mathbb{D}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^\top$, where $A^\top$ denotes the transposed A and $\nabla\mathbf{u}$ the gradient of N vector of function $\mathbf{u} = (u_1,\dots,u_N)$, and $\mathrm{div}\mathbf{u} = \sum_{j=1}^N \partial_j u_j$, where $\partial_j = \partial/\partial x_j$, and $\mathbb{I}$ denotes the $N\times N$ identity matrix. For any $N\times N$ matrix of functions $A = (A_{jk})$, $\mathrm{Div}\,A$ denotes the N vector of functions whose i-th component is $\sum_{j=1}^N \partial_j A_{ij}$.

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The equations in (1) are obtained by linearization at $\rho = \eta_0$ and $\mathbf{u} = 0$ of the Navier-Stokes equations describing the viscous compressible fluid motion without surface tension in a time dependent domain Ω_t with free boundary conditions on its boundary $\partial\Omega_t$, which read as

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0 & \text{in } \Omega_t, \\ \rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P(\rho)\mathbb{I}) = \mathbf{G} & \text{in } \Omega_t, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P(\rho)\mathbb{I})\mathbf{n}_t = P(\rho_*)\mathbf{n}_t & \text{on } \partial\Omega_t, \\ V_{\partial\Omega_t} = \mathbf{n}_t\cdot\mathbf{u} & \text{on } \partial\Omega_t, \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{3}$$

for $t \in (0, T)$. Here, $\mathbf{n}_t$ denotes the unit outer normal to $\partial\Omega_t$ and $V_{\partial\Omega_t}$ denotes the evolution speed of $\partial\Omega_t$ in the normal direction. Thus, $V_{\partial\Omega_t} = \mathbf{n}_t\cdot\mathbf{u}$ is non-slip condition which implies that the free surface is advected with the fluid. In other words, this boundary condition ensures that fluid particles do not cross the free surface $\partial\Omega_t$. This condition is called a kinematic boundary condition.

The purpose of this paper is to prove the L_1 in time maximal regularity of solutions ρ and $\mathbf{u}$ of equations (1). The local well-posedness in Ω and the global well-posedness for small initial data in a bounded domain for (3) will be proved as a direct application of L_1 maximal regularity of this paper, and the global well-posedness for small initial data in exterior domains for (3) will be proved some combination of the L_1 maximal regularity and decay properties of solutions to (1) with $\eta_0 = \rho_*$. The topics for the nonlinear equations (3) will be treated in a forthcoming paper.

To state the main result, we introduce the operator $\Lambda_\gamma^{1/2}$ which is defined by

$$\Lambda_\gamma^{1/2} f = \mathcal{L}^{-1}[\lambda^{1/2}\mathcal{L}[f](\lambda)](t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{\lambda t}\lambda^{1/2}\mathcal{L}[f](\lambda)\,d\tau \quad (\lambda = \gamma + i\tau \in \mathbb{C}).$$

Here $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the inverse Laplace transform defined by

$$\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t)\,dt, \quad \mathcal{L}^{-1}[f](t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{\lambda t} f(\tau)\,d\tau \quad (\lambda = \gamma + i\tau \in \mathbb{C}).$$

In this paper, we shall prove the following theorem.

THEOREM 1.1. *Let $1 < q < \infty$. Assume that the following conditions* (1) *or* (2) *holds.*

(1) *If $\eta_0(x) = \rho_*$, then $-1 + 1/q < s < 1/q$.*

(2) *If $\tilde\eta_0(x) \not\equiv 0$ and $\tilde\eta_0(x) \in B^{s+1}_{q,1}(\Omega)$, then $-\min(1 - 1/q, N/q) < s < 1/q$.*

Then, there exists a large constant $\gamma_0 > 0$ such that for any initial data $(\rho_0, \mathbf{u}_0) \in \mathcal{H}^s_{q,1}(\Omega) := B^{s+1}_{q,1}(\Omega) \times B^s_{q,1}(\Omega)^N$, and right members F, $\mathbf{G}$, and $\mathbf{H}$ satisfying the conditions:

$$e^{-\gamma t}F \in L_1(\mathbb{R}, B^{s+1}_{q,1}(\Omega)), \quad e^{-\gamma t}\mathbf{G} \in L_1(\mathbb{R}, B^s_{q,1}(\Omega)^N),$$

$$e^{-\gamma t}\mathbf{H} \in W^{1/2}_1(\mathbb{R}, B^s_{q,1}(\Omega)^N) \cap L_1(\mathbb{R}, B^{s+1}_{q,1}(\Omega)^N),$$

for some $\gamma \geq \gamma_0$, then problem (1) admits unique solutions ρ and $\mathbf{u}$ with

$$\begin{aligned}
&e^{-\gamma t}\rho \in W^1_1((0,\infty), B^{s+1}_{q,1}(\Omega)),\\
&e^{-\gamma t}\mathbf{u} \in L_1((0,\infty), B^s_{q,1}(\Omega)^N) \cap W^1_1((0,\infty), B^{s+2}_{q,1}(\Omega)^N)
\end{aligned}$$

possessing the estimate:

$$\begin{aligned}
&\|e^{-\gamma t}\rho\|_{W^1_1((0,\infty),B^{s+1}_{q,1}(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{u}\|_{L_1((0,\infty),B^s_{q,1}(\Omega))} + \|e^{-\gamma t}\Lambda^{1/2}_\gamma\mathbf{u}\|_{L_1((0,\infty),B^{s+1}_{q,1}(\Omega))}\\
&\quad + \|e^{-\gamma t}\mathbf{u}\|_{L_1((0,\infty),B^{s+2}_{q,1}(\Omega))}\\
&\leq C(\|\rho_0\|_{B^{s+1}_{q,1}(\Omega)} + \|\mathbf{u}_0\|_{B^s_{q,1}(\Omega)} + \|e^{-\gamma t}F\|_{L_1(\mathbb{R},B^{s+1}_{q,1}(\Omega))} + \|e^{-\gamma t}\mathbf{G}\|_{L_1(\mathbb{R},B^s_{q,1}(\Omega))}\\
&\quad + \|e^{-\gamma t}\Lambda^{1/2}_\gamma\mathbf{H}\|_{L_1(\mathbb{R},B^s_{q,1}(\Omega))} + \|e^{-\gamma t}\mathbf{H}\|_{L_1(\mathbb{R},W^{s+1}_{q,1}(\Omega))}).
\end{aligned}$$

Here, C depends on γ_0 but is independent of γ whenever $\gamma \geq \gamma_0$.

Theorem 1.1 follows from the spectral properties of solutions to the generalized resolvent problem:

$$\begin{cases}
\lambda\rho + \eta_0\mathrm{div}\mathbf{v} = f & \text{in } \Omega,\\
\eta_0\lambda\mathbf{v} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I} - P'(\eta_0)\rho\mathbb{I}) = \mathbf{g} & \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I} - P'(\eta_0)\rho\mathbb{I})\mathbf{n} = \mathbf{h} & \text{on } \partial\Omega
\end{cases} \tag{4}$$

where the spectral parameter λ runs through the parabolic sector $\Sigma_{\epsilon,\lambda_0}$ for $0 < \epsilon < \pi/2$ and large $\lambda_0 > 0$, where

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}, \quad \Sigma_{\epsilon,\lambda_0} = \{\lambda \in \Sigma_\epsilon \mid |\lambda| \geq \lambda_0\}.$$

And the spectral properties of solutions to equations (4) will be derived as perturbation for large λ from the spectral properties of solutions to Lamé equations, which read as

$$\begin{cases}
\eta_0\lambda\mathbf{v} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I}) = \mathbf{g} & \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I})\mathbf{n} = \mathbf{h} & \text{on } \partial\Omega.
\end{cases} \tag{5}$$

In fact, setting $\rho = \lambda^{-1}(f - \eta_0\mathrm{div}\mathbf{u})$ in the first equation of (4) and inserting this formula into the second equations in (4) imply that

$$\begin{cases}
\eta_0\lambda\mathbf{v} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{v}\mathbb{I}) = \mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f) & \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I} + \lambda^{-1}P'(\eta_0)\mathrm{div}\mathbf{v}\mathbb{I})\mathbf{n} = \mathbf{h} - \lambda^{-1}P'(\eta_0)f & \text{on } \partial\Omega.
\end{cases}$$

From this observation, we see that (4) can be regarded as a perturbation from (5) for large λ. Thus, the main part of this paper is devoted to analysis of the spectral properties of solutions to (5).

1.2. Short History

Mathematical studies on the compressible Navier-Stokes equations started the uniqueness results by [**18, 23, 37, 43, 63**]. After these works, concerning the study of the local and global well-posedness for the Cauchy problem and the initial boundary value problem with non-slip conditions (Dirichlet conditions), many studies have been done. The local well-posedness has been studied by Tani [**57**] in the Hölder space and by Solonnikov [**53**] in the Sobolev-Slobodetskii spaces. Danchin [**6**] improved Solonnikov's result in the critical space in the Cauchy problem case. Matsumura and Nishida [**33, 34**] made a breakthrough in proving the global well-posedness for small initial data using the energy method. This result was extended to the optimal regularity of initial data in the L_2 space by Kawashita [**27**]. Kobayashi and Shibata [**28**] improved the decay properties of solutions in the exterior domains combining the energy method and L_p-L_q decay properties of solutions to the linearized equations, where the condition: $1 < p \leq 2 \leq q \leq \infty$ is assumed. In the no restrictions of exponents case, so called the diffusion wave properties has been studied by Hopf and Zumbrun [**21**] and Liu and Wang [**32**]. Kobayashi and Shibata [**28, 29**] improved their results. In the half space case, the decay properties were studied by Kagei and Kobayashi [**24, 25**]. The global well-posedness results were extensively studied in the energy spaces of exterior domains by [**48, 49, 62, 64**] and in the critical space of the whole space by [**1, 5, 10, 4, 19, 20, 39**]. Valli [**61**] , Kagei and Tsuda [**26**], and Tsuda [**60**] studied time periodic solutions in the bounded domains and in $\mathbb{R}^N$, respectively. The analytic semigroup approach using Lagrange transformation was started by Ströhmer [**55, 56**] and the L_p-L_q maximal regularity of Stokes semigroup in general domains was proved by Enomoto and Shibata [**14**] and the global well-posedness in the maximal L_p-L_q regularity class was proved by Shibata [**46**] in exterior domains. More references can be found in a survey paper [**48**], and also references of the papers mentioned above.

Concerning the viscous compressible fluid motion without surface tension, the local well-posedness of the free boundary problem without surface tension for compressible viscous fluid flow in the multi-dimensional case was first proved by Secchi and Valli [**42**] in the L_2 framework and by Tani [**58**] in the Hölder spaces, respectively. The gaseous stars case was established by Secchi [**40, 41**]. Enomoto, von Below and Shibata [**15**] proved the local well-posedness in the maximal L_p-L_q regularity class, where they used the L_p-L_q maximal regularity theory for the linearized equations established by Götz and Shibata [**16**]. The global well-posedness has been established by Shibata [**44**] in the bounded domain case. More recently, Shibata and Zhang [**52**] prove the global well-posedness for small data in exterior domains combining the L_p-L_q maximal regularity theorem with L_p-L_q decay properties of solutions to the linearized equations obtained in Shibata and Zhang [**51**].

Concerning the viscous compressible fluid motion with surface tension, the local well-posedness has been studied by Denisova and Solonnikov [**12, 13**] in the Hölder space framework, and the global well-posedness was proved by Solonnikov and Tani [**54**], Zadrzynska and Zajaczkowski [**65**], and Zajaczkowski [**66**] in the bounded domains and L_2 framework, where the reference domain is close to a ball, respectively, the initial density is close to a positive constant and the initial velocity is small. Concerning the L_p-L_q approach to the surface tension problem has been done by Zhang [**67**]

On the other hand, L_1 in time maximal regularity theorem was first investigated by Danchin and Mucha [**8**] for the viscous incompressible fluid with non-slip boundary conditions in exterior domains. The global well-posedness for the free boundary problem of the viscous incompressible fluid in the half-space has been studied first by Danchin et al [**7**] using the extension of Da Prato and Grisvard [**11**] L_1 in time maximal regularity theorem for abstract evolution equations under some regularity assumption of initial data. The restriction in [**7**] was removed by Ogawa and Shimizu [**38**] and Shibata and Watanabe [**50**]. The argument based on Littlewood-Payley decomposition in the half space plays an essential role in [**38**], while the real interpolation method due to Shibata [**47**] was used in [**50**]. The method due to Shibata [**47**] is based on the spectral analysis of linearized equations and so L_1 in time maximal regularity can be obtained for initial boundary value problems of parabolic and hyperbolic- parabolic system of equations appearing in the mathematical fluid mechanics, like Navier-Stokes equations both in the viscous incompressible fluid case and in the viscous compressible fluid case. In fact, the L_1 in time maximal regularity was obtained for the compressible Navier-Stokes equations with non-slip boundary conditions by Kuo [**30**] and Kuo and Shibata [**31**]. In this paper, we consider the L_1 maximal regularity in the free boundary condition case. This paper is a continuation of the study of free boundary problem for the compressible viscous fluid flows in L_p-L_q maximal regularity framework due to [**16, 15**]

Why is the L_1 maximal regularity important ? Assuming that the velocity field $\mathbf{u}$ is expected to be found as an element of the function space $L_p((0,T), W^2_q(\Omega)) \cap W^1_p((0,T), L_q(\Omega))$, the trace theorem tells us that $\mathbf{u}|_{t=0} \in B^{2(1-1/p)}_{q,p}(\Omega)$. Thus, L_1 maximal regularity gives the best order regularity class of the initial data. Moreover, without surface tension case, we usually use the Lagrange transform, because of the lack of regularity of functions representing the free surface. Thus, L_1 maximal regularity is the best space to treat the Lagrange transform.

Main idea. Finally, we shall explain briefly the main idea of showing L_1 maximal regularity for semigroup, which has been given in [**47**]. Let X be a Banach space and $D(A)$ is its subspace. Let $A : D(A) \to X$ be a linear operator satisfying the standard resolvent estimates:

$$\|\lambda(\lambda - A)^{-1} f\|_X \leq C\|f\|_X$$

for any $f \in X$ and $\lambda \in \Sigma_{\epsilon,\gamma}$ where $0 < \epsilon < \pi/2$, $\gamma > 0$, and

$$\Sigma_{\epsilon,\gamma} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}.$$

Then, we know the generation of analytic semigroup $\{T(t)\}_{t\geq 0}$ and

$$\|\partial_t T(t) f\|_X \leq Ce^{\gamma t} t^{-1}\|f\|_X$$

for $t > 0$. In order to obtain L_1 maximal regularity of $\{T(t)\}_{t\geq 0}$, we assume that A satisfies additionally the following two estimates:

$$\|\lambda(\lambda - A)^{-1} f\|_X \leq C|\lambda|^{-\sigma/2}\|f\|_{X_{+\sigma}}, \tag{6}$$

$$\|\lambda\partial_\lambda(\lambda - A)^{-1} f\|_X \leq C|\lambda|^{-(1-\sigma/2)}\|f\|_{X_{-\sigma}}, \tag{7}$$

for any $f \in X_{-\sigma}$ and $\lambda \in \Sigma_{\epsilon,\gamma}$. Here, $X_{\pm\sigma}$ are some Banach spaces such that $X_{+\sigma} \hookrightarrow X \hookrightarrow X_{-\sigma}$ and $(X_{-\sigma}, X_{+\sigma})_{1/2,1} = X$ ($(\cdot,\cdot)_{\theta,r}$ denotes the real interpolation functor.). Then, using standard argument in the analytic semigroup theorem, we see that (6) implies that

$$\|\partial_t T(t)f\|_X \leq Ce^{\gamma t}t^{-1+\sigma/2}\|f\|_{X_{+\sigma}} \quad (t > 0), \tag{8}$$

and (7) implies that

$$\|\partial_t T(t)f\|_X \leq Ce^{\gamma t}t^{-1-\sigma/2}\|f\|_{X_{-\sigma}} \quad (t > 0). \tag{9}$$

Thus, real interpolating (8) and (9) gives that

$$\int_0^\infty e^{-\gamma t}\|\partial_t T(t)f\|_X \, \mathrm{d}t \leq C\|f\|_{(X_{-\sigma},X_{+\sigma})_{1/2,1}} = C\|f\|_X.$$

In [**30**] and [**31**], how to derive (6) and (7) from standard resolvent estimates are discussed for the Dirichlet boundary condition case, In this paper, we extend ideas due to [**30**] and [**31**] to the free boundary condition case. Since free boundary condition has the non-homogeneous right member, that is $\mathbf{h} \neq 0$ in (4), the argument of this paper is more complicated and technical compared with zero-Dirichlet boundary condition case in [**30**] and [**31**]. Essentially, instead of resolvent $\lambda - A$, we have to consider an inverse operator for generalized resolvent problem (4) with non-zero $(f, \mathbf{g}, \mathbf{h})$. Specifically, the key is that generalized resolvent operator satisfies the conditions that corresponds from generalized resolvent estimates to conditions like (6) and (7). We shall show this for the Lamé equations with free boundary conditions. Then, the stokes equations can be treated as a perturbation from Lamé equations, which is the main reason why we choose $\gamma > 0$ so large in our argument. Another reason why we have to take $\gamma > 0$ so large is that the generalized resolvent problem will be treated in bent-half space as a perturbation from half-space. When Ω is bounded, we will be able to treat the case where λ is in some neighborhood of 0 using completely different argument from this paper. In fact, we will be able to prove that 0 is the resolvent for Stokes equations with null free boundary conditions. Moreover, when Ω is exterior domain, we will be able to treat the generalized resolvent problem near $\lambda = 0$ by using the cut-off technique combining the results in the whole space with these in bounded domains. But, dealing with the case where $\lambda = 0$ will be future task. The arguments in this paper are very general, and so applicable for L_1 maximal regularity of initial boundary value problems of parabolic type or hyperbolic-parabolic type.

2. Preparations for latter sections

2.1. Symbols used throughout the paper

Let us explain the symbols used in this paper. Let $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{C}$ be the set of all real, natural, complex numbers, respectively, while $\mathbb{Z}$ denotes the set of all integers. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}_0^N$ and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\partial_x^\kappa = \partial^\kappa = \partial^{|\kappa|}/\partial^{\kappa_1}x_1 \cdots \partial^{\kappa_N}x_N$ stands for standard partial derivatives with respect to

x of order κ. For the dual variable $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$, $D_\xi^\kappa = \partial^{|\kappa|}/\partial^{\kappa_1}\xi_1 \cdots \partial^{\kappa_N}\xi_N$. For differentiations, we use symbols $\nabla f = \{\partial^\kappa f \mid |\kappa| = 1\}, \bar\nabla f = \{\partial^\kappa f \mid |\kappa| \leq 1\}$, $\nabla^2 f = \{\partial^\kappa f \mid |\kappa| = 2\}$, $\bar\nabla^2 f = \{\partial^\kappa f \mid |\kappa| \leq 2\}$.

Let $\mathbb{R}^N_+$ and $\mathbb{R}^N_0$ denote the half space and its boundary defined by

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\}, \quad \mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

and $\mathbf{n}_0 = (0, \ldots, 0, -1)$ denotes the unit outer normal to $\partial\mathbb{R}^N_+$. For $N \in \mathbb{N}$ and a Banach space X, let $\mathcal{S}(\mathbb{R}^N; X)$ be the Schwartz class of X-valued rapidly decreasing functions on $\mathbb{R}^N$. Let $\mathcal{S}'(\mathbb{R}^N; X)$ denote the space of X-valued tempered distributions, which means the set of all continuous linear mappings from $\mathcal{S}(\mathbb{R}^N)$ to X. The Fourier transform $\mathcal{F}[f]$ and the inverse Fourier transform $\mathcal{F}^{-1}_\xi[g]$ are defined by

$$\mathcal{F}[f](\xi) := \int_{\mathbb{R}^N} f(x)e^{-ix\cdot\xi}\,dx, \qquad \mathcal{F}^{-1}_\xi[g](x) := \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} g(\xi)e^{ix\cdot\xi}\,d\xi,$$

respectively. In addition, the partial Fourier transform $\mathcal{F}'[f(\,\cdot\,, x_N)] = \hat f(\xi', x_N)$ and the partial inverse Fourier transform $\mathcal{F}^{-1}_{\xi'}$ are defined by

$$\begin{aligned}\mathcal{F}'[f(\,\cdot\,, x_N)](\xi') &:= \hat f(\xi', x_N) = \int_{\mathbb{R}^{N-1}} f(x', x_N)e^{-ix'\cdot\xi'}\,dx',\\ \mathcal{F}^{-1}_{\xi'}[g(\,\cdot\,, x_N)](x') &:= \frac{1}{(2\pi)^{N-1}}\int_{\mathbb{R}^{N-1}} g(\xi', x_N)e^{ix'\cdot\xi'}\,d\xi',\end{aligned}$$

respectively, where $x' = (x_1, \cdots, x_{N-1}) \in \mathbb{R}^{N-1}$ and $\xi' = (\xi_1, \cdots, \xi_{N-1}) \in \mathbb{R}^{N-1}$. The Laplace transform $\mathcal{L}[f](\lambda)$ and inverse Laplace transform $\mathcal{L}^{-1}[g](t)$ are defined by

$$\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t}f(t)\,dt, \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{\lambda t}g(\lambda)\,d\tau \quad (\lambda = \gamma + i\tau),$$

respectively.

For a domain D and a Banach space X, $L_p(D, X)$, $W^m_p(D, X)$ ($m \in \mathbb{N}$) and $W^s_p(D, X)$ ($s > 0$, $s \notin \mathbb{N}$) stand for standard X valued Lebesgue spaces, Sobolev spaces and Sobolev-Slobodetskii spaces, while $\|\cdot\|_{L_p(D,X)}$, $\|\cdot\|_{W^m_p(D,X)}$ and $\|\cdot\|_{W^s_p(D,X)}$ denote their norms. In particular, we write

$$\begin{aligned}\|e^{-\gamma t}f\|_{L_p((0,T),X)} &= \Big(\int_0^T (e^{-\gamma t}\|f(t)\|_X)^p\,dt\Big)^{1/p},\\ \|e^{-\gamma t}\Lambda^{1/2}_\gamma f\|_{L_p(\mathbb{R},X)} &= \Big(\int_{\mathbb{R}} (e^{-\gamma t}\|\Lambda^{1/2}_\gamma f(t)\|_X)^p\,dt\Big)^{1/p}.\end{aligned}$$

We set

$$W^{1/2}_p(\mathbb{R}, X) = \{f \mid \|e^{-\gamma t}f\|_{W^{1/2}_p(\mathbb{R},X)} = \|e^{-\gamma t}f\|_{L_p(\mathbb{R},X)} + \|e^{-\gamma t}\Lambda^{1/2}_\gamma f\|_{L_p(\mathbb{R},X)} < \infty\}.$$

When $X = \mathbb{R}^N$, we omit $X = \mathbb{R}^N$, namely, we write $L_p(D)$, $W^m_p(D)$, $W^s_p(D)$, $\|\cdot\|_{L_p(D)}$, $\|\cdot\|_{W^m_p(D)}$ and $\|\cdot\|_{W^s_q(D)}$. In particular, $W^0_p(D) = L_p(D)$ for the notational simplicity.

For any domain D, the functional space $\mathcal{H}^\nu_{q,1}(D)$ of data $(\mathbf{g}, \mathbf{h})$ for spectral problems is defined by

$$\mathcal{H}^\nu_{q,1}(D) = B^\nu_{q,1}(D)^N \times B^{\nu+1}_{q,1}(D)^N,$$
$$\|(\mathbf{g}, \mathbf{h})\|_{\mathcal{H}^\nu_{q,1}(D)} = \|\mathbf{g}\|_{B^\nu_{q,1}(D)} + \|\mathbf{h}\|_{B^{\nu+1}_{q,1}(D)} \quad \text{for } (\mathbf{g}, \mathbf{h}) \in \mathcal{H}^\nu_{q,1}(D).$$

For $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}^\nu_{q,1}(D)$, $H_i = \{H_{ij} \mid j = 1, \dots, N\}$ $(i = 1, 2)$ are respective corresponding variables to $\mathbf{g} = (g_1, \dots, g_N)$ and $\lambda^{1/2}\mathbf{h} = (\lambda^{1/2}h_1, \dots, \lambda^{1/2}h_N)$. And, $\{H_{3ij} \mid i, j = 1, \dots, N\}$ are corresponding variables to $\partial_i h_j$, and set $H_3 = \{H_{3ij} \mid i, j = 1, \dots, N\}$. Let $m(N) = 2N + N^2$. Set $H = (H_1, H_2, H_3)$, which is an $m(N)$ vector of functions.

For a domain D in $\mathbb{R}^N$ and $N \geq 2$, we set $(\mathbf{f}, \mathbf{g})_D = \int_D \mathbf{f}(x) \cdot \mathbf{g}(x)\,dx$ for N-vector functions $\mathbf{f}$ and $\mathbf{g}$ on D, where we will write $(\mathbf{f}, \mathbf{g}) = (\mathbf{f}, \mathbf{g})_D$ for short if there is no confusion.

For a Banach space X, $\|\cdot\|_X$ denotes its norm. For Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y, and we write $\mathcal{L}(X) = \mathcal{L}(X, X)$. Let $\mathrm{Hol}\,(U, X)$ denote the set of all X valued holomorphic functions defined on a domain $U \subset \mathbb{C}$. The n product space of X is written as $X^n = \{x = (x_1, \dots, x_n) \mid x_i \in X (i = 1, \dots, n)\}$, while its norm is denoted by $\|x\|_X = \sum_{i=1}^n \|x_i\|_X$. $X \hookrightarrow Y$ means that X is continuously imbedded into Y, that is $X \subset Y$ and $\|x\|_Y \leq C\|x\|_X$ with some constant C.

For any interpolation couple (X, Y) of Banach spaces X and Y, the maps $(X, Y) \to (X, Y)_{\theta,p}$ and $(X, Y) \to (X, Y)_{[\theta]}$ denote the real interpolation functor for each $\theta \in (0, 1)$ and $p \in [1, \infty]$ and the complex interpolation functor for each $\theta \in (0, 1)$, respectively. By $C > 0$ we will often denote a generic constant that does not depend on the quantities at stake. And, by $C_{a,b,c,\cdots}$ we denote generic constants depending on the quantities a, b, $c, \cdots$. C and $C_{a,b,c,\cdots}$ may change from line to line.

2.2. Definition of Besov spaces and some properties

To define Besov pace $B^s_{q,r}$, we introduce Littlewood-Paley decomposition. Let $\phi \in \mathcal{S}(\mathbb{R}^N)$ with $\mathrm{supp}\,\phi = \{\xi \in \mathbb{R}^N \mid 1/2 \leq |\xi| \leq 2\}$ such that $\sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) = 1$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$. Then, define

$$\phi_k := \mathcal{F}^{-1}_\xi[\phi(2^{-k}\xi)] \quad (k \in \mathbb{Z}), \qquad \psi = 1 - \sum_{k\in\mathbb{N}} \phi_k.$$

For $1 \leq q, r \leq \infty$ and $s \in \mathbb{R}$ we denote

$$\|f\|_{B^s_{q,r}(\mathbb{R}^N)} := \begin{cases} \|\psi * f\|_{L_q(\mathbb{R}^N)} + \Bigl(\sum_{k\in\mathbb{N}} \bigl(2^{sk}\|\phi_k * f\|_{L_q(\mathbb{R}^N)}\bigr)^r\Bigr)^{1/r} & \text{if } 1 \leq r < \infty, \\ \|\psi * f\|_{L_q(\mathbb{R}^N)} + \sup_{k\in\mathbb{N}} \bigl(2^{sk}\|\phi_k * f\|_{L_q(\mathbb{R}^N)}\bigr) & \text{if } r = \infty. \end{cases}$$

Here, $f * g$ means the convolution between f and g. The inhomogeneous Besov spaces $B^s_{q,r}(\mathbb{R}^N)$ is defined as the sets of all $f \in \mathcal{S}'(\mathbb{R}^N)$ such that $\|f\|_{B^s_{q,r}(\mathbb{R}^N)} < \infty$. In particular, we define $B^s_{q,\infty-}(\mathbb{R}^N)$ by

$$B^s_{q,\infty-}(\mathbb{R}^N) = \{f \in B^s_{q,\infty}(\mathbb{R}^N) \mid \lim_{j\to\infty} 2^{js}\|\phi_j * f\|_{L_q(\mathbb{R}^N)} = 0\}.$$

In this paper, we use the following conventions: $r < \infty- < \infty$ for $r \in \mathbb{R}$ and $1/\infty- = 1/\infty = 0$. In particular, for the Hölder conjugate we know that $B^s_{q,r}(\mathbb{R}^N)^* = B^{-s}_{q',r'}(\mathbb{R}^N)$ $B^s_{q,1}(\mathbb{R}^N)^* = B^{-s}_{q',\infty}(\mathbb{R}^N)$, and $B^s_{q,\infty-}(\mathbb{R}^N) = B^{-s}_{q',1}(\mathbb{R}^N)$ for $1 < q, r < \infty$, where $q' = q/(q-1)$ and $r' = r/(r-1)$.

For any domain D in $\mathbb{R}^N$, $B^s_{q,1}(D)$ is defined by the restriction of $B^s_{q,1}(\mathbb{R}^N)$, that is

$$\begin{aligned} B^s_{q,1}(D) &= \{f \in \mathcal{D}'(D) \mid \text{there exists a } g \in B^s_{q,1}(\mathbb{R}^N) \text{ such that } g|_D = f\}, \\ \|f\|_{B^s_{q,1}(D)} &= \inf\{\|g\|_{B^s_{q,1}(\mathbb{R}^N)} \mid g \in B^s_{q,1}(\mathbb{R}^N),\ \ g|_D = f\}. \end{aligned} \tag{10}$$

Here, $\mathcal{D}'(D)$ denotes the set of all distributions on D and $g|_D$ denotes the restriction of g to D. Notice that $W^s_q(D) = B^s_{q,q}(D)$, which are called a Sobolev-Slobodeckij spaces. In particular, if s is a non-negative integer, W^s_q is a usual Sobolev space.

It is well-known that if D satisfies the cone property, then $B^s_{q,r}(D)$ may be *characterized* by means of real interpolation. In fact, for $-\infty < s_0 < s_1 < \infty$, $1 < q < \infty$, $1 \le r \le \infty$ and $0 < \theta < 1$, it follows that

$$B^{\theta s_0 + (1-\theta)s_1}_{q,r}(D) = \big(W^{s_0}_q(D), W^{s_1}_q(D)\big)_{\theta,r}$$

cf. [**36**, Theorem 8], [**59**, Theorem 2.4.2]. If D is a uniform C^1 domain, then D satisfies the cone property.

2.3. The estimate for the product of 2 functions and some composite functions using Besov norms

The following lemma is concerned the estimate of product of two functions using Besov norms.

LEMMA 1. *Let $1 \le q \le q_1 \le \infty$ and $1 \le r \le \infty$. Let q' be the Hölder conjugate of q. If $s \in \mathbb{R}$ satisfies*

$$\begin{cases} -N/q_1 < s < N/q_1 & \text{if } 1/q + 1/q_1 \le 1 \\ -N/q' < s < N/q_1 & \text{if } \ 1/q + 1/q_1 > 1 \end{cases}$$

then for every $u \in \mathcal{B}^s_{q,r}(\mathbb{R}^N)$ and $v \in \mathcal{B}^{N/q_1}_{q_1,\infty}(\mathbb{R}^N) \cap L_\infty(\mathbb{R}^N)$, there holds

$$\|uv\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N)} \le C\|u\|_{\mathcal{B}^s_{q,r}(\mathbb{R}^N)}\|v\|_{\mathcal{B}^{N/q_1}_{q_1,\infty}(\mathbb{R}^N)\cap L_\infty(\mathbb{R}^N)} \tag{11}$$

for some constant $C > 0$.

PROOF. For the proof, refer Abidi-Paicu [**1**] and Haspot [**19**]. □

REMARK 2. (1) *For any domain D in $\mathbb{R}^N$, the Besov spaces on D are defined by restriction of elements on $\mathbb{R}^N$ to D (cf.* (10)*), and therefore Lemma 1 holds on any domain D.*
(2) *As is known $B^{N/q_1}_{q_1,1}(D) \cap L_\infty(D) \hookrightarrow B^{N/q_1}_{q_1,1}(D)$, and therefore from* (11) *it follows that*

$$\|uv\|_{\mathcal{B}^s_{q,r}(D)} \leq C\|u\|_{\mathcal{B}^s_{q,r}(D)}\|v\|_{\mathcal{B}^{N/q_1}_{q_1,1}(D)}$$

LEMMA 3. *Let D be a domain in $\mathbb{R}^N$ and $1 < q < \infty$. If $-\min(1-1/q, N/q) < s < 1/q$, then for any $u \in B^s_{q,1}(D)$ and $v \in B^{N/q}_{q,\infty}(D) \cap L_\infty(D)$, there holds*

$$\|uv\|_{B^s_{q,1}(D)} \leq C\|u\|_{B^s_{q,1}(D)}\|v\|_{B^{N/q}_{q,\infty}(D)\cap L_\infty(D)}. \tag{12}$$

for some constant $C > 0$.

REMARK 4. *Instead of* (12) *there holds*

$$\|uv\|_{B^s_{q,1}(D)} \leq C\|u\|_{B^s_{q,1}(D)}\|v\|_{B^{N/q}_{q,1}(D)}. \tag{13}$$

PROOF. If we choose $q_1 = q$ in Lemma 1, then s should satisfy

$$\begin{aligned} -N/q < s < N/q \quad & \text{if } 2/q \leq 1, \\ -N/q' < s < N/q \quad & \text{if } 2/q > 1 \end{aligned}$$

for (12) to hold. If $-1+1/q < s < 1/q$ and $2/q \leq 1$, then (13) holds when $-N/q \leq -1+1/q < s < 1/q$ or $-1+1/q \leq -N/q < s < 1/q$. Thus, when $2/q \leq 1$, (13) holds when $-\min(1-1/q, N/q) < s < 1/q$. Since $-N/q' < -1+1/q$ holds automatically, (13) holds when $-1+1/q < s < 1/q$ and $2/q > 1$. This completes the proof of Lemma 3. □

As a limiting case of Lemma 1, we know the following lemma.

LEMMA 5. *Let D be a domain in $\mathbb{R}^N$ and $1 < q < \infty$. Then, $B^{N/q}_{q,1}(D)$ is a Banach algebra, that is for any u, $v \in B^{N/q}_{q,1}(D)$, there holds*

$$\|uv\|_{B^{N/q}_{q,1}(D)} \leq C_{D,q,N}\|u\|_{B^{N/q}_{q,1}(D)}\|v\|_{B^{N/q}_{q,1}(D)}.$$

PROOF. For the proof, refer Abidi-Paicu [**1**] and Haspot [**19**]. □

LEMMA 6. *Let D be a domain in $\mathbb{R}^N$. Let $1 < q < \infty$ and $-1+1/q < s < 1/q$. Let β be a number such that $1 \leq \beta < \min(qN, q'N)$ where q' be the Hölder conjugate of q. Then, there holds*

$$\|uv\|_{\mathcal{B}^s_{q,r}(D)} \leq C\|u\|_{\mathcal{B}^s_{q,r}(D)}\|v\|_{\mathcal{B}^{\beta/N}_{\beta,\infty}(D)\cap L_\infty(D)} \tag{14}$$

for any $u \in \mathcal{B}^s_{q,r}(D)$ and $v \in \mathcal{B}^{N/q}_\beta(D) \cap L_\infty(D)$.

PROOF. When $1/q + 1/\beta \leq 1$, that is $1/\beta \leq 1/q'$, using Lemma 1 with $q_1 = \beta$, we have (14) provided $-N/\beta < s < N/\beta$. Since $\beta < \min(qN, q'N)$, we have $1/q' < N/\beta$, and so $-N/\beta < -1/q' = -1+1/q$. Moreover, $1/q < N/\beta$. Thus, if $-1+1/q < s < 1/q$, then $-N/\beta < s < N/\beta$.

When $1/q + 1/\beta > 1$, that is $1/\beta > 1/q'$, using Lemma 1 with $q_1 = \beta$, we have (14) provided $-N/q' < s < N/\beta$. If $-1+1/q < s < 1/q$, from $-N/q' < -1/q'$ and $1/q < N/\beta$ it follows that $-N/q' < s < N/\beta$. This completes the proof of the lemma. □

We now use the following lemma for the Besov norm estimate of composite functions cf. [**19**, Proposition 2.4] and [**2**, Theorem 2.87].

Lemma 7. *Let $1 < q < \infty$. Let I be an open interval of $\mathbb{R}$. Let $\omega > 0$ and let $\tilde\omega$ be the smallest integer such that $\tilde\omega \geq \omega$. Let $F : I \to \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in BC^{\tilde\omega}(I, \mathbb{R})$ that is $F' \in C^{\tilde\omega}(I, \mathbb{R})$ and $\|F'\|_{BC^{\tilde\omega}(I,\mathbb{R})} := \sum_{\ell=0}^{\tilde\omega} \sup_{t\in I} \|\partial_t^\ell F'\|_{L_\infty(\mathbb{R})} < \infty$. Assume that $v \in B^\omega_{q,1}$ has valued in $J \subset I$. Then, $F(v) \in B^\omega_{q,1}$ and there exists a constant C depending only on ν, I, J, and N, such that*

$$\|F(v)\|_{B^\omega_{q,1}} \leq C(1 + \|v\|_{L_\infty})^{\tilde\omega} \|F'\|_{BC^{\tilde\omega}(I,\mathbb{R})} \|v\|_{B^\omega_{q,1}}.$$

2.4. Fourier multiplier theorems in $\mathbb{R}^N$

To estimate solutions of the equations in $\mathbb{R}^N$, we use the following Fourier multiplier theorem of Mihlin - Hörmander type [**35, 22**], which is stated as follows: Let $m(\xi)$ be a $C^\infty(\mathbb{R}^N)$ function such that for any multi-index $\kappa \in \mathbb{N}_0^N$ there exists a constant C_κ such that

$$|D^\kappa_\xi m(\xi)| \leq C_\kappa |\xi|^{-|\kappa|}.$$

We call m a multiplier symbol of order 0. Set $[m] = \max_{|\kappa|\leq N} C_\kappa$. For any multiplier symbol of order 0, we define an operator T_m by

$$T_m f = \mathcal{F}^{-1}[m\mathcal{F}[f]].$$

We call T_m the Fourier multiplier with symbol m. Then, for any $1 < p < \infty$, there exists a constant C_p depending on p such that there holds

$$\|T_m f\|_{L_p(\mathbb{R}^N)} \leq C_p[m]\|f\|_{L_p(\mathbb{R}^N)}.$$

We extend this result to the Besov space case as follows:

Proposition 8. *Let $1 < q < \infty$, $1 \leq r \leq \infty$ and $s \in \mathbb{R}$. Let $m(\xi)$ be a multiplier symbol of order 0 and let T_m be the Fourier multiplier with symbol m. Then, there exists a constant $C_{q,r}$ depending on q and r such that for any $f \in B^s_{q,r}(\mathbb{R}^n)$, there holds*

$$\|T_m f\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C_{q,r}[m]\|f\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

Proof. First of all, we recall the definition of Besov spaces in Subsection 2.2. Since $\phi_k * (T_m f) = \mathcal{F}^{-1}[m\mathcal{F}[\phi_k * f]]$, by the standard Fourier multiplier theorem of Mihlin-Hörmander type, we have

$$\|\phi_k * (T_m f)\|_{L_q(\mathbb{R}^N)} \leq C[m]\|\phi_k * f\|_{L_q(\mathbb{R}^N)}.$$

Similarly, we have

$$\|\psi * (T_m f)\|_{L_q(\mathbb{R}^N)} \leq C[m]\|\psi * f\|_{L_q(\mathbb{R}^N)}.$$

Thus, by the definition of the Besov norm, we have

$$\|T_m f\|_{B^s_{q,r}(\mathbb{R}^N)} \leq C[m]\|f\|_{B^s_{q,r}(\mathbb{R}^N)}.$$

This completes the proof of Proposition 8. □

2.5. Symbol classes and estimates of the integral operators in $\mathbb{R}^N_+$

To state main results of this subsection, we introduce a symbol class. In the following, let $m(\lambda, \xi')$ be a function defined on $\Sigma_{\epsilon,\lambda_0} \times (\mathbb{R}^{N-1}\setminus\{0\})$ such that for each $\xi' \in \mathbb{R}^{N-1}\setminus\{0\}$ $m(\lambda, \xi')$ is holomorphic with respect to $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and for each $\lambda \in \Sigma_{\epsilon,\lambda_0}$ $m(\lambda, \xi')$ is a C^∞ function with respect to $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$. Let $\ell \in \mathbb{Z}$. We say that $m(\lambda, \xi')$ is an order ℓ symbol if for any $\kappa' \in \mathbb{N}_0^{N-1}$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$ there exists a constant $C_{\kappa'}$ being depending on κ', ϵ, λ_0 and ℓ such that

$$|D^{\kappa'}_{\xi'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{\ell-|\kappa'|}.$$

Let $\mathbf{M}_\ell$ be the set of all order ℓ symbols. Let

$$\|m\| = \max_{|\kappa'|\leq N} C_{\kappa'}.$$

Let

$$A = \sqrt{(\alpha+\beta)^{-1}\lambda + |\xi'|^2}, \quad B = \sqrt{\alpha^{-1}\lambda + |\xi'|^2}.$$

Here, A and B are characteristic roots of the Lamé equations given in (37) in Sect. 4.1 below. The solution formulas of equations (37) will be given in (38). There exist two constants $d_1 < d_2$ depending on $0 < \epsilon < \pi/2$ such that

$$d_1(|\lambda|^{1/2} + |\xi'|) \leq \operatorname{Re} E \leq |E| \leq d_2(|\lambda|^{1/2} + |\xi'|) \tag{15}$$

for any $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$, where $E \in \{A, B\}$. We can show the following two propositions using the same argument as in the proof of Lemma 4.4 in Enomoto and Shibata [**14**].

Proposition 9. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$, $\lambda_0 > 0$, and $\lambda \in \Sigma_{\epsilon,\lambda_0}$. Let $m_0(\lambda, \xi') \in \mathbf{M}_0$. Set*

$$M(x_N) = \frac{e^{-Bx_N} - e^{-Ax_N}}{B - A}. \tag{16}$$

Define the integral operators L_i $(i = 1, 2, 3, 4)$ by the following formulas:

$$L_1(\lambda)f = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda, \xi')Be^{-B(x_N+y_N)}\mathcal{F}'[f](\xi', y_N)\right](x')\,dy_N,$$
$$L_2(\lambda)f = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda, \xi')B^2M(x_N+y_N)\mathcal{F}'[f](\xi', y_N)\right](x')\,dy_N,$$
$$L_3(\lambda)f = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda, \xi')B^2\partial_\lambda(Be^{-B(x_N+y_N)})\mathcal{F}'[f](\xi', y_N)\right](x')\,dy_N,$$

$$L_4(\lambda)f = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\left[m_0(\lambda,\xi')B^2\partial_\lambda(B^2M(x_N+y_N))\mathcal{F}'[f](\xi',y_N)\right](x')\,dy_N,$$

respectively. Then for every $f \in L_q(\mathbb{R}^N_+)$, it holds

$$\|L_i(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C_q\|m_0\|\|f\|_{L_q(\mathbb{R}^N_+)} \quad (i=1,2,3,4).$$

2.6. Estimates of operator valued holomorphic functions with respect to Besov norms

We consider two operator valued holomorphic functions $Q_i(\lambda)$ $(i=1,2)$ defined on Σ_ϵ acting on $f \in C_0^\infty(\mathbb{R}^N_+)$. We denote the dual operator of $Q_i(\lambda)$ by $Q_i(\lambda)^*$ which satisfies the equality:

$$(Q_i(\lambda)f,\varphi)_{\mathbb{R}^N_+} = (f, Q_i(\lambda)^*\varphi)_{\mathbb{R}^N_+} \quad (i=1,2)$$

for any f and $\varphi \in C_0^\infty(\mathbb{R}^N_+)$. Here, $(f,g) = \int_{\mathbb{R}^N_+} f(x)g(x)\,dx$. Let $Q_i(\lambda)$ satisfy the following assumptions.

Assumption 10. *Let $1<q<\infty$ and $q'=q/(q-1)$. For any $f \in C_0^\infty(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, the following estimates hold:*

$$\|Q_1(\lambda)f\|_{W^i_q(\mathbb{R}^N_+)} \le C\|f\|_{W^i_q(\mathbb{R}^N_+)}, \tag{17}$$

$$\|Q_1(\lambda)f\|_{L_q(\mathbb{R}^N_+)} \le C|\lambda|^{-1/2}\|f\|_{W^1_q(\mathbb{R}^N_+)}, \tag{18}$$

$$\|Q_1(\lambda)^*f\|_{W^i_{q'}(\mathbb{R}^N_+)} \le C\|f\|_{W^i_{q'}(\mathbb{R}^N_+)}, \tag{19}$$

$$\|Q_1(\lambda)^*f\|_{L_{q'}(\mathbb{R}^N_+)} \le C|\lambda|^{-1/2}\|f\|_{W^1_{q'}(\mathbb{R}^N_+)}, \tag{20}$$

$$\|Q_2(\lambda)f\|_{W^i_q(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|f\|_{W^i_q(\mathbb{R}^N_+)}, \tag{21}$$

$$\|Q_2(\lambda)f\|_{W^1_q(\mathbb{R}^N_+)} \le C|\lambda|^{-1/2}\|f\|_{L_q(\mathbb{R}^N_+)}, \tag{22}$$

$$\|Q_2(\lambda)^*f\|_{W^i_{q'}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|f\|_{W^i_{q'}(\mathbb{R}^N_+)}, \tag{23}$$

$$\|Q_2(\lambda)^*f\|_{W^1_{q'}(\mathbb{R}^N_+)} \le C|\lambda|^{-1/2}\|f\|_{L_{q'}(\mathbb{R}^N_+)} \tag{24}$$

for $i=0,1$, where $W^0_r(\mathbb{R}^N_+) = L_r(\mathbb{R}^N_+)$.

The following theorem has been proved in [**47**] and [**50**].

Theorem 2.1. *Let $1<q<\infty$ and $-1+1/q<s<1/q$. Let $\sigma>0$ be a number such that $-1+1/q<s-\sigma<s+\sigma<1/q$ and let $\nu \in \{s-\sigma, s, s+\sigma\}$. Let $Q_i(\lambda)$ $(i=1,2)$ be operator valued holomorphic functions defined on $\Sigma_{\epsilon,\lambda_0}$ acting on $C_0^\infty(\mathbb{R}^N_+)$ functions. Then, for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $f \in C_0^\infty(\mathbb{R}^N_+)$, the following assertions hold.*
(1) If $Q_1(\lambda)$ satisfies (17) and (19), then there holds

$$\|Q_1(\lambda)f\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \le C\|f\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}.$$

If $Q_1(\lambda)$ satisfies (18) and (20) in addition, then there holds

$$\|Q_1(\lambda)f\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-\sigma/2}\|f\|_{B^{s+\sigma}_{q.r}(\mathbb{R}^N_+)}.$$

(2) *If* $Q_2(\lambda)$ *satisfies* (21) *and* (23), *then there holds*

$$\|Q_2(\lambda)f\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|f\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}.$$

If $Q_2(\lambda)$ *satisfies* (22) *and* (24) *in addition, then there holds*

$$\|Q_2(\lambda)f\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\sigma/2)}\|f\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.$$

3. L_1 integrability of Laplace inverse transform.

In this section, we consider the L_1 integrability of solutions to equations (1). The equations (1) is treated as a perturbation of Lamé equations with free boundary conditions. In particular, one of main issues is to prove the L_1 integrability of solutions in time, which is represented by the Laplace inverse transform of solutions to the corresponding generalized resolvent problem. Thus, in this section, we introduce a solution operator $\mathcal{L}_\Omega(\lambda)$ to the Lamé equations with free boundary conditions (5). We will construct a solution operator $\mathcal{L}_\Omega(\lambda)$ which has two properties. One is stated in the following definition.

Definition 11. *Let* $\lambda_{lame} > 0$ *and* $0 < \epsilon < \pi/2$. *Let* $1 < q < \infty$ *and* $-1 + 1/q < s < 1/q$. *Let* $\sigma > 0$ *be a small number such that* $-1 + 1/q < s - \sigma < s + \sigma < 1/q$. *Let* $\nu \in \{s - \sigma, s, s + \sigma\}$. *Let* $\mathcal{L}_\Omega(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_{lame}}, \mathcal{L}(B^s_{q,1}(\Omega)^{m(N)}, B^{s+2}_{q,1}(\Omega)^N))$. *We say that* $\mathcal{L}_\Omega$ *has* (s, σ, q) *properties in* Ω *if for any* $\lambda \in \Sigma_{\epsilon,\lambda_{lame}}$ *there hold*

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial^\ell_\lambda\mathcal{L}_\Omega(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C|\lambda|^{-\ell}\|H\|_{B^\nu_{q,1}(\Omega)} \quad (\ell = 0, 1),\\
\|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{L}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-\sigma/2}\|H\|_{B^{s+\sigma}_{q,1}(\Omega)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{L}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)},\\
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{L}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)}
\end{aligned} \tag{25}$$

provided that $H \in B^{s+\sigma}_{q,1}(\Omega)^{m(N)}$.

Remark 12. *Since* $s-\sigma < s < s+\sigma$, *that* $H \in B^{s+\sigma}_{q,1}(\Omega)$ *implies that* $H \in B^\nu_{q,1}(\Omega)$ *for* $\nu = s$ *and* $\nu = s - \sigma$.

For equations (5), we shall prove the existence of operators having the L_1 spectrum properties as follows.

Theorem 3.1. *Let* $1 < q < \infty$ *and* $0 < \epsilon < \pi/2$. *Assume that the following* (1) *or* (2) *holds:*

(1) *If* $\eta_0 = \rho_*$, *then* $-1 + 1/q < s < 1/q$. *In this case,* $\sigma > 0$ *is a number such that* $-1 + 1/q < s - \sigma < s + \sigma < 1/q$.

(2) *If* $\tilde\eta_0 \not\equiv 0$ *and* $\tilde\eta_0 \in B^{s+1}_{q,1}(\Omega)$, *then* $-\min(1 - 1/q, N/q) < s < 1/q$. *In this case,* $\sigma > 0$ *is a number such that* $-\min(1 - 1/q, N/q) < s - \sigma < s + \sigma < 1/q$.

Then, there exists a positive constant λ_{lame} and an operator $\mathcal{L}_\Omega(\lambda)$ with

$$\mathcal{L}_\Omega(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_{lame}}, \mathcal{L}(B^s_{q,1}(\Omega)^{m(N)}, B^{s+2}_{q,1}(\Omega)^N))$$

having (s,σ,q) properties such that $\mathbf{u} = \mathcal{L}_\Omega(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ is a unique solution of equations (5) *for $\lambda \in \Sigma_{\epsilon,\lambda_{lame}}$ and $(\mathbf{g},\mathbf{h}) \in \mathcal{H}^s_{q,1}(\Omega)$. Here and in the following, $\mathcal{O}_\lambda$ is an operation defined by*

$$\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}) = (\mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}).$$

Let $L_\Omega(t)$ be the Laplace inverse transform of $\mathcal{L}_\Omega$, which is defined by

$$L_\Omega(t)H = \mathcal{L}^{-1}[\mathcal{L}_\Omega(\lambda)H](t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{(\gamma+i\tau)t}\mathcal{L}_\Omega(\gamma+i\tau)H\,d\tau.$$

Then, we have the following proposition about $L_\Omega(t)$.

Proposition 13. *Let $0 < \epsilon < \pi/2$ and $\lambda_{lame} > 0$. Let q, s and σ be numbers given in Theorem 3.1 and let $\mathcal{L}_\Omega(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_{lame}}, \mathcal{L}(B^s_{q,1}(\Omega)^{m(N)}, B^{s+2}_{q,1}(\Omega)^N))$ be the operator having the (s,σ,q) properties. Then, $L_\Omega(t)H$ and $\Lambda^{1/2}_\gamma L_\Omega(t)H$ vanish for $t<0$. Moreover, $e^{-\gamma t}L_\Omega(t)H \in L_1(\mathbb{R}, B^{s+2}_{q,1}(\Omega)^N)$ and $e^{-\gamma t}\Lambda^{1/2}_\gamma L_\Omega(t)H \in L_1(\mathbb{R}, B^{s+1}_{q,1}(\Omega)^N)$, and there holds*

$$\int_{\mathbb{R}} e^{-\gamma t}\|L_\Omega(t)H\|_{B^{s+2}_{q,1}(\Omega)}\,dt + \int_{\mathbb{R}} e^{-\gamma t}\|\Lambda^{1/2}_\gamma L_\Omega(t)H\|_{B^{s+1}_{q,1}(\Omega)}\,dt \le C\|H\|_{B^s_{q,1}(\Omega)} \tag{26}$$

for any $H \in B^s_{q,1}(\Omega)^{m(N)}$.

If for any G with $e^{-\gamma t}G \in L_1(\mathbb{R}, B^s_q(\Omega))$, we define $L_\Omega(t)G(t)$ by

$$L_\Omega(t)G(t) = \mathcal{L}^{-1}[\mathcal{L}_\Omega(\lambda)\mathcal{L}[G](\lambda)](t),$$

then, there holds

$$\begin{aligned}\int_{\mathbb{R}} e^{-\gamma t}(\|L_\Omega(t)G(t)\|_{B^{s+2}_{q,1}(\Omega)} + \|\Lambda^{1/2}_\gamma L_\Omega(t)G(t)\|_{B^{s+1}_{q,1}(\Omega)})\,dt \\ \le C\int_{\mathbb{R}} e^{-\gamma t}\|G(t)\|_{B^s_{q,1}(\Omega)}\,dt.\end{aligned} \tag{27}$$

Proof. Since $C^\infty_0(\Omega)$ is dense in $B^{s+\sigma}_{q,1}(\Omega)$ and $B^s_{q,1}(\Omega)$, we may assume that $H \in C^\infty_0(\Omega)^{m(N)}$ below. First, we shall show that

$$L_\Omega(t)H = 0 \quad \text{for } t<0, \quad \Lambda^{1/2}_\gamma L_\Omega(t)H = 0 \quad \text{for } t<0. \tag{28}$$

To prove (28), we represent $L_\Omega(t)$ by using the contour integral in the complex plane $\mathbb{C}$. Let C_R be a contour defined by

$$C_R = \{\lambda \in \mathbb{C} \mid \lambda = Re^{i\theta},\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}.$$

Let $\gamma > \lambda_{lame}$. By the Cauchy theorem in theory of one complex variable, we have

$$0=\int_{-R}^{R}e^{(\gamma+i\tau)t}\mathcal{L}_\Omega(\gamma+i\tau)H\,d\tau+\int_{C_R+\gamma}e^{\lambda t}\mathcal{L}_\Omega(\lambda)H\,d\lambda. \tag{29}$$

Using (25), we know that

$$\|\mathcal{L}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)}\leq C|\lambda|^{-1}\|H\|_{B^s_{q,1}(\Omega)}.$$

Thus, for $t<0$ we have

$$\Big\|\int_{C_R+\gamma}e^{\lambda t}\mathcal{L}_\Omega(\lambda)H\,d\lambda\Big\|_{B^s_{q,1}(\Omega)}\leq Ce^{\gamma t}\int_0^{\pi/2}e^{-|t|R\cos\theta}\,d\theta\|H\|_{B^s_{q,1}(\Omega)}.$$

Since $|e^{-|t|R\cos\theta}|\leq 1$, by Lebesgue's dominated convergence theorem,

$$\lim_{R\to\infty}\int_0^{\pi/2}e^{-|t|R\cos\theta}\,d\theta=\int_0^{\pi/2}\lim_{R\to\infty}e^{-|t|R\cos\theta}\,d\theta=0.$$

Therefore, letting $R\to\infty$ in (29), we have

$$0=\frac{1}{2\pi i}\int_{\mathbb{R}}e^{(\gamma+i\tau)t}\mathcal{L}_\Omega(\gamma+i\tau)H\,d\tau=L_\Omega(t)H,$$

which proves the first part of (28).

To prove the second part of (28), we use the similar argument. Notice that

$$\Lambda^{1/2}_\gamma f=\frac{1}{2\pi}\int_{\mathbb{R}}(\gamma+i\tau)^{1/2}e^{(\gamma+i\tau)t}\mathcal{L}[f](\gamma+i\tau)\,d\tau=\frac{1}{2\pi i}\int_{\mathrm{Re}\,\lambda=\gamma}\lambda^{1/2}e^{\lambda t}\mathcal{L}[f](\lambda)\,d\lambda.$$

By the Cauchy theorem in theory of one complex variable, we have

$$0=\int_{-R}^{R}e^{(\gamma+i\tau)t}(\gamma+i\tau)^{1/2}\mathcal{L}_\Omega(\gamma+i\tau)H\,d\tau+\int_{C_R+\gamma}e^{\lambda t}\lambda^{1/2}\mathcal{L}_\Omega(\lambda)H\,d\lambda.$$

Using (25), we know that

$$\|\lambda^{1/2}\mathcal{L}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)}\leq C|\lambda|^{-1/2}\|H\|_{B^s_{q,1}(\Omega)}.$$

Thus, for $t<0$ we have

$$\Big\|\int_{C_R+\gamma}e^{\lambda t}\lambda^{1/2}\mathcal{L}_\Omega(\lambda)H\,d\lambda\Big\|_{B^s_{q,1}(\Omega)}\leq Ce^{\gamma t}\int_0^{\pi/2}(R+\gamma)^{1/2}e^{-|t|R\cos\theta}\,d\theta=(*)$$

Using the change of variable $\theta=\pi/2-\tau$ and the inequality: $\sin\tau\geq(2/\pi)\tau$ for $\tau\in(0,\pi/2)$, we have

$$(*)=Ce^{\gamma t}\int_0^{\pi/2}(R+\gamma)^{1/2}e^{-(2|t|R/\pi)\tau}\,d\tau\leq Ce^{\gamma t}(R+\gamma)^{1/2}((|t|R/\pi)^{-1}\to 0\quad(R\to\infty).$$

Therefore, we have

$$0 = \frac{1}{2\pi i}\int_{\mathbb{R}} (\gamma+i\tau)^{1/2} e^{(\gamma+i\tau)t}\mathcal{L}_\Omega(\gamma+i\tau)H\,d\tau = \Lambda_\gamma^{1/2}L_\Omega(t)H,$$

which proves the second part of (28).

We next consider the case where $t>0$. Let $\Gamma_\pm$ be contours defined by

$$\Gamma_\pm = \{\lambda = re^{\pm i(\pi-\epsilon)} \mid r \in (0,\infty)\}.$$

We shall show that for $t>0$ $L_\Omega(t)$ is represented by

$$L_\Omega(t)H = \frac{1}{2\pi i}\int_{\Gamma_+\cup\Gamma_-+\gamma} e^{\lambda t}\mathcal{L}_\Omega(\lambda)H\,d\lambda. \tag{30}$$

In fact, for $R>0$, define $\tilde C_{R\pm}$ and $\Gamma_{R\pm}$ by

$$\tilde C_{R\pm} = \{\lambda = Re^{i\theta} \mid \pi/2 < \pm\theta < \pi-\epsilon\}, \quad \Gamma_{R\pm} = \{\lambda\in\Gamma_\pm \mid |\lambda| < R\}.$$

By the Cauchy theorem in theory of one complex variable, we have

$$\begin{aligned}&\frac{1}{2\pi}\int_{-R}^{R} e^{i\tau t}\mathcal{L}_\Omega(\gamma+i\tau)H\,d\tau\\ &\quad+\frac{1}{2\pi i}\Big\{\int_{\tilde C_{R+}+\gamma} - \int_{\Gamma_{R+}+\gamma} - \int_{\Gamma_{R-}+\gamma} + \int_{\tilde C_{R-}+\gamma}\Big\} e^{\lambda t}\mathcal{L}_\Omega(\lambda)H\,d\lambda = 0.\end{aligned} \tag{31}$$

Using (25) and the change of variable: $\theta = \tau+\pi/2$, for $R>\gamma$ we have

$$\begin{aligned}\Big\|\int_{\tilde C_{R+}+\gamma} e^{\lambda t}\mathcal{L}_\Omega(\lambda)H\,d\lambda\Big\|_{B^s_{q,1}(\Omega)} &\le \frac{e^{\gamma t}}{2\pi}\int_{\pi/2}^{\pi-\epsilon} e^{R\cos\theta t}\frac{R}{|Re^{i\theta}+\gamma|}\,d\theta\|H\|_{B^s_{q,1}(\Omega)}\\ &\le e^{\gamma t}\frac{R}{R-\gamma}\int_0^{\pi/2-\epsilon} e^{-R\sin\theta t}\,d\theta\|H\|_{B^s_{q,1}(\Omega)}\\ &\le \frac{e^{\gamma t}}{2\pi}\frac{R}{R-\gamma}\int_0^{\pi/2} e^{-\frac{2Rt}{\pi}\tau}\,d\tau\|H\|_{B^s_{q,1}(\Omega)}\\ &\le \frac{e^{\gamma t}}{2\pi}\frac{R}{R-\gamma}\frac{\pi}{2Rt}\|H\|_{B^s_{q,1}(\Omega)}.\end{aligned}$$

Thus, for $t>0$

$$\lim_{R\to\infty}\int_{\tilde C_{R+}+\gamma} e^{\lambda t}\mathcal{L}_\Omega(\lambda)H = 0.$$

Analogously, we have

$$\lim_{R\to\infty}\int_{\tilde C_{R-}+\gamma} e^{\lambda t}\mathcal{L}_\Omega(\lambda)H = 0.$$

Combining these facts and letting $R\to\infty$ in (31) yield (30).

Using the representation formula (30), we shall show that

$$\|\bar\nabla^2 L_\Omega(t)H\|_{B^s_{q,1}(\Omega)} \leq Ce^{\gamma t}t^{-(1-\sigma/2)}\|H\|_{B^{s+\sigma}_{q,1}(\Omega)}, \tag{32}$$
$$\|\bar\nabla^2 L_\Omega(t)H\|_{B^s_{q,1}(\Omega)} \leq Ce^{\gamma t}t^{-(1+\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)}. \tag{33}$$

In fact, noticing that $|e^{\lambda t}| = e^{-tr\cos\epsilon}$ and $|\gamma + re^{\pm i(\pi-\epsilon)}| \geq (1-\cos\epsilon)^{1/2}r$ for $\lambda \in \Gamma_+ \cup \Gamma_- + \gamma$ in (30) and using (25), we have

$$\begin{aligned}
&\|\bar\nabla^2 L_\Omega(t)H\|_{B^s_{q,1}(\Omega)} \\
&\leq Ce^{\gamma t}\int_0^\infty e^{-tr\cos\epsilon}((1-\cos\epsilon)^{1/2}r)^{-\sigma/2}\,dr \\
&\leq Ce^{\gamma t}t^{-1+\sigma/2}\int_0^\infty e^{-\tau\cos\epsilon}((1-\cos\epsilon)^{1/2}\tau)^{-\sigma/2}\,d\tau\|H\|_{B^{s+\sigma}_{q,1}(\Omega)}.
\end{aligned}$$

From this (32) follows.

Using integration by parts concerning λ in (30), we have

$$\bar\nabla^2 L_\Omega(t)H = \frac{-1}{2\pi it}\int_{\Gamma_+\cup\Gamma_-+\gamma} e^{\lambda t}\bar\nabla^2\partial_\lambda\mathcal{L}_\Omega(\lambda)H\,d\lambda.$$

Using (25), we have

$$\begin{aligned}
\|\bar\nabla^2 L_\Omega(t)H\|_{B^s_{q,1}(\Omega)} &\leq \Big\|\frac{1}{2\pi it}\int_{\Gamma_+\cup\Gamma_-+\gamma} e^{\lambda t}\bar\nabla^2\partial_\lambda\mathcal{L}_\Omega(\lambda)H\,d\lambda\Big\|_{B^s_{q,1}} \\
&\leq Ce^{\gamma t}t^{-1}\int_0^\infty e^{-tr\cos\epsilon}((1-\cos\epsilon)^{1/2}r)^{-(1-\sigma/2)}\,dr\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} \\
&= Ce^{\gamma t}t^{-1-\sigma/2}\int_0^\infty e^{-\tau\cos\epsilon}((1-\cos\epsilon)^{1/2}\tau)^{-(1-\sigma/2)}\,d\tau\|H\|_{B^{s-\sigma}_{q,1}(\Omega)}.
\end{aligned}$$

Therefore, we have (33).

Now, we shall estimate $\Lambda_\gamma^{1/2}L_\Omega(t)H$. We shall show that

$$\|\bar\nabla\Lambda_\gamma^{1/2}L_\Omega(t)H\|_{B^s_{q,1}(\Omega)} \leq Ce^{\gamma t}t^{-(1-\sigma/2)}\|H\|_{B^{s+\sigma}_{q,1}(\Omega)}, \tag{34}$$
$$\|\bar\nabla\Lambda_\gamma^{1/2}L_\Omega(t)H\|_{B^s_{q,1}(\Omega)} \leq Ce^{\gamma t}t^{-(1+\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)}. \tag{35}$$

In fact, for $\gamma > \gamma_b$ and $t > 0$ from (30) we have

$$\Lambda_\gamma^{1/2}L(t)H = \frac{1}{2\pi i}\int_{\Gamma_+\cup\Gamma_-+\gamma}\lambda^{1/2}e^{\lambda t}\mathcal{L}(\lambda)H\,d\lambda. \tag{36}$$

Using the assumption (25) and the same argument as in the proof of (32), we have (34).

Using integration by parts in (36), we have

$$\begin{aligned}
\bar\nabla\Lambda_\gamma^{1/2}L_\Omega(t)H &= \frac{-1}{2\pi it}\int_{\Gamma_+\cup\Gamma_-+\gamma} e^{\lambda t}\bar\nabla\partial_\lambda(\lambda^{1/2}\mathcal{L}_\Omega(\lambda)H)\,d\lambda \\
&= \frac{-1}{2\pi it}\int_{\Gamma_+\cup\Gamma_-+\gamma} e^{\lambda t}((1/2)\bar\nabla\lambda^{-1/2}\mathcal{L}_\Omega(\lambda)H + \bar\nabla^{1/2}\lambda^{1/2}\partial_\lambda(\mathcal{L}_\Omega(\lambda)H))\,d\lambda.
\end{aligned}$$

Using (25) and the same argument as in the proof of (33), we have (35).

Now, we shall prove (26) by using (32), (33), (34), and (35). We write

$$\begin{aligned}\int_0^\infty & e^{-\gamma t}\|(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)H\|_{B^s_{q,1}(\Omega)}\,dt \\ &= \sum_{j\in\mathbb{Z}}\int_{2^j}^{2^{(j+1)}} e^{-\gamma t}\|(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)H\|_{B^s_{q,1}(\Omega)}\,dt \\ &\leq \sum_{j\in\mathbb{Z}} 2^j \sup_{t\in(2^j,2^{j+1})}(e^{-\gamma t}\|(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)H\|_{B^s_{q,1}(\Omega)}).\end{aligned}$$

Setting $a_j = \sup_{t\in(2^j,2^{j+1})} e^{-\gamma t}\|(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)H\|_{B^s_{q,1}(\Omega)}$, we have

$$\int_0^\infty e^{-\gamma t}\|(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)H\|_{B^s_{q,1}(\Omega)}\,dt \leq 2\|(2^j a_j)\|_{\ell_1} = 2\|(a_j)\|_{\ell^1_1}.$$

Here and in the following, ℓ^s_q denotes the set of all sequences $(2^{js}a_j)_{j\in\mathbb{Z}}$ such that

$$\begin{aligned}\|(a_j)\|_{\ell^s_q} &= \Big\{\sum_{j\in\mathbb{Z}}(2^{js}|a_j|)^q\Big\}^{1/q} < \infty \quad \text{for } 1\leq q<\infty, \\ \|(a_j)\|_{\ell^s_\infty} &= \sup_{j\in\mathbb{Z}} 2^{js}|a_j| < \infty \quad \text{for } q=\infty.\end{aligned}$$

By (32) and (33), we have

$$\sup_{j\in\mathbb{Z}} 2^{j(1-\sigma/2)}a_j \leq C\|H\|_{B^{s+\sigma}_{q,1}(\Omega)}, \quad \sup_{j\in\mathbb{Z}} 2^{j(1+\sigma/2)}a_j \leq C\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} \quad (H\in B^s_{q,1}(\Omega)).$$

Namely, we have

$$\|(a_j)_{j\in\mathbb{Z}}\|_{\ell^{1-\sigma/2}_\infty} \leq C\|H\|_{B^{s+\sigma}_{q,1}(\Omega)}, \quad \|(a_j)_{j\in\mathbb{Z}}\|_{\ell^{1+\sigma/2}_\infty} \leq C\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} \quad (H\in B^s_{q,1}(\Omega)).$$

According to [**3**, 5.6.1.Theorem], we know that $\ell^1_1 = (\ell^{1-\sigma/2}_\infty, \ell^{1+\sigma/2}_\infty)_{1/2,1}$, where $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation functor, and therefore we have

$$\int_0^\infty e^{-\gamma t}\|(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)H\|_{B^s_{q,1}(\Omega)}\,dt \leq C\|H\|_{(B^{s+\sigma}_{q,1}(\Omega), B^{s-\sigma}_{q,1}(\Omega))_{1/2,1}} = C\|H\|_{B^s_{q,1}(\Omega)}$$

for any $H\in B^s_{q,1}(\Omega)^{m(N)}$. This completes the proof of (26).

To prove (27), we write

$$\begin{aligned}(\bar\nabla^2, \bar\nabla\Lambda_\gamma^{1/2})L_\Omega(t)G(t) &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{\lambda t}(\bar\nabla^2, \bar\nabla\lambda^{1/2})\mathcal{L}_\Omega(\lambda)\mathcal{L}[G](\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{\lambda t}(\bar\nabla^2, \bar\nabla\lambda^{1/2})\mathcal{L}_\Omega(\lambda)\Big(\int_{\mathbb{R}} e^{-\lambda s}G(s)\,ds\Big)\,d\lambda \\ &= \int_{\mathbb{R}}\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{\lambda(t-s)}(\bar\nabla^2, \bar\nabla\lambda^{1/2})\mathcal{L}_\Omega(\lambda)G(s)\,d\lambda\Big)\,ds\end{aligned}$$

$$= \int_{\mathbb{R}} (\bar\nabla^2, \bar\nabla \Lambda_\gamma^{1/2}) L_\Omega(t-s) G(s)\, ds$$
$$= \int_{-\infty}^{t} (\bar\nabla^2, \bar\nabla \Lambda_\gamma^{1/2}) L_\Omega(t-s) G(s)\, ds.$$

Here, $\lambda = \gamma + i\tau$ and we have used the fact that $L_\Omega(t) = 0$ and $\Lambda_\gamma^{1/2} L_\Omega(t) = 0$ for $t < 0$. By Fubini's theorem, we have (27). This completes the proof of Proposition 13. □

To treat the perturbation term, we introduce one more definition.

Definition 14. *Let $\lambda_0 > 0$ and $0 < \epsilon < \pi/2$. Let X be a Banach space and $\mathcal{M}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X))$. We say that $\mathcal{M}(\lambda)$ has a generalized resolvent properties if there hold*

$$\|\partial_\lambda^\ell \mathcal{M}(\lambda) f\|_X \le C|\lambda|^{-\ell-1}\|f\|_X \quad \text{for } f \in X \text{ and } \ell = 0, 1.$$

Let $M(t)$ be the Laplace inverse transform of $\mathcal{M}(\lambda)$ defined by

$$M(t)f = \mathcal{L}^{-1}[\mathcal{M}(\lambda)f] = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{(\gamma+i\tau)t}\mathcal{M}(\gamma+i\tau) f\, d\tau.$$

Then, we have the following proposition about the L_1 integrability of $M(t)$.

Proposition 15. *Let $\lambda_0 > 0$ and $0 < \epsilon < \pi/2$. Let X be a Banach space and $\mathcal{M}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(X))$. If $\mathcal{M}(\lambda)$ has generalized resolvent properties, then, for $f \in X$, $M(t) = 0$ for $t < 0$, and for any $\gamma > \lambda_0$ it holds that*

$$\int_{\mathbb{R}} e^{-\gamma t}\|M(t)f\|_X\, dt \le C\|f\|_X$$

with some constant C depending on λ_0.

Moreover, if we define $M(t)g(t)$ by

$$M(t)g(t) = \mathcal{L}^{-1}[\mathcal{M}(\lambda)\mathcal{L}[g](\lambda)]$$

for $g(t)$ with $e^{-\gamma t}g(t) \in L_1(\mathbb{R}, X)$, then there holds

$$\int_{\mathbb{R}} e^{-\gamma t}\|M(t)g(t)\|_X\, dt \le C\int_{\mathbb{R}} e^{-\gamma t}\|g(t)\|_X\, dt.$$

with some constant C depending on λ_0.

Proof. For $\lambda \in \Sigma_{\epsilon,\lambda_0}$, we have

$$\|\mathcal{M}(\lambda)f\|_X \le C|\lambda|^{-1}\|f\|_X \le C\lambda_0^{-(1-\sigma/2)}|\lambda|^{-\sigma/2}\|f\|_X,$$
$$\|\partial_\lambda\mathcal{M}(\lambda)f\|_X \le C|\lambda|^{-2}\|f\|_X \le C\lambda_0^{-(1+\sigma/2)}|\lambda|^{-(1-\sigma/2)}\|f\|_X$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$. Thus, employing the same argument as in the proof of Proposition 13, we can prove Proposition 15. This completes the proof. □

It is stated in the last section 6 to prove Theorem 1.1 using Propositions 13 and 15.

4. The spectral analysis of the Lamé equations

In view of Proposition 13, to prove the L_1 integrability of solutions to the evolution equations (1), we start with proving Theorem 3.1. Our proof is divided into several steps.

4.1. Estimate of solutions in $\mathbb{R}^N$

In this subsection, we consider the Lamé equations in $\mathbb{R}^N$:

$$\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I}) = \mathbf{g} \quad \text{in } \mathbb{R}^N \tag{37}$$

for $\lambda \in \Sigma_\epsilon$ with $0 < \epsilon < \pi/2$. Notice that $\mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I}) = \alpha\Delta\mathbf{u} + \beta\nabla\mathrm{div}\mathbf{u}$. We shall prove the following theorem.

THEOREM 4.1. *Let $1 < q < \infty$, $-1+1/q < s < 1/q$ and $\lambda_0 > 0$. Let $\sigma > 0$ be a small number such that $-1+1/q < s-\sigma < s+\sigma < 1/q$. Then, there exists an operator $\mathcal{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_\epsilon, \mathcal{L}(B^s_{q,1}(\mathbb{R}^N)^N, B^{s+2}_{q,1}(\mathbb{R}^N)^N))$ having (s,σ,q) properties such that for any $\mathbf{g} \in B^s_{q,1}(\mathbb{R}^N)^N$, $\mathbf{u} = \mathcal{S}(\lambda)\mathbf{g}$ is a unique solution of equations* (37).

PROOF. Let $\nu \in \{s-\sigma, s, s+\sigma\}$. Applying the divergence to equations (37) gives

$$\lambda\mathrm{div}\mathbf{u} - (\alpha+\beta)\Delta\mathrm{div}\mathbf{u} = \mathrm{div}\mathbf{g} \quad \text{in } \mathbb{R}^N.$$

Using the Fourier transform $\mathcal{F}$ and its inverse transform $\mathcal{F}^{-1}$, we have

$$\mathrm{div}\,\mathbf{u} = \mathcal{F}^{-1}\Big[\frac{i\xi\cdot\mathcal{F}[\mathbf{g}](\xi)}{\lambda+(\alpha+\beta)|\xi|^2}\Big].$$

Inserting this formula into (37) gives that

$$\mathbf{u} = \mathcal{F}^{-1}\Big[\frac{\mathcal{F}[\mathbf{g}](\xi)}{\lambda+\alpha|\xi|^2}\Big] + \beta\mathcal{F}^{-1}\Big[\frac{i\xi i\xi\cdot\mathcal{F}[\mathbf{g}](\xi)}{(\lambda+\alpha|\xi|^2)(\lambda+(\alpha+\beta)|\xi|^2)}\Big]. \tag{38}$$

As we know well, there exist positive constants c_1 and c_2 depending on α, β and ϵ such that for any $\lambda \in \Sigma_\epsilon$ there hold:

$$\begin{aligned} c_1(|\lambda|^{1/2}+|\xi|) \leq& |\lambda+\alpha|\xi|^2| \leq c_2(|\lambda|^{1/2}+|\xi|), \\ c_1(|\lambda|^{1/2}+|\xi|) \leq& |\lambda+(\alpha+\beta)|\xi|^2| \leq c_2(|\lambda|^{1/2}+|\xi|). \end{aligned} \tag{39}$$

Moreover,

$$\partial_\lambda\mathbf{u} = \mathcal{F}^{-1}\Big[\partial_\lambda\frac{\mathcal{F}[\mathbf{g}](\xi)}{\lambda+\alpha|\xi|^2}\Big] + \beta\mathcal{F}^{-1}\Big[\partial_\lambda\frac{i\xi i\xi\cdot\mathcal{F}[\mathbf{g}](\xi)}{(\lambda+\alpha|\xi|^2)(\lambda+(\alpha+\beta)|\xi|^2)}\Big]$$

and we see that

$$\begin{aligned} c_1(|\lambda|^{1/2}+|\xi|)^{-4} \leq |\partial_\lambda(\lambda+\alpha|\xi|^2)^{-1}| \leq c_2(|\lambda|^{1/2}+|\xi|)^{-4}, \\ c_1(|\lambda|^{1/2}+|\xi|)^{-6} \leq |\partial_\lambda((\lambda+\alpha|\xi|^2)(\lambda+(\alpha+\beta)|\xi|^2))^{-1}| \leq c_2(|\lambda|^{1/2}+|\xi|)^{-6}. \end{aligned}$$

Here, if necessary, we choose different c_2 from (39). Thus, applying the Fourier multiplier theorem of Mikhlin-Hörmander type, we have

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda^\ell \mathbf{u}\|_{B^\nu_{q,1}(\mathbb{R}^N)} \leq C|\lambda|^{-\ell}\|\mathbf{g}\|_{B^\nu_{q,1}(\mathbb{R}^N)} \qquad (\ell = 0, 1).$$

For $\mathbf{g} \in B^{s+\sigma}_{q,1}(\mathbb{R}^N)$, we write

$$\begin{aligned}
&\lambda^{1/2}\lambda^{\sigma/2}\bar\nabla\mathbf{u}\\
&\quad = \mathcal{F}^{-1}\Big[\frac{\lambda^{1/2+\sigma/2}(1, i\xi)(1+|\xi|^2)^{\sigma/2}\mathcal{F}[\mathbf{g}](\xi)}{(\lambda+\alpha|\xi|^2)(1+|\xi|^2)^{\sigma/2}}\Big]\\
&\qquad + \beta\mathcal{F}^{-1}\Big[\frac{\lambda^{1/2+\sigma/2}(1, i\xi)i\xi i\xi\cdot((1+|\xi|^2)^{\sigma/2}\mathcal{F}[\mathbf{g}](\xi))}{(\lambda+\alpha|\xi|^2)(\lambda+(\alpha+\beta)|\xi|^2)(1+|\xi|^2)^{\sigma/2}}\Big],\\
&\lambda^{1/2}\lambda^{1-\sigma/2}\bar\nabla\partial_\lambda\mathbf{u}\\
&\quad = \mathcal{F}^{-1}\Big[\lambda^{3/2-\sigma/2}(1, i\xi)\{(1+|\xi|^2)^{-\sigma/2}\mathcal{F}[\mathbf{g}](\xi)\}\Big(\partial_\lambda\frac{1}{(\lambda+\alpha|\xi|^2)}\Big)(1+|\xi|^2)^{\sigma/2}\Big]\\
&\qquad + \beta\mathcal{F}^{-1}\Big[\lambda^{3/2-\sigma/2}(1, i\xi)i\xi i\xi\cdot\{(1+|\xi|^2)^{-\sigma/2}\mathcal{F}[\mathbf{g}](\xi)\}\\
&\qquad\qquad \times\Big(\partial_\lambda\frac{1}{(\lambda+\alpha|\xi|^2))(\lambda+(\alpha+\beta)|\xi|^2)}\Big)(1+|\xi|^2)^{\sigma/2}\Big].
\end{aligned}$$

Applying the Fourier multiplier theorem of Mikhilin-Hörmander type, we have

$$\begin{aligned}
\|\lambda^{1/2}\bar\nabla\mathbf{u}\|_{B^s_{q,1}(\mathbb{R}^N)} &\leq C(1+\lambda_0^{-1/2})|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N)},\\
\|\lambda^{1/2}\bar\nabla\partial_\lambda\mathbf{u}\|_{B^s_{q,1}(\mathbb{R}^N)} &\leq C(1+\lambda_0^{-1/2})|\lambda|^{-(1-\sigma/2)}\|\mathbf{g}\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N)}.
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
\|\bar\nabla^2\mathbf{u}\|_{B^s_{q,1}(\mathbb{R}^N)} &\leq C(1+\lambda_0^{-1/2}+\lambda_0^{-1})|\lambda|^{-\sigma/2}\|\mathbf{g}\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N)},\\
\|\bar\nabla^2\partial_\lambda\mathbf{u}\|_{B^s_{q,1}(\mathbb{R}^N)} &\leq C(1+\lambda_0^{-1/2}+\lambda_0^{-1})|\lambda|^{-(1-\sigma/2)}\|\mathbf{g}\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathbf{u}\|_{B^s_{q,1}(\mathbb{R}^N)} &\leq C(1+\lambda_0^{-1/2})|\lambda|^{-(1-\sigma/2)}\|\mathbf{g}\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N)}.
\end{aligned}$$

Define $\mathcal{S}(\lambda)$ by $\mathcal{S}(\lambda)\mathbf{g} = \mathbf{u}$, and then we see that $\mathcal{S}(\lambda)$ has (s, σ, q) properties and $\mathbf{u} = \mathcal{S}(\lambda)\mathbf{g}$ is a solution of equations (37). The uniqueness follows from the existence of solutions to the dual problem. This completes the proof of Theorem 4.1. □

4.2. Solution formulas of Lamé equations in the half-space

In this subsection, we find solution formulas of the Lamé equations in the half-space with free boundary conditions which reads as

$$\begin{aligned}
\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I}) = 0 &\qquad \text{in } \mathbb{R}^N_+,\\
(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n}_0 = \mathbf{h} &\qquad \text{on } \partial\mathbb{R}^N_+.
\end{aligned} \tag{40}$$

Since $\mathrm{Div}\,\mathbb{D}(\mathbf{u}) = \Delta\mathbf{u} + \nabla\mathrm{div}\mathbf{u}$, (40) is rewritten as follows:

$$
\begin{aligned}
&\lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\mathbf{u} = 0 && \text{in } \mathbb{R}^N_+,\\
&-\alpha(\partial_N u_j + \partial_j u_N)|_{x_N=0} = h_j|_{x_N=0},\\
&-(2\alpha\partial_N u_N + (\beta-\alpha)\mathrm{div}\mathbf{u})|_{x_N=0} = h_N|_{x_N=0}
\end{aligned}
$$

for $j = 1, \ldots, N-1$. In this subsection, indices j and J run from 1 through $N-1$ and from 1 through N, respectively. We apply the partial Fourier transform $\mathcal{F}'$ with respect to $x' = (x_1, \ldots, x_{N-1})$ variables and then equations (40) are transformed to the following system of ordinary differential equations:

$$
\begin{aligned}
&(\lambda + \alpha|i\xi'|^2)\mathcal{F}'[u_j] - \alpha\partial_N^2\mathcal{F}'[u_j] - \beta i\xi_j(i\xi'\cdot\mathcal{F}'[\mathbf{u}'] + \partial_N\mathcal{F}'[u_N]) = 0 \qquad (x_N > 0),\\
&(\lambda + \alpha|i\xi'|^2)\mathcal{F}'[u_N] - \alpha\partial_N^2\mathcal{F}'[u_N] - \beta\partial_N(i\xi'\cdot\mathcal{F}'[\mathbf{u}'] + \partial_N\mathcal{F}'[u_N]) = 0 \quad (x_N > 0),\\
&(\partial_N\mathcal{F}'[u_j] + i\xi_j\mathcal{F}'[u_N])|_{x_N=0} = -\alpha^{-1}\mathcal{F}'[h_j](\xi', 0),\\
&(\alpha(\partial_N\mathcal{F}'[u_N] - i\xi'\cdot\mathcal{F}'[\mathbf{u}']) + \beta(i\xi'\cdot\mathcal{F}'[\mathbf{u}'] + \partial_N\mathcal{F}'[u_N]))|_{x_N=0} = -\mathcal{F}'[h_N](\xi', 0).
\end{aligned}
\tag{41}
$$

To obtain solutions formula, we define $\mathcal{F}[u_J]$ by

$$
\mathcal{F}[u_J](\xi', x_N) = m_J e^{-Bx_N} + n_J(e^{-Ax_N} - e^{-Bx_N}).
$$

From the first two equations in (41), we have

$$
\begin{aligned}
&\alpha(B^2 - A^2)n_j - \beta i\xi_j(i\xi'\cdot\mathbf{n}' - An_N) = 0, \quad \beta i\xi_j(i\xi'\cdot(\mathbf{m}'-\mathbf{n}') - B(m_N - n_N)) = 0,\\
&\alpha(B^2 - A^2)n_N + \beta A(i\xi'\cdot\mathbf{n}' - An_N) = 0, \quad \beta B(i\xi'\cdot(\mathbf{m}'-\mathbf{n}') - B(m_N - n_N)) = 0.
\end{aligned}
\tag{42}
$$

From the boundary conditions it follows that

$$
B(m_j - n_j) + An_j - i\xi_j m_N = \alpha^{-1}\hat{h}_j, \tag{43}
$$

$$
\alpha(B(m_N - n_N) + An_N + i\xi'\cdot\mathbf{m}') + \beta(B(m_N - n_N) + An_N - i\xi'\cdot\mathbf{m}') = \hat{h}_N. \tag{44}
$$

Setting $\ell = \alpha^{-1}\beta(i\xi'\cdot\mathbf{n}' - An_N)$, from (42) we have

$$
n_j = \frac{i\xi_j}{B^2 - A^2}\ell, \quad n_N = -\frac{A}{B^2 - A^2}\ell, \tag{45}
$$

$$
i\xi'\cdot\mathbf{m}' - i\xi'\cdot\mathbf{n}' - B(m_N - n_N) = 0. \tag{46}
$$

Since $-B(m_N - n_N) + i\xi'\cdot\mathbf{m}' = i\xi'\cdot\mathbf{n}'$ as follows from (46), the last boundary condition in (44) becomes

$$
Bm_N + (A - B)n_N + i\xi'\cdot\mathbf{m}' - \ell = \alpha^{-1}\hat{h}_N. \tag{47}
$$

From (45) we obtain

$$
i\xi'\cdot\mathbf{n}' = \frac{-|\xi'|^2}{B^2 - A^2}\ell. \tag{48}
$$

Substituting (45) and (48) into (46), we get

$$m_N = B^{-1}\Big(i\xi'\cdot\mathbf{m}' + \frac{|\xi'|^2 - AB}{B^2 - A^2}\ell\Big). \tag{49}$$

From (43), (48) and (49), we have

$$\alpha^{-1}Bi\xi'\cdot\hat{\mathbf{h}}' = (B^2 + |\xi'|^2)i\xi'\cdot\mathbf{m}' + |\xi'|^2\frac{B^2 + |\xi'|^2 - 2AB}{B^2 - A^2}\ell. \tag{50}$$

Substituting (45) and (49) into (47), we obtain

$$\alpha^{-1}\hat{h}_N = 2i\xi'\cdot\mathbf{m}' + \frac{|\xi'|^2 - B^2}{B^2 - A^2}\ell. \tag{51}$$

Let L be a 2×2 matrix defined by

$$L = \begin{pmatrix} B^2 + |\xi'|^2 & |\xi'|^2\dfrac{B^2 + |\xi'|^2 - 2AB}{B^2 - A^2} \\ \\ 2 & \dfrac{|\xi'|^2 - B^2}{B^2 - A^2} \end{pmatrix}.$$

From (50) and (51), it follows that

$$L\begin{pmatrix} i\xi'\cdot\mathbf{m}' \\ \ell \end{pmatrix} = \begin{pmatrix} \alpha^{-1}Bi\xi'\cdot\hat{\mathbf{h}}' \\ \alpha^{-1}\hat{h}_N \end{pmatrix}.$$

We observe that

$$\det L = \frac{4AB|\xi'|^2 - (B^2 + |\xi'|^2)^2}{B^2 - A^2}.$$

Thus, setting

$$\mathcal{L} = 4AB|\xi'|^2 - (B^2 + |\xi'|^2)^2,$$

we have

$$L^{-1} = \frac{B^2 - A^2}{\mathcal{L}}\begin{pmatrix} \dfrac{|\xi'|^2 - B^2}{B^2 - A^2} & -\dfrac{|\xi'|^2(B^2 + |\xi'|^2 - 2AB)}{B^2 - A^2} \\ \\ -2 & (B^2 + |\xi'|^2) \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} i\xi'\cdot\mathbf{m}' &= \frac{1}{\alpha\mathcal{L}}((|\xi'|^2 - B^2)Bi\xi'\cdot\hat{\mathbf{h}}' - |\xi'|^2(B^2 + |\xi'|^2 - 2AB)\hat{h}_N), \\ \ell &= \frac{B^2 - A^2}{\alpha\mathcal{L}}(-2Bi\xi'\cdot\hat{\mathbf{h}}' + (B^2 + |\xi'|^2)\hat{h}_N). \end{aligned}$$

From (43), (45) and (49), we obtain

$$
\begin{aligned}
m_N &= \frac{1}{\alpha\mathcal{L}}((-|\xi'|^2 - B^2 + 2AB)i\xi' \cdot \hat{\mathbf{h}}' - A(B^2 - |\xi'|^2)\hat{h}_N),\\
n_j &= \frac{i\xi_j}{\alpha\mathcal{L}}(-2Bi\xi' \cdot \hat{\mathbf{h}}' + (B^2 + |\xi'|^2)\hat{h}_N),\\
n_N &= \frac{A}{\alpha\mathcal{L}}(2Bi\xi' \cdot \hat{\mathbf{h}}' - (B^2 + |\xi'|^2)\hat{h}_N),\\
m_j &= \frac{1}{\alpha B}\hat{h}_j + \frac{i\xi_j}{\alpha\mathcal{L}B}((-3B^2 - |\xi'|^2 + 4AB)i\xi' \cdot \hat{\mathbf{h}}' - B(2AB - B^2 - |\xi'|^2)\hat{h}_N).
\end{aligned}
$$

By (16) we have solution formulas:

$$
\begin{aligned}
\mathcal{F}[u_j](\xi', x_N) &= \frac{1}{\alpha B}\hat{h}_j e^{-Bx_N}\\
&\quad + e^{-Bx_N}\frac{i\xi_j}{\alpha\mathcal{L}B}((-3B^2 - |\xi'|^2 + 4AB)i\xi' \cdot \hat{\mathbf{h}}' - B(2AB - B^2 - |\xi'|^2)\hat{h}_N)\\
&\quad + M(x_N)\frac{i\xi_j(B-A)}{\alpha\mathcal{L}}(-2Bi\xi' \cdot \hat{\mathbf{h}}' + (B^2 + |\xi'|^2)\hat{h}_N),\\
\mathcal{F}[u_N](\xi', x_N) &= e^{-Bx_N}\frac{1}{\alpha\mathcal{L}}((-|\xi'|^2 - B^2 + 2AB)i\xi' \cdot \hat{\mathbf{h}}' - A(B^2 - |\xi'|^2)\hat{h}_N)\\
&\quad + M(x_N)\frac{A(B-A)}{\alpha\mathcal{L}}(2Bi\xi' \cdot \hat{\mathbf{h}}' - (B^2 + |\xi'|^2)\hat{h}_N).T
\end{aligned}
$$

Using Volevich's trick and the identity:

$$
\partial_N M(x_N) = -e^{-Bx_N} - AM(x_N),
$$

we have

$$
\begin{aligned}
&u_j(x)\\
&= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-B(x_N+y_N)}\Big\{\frac{\lambda^{1/2}}{\alpha^2B^3}\mathcal{F}'[\lambda^{1/2}h_j](\xi', y_N) - \sum_{\ell=1}^{N-1}\frac{i\xi_\ell}{\alpha B^3}\mathcal{F}'[\partial_\ell h_j](\xi', y_N)\\
&\qquad\qquad - \frac{1}{\alpha B^2}\mathcal{F}'[\partial_N h_j](\xi', y_N)\Big\}\Big](x')\,dy_N\\
&\quad + \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-B(x_N+y_N)}\Big\{\frac{1}{\alpha\mathcal{L}B}(-3B^2 - |\xi'|^2 + 4AB)i\xi' \cdot \mathcal{F}'[\partial_j\mathbf{h}'](\xi', y_N)\\
&\qquad\qquad - B(2AB - B^2 - |\xi'|^2)\mathcal{F}'[\partial_j h_N](\xi', y_N))\\
&\qquad - \frac{i\xi_j}{\alpha\mathcal{L}B^2}((-3B^2 - |\xi'|^2 + 4AB)i\xi' \cdot \mathcal{F}'[\partial_N\mathbf{h}'](\xi', y_N)\\
&\qquad\qquad - B(2AB - B^2 - |\xi|^2)\mathcal{F}'[\partial_N h_N](\xi', y_N))\Big\}\Big](x')\,dy_N\\
&\quad + \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[(e^{-B(x_N+y_N)} + AM(x_N+y_N))\frac{B-A}{\alpha\mathcal{L}}\\
&\qquad\qquad \times(-2Bi\xi' \cdot \mathcal{F}'[\partial_j\mathbf{h}'](\xi', y_N) + (B^2 + |\xi'|^2)\mathcal{F}'[\partial_j h_N](\xi', y_N))\\
&\qquad - M(x_N+y_N)\frac{i\xi_j(B-A)}{\alpha\mathcal{L}}\\
&\qquad\qquad \times(-2Bi\xi' \cdot \mathcal{F}'[\partial_N\mathbf{h}'](\xi', y_N) + (B^2 + |\xi'|^2)\mathcal{F}'[\partial_N h_N](\xi', y_N))\Big](x')\,dy_N; \quad (52)
\end{aligned}
$$

$$
\begin{aligned}
&u_N(x)\\
&= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[Be^{-B(x_N+y_N)}\frac{1}{\alpha\mathcal{L}}((-|\xi'|^2 - B^2 + 2AB)\mathcal{F}'[\mathrm{div}'\mathbf{h}'](\xi', y_N)\\
&\quad - \frac{A\lambda^{1/2}}{\alpha B^2}(B^2 - |\xi'|^2)\mathcal{F}[\lambda^{1/2}h_N](\xi', y_N) + \sum_{\ell=1}^{N-1}\frac{Ai\xi_\ell}{\alpha B^2}(B^2 - |\xi'|^2)\mathcal{F}[\partial_\ell h_N](\xi', y_N))\\
&\quad - Be^{-B(x_N+y_N)}\frac{1}{\alpha B\mathcal{L}}((-|\xi'|^2 - B^2 + 2AB)i\xi'\cdot\mathcal{F}'[\partial_N\mathbf{h}'](\xi', y_N)\\
&\quad - A(B^2 - |\xi'|^2)\mathcal{F}'[\partial_N h_N](\xi', y_N))\\
&\quad + (e^{-B(x_N+y_N)} + AM(x_N + y_N))\frac{A(B-A)}{\alpha\mathcal{L}}\{2B\mathcal{F}'[\mathrm{div}'\mathbf{h}'](\xi', y_N)\\
&\quad - (B^2 + |\xi'|^2)(\frac{\lambda^{1/2}}{\alpha B^2}\mathcal{F}'[\lambda^{1/2}h_N](\xi', y_N) - \sum_{\ell=1}^{N-1}\frac{i\xi_\ell}{B^2}\mathcal{F}'[\partial_\ell h_N](\xi', y_N))\}\\
&\quad - M(x_N + y_N)\frac{A(B-A)}{\alpha\mathcal{L}}\\
&\qquad \times (2Bi\xi'\cdot\mathcal{F}'[\partial_N\mathbf{h}'](\xi', y_N) - (B^2 + |\xi'|^2)\mathcal{F}'[\partial_N h_N](\xi', y_N))\Big](x')\,dy_N. \qquad (53)
\end{aligned}
$$

Definition 16. *A class of* $\mathbf{N}_\ell$ *is the set of all symbols* $m(\lambda, \xi') \in \mathbf{M}_\ell$ *such that* $\partial_\lambda m(\lambda, \xi') \in \mathbf{M}_{\ell-2}$.

Lemma 17. *Let* $\mathcal{L} = 4AB|\xi'|^2 - (B^2 + |\xi'|^2)^2$ *and* $M = \mathcal{L}/\lambda$. *Then, there holds* $M^{-1} \in \mathbf{N}_{-2}$.

Proof. Let $r > 0$ be a sufficiently small positive number determined later. First, we consider the case where $|\lambda|/|\xi'|^2 \leq r$ and $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$. The Taylor expansion of $(1+t)^{1/2}$ is that

$$(1+t)^{1/2} = 1 + \frac{t}{2} - \frac{t^2}{4}\int_0^1 (1-\theta)(1+\theta t)^{-3/2}\,d\theta.$$

Thus, setting $\gamma_A = (\alpha+\beta)^{-1}$ and $\gamma_B = \alpha^{-1}$, for $E \in \{A, B\}$, we have

$$E = |\xi'|\sqrt{1 + \gamma_E\lambda|\xi'|^{-2}} = |\xi'|\Big\{1 + \frac{\gamma_E}{2}\lambda|\xi'|^{-2} + \gamma_E^2 z_E(\lambda|\xi'|^{-2})^2\Big\} \qquad (54)$$

where we have set

$$z_E = -\frac{1}{4}\int_0^1 (1-\theta)(1+\theta\gamma_E\lambda|\xi'|^{-2})^{-3/2}\,d\theta.$$

Thus,

$$AB = |\xi'|^2\Big\{1 + \frac{1}{2}(\gamma_A + \gamma_B)\lambda|\xi'|^{-2} + y_{AB}\Big\}, \qquad (55)$$

where we have set

$$
\begin{aligned}
y_{AB} &= \frac{\gamma_A\gamma_B}{4}\lambda^2|\xi'|^{-4} + \gamma_A^2 z_A\lambda^2|\xi'|^{-4} + \gamma_B^2 z_B\lambda^2|\xi'|^{-4} \\
&\quad + \frac{\gamma_B\gamma_A^2}{2}z_A\lambda^3|\xi'|^{-6} + \frac{\gamma_A\gamma_B^2}{2}z_B\lambda^3|\xi'|^{-6} + \gamma_A^2\gamma_B^2 z_A z_B\lambda^4|\xi'|^{-8}. \qquad (56)
\end{aligned}
$$

First, we shall show that

$$
\begin{aligned}
|D_{\xi'}^{\kappa'} z_E| &\leq C_{\kappa'}|\xi'|^{-|\kappa'|}, \qquad (57)\\
|D_{\xi'}^{\kappa'} \partial_\lambda z_E| &\leq C_{\kappa'}|\xi'|^{-2-|\kappa'|}, \qquad (58)\\
|D_{\xi'}^{\kappa'} \lambda^{-1} y_{AB}| &\leq C|\xi'|^{-2-|\kappa'|}, \qquad (59)\\
|D_{\xi'}^{\kappa'} \partial_\lambda(\lambda^{-1} y_{AB})| &\leq C|\xi'|^{-4-|\kappa'|} \qquad (60)
\end{aligned}
$$

where $E \in \{A, B\}$. To prove (57), we observe that for any $\kappa' \in \mathbb{N}_0^{N-1}$, $\ell \in \mathbb{N}$, and $E \in \{A, B\}$, we have

$$
|D_{\xi'}^{\kappa'}(1 + \theta\gamma_E\lambda|\xi|^{-2})^{-\ell/2}| \leq C_{\kappa'}|\xi'|^{-|\kappa'|}. \qquad (61)
$$

In fact, by the Bell formula and (15), we have

$$
\begin{aligned}
&|D_{\xi'}^{\kappa'}(1 + \theta\gamma_E\lambda|\xi'|^{-2})^{-\ell/2}| \\
&\leq C_{\kappa'}\sum_{m=1}^{|\kappa'|}|1 + \theta\gamma_E\lambda|\xi'|^{-2}|^{-\ell/2-m}\sum_{\kappa'_1+\cdots+\kappa'_m=\kappa'}|D_{\xi'}^{\kappa'_1}\theta\gamma_E\lambda|\xi'|^{-2}|\cdots|D_{\xi'}^{\kappa'_m}\theta\gamma_E\lambda|\xi'|^{-2}| \\
&\leq C_{\kappa'}\sum_{m=1}^{|\kappa'|}|1 + \theta\gamma_E\lambda|\xi'|^{-2}|^{-\ell/2-m}|\theta\gamma_E\lambda|\xi'|^{-2}|^m|\xi'|^{-|\kappa'|} \\
&\leq C_{\kappa'}|\xi'|^{-|\kappa'|}.
\end{aligned}
$$

Thus, we have (61), and so (57). Since

$$
\partial_\lambda z_E = \frac{3}{8}\int_0^1 (1-\theta)(1 + \theta\gamma_E\lambda|\xi'|^{-2})^{-5/2}\theta\, d\theta\gamma_E|\xi'|^{-2},
$$

by Leibniz's rule and (61), we have (58). From (56), it follows that

$$
\begin{aligned}
\lambda^{-1}y_{AB} &= \frac{\gamma_A\gamma_B}{4}\lambda|\xi'|^{-4} + \gamma_A^2 z_A\lambda|\xi'|^{-4} + \gamma_B^2 z_B\lambda|\xi'|^{-4} \\
&\quad + \frac{\gamma_B\gamma_A^2}{2}z_A\lambda^2|\xi'|^{-6} + \frac{\gamma_A\gamma_B^2}{2}z_B\lambda^2|\xi'|^{-6} + \gamma_A^2\gamma_B^2 z_A z_B\lambda^3|\xi'|^{-8}.
\end{aligned}
$$

Thus, using (57) and Leibniz's rule, we have

$$
\begin{aligned}
|D_{\xi'}^{\kappa'}\lambda^{-1}y_{AB}| &\leq C_{\kappa'}(|\lambda||\xi'|^{-4-|\kappa'|} + |\lambda|^2|\xi'|^{-6-|\kappa'|} + |\lambda|^3|\xi'|^{-8-|\kappa'|}) \\
&\leq C_{\kappa'}(r + r^2 + r^3)|\xi'|^{-2-|\kappa'|},
\end{aligned}
$$

which shows (59). Since

$$\begin{aligned}\partial_\lambda(\lambda^{-1}y_{AB}) = \frac{\gamma_A\gamma_B}{4}|\xi'|^{-4} &+ \gamma_A^2((\partial_\lambda z_A)\lambda + z_A)|\xi'|^{-4} + \gamma_B^2((\partial_\lambda z_B)\lambda + z_B)|\xi'|^{-4} \\ &+ \frac{\gamma_B\gamma_A^2}{2}((\partial_\lambda z_A)\lambda^2 + 2z_A\lambda)|\xi'|^{-6} + \frac{\gamma_A\gamma_B^2}{2}((\partial_\lambda z_B)\lambda^2 + 2z_B\lambda)|\xi'|^{-6} \\ &+ \gamma_A^2\gamma_B^2((\partial_\lambda z_A)z_B\lambda^3 + z_A(\partial_\lambda z_B)\lambda^3 + 3z_Az_B\lambda^2)|\xi'|^{-8}.\end{aligned}$$

Thus, by Leibniz's rule, (57) and (58), we have

$$\begin{aligned}|D_{\xi'}^{\kappa'}(\lambda^{-1}y_{AB})| &\leq C_{\kappa'}(|\xi'|^{-4-|\kappa'|} + |\lambda||\xi'|^{-6-|\kappa'|} + |\lambda|^2|\xi'|^{-8-|\kappa'|} + |\lambda|^3|\xi'|^{-10-|\kappa'|}) \\ &\leq C_{\kappa'}|\xi'|^{-4-|\kappa'|}.\end{aligned}$$

This shows (60).

Since $(B^2 + |\xi'|^2)^2 = (\alpha^{-1}\lambda + 2|\xi'|^2)^2 = \alpha^{-2}\lambda^2 + 4\gamma_B\lambda|\xi'|^2 + 4|\xi'|^4$, we have

$$\mathcal{L} = 4AB|\xi'|^2 - (B^2 + |\xi'|^2)^2 = 2c_0\lambda|\xi'|^2 - \alpha^{-2}\lambda^2 + 4|\xi'|^4 y_{AB}$$

with $c_0 = \gamma_A - \gamma_B = (\alpha+\beta)^{-1} - \alpha^{-1}$. Thus, we have

$$M = \frac{\mathcal{L}}{\lambda} = 2c_0|\xi'|^2 - \alpha^{-2}\lambda + 4|\xi'|^4\lambda^{-1}y_{AB}. \tag{62}$$

Thus, by (59) and Leibniz's rule, we have

$$|D_{\xi'}^{\kappa'}M| \leq C_{\kappa'}|\xi'|^{2-|\kappa'|}. \tag{63}$$

In particular,

$$|M| \geq 2|c_0||\xi'|^2 - C(|\lambda| + |\lambda|^2|\xi'|^{-2} + |\lambda|^3|\xi'|^{-4}) \geq 2|c_0| - C|\xi'|^2(r + r^2 + r^3).$$

Thus, choosing $r > 0$ so small that $C'(r + r^2 + r^3) \leq |c_0|$, we have

$$|M| \geq |c_0||\xi'|^2. \tag{64}$$

By (63), (64) and Bell's formula, we have

$$|D_{\xi'}^{\kappa'}M^{-1}| \leq C_{\kappa'}\sum_{\ell=1}^{|\kappa'|}|M|^{-\ell-1}\sum_{\kappa_1'+\cdots+\kappa_\ell'=\kappa'}|D_{\xi'}^{\kappa_1'}M|\cdots|D_{\xi'}^{\kappa_\ell'}M| \leq C_{\kappa'}|\xi'|^{-2-|\kappa'|}. \tag{65}$$

We now consider $\partial_\lambda M^{-1} = -M^{-2}\partial_\lambda M$. From (62) it follows that

$$\partial_\lambda M = -\alpha^{-2} + 4|\xi'|^4\partial_\lambda(\lambda^{-1}y_{AB}).$$

Thus, by (60), we have

$$|D_{\xi'}^{\kappa'}(\partial_\lambda M)| \leq C|\xi'|^{-|\kappa'|}. \tag{66}$$

Since $\partial_\lambda M^{-1} = -M^{-2}\partial_\lambda M$, by Leibniz's rule, (65), and (66), we have

$$|D^{\kappa'}_{\xi'}\partial_\lambda M^{-1}| \leq C_{\kappa'}|\xi|^{-4-|\kappa'|}. \tag{67}$$

Noting that $|\xi'| \geq (1/2)(|\xi'| + r^{-1}|\lambda|^{1/2})$, by (65) and (67), we have

$$|D^{\kappa'}_{\xi'}M^{-1}| \leq C(|\lambda|^{1/2} + |\xi'|)^{-2-|\kappa'|},$$
$$|D^{\kappa'}_{\xi'}(\partial_\lambda M^{-1})| \leq C(|\lambda|^{1/2} + |\xi'|)^{-4-|\kappa'|}.$$

Next, we consider the case where $|\xi'|^2/|\lambda| \leq r$ and $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$. In this case, we have

$$A = ((\alpha+\beta)^{-1}\lambda)^{1/2}(1 + |\xi'|^2(\alpha+\beta)\lambda^{-1})^{1/2},$$
$$B = (\alpha^{-1}\lambda)^{1/2}(1 + |\xi'|^2\alpha\lambda^{-1})^{1/2},$$

and so

$$\mathcal{L} = 4(\alpha+\beta)^{-1/2}\alpha^{-1/2}\lambda|\xi'|^2(1 + O(|\xi'|^2/\lambda)) - \alpha^{-2}\lambda^2 - 4\alpha^{-1}\lambda|\xi'|^2 - 4|\xi'|^4$$

This implies that

$$|\mathcal{L}| \geq \alpha^{-2}|\lambda|^2 - Cr|\lambda|^2$$

for some constant C. Choosing $r > 0$ so small that $Cr \leq \alpha^{-2}/2$, we have

$$|\mathcal{L}| \geq (\alpha^{-2}/2)|\lambda|^2.$$

Thus, we have

$$|M| \geq (\alpha^{-2}/2)|\lambda| \geq c_1(|\lambda|^{1/2} + |\xi'|)^2$$

with some positive number c_1, where we have used the fact that $|\lambda|^{1/2} \geq (1/2)(|\lambda|^{1/2} + r^{-1/2}|\xi'|)$. We know that

$$|D^{\kappa'}|\xi'|^2| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{2-|\kappa'|}.$$

Moreover, for $E \in \{A, B\}$ we know that

$$|D^{\kappa'}_{\xi'}E| \leq C(|\lambda|^{1/2} + |\xi'|)^{1-|\kappa'|},$$
$$|D^{\kappa'}_{\xi'}(\partial_\lambda E)| \leq C(|\lambda|^{1/2} + |\xi'|)^{-1-|\kappa'|}.$$

Thus, we have

$$|D^{\kappa'}_{\xi'}\mathcal{L}| \leq C(|\lambda|^{1/2} + |\xi'|)^{4-|\kappa'|}.$$

Recalling that $M = \mathcal{L}/\lambda$, we have

$$|D^{\kappa'}_{\xi'}M| \leq C(|\lambda|^{1/2} + |\xi'|)^{2-|\kappa'|}.$$

By Bell's formula,

$$|D_{\xi'}^{\kappa'} M^{-1}| \leq C_{\kappa'} \sum_{\ell=1}^{|\kappa'|} (|\lambda|^{1/2} + |\xi'|)^{-2(\ell+1)} (|\lambda|^{1/2} + |\xi'|)^{2\ell - |\kappa'|}.$$

Namely, we have

$$|D_{\xi'}^{\kappa'} M^{-1}| \leq C_{\kappa'} (|\lambda|^{1/2} + |\xi'|)^{-2-|\kappa'|}. \tag{68}$$

We observe that $\partial_\lambda M^{-1} = -M^{-2}(\partial_\lambda M)$ and $\partial_\lambda M = \lambda^{-1}\partial_\lambda \mathcal{L} - \lambda^{-2}\mathcal{L}$. Since

$$\partial_\lambda \mathcal{L} = 4((\partial_\lambda A)B + A(\partial_\lambda B))|\xi'|^2 - 2(\alpha^{-1}\lambda + 2|\xi'|^2)\alpha^{-1},$$

we have

$$|D_{\xi'}^{\kappa'} \partial_\lambda \mathcal{L}| \leq C_{\kappa'} (|\lambda|^{1/2} + |\xi|)^{2-|\kappa'|},$$

and so

$$|D_{\xi'}^{\kappa'} (\partial_\lambda M)| \leq C_{\kappa'} (|\lambda|^{1/2} + |\xi'|)^{-|\kappa'|}.$$

Therefore, we have

$$|D_{\xi'}^{\kappa'} (\partial_\lambda M^{-1})| \leq C_{\kappa'} (|\lambda|^{1/2} + |\xi'|)^{-4-|\kappa'|}. \tag{69}$$

Finally, we consider the case where

$$r|\lambda| \leq |\xi'|^2 \leq r^{-1}|\lambda|, \quad (\lambda, \xi') \in \Sigma_{\epsilon,\lambda_0} \times \mathbb{R}^{N-1}. \tag{70}$$

First, we shall show that

$$\mathcal{L} \neq 0 \quad \text{for any } (\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\}). \tag{71}$$

We know that the uniqueness of solutions of the forms $\hat{u}_J = a_J e^{-Bx_N} + b_J(e^{-Bx_N} - e^{-Ax_N})$ to equations (41) guarantees $\mathcal{L} \neq 0$. Thus, we shall show the uniqueness. Let us write $\hat{u}_J = v_J$ for the notational simplicity and we assume that v_J satisfy equations:

$$\begin{aligned}
&(\lambda + \alpha|i\xi'|^2)v_j - \alpha\partial_N^2 v_j - \beta i\xi_j(i\xi' \cdot \mathbf{v}' + \partial_N v_N) = 0 \qquad (x_N > 0),\\
&(\lambda + \alpha|i\xi'|^2)v_N - \alpha\partial_N^2 v_N - \beta\partial_N(i\xi' \cdot \mathbf{v}' + \partial_N v_N) = 0 \quad (x_N > 0),\\
&(\partial_N v_j + i\xi_j v_N)|_{x_N=0} = 0,\\
&(\alpha(\partial_N v_N - i\xi' \cdot \mathbf{v}') + \beta(i\xi' \cdot \mathbf{v}' + \partial_N v_N))|_{x_N=0} = 0.
\end{aligned}$$

Here, $\mathbf{v}' = (v_1, \ldots, v_{N-1})$. Notice that

$$\begin{aligned}
0 &= \lambda v_j + \alpha|\xi'|^2 v_j - \alpha\partial_N^2 v_j - \beta i\xi_j(i\xi' \cdot \mathbf{v}' + \partial_N v_N)\\
&= \lambda v_j - \alpha\sum_{k=1}^{N-1} i\xi_k(i\xi_k v_j + i\xi_j v_k) - \alpha\partial_N(\partial_N v_j + i\xi_j v_N)\\
&\qquad\qquad - (\beta - \alpha)i\xi_j(i\xi' \cdot \mathbf{v}' + \partial_N v_N),
\end{aligned} \tag{72}$$

$$
\begin{aligned}
0 &= \lambda v_N + \alpha|\xi'|^2 v_N - \alpha\partial_N^2 v_N - \beta\partial_N(i\xi'\cdot\mathbf{v}' + \partial_N v_N)\\
&= \lambda v_N - \alpha\sum_{k=1}^{N-1} i\xi_k(i\xi_k v_N + \partial_N v_k) - 2\alpha\partial_N^2 v_N - (\beta-\alpha)\partial_N(i\xi'\cdot\mathbf{v}' + \partial_N v_N).
\end{aligned}
\tag{73}
$$

Since v_j decay exponentially as $x_N \to \infty$, multiplying (72) with $\overline{v_j}$ and (73) with $\overline{v_N}$, and using integration by parts

$$
\begin{aligned}
0 =&\lambda\|\mathbf{v}\|^2 + \alpha\sum_{j,k=1}^{N-1}\|i\xi_k v_j\|^2 + \alpha\sum_{j=1}^{N-1}\|\partial_N v_j\|^2 + \alpha\|i\xi'\cdot\mathbf{v}'\|^2\\
&+ \alpha\sum_{j=1}^{N-1}(i\xi_j v_N, \partial_N v_j) + \alpha\sum_{j=1}^{N-1}\|i\xi_j v_N\|^2 + \alpha\sum_{j=1}^{N-1}(\partial_N v_j, i\xi_j v_N) + 2\alpha\|\partial_N v_N\|^2\\
&+ (\beta-\alpha)(\|i\xi'\cdot\mathbf{v}'\|^2 + (\partial_N v_N, i\xi'\cdot\mathbf{v}') + (i\xi'\cdot\mathbf{v}', \partial_N v_N) + \|\partial_N v_N\|^2).
\end{aligned}
\tag{74}
$$

Here $(f,g) = \int_{\mathbb{R}_+} f(x_N)\overline{g(x_N)}\,dx_N$ and $\|f\|^2 = (f,f)$. Taking the real part and the imaginary part of (74) yields

$$
(\operatorname{Im}\lambda)\|\mathbf{v}\|^2 = 0, \tag{75}
$$

$$
\begin{aligned}
&(\operatorname{Re}\lambda)\|\mathbf{v}\|^2 + \alpha\sum_{j,k=1}^{N-1}\|i\xi_k v_j\|^2\\
&\quad + \alpha\sum_{j=1}^{N-1}(\|i\xi_j v_N\|^2 + \|\partial_N v_j\|^2 + (i\xi_j v_N, \partial_N v_j) + (\partial_N v_j, i\xi_j v_N))\\
&\quad + \alpha\|i\xi'\cdot\mathbf{v}'\|^2 + 2\alpha\|\partial_N v_N\|^2 + \beta(\|i\xi'\cdot\mathbf{v}'\|^2 + \|\partial_N v_N\|^2 + (\partial_N v_N, i\xi'\cdot\mathbf{v}')\\
&\quad + (i\xi'\cdot\mathbf{v}', \partial_N v_N)) - \alpha(\|i\xi'\cdot\mathbf{v}'\|^2 + (\partial_N v_N, i\xi'\cdot\mathbf{v}')\\
&\qquad\qquad + (i\xi'\cdot\mathbf{v}', \partial_N v_N) + \|\partial_N v_N\|^2) = 0.
\end{aligned}
\tag{76}
$$

If $\operatorname{Im}\lambda \neq 0$, by (75), we have $\mathbf{v} = 0$. If $\operatorname{Im}\lambda = 0$, then $\operatorname{Re}\lambda > 0$. Thus, by (76), we have

$$
\begin{aligned}
0 \geq &\operatorname{Re}\lambda\|\mathbf{v}\|^2 + \alpha\sum_{j,k=1}^{N-1}\|i\xi_k v_j\|^2 + \alpha\sum_{j=1}^{N-1}(\|\partial_N v_j\|^2 - 2\|i\xi_j v_N\|\|\partial_N v_j\| + \|i\xi_j v_N\|^2)\\
&+ \alpha\|i\xi'\cdot\mathbf{v}'\|^2 + 2\alpha\|\partial_N v_N\|^2 - \alpha\|i\xi'\cdot\mathbf{v}'\|^2 - \alpha\|\partial_N v_N\|^2 - 2\alpha\|\partial_N v_N\|\|i\xi'\cdot\mathbf{v}'\|\\
&+ \beta(\|i\xi'\cdot\mathbf{v}'\|^2 + \|\partial_N v_N\|^2 - 2\|\partial_N v_N\|\|i\xi'\cdot\mathbf{v}'\|).
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
&\alpha(\|\partial_N v_j\|^2 - 2\|i\xi_j v_N\|\|\partial_N v_j\| + \|i\xi_j v_N\|^2) \geq \alpha(\|\partial_N v_j\| - \|i\xi_j v_N\|)^2 \geq 0,\\
&\beta(\|i\xi'\cdot\mathbf{v}'\|^2 + \|\partial_N v_N\|^2 - 2\|\partial_N v_N\|\|i\xi'\cdot\mathbf{v}'\| \geq \beta(\|i\xi'\cdot\mathbf{v}'\| - \|\partial_N v_N\|)^2 \geq 0.
\end{aligned}
$$

Moreover, noting that $\sum_{j,k=1}^{N-1}\|i\xi_k v_j\|^2 = |\xi'|^2\|\mathbf{v}'\|^2$ and $\|i\xi'\cdot\mathbf{v}'\| \leq |\xi'|\|\mathbf{v}'\|$, we have

$$\begin{aligned}
\alpha \sum_{j,k=1}^{N-1} \|i\xi_k v_j\|^2 + \alpha \|i\xi' \cdot \mathbf{v}'\|^2 + 2\alpha \|\partial_N v_N\|^2 \\
- \alpha \|i\xi' \cdot \mathbf{v}'\|^2 - \alpha \|\partial_N v_N\|^2 - 2\alpha \|\partial_N v_N\| \|i\xi' \cdot \mathbf{v}'\| \\
\geq \alpha (|\xi'|^2 \|\mathbf{v}'\|^2 + \|\partial_N v_N\|^2 - 2\|\partial_N v_N\| |\xi'| \|\mathbf{v}'\|) \\
\geq \alpha (|\xi'| \|\mathbf{v}'\| - \|\partial_N v_N\|)^2 \geq 0.
\end{aligned}$$

Thus, $0 \geq \operatorname{Re} \lambda \|\mathbf{v}\|^2$, which yields $v = 0$. In particular, we have proved (71).

Let

$$\begin{aligned}
&\tilde{\lambda} = \frac{\lambda}{(|\lambda|^{1/2} + |\xi'|)^2}, \quad \tilde{\xi}' = \frac{\xi'}{|\lambda|^{1/2} + |\xi'|}, \\
&\tilde{A} = ((\alpha + \beta)^{-1} \tilde{\lambda} + |\tilde{\xi}'|^2)^{1/2}, \quad \tilde{B} = (\alpha^{-1} \tilde{\lambda} + |\tilde{\xi}'|^2)^{1/2}, \\
&\tilde{\mathcal{L}} = 4\tilde{A}\tilde{B}|\tilde{\xi}'|^2 - (\tilde{B}^2 + |\tilde{\xi}'|^2)^2.
\end{aligned}$$

We have $\mathcal{L} = (|\lambda|^{1/2} + |\xi'|)^4 \tilde{\mathcal{L}}$. Notice that $\tilde{\lambda} \in \Sigma_\epsilon$ provided that $\lambda \in \Sigma_\epsilon$. Moreover, from (70) it follows that

$$\frac{r^{1/2}}{1 + r^{1/2}} \leq |\tilde{\xi}| \leq \frac{1}{1 + r^{1/2}}, \quad \frac{r}{(1 + r^{1/2})^2} \leq |\tilde{\lambda}| \leq \frac{1}{(1 + r^{1/2})^2}. \tag{77}$$

Thus, we set

$$U = \{(\tilde{\lambda}, \tilde{\xi}') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\}) \mid (77) \text{ holds}\}.$$

Since U is a compact set and $\tilde{L} \neq 0$ for $(\tilde{\lambda}, \tilde{\xi}') \in U$, there exists a constant $c_2 > 0$ such that

$$\inf_{(\tilde{\lambda}, \tilde{\xi}') \in U} |\tilde{\mathcal{L}}| = c_2 > 0.$$

Thus, we have

$$|\mathcal{L}| \geq c_2 (|\lambda|^{1/2} + |\xi'|)^4$$

for $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$ satisfying the condition (71). Employing the same argument as in the proof of (68) and (69), we have

$$\begin{aligned}
|D_{\xi'}^{\kappa'} M^{-1}| &\leq C_{\kappa'} (|\lambda|^{1/2} + |\xi'|)^{-2-|\kappa'|}, \\
|D_{\xi'}^{\kappa'} (\partial_\lambda M^{-1})| &\leq C_{\kappa'} (|\lambda|^{1/2} + |\xi'|)^{-4-|\kappa'|}.
\end{aligned}$$

Thus, we have proved $M^{-1} \in \mathbf{M}_{-2}$ and $\partial_\lambda M^{-1} \in \mathbf{M}_{-4}$. Namely, $M^{-1} \in \mathbf{N}_{-2}$. □

Lemma 18. *Let $m_1 = -3B^2 - |\xi'|^2 + 4AB$, $m_2 = 2AB - B^2 - |\xi'|^2$ and $m_3 = B - A$ be symbols appearing in the solution formula. Then, $m_i/\lambda \in \mathbf{N}_0$ for $i = 1, 2$ and $m_3/\lambda \in \mathbf{N}_{-1}$.*

Proof. To prove the lemma, we use the symbols given in the proof of Lemma 17. First, we consider the case where $|\lambda|/|\xi'|^2 \leq r$ and $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$ with some small $r > 0$. Using (55), we write $m_1/\lambda = (2\gamma_A - \gamma_B) + 4|\xi'|^2\lambda^{-1}y_{AB}$. By (59), (60) and Leibniz's rule, we have

$$|D_{\xi'}^{\kappa'}(m_1/\lambda)| \leq C_{\kappa'}|\xi'|^{-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda(m_1/\lambda)| \leq C_{\kappa'}|\xi'|^{-2-|\kappa'|}.$$

Analogously, using (55) we write $m_2/\lambda = \gamma_A + 2|\xi'|^2\lambda^{-1}y_{AB}$. By (59), (60) and Leibniz's rule, we have

$$|D_{\xi'}^{\kappa'}(m_2/\lambda)| \leq C_{\kappa'}|\xi'|^{-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda(m_2/\lambda)| \leq C_{\kappa'}|\xi'|^{-2-|\kappa'|}.$$

Using (54), we have $m_3/\lambda = (\gamma_B - \gamma_A)|\xi'|^{-1}/2 + (\gamma_B^2 z_B - \gamma_A^2 z_A)\lambda|\xi'|^{-3}$. Since

$$|D_{\xi'}^{\kappa'}(\lambda|\xi'|^{-3})| \leq C_{\kappa'}|\lambda||\xi'|^{-3-|\kappa'|} \leq C_{\kappa'}r|\xi'|^{-1-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda(\lambda|\xi'|^{-3})| \leq C_{\kappa'}|\xi'|^{-3-|\kappa'|}.$$

Using this, (57), (58) and Leibniz's rule, we have

$$|D_{\xi'}^{\kappa'}(m_3/\lambda)| \leq C_{\kappa'}|\xi'|^{-1-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda(m_3/\lambda)| \leq C_{\kappa'}|\xi'|^{-3-|\kappa'|}.$$

We now consider the case where $|\lambda|/|\xi'|^2 \geq r$ and $(\lambda, \xi') \in \Sigma_\epsilon \times (\mathbb{R}^{N-1} \setminus \{0\})$. In view of (15), we see easily that

$$|D_{\xi'}^{\kappa'}E| \leq C_{\kappa'}(|\lambda^{1/2} + |\xi'|)^{1-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda E| \leq C_{\kappa'}(|\lambda^{1/2} + |\xi'|)^{-1-|\kappa'|}$$

for $E \in \{A, B\}$. Noticing that $|\lambda|^{1/2} \geq (1/2)(|\lambda|^{1/2} + r^{1/2}|\xi'|)$, we see easily that

$$|D_{\xi'}^{\kappa'}(m_i/\lambda)| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda(m_i/\lambda)| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{-2-|\kappa'|}$$

for $i = 1, 2$. And

$$|D_{\xi'}^{\kappa'}(m_3/\lambda)| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{-1-|\kappa'|},$$
$$|D_{\xi'}^{\kappa'}\partial_\lambda(m_3/\lambda)| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{-3-|\kappa'|}.$$

Combining these results, we have proved that $m_i/\lambda \in \mathbf{N}_0$ $(i = 1, 2)$ and $m_3/\lambda \in \mathbf{N}_{-1}$ This completes the proof. □

4.3. Estimates of solution formulas in the half-space

Let $\tilde{H} = (H_2, H_3)$ be an $N + N^2$ vertical vector such that H_2 and H_3 correspond to $\lambda^{1/2}\mathbf{h} = (\lambda^{1/2}h_1, \ldots, \lambda^{1/2}h_N)^\top$ and $\nabla\mathbf{h} = (\partial_i h_j \mid i, j = 1, \ldots, N)^\top$, respectively. Here, $\mathbf{k}^\top$ denotes the transposed vector of $\mathbf{k} = (k_1, \ldots, k_N)$. In the solution formulas (52) and (53), writing

$$\frac{-3B^2 - |\xi'|^2 + 4AB}{\mathcal{L}} = \frac{-3B^2 - |\xi'|^2 + 4AB}{\lambda M}, \quad \frac{2AB - B^2 - |\xi'|^2}{\mathcal{L}} = \frac{2AB - B^2 - |\xi'|^2}{\lambda M},$$
$$\frac{B^2 - |\xi'|^2}{\mathcal{L}} = \frac{1}{\alpha M}, \qquad \frac{B - A}{\mathcal{L}} = \frac{B - A}{\lambda M},$$

by Lemmas 17 and 18, we see that the first three terms belong to $\mathbf{N}_{-2}$ and the last term belongs to $\mathbf{N}_{-3}$. Thus, all the symbols appearing in the solutions formulas (52) and (53) belong to $\mathbf{N}_{-2}$. Using this fact, we see that there exist $N \times N$ matrices $\mathcal{M}_1(\lambda, \xi')$ and $\mathcal{M}_2(\lambda, \xi')$ of $\mathbf{N}_{-2}$ symbols such that setting

$$\mathcal{T}(\lambda)\tilde{H} = \int_0^\infty \mathcal{F}^{-1}_{\xi'}[Be^{-B(x_N+y_N)}\mathcal{M}_1(\lambda, \xi')\mathcal{F}'[\tilde{H}](\xi', y_N)](x')\,dy_N$$
$$+ \int_0^\infty \mathcal{F}^{-1}_{\xi'}[B^2 M(x_N + y_N)\mathcal{M}_2(\lambda, \xi')\mathcal{F}'[\tilde{H}](\xi', y_N)](x')\,dy_N$$

we have $\mathbf{u} = \mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$ is a solution of equations (40). Here, $\mathcal{T}(\lambda)\tilde{H}$ and $\tilde{H}$ are both vertical vectors. Moreover, $\mathcal{T}(\lambda)$ has the following properties.

THEOREM 4.2. *Let $1 < q < \infty$, $-1 + 1/q < s < 1/q$, $0 < \epsilon < \pi/2$, $\lambda_0 > 0$ and let σ be a small positive number such that $-1 + 1/q < s - \sigma < s < s + \sigma < 1/q$. Then, $\mathcal{T}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(B^s_{q,1}(\mathbb{R}^N_+)^{N+N^2}, B^{s+2}_{q,1}(\mathbb{R}^N_+)^N))$ has (s, σ, q) properties.*

PROOF. To prove the theorem, we shall use Theorem 2.1 in Subsection 2.6. Notice that $\|f\|_{W^1_q(\mathbb{R}^N_+)} = \|\bar{\nabla} f\|_{L_q(\mathbb{R}^N_+)}$ and $\|f\|_{W^2_q(\mathbb{R}^N_+)} = \|\bar{\nabla}^2 f\|_{L_q(\mathbb{R}^N_+)}$. In what follows, we may assume that $\tilde{H} \in C^\infty_0(\mathbb{R}^N_+)^{N+N^2}$. In fact, $C^\infty_0(\mathbb{R}^N_+)$ is dense in $B^s_{q,1}(\mathbb{R}^N_+)$ for $1 < q < \infty$ and $-1 + 1/q < s < 1/q$ (cf. Proposition 2.24, Lemma 2.32, and Corollaries 2.26 and 2.34 in [**17**]). Using the formulas:

$$\partial^\ell_N M(x_N) = (-1)^\ell(A^\ell M(x_N) + \frac{A^\ell - B^\ell}{A - B}e^{-Bx_N}) \quad (\ell \geq 1),$$

and setting

$$\mathcal{M}^{(0)}_1(\lambda) = \mathcal{M}_1(\lambda, \xi'), \quad \mathcal{M}^{(\ell)}_1(\lambda) = (-B)^\ell\mathcal{M}_1(\lambda, \xi') + (-1)^\ell\frac{A^\ell - B^\ell}{A - B}B\mathcal{M}_2(\lambda, \xi') \quad (\ell \geq 1),$$
$$\mathcal{M}^{(0)}_2(\lambda) = \mathcal{M}_2(\lambda, \xi'), \quad \mathcal{M}^{(\ell)}_2(\lambda) = (-1)^\ell A^\ell\mathcal{M}_2(\lambda, \xi') \quad (\ell \geq 1)$$

for the notational simplicity, we write

$$\partial^\ell_N\mathcal{T}(\lambda)\tilde{H} = \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\mathcal{M}^{(\ell)}_1(\lambda)\mathcal{F}'[\tilde{H}](\xi', y_N)Be^{-B(x_N+y_N)}$$

$$+ \mathcal{M}_2^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N. \tag{78}$$

Using these symbols, we can write

$$\begin{aligned}\lambda^k\partial_{x'}^{\kappa'}\partial_N^\ell\mathcal{T}(\lambda)\tilde H = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[&\lambda^k(i\xi')^{\kappa'}\mathcal{M}_1^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)Be^{-B(x_N+y_N)} \\ &+\lambda^k(i\xi')^{\kappa'}\mathcal{M}_2^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N.\end{aligned}$$

If $2k+|\kappa'|+\ell \leq 2$, then $\lambda^k(i\xi')^{\kappa'}\mathcal{M}_1^{(\ell)}(\lambda) \in \mathbf{M}_0$ and $\lambda^k(i\xi')^{\kappa'}\mathcal{M}_2^{(\ell)}(\lambda) \in \mathbf{M}_0$. Thus, by Proposition 9 we have

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)\tilde H\|_{L_q(\mathbb{R}^N_+)} \leq C\|\tilde H\|_{L_q(\mathbb{R}^N_+)}. \tag{79}$$

To obtain the $W^1_q(\mathbb{R}^N_+)$ estimate, noting that $\tilde H \in C^\infty_0(\mathbb{R}^N_+)^{N+N^2}$, using the formulas:

$$\begin{aligned}&\partial_N(-B)^{-1}e^{-B(x_N+y_N)} = e^{-B(x_N+y_N)}, \\ &\partial_N((AB)^{-1}e^{-B(x_N+y_N)} - A^{-1}M(x_N+y_N)) = M(x_N+y_N)\end{aligned}$$

and setting

$$\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda) = (B^{-1}\mathcal{M}_1^{(\ell)}(\lambda) - A^{-1}\mathcal{M}_2^{(\ell)}(\lambda)), \quad \tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda) = A^{-1}\mathcal{M}_2^{(\ell)}(\lambda),$$

by integration by parts, we have

$$\begin{aligned}\partial_N^\ell\mathcal{T}(\lambda)\tilde H = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[&\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)Be^{-B(x_N+y_N)} \\ &+ \tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N.\end{aligned} \tag{80}$$

Thus, we have

$$\begin{aligned}\lambda^k\partial_{x'}^{\kappa'}\partial_N^\ell\mathcal{T}(\lambda)\tilde H = \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[&\lambda^k(i\xi')^{\kappa'}\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)Be^{-B(x_N+y_N)} \\ &+ \lambda^k(i\xi')^{\kappa'}\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N.\end{aligned}$$

If $2k+|\kappa'|+\ell \leq 3$, both $\lambda^k(i\xi')^{\kappa'}\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda)$ and $\lambda^k(i\xi')^{\kappa'}\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda)$ belong to $\mathbf{M}_0$, and so by Proposition 9, we have

$$\begin{aligned}&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)\tilde H\|_{W^1_q(\mathbb{R}^N_+)} \leq C\|\tilde H\|_{W^1_q(\mathbb{R}^N_+)}, \\ &\|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)\tilde H\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|\tilde H\|_{W^1_q(\mathbb{R}^N_+)}.\end{aligned} \tag{81}$$

We next consider $\mathcal{T}(\lambda)^*$, which is defined by exchanging $\mathcal{F}'$ and $\mathcal{F}_{\xi'}^{-1}$ in the formula of $\mathcal{T}(\lambda)$. Namely,

$$\mathcal{T}(\lambda)^*\tilde H = \int_0^\infty \mathcal{F}'\Big[\mathcal{M}_1^{(0)}(\lambda)\mathcal{F}_{\xi'}^{-1}[\tilde H](\xi', y_N)Be^{-B(x_N+y_N)}$$

$$+ \mathcal{M}_2^{(0)}(\lambda)\mathcal{F}_{\xi'}^{-1}[\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N.$$

Obvisouly, for $\tilde H, \tilde G \in C_0^\infty(\mathbb{R}^N_+)^{N+N^2}$, we see that

$$((\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)\tilde H, \tilde G) = (\tilde H, (\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}^*(\lambda)\tilde G).$$

In particular, $(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)^* = ((\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2\mathcal{T}(\lambda))^*$.

Employing the same argument as in the proof of (79) and (81), we have

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)^*\tilde H\|_{L_{q'}(\mathbb{R}^N_+)} &\le C\|\tilde H\|_{L_{q'}(\mathbb{R}^N_+)},\\
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)^*\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)} &\le C\|\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)},\\
\|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)^*\tilde H\|_{L_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1/2}\|\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)}.
\end{aligned} \tag{82}$$

In view of (79), (81) and (82), the assertion (1) of Theorem 2.1 implies that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$, there hold

$$\begin{aligned}
&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)\tilde H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \le C\|\tilde H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \quad (\nu \in \{s-\sigma, s, s+\sigma\}),\\
&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{T}(\lambda)\tilde H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-\frac{\sigma}{2}}\|\tilde H\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)}.
\end{aligned}$$

We now consider $\partial_\lambda\mathcal{T}(\lambda)\tilde H$. From (78) it follows that

$$\begin{aligned}
\partial_N^\ell\partial_\lambda\mathcal{T}(\lambda)\tilde H &= \partial_\lambda\partial_N^\ell\mathcal{T}(\lambda)\tilde H\\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[(\partial_\lambda\mathcal{M}_1^{(\ell)}(\lambda))\mathcal{F}'[\tilde H](\xi', y_N)Be^{-B(x_N+y_N)}\\
&\qquad + B^{-2}\mathcal{M}_1^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)B^2\partial_\lambda(Be^{-B(x_N+y_N)})\\
&\qquad + (\partial_\lambda\mathcal{M}_2^{(\ell)}(\lambda))\mathcal{F}'[\tilde H](\xi', y_N)B^2M(x_N+y_N)\\
&\qquad + B^{-2}\mathcal{M}_2^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)B^2\partial_\lambda(B^2M(x_N+y_N))\Big](x')\,dy_N.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\lambda^k\partial_{x'}^{\kappa'}\partial_N^\ell\partial_\lambda\mathcal{T}(\lambda)\tilde H\\
&\quad= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[\lambda^k(i\xi')^{\kappa'}(\partial_\lambda\mathcal{M}_1^{(\ell)}(\lambda))\mathcal{F}'[\tilde H](\xi', y_N)Be^{-B(x_N+y_N)}\\
&\quad + \lambda^k(i\xi')^{\kappa'}B^{-2}\mathcal{M}_1^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)B^2\partial_\lambda(Be^{-B(x_N+y_N)})\\
&\quad + \lambda^k(i\xi')^{\kappa'}(\partial_\lambda\mathcal{M}_2^{(\ell)}(\lambda))\mathcal{F}'[\tilde H](\xi', y_N)B^2M(x_N+y_N)\\
&\quad + \lambda^k(i\xi')^{\kappa'}B^{-2}\mathcal{M}_2^{(\ell)}(\lambda)\mathcal{F}'[\tilde H](\xi', y_N)B^2\partial_\lambda(B^2M(x_N+y_N))\Big](x')\,dy_N.
\end{aligned}$$

If $2k + |\kappa'| + \ell \le 4$, the following symbols:

$$\begin{aligned}
&\lambda^k(i\xi')^{\kappa'}(\partial_\lambda\mathcal{M}_1^{(\ell)}(\lambda)), \quad \lambda^k(i\xi')^{\kappa'}B^{-2}\mathcal{M}_1^{(\ell)}(\lambda),\\
&\lambda^k(i\xi')^{\kappa'}(\partial_\lambda\mathcal{M}_2^{(\ell)}(\lambda)), \quad \lambda^k(i\xi')^{\kappa'}B^{-2}\mathcal{M}_2^{(\ell)}(\lambda)
\end{aligned}$$

belong to $\mathbf{M}_0$, and so by Proposition 9 we have

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda \mathcal{T}(\lambda)\tilde H\|_{L_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\tilde H\|_{L_q(\mathbb{R}^N_+)}. \tag{83}$$

Moreover, from (80) it follows that

$$\begin{aligned}
\partial_N^\ell\partial_\lambda\mathcal{T}(\lambda)\tilde H &= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[(\partial_\lambda\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda))\mathcal{F}'[\partial_N\tilde H](\xi', y_N)Be^{-B(x_N+y_N)}\\
&+ B^{-2}\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2\partial_\lambda(Be^{-B(x_N+y_N)})\\
&+ (\partial_\lambda\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda))\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N\\
&+ B^{-2}\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2\partial_\lambda(B^2M(x_N+y_N))\Big](x')\,dy_N.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\lambda^k\partial_{x'}^{\kappa'}\partial_N^\ell\partial_\lambda\mathcal{T}(\lambda)\tilde H\\
&= \int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\lambda^k(i\xi')^{\kappa'}(\partial_\lambda\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda))\mathcal{F}'[\partial_N\tilde H](\xi', y_N)Be^{-B(x_N+y_N)}\\
&+ \lambda^k(i\xi')^{\kappa'}B^{-2}\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2\partial_\lambda(Be^{-B(x_N+y_N)})\\
&+ \lambda^k(i\xi')^{\kappa'}(\partial_\lambda\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda))\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2M(x_N+y_N)\Big](x')\,dy_N\\
&+ \lambda^k(i\xi')^{\kappa'}B^{-2}\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda)\mathcal{F}'[\partial_N\tilde H](\xi', y_N)B^2\partial_\lambda(B^2M(x_N+y_N))\Big](x')\,dy_N.
\end{aligned}$$

If $2k+|\kappa'|+\ell \leq 5$, the following symbols:

$$\begin{aligned}
&\lambda^k(i\xi')^{\kappa'}(\partial_\lambda\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda)), \quad \lambda^k(i\xi')^{\kappa'}B^{-2}\tilde{\mathcal{M}}_1^{(\ell-1)}(\lambda),\\
&\lambda^k(i\xi')^{\kappa'}(\partial_\lambda\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda)), \quad \lambda^k(i\xi')^{\kappa'}B^{-2}\tilde{\mathcal{M}}_2^{(\ell-1)}(\lambda))
\end{aligned}$$

all belong to $\mathbf{M}_0$, and so by Proposition 9 we have

$$\begin{aligned}
&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)\tilde H\|_{W^1_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|\tilde H\|_{W^1_q(\mathbb{R}^N_+)},\\
&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)\tilde H\|_{W^1_q(\mathbb{R}^N_+)} \leq C|\lambda|^{-1/2}\|\tilde H\|_{L_q(\mathbb{R}^N_+)}.
\end{aligned} \tag{84}$$

Since $\partial_\lambda\mathcal{T}_1(\lambda)^*$ is obtained by exchanging $\mathcal{F}^{-1}_{\xi'}$ and $\mathcal{F}'$, that is

$$\begin{aligned}
\partial_\lambda\mathcal{T}(\lambda)^*\tilde H = \int_0^\infty \mathcal{F}'\Big[&(\partial_\lambda\mathcal{M}_1^{(0)}(\lambda))\mathcal{F}^{-1}_{\xi'}[\tilde H](\xi', y_N)Be^{-B(x_N+y_N)}\\
&+ B^{-2}\mathcal{M}_1^{(0)}(\lambda)\mathcal{F}^{-1}_{\xi'}[\tilde H](\xi', y_N)B^2\partial_\lambda(Be^{-B(x_N+y_N)})\\
&+ (\partial_\lambda\mathcal{M}_2^{(0)}(\lambda))\mathcal{F}^{-1}_{\xi'}[\tilde H](\xi', y_N)B^2M(x_N+y_N)\\
&+ B^{-2}\mathcal{M}_2^{(0)}(\lambda)\mathcal{F}^{-1}_{\xi'}[\tilde H](\xi', y_N)B^2\partial_\lambda(B^2M(x_N+y_N))\Big](x')\,dy_N.
\end{aligned}$$

Employing the same argument, we see that

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)^*\tilde H\|_{L_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\tilde H\|_{L_{q'}(\mathbb{R}^N_+)},\\
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)^*\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)},\\
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)^*\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1/2}\|\tilde H\|_{L_{q'}(\mathbb{R}^N_+)}.
\end{aligned} \tag{85}$$

Since $\tilde H \in C_0^\infty(\mathbb{R}^N_+)^{N+N^2}$, we see that $((\lambda, \lambda^{1/2}\bar\nabla.\bar\nabla^2)(\partial_\lambda\mathcal{T}(\lambda))^* = (\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)(\partial_\lambda\mathcal{T}(\lambda))^*$. Thus, in view of (83), (84) and (85), by Theorem 2.1, we see that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ there holds

$$\begin{aligned}
&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)\tilde H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|\tilde H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \quad (\nu \in \{s-\sigma, s, s+\sigma\}),\\
&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}(\lambda)\tilde H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\tilde H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.
\end{aligned}$$

Finally, we shall see that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ there holds

$$\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)\tilde H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \le C|\lambda|^{-(1-\frac{\sigma}{2})}\|\tilde H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}. \tag{86}$$

In fact, by (79), we have

$$\begin{aligned}
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)\tilde H\|_{L_q(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\tilde H\|_{L_q(\mathbb{R}^N_+)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)\tilde H\|_{W^1_q(\mathbb{R}^N_+)} &\le C|\lambda|^{-1/2}\|\tilde H\|_{L_q(\mathbb{R}^N_+)}.
\end{aligned}$$

And, by (81), we have

$$\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)\tilde H\|_{W^1_q(\mathbb{R}^N_+)} \le C|\lambda|^{-1}\|\tilde H\|_{W^1_q(\mathbb{R}^N_+)}.$$

Moreover, by (82), we have

$$\begin{aligned}
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)^*\tilde H\|_{L_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\tilde H\|_{L_{q'}(\mathbb{R}^N_+)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)^*\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1/2}\|\tilde H\|_{L_{q'}(\mathbb{R}^N_+)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{T}(\lambda)^*\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)} &\le C|\lambda|^{-1}\|\tilde H\|_{W^1_{q'}(\mathbb{R}^N_+)}.
\end{aligned}$$

Therefore, by Theorem 2.1, we see that (86) holds. This completes the proof of Theorem 4.2. □

Finally, we consider the full equations:

$$\begin{aligned}
\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I}) &= \mathbf{g} && \text{in } \mathbb{R}^N_+,\\
(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u})\mathbf{n}\mathbb{I} &= \mathbf{h} && \text{on } \partial\mathbb{R}^N_+.
\end{aligned} \tag{87}$$

Combining Theorems 4.1 and 4.2, we have the following corollary immediately.

COROLLARY 19. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. Let $\sigma > 0$ be a small constant such that $-1 + 1/q < s - \sigma < s + \sigma < 1/q$. Then, there exists an operator $\mathcal{S}_{\mathbb{R}^N_+}(\lambda)$ with*

$$\mathcal{S}_{\mathbb{R}^N_+}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(B^s_{q,1}(\mathbb{R}^N_+)^{m(N)}, B^{s+2}_{q,1}(\mathbb{R}^N_+)^N))$$

haviing (s,σ,q) properties such that for any $(\mathbf{g},\mathbf{h}) \in \mathcal{H}^s_{q,1}(\mathbb{R}^N_+)$, $\mathbf{u} = \mathcal{S}_{\mathbb{R}^N_+}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ is a unique solution of equations (87)*. Here and in the following, an operation $\mathcal{O}_\lambda$ is defined by*

$$\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}) = (\mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}).$$

4.4. Spectral analysis of Lamé equations in a bent half space

Let $x_0 \in \partial\Omega$. Here and in the following, we write $B_r(x_0) = \{y \in \mathbb{R}^N \mid |y - x_0| < r\}$ and $B_r = \{x \in \mathbb{R}^N \mid |x| < r\}$ for $r > 0$. As was seen in [**14**, Appendix] or in [**45**, Subsec. 3.2.1], there exist a constant $d > 0$, a diffeomorphism of C^3 class $\Phi : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto y = \Phi(x)$ and its inverse map $\Phi^{-1} : \mathbb{R}^N \to \mathbb{R}^N$, $y \mapsto x = \Phi^{-1}(y)$ such that $\Phi^{-1}(B_d(x_0) \cap \Omega) \subset \mathbb{R}^N_+$, $\Phi^{-1}(B_d(x_0) \cap \partial\Omega) \subset \mathbb{R}^N_0$ and

$$\nabla\Phi = \mathcal{A} + \mathcal{B}(x), \quad \nabla\Phi^{-1}(y) = \mathcal{A}_- + \mathcal{B}_-(y)$$

where $\mathcal{A}$ and $\mathcal{A}_-$ are $N \times N$ orthogonal matrices of constant coefficients such that $\mathcal{A}\mathcal{A}_- = \mathcal{A}_-\mathcal{A} = I$ and $\mathcal{B}(x)$ and $\mathcal{B}_-(y)$ are $N \times N$ matrices of C^2 functions. Moreover, we may assume that for any small constant $M_1 > 0$ we can choose $0 < d < 1$ such that

$$\|(\mathcal{B}, \mathcal{B}_-)\|_{L_\infty(\mathbb{R}^N)} \leq M_1. \tag{88}$$

For such d, we may assume that $\mathrm{supp}\,\mathcal{B}, \mathrm{supp}\,\mathcal{B}_- \subset B_d(x_0)$. Furthermore, we may assume that there exist constants D and M_2 such that D is independent of M_1, but M_2 depends on M_1, and

$$\begin{aligned} \|\nabla(\mathcal{B}, \mathcal{B}_-)\|_{L_\infty(\mathbb{R}^N)} &\leq D, \\ \|\nabla^2(\mathcal{B}, \mathcal{B}_-)\|_{L_\infty(\mathbb{R}^N)} &\leq M_2. \end{aligned} \tag{89}$$

We may assume that $M_1 < 1 \leq D \leq M_2$.

Let

$$\Omega_+ = \Phi(\mathbb{R}^N_+), \quad \Gamma_+ = \Phi(\mathbb{R}^N_0).$$

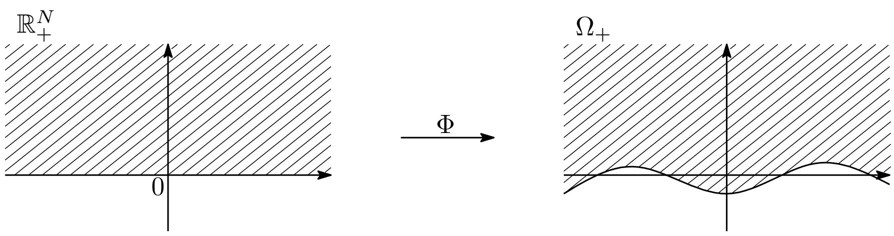

Figure 1. $\mathbb{R}^N_+$ and Ω_+

Let $\mathbf{n}_+$ denote the unit outer normal to Γ_+. In this section, first we consider the Lamé equations

$$
\begin{aligned}
\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I}) &= \tilde{\mathbf{g}} && \text{in } \Omega_+, \\
(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n}_+ &= \tilde{\mathbf{h}} && \text{on } \partial\Omega_+.
\end{aligned}
\tag{90}
$$

We shall show the following theorem.

Theorem 4.3. *Let $x_0 \in \partial\Omega$. Let Φ and Φ^{-1} be a C^3 diffeomorphism on $\mathbb{R}^N$ and its inverse given above. Let $1 < q < \infty$, $-1+1/q < s < 1/q$, and $0 < \epsilon < \pi/2$. Let σ be a small positive number such that $-1+1/q < s-\sigma < s+\sigma < 1/q$. Let $\nu \in \{s-\sigma, s, s+\sigma\}$. Then, there exist a small constant $d > 0$, a large constant $\lambda_1 \geq \max(1, \lambda_0)$ and an operator $\mathcal{S}_{\Omega_+}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}(B^\nu_{q,1}(\Omega_+)^{m(N)}, B^{\nu+2}_{q,1}(\Omega_+)^N))$ having (s,σ,q) properties such that for any $(\tilde{\mathbf{g}}, \tilde{\mathbf{h}}) \in \mathcal{H}^\nu_{q,1}(\Omega_+)$, $\mathbf{u} = \mathcal{S}_{\Omega_+}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ is a unique solution of equations* (90).

Proof. The theorem will be proved using some perturbation method from the half space, and so we have to choose $\lambda_1 > 0$ large enough below. First, we shall reduce problem (90) to that in the half-space $\mathbb{R}^N_+$. Let a_{jk} and $b_{jk}(x)$ be the (j,k)th components of $\mathcal{A}_-$ and $\mathcal{B}_-(\Phi(x))$, and then we have

$$
\frac{\partial}{\partial y_j} = \sum_{k=1}^N (a_{kj} + b_{kj}(x))\frac{\partial}{\partial x_k} \quad (j = 1, \dots, N). \tag{91}
$$

Since $y = \Phi(x)$ denotes the point of Γ_+ for $x \in \mathbb{R}^N_0$, $dy = \nabla\Phi(x)dx$. Thus, $\mathbf{n}_+ = \nabla\Phi(x)\mathbf{n}_0/|\nabla\Phi(x)\mathbf{n}_0|$. Recalling that $\mathbf{n}_0 = (0,\dots,0,-1)^t$ and denoting the (j,k)th component of $\nabla\Phi = \mathcal{A} + \mathcal{B}(x)$ by $a_{kj} + \tilde{b}_{kj}(x)$, we have

$$
\mathbf{n}_+ = -(a_{N1} + \tilde{b}_{N1}(x), \dots, a_{NN} + \tilde{b}_{NN}(x))/|\mathbf{n}_+|
$$

with $|\mathbf{n}_+| = (\sum_{j=1}^N (a_{Nj} + \tilde{b}_{Nj})^2)^{1/2}$. Let $\mathcal{D} = |\mathbf{n}_+|$. Notice that

$$
\sum_{j=1}^N a_{jk}a_{j\ell} = \sum_{j=1}^N a_{kj}a_{\ell j} = \delta_{k\ell}, \tag{92}
$$

$$
\sum_{j=1}^N (2|\tilde{b}_{Nj}||a_{Nj}| + |\tilde{b}_{Nj}|^2) \leq CM_1. \tag{93}
$$

Thus,

$$
1 - CM_1 \leq \mathcal{D}^2 \leq 1 + CM_1.
$$

Let $\tilde{\mathbf{v}}(x) = \mathbf{v}(y)$. We write $\tilde{\mathbf{v}}(x) = (\tilde{v}_1(x), \dots, \tilde{v}_N(x))^\top$ and $\mathbf{v}(y) = (v_1(y), \dots, v_N(y))^\top$, where $A^\top$ denotes the transposed A for any vector or matrix A. By (91) we have

$$
\mathrm{div}_y\mathbf{v}(y) = \sum_{\ell,m=1}^N (a_{m\ell} + b_{m\ell}(x))\frac{\partial\tilde{v}_\ell}{\partial x_m}. \tag{94}
$$

Moreover, we set $\tilde{v}_\ell = \sum_{k=1}^N a_{k\ell}w_k$, and $\mathbf{w} = (w_1, \dots, w_N)^\top$. From (91), (92) and (94), we have

$$
\begin{aligned}
\tilde g_j &= \lambda v_j - \sum_{k=1}^N \frac{\partial}{\partial y_k}(\alpha D_{jk}(\mathbf{v}) + (\beta-\alpha)\delta_{jk}\mathrm{div}\mathbf{v}) \\
&= \lambda\sum_{n=1}^N a_{nj}w_n - \alpha\sum_{\ell,n=1}^N a_{\ell j}\frac{\partial}{\partial x_n}D_{\ell n}(\mathbf{w}) - (\beta-\alpha)\sum_{n=1}^N a_{nj}\frac{\partial}{\partial x_n}\mathrm{div}\mathbf{w} - R^1_j\mathbf{w}.
\end{aligned}
$$

Here, we have set

$$
\begin{aligned}
R^1_j\mathbf{w} &= \alpha\sum_{p,k=1}^N a_{pk}\frac{\partial}{\partial x_p}\Big(\sum_{\ell,n=1}^N (b_{\ell j}a_{nk} + b_{\ell k}a_{nj})\frac{\partial w_n}{\partial x_\ell}\Big) \\
&\quad + \alpha\sum_{p,k=1}^N b_{pk}\frac{\partial}{\partial x_p}\Big(\sum_{\ell,n=1}^N (a_{\ell j}a_{nk} + a_{\ell k}a_{nj} + b_{\ell j}a_{nk} + b_{\ell k}a_{nj})\frac{\partial w_n}{\partial x_\ell}\Big) \\
&\quad + (\beta-\alpha)\Big(\sum_{p=1}^N a_{pj}\frac{\partial}{\partial x_p}(\sum_{\ell,m,n=1}^N a_{\ell m}b_{nm}\frac{\partial w_\ell}{\partial x_n}) \\
&\qquad\qquad + \sum_{p=1}^N b_{pj}\frac{\partial}{\partial x_p}(\mathrm{div}\mathbf{w} + \sum_{\ell,m,n=1}^N a_{\ell m}b_{nm}\frac{\partial w_\ell}{\partial x_n})\Big).
\end{aligned}
$$

Using (92), we have

$$
\sum_{j=1}^N a_{sj}\tilde g_j = \lambda w_s - \alpha\sum_{n=1}^N\frac{\partial}{\partial x_n}D_{sn}(\mathbf{w}) - (\beta-\alpha)\sum_{n=1}^N\delta_{sn}\frac{\partial}{\partial x_n}\mathrm{div}\mathbf{w} - \sum_{j=1}^N a_{sj}R^1_j\mathbf{w}. \tag{95}
$$

We next consider the boundary conditions.

$$
\begin{aligned}
\tilde h_j &= \sum_{k=1}^N(\alpha D_{jk}(\mathbf{v}) + (\beta-\alpha)\delta_{jk}\mathrm{div}\mathbf{v})n_k \\
&= -(\sum_{\ell=1}^N a_{\ell j}\alpha D_{N\ell}(\mathbf{w}) + (\beta-\alpha)a_{Nj}\mathrm{div}\mathbf{w} + R^2_j\mathbf{w})
\end{aligned}
$$

where we have set

$$
\begin{aligned}
R^2_j\mathbf{w} &= (\mathcal{D}^{-1}-1)(\alpha\sum_{k,\ell,n=1}^N (a_{\ell j}a_{nk} + a_{\ell k}a_{nj})\frac{\partial w_n}{\partial x_\ell}a_{Nk} + (\beta-\alpha)\delta_{jk}a_{Nk}\mathrm{div}\mathbf{w}) \\
&\quad + \mathcal{D}^{-1}\alpha\sum_{k,\ell,n=1}^N (a_{\ell j}a_{nk} + a_{\ell k}a_{nj})\frac{\partial w_n}{\partial x_\ell}\tilde b_{Nk} \\
&\quad + \mathcal{D}^{-1}\alpha\sum_{k,\ell,n=1}^N (b_{\ell j}a_{nk} + b_{\ell k}a_{nj})\frac{\partial w_n}{\partial x_\ell}(a_{Nk} + \tilde b_{Nk})
\end{aligned}
$$

$$+\mathcal{D}^{-1}(\beta-\alpha)(\mathrm{div}\mathbf{w}\tilde b_{Nj}+\sum_{\ell,m,n=1}^N a_{n\ell}b_{m\ell}\frac{\partial w_n}{\partial x_\ell}(a_{Nj}+\tilde b_{Nj})).$$

Using (92), we have

$$\sum_{j=1}^N a_{sj}\tilde h_j=-(\alpha D_{Ns}(\mathbf{w})+(\beta-\alpha)\delta_{Ns}\mathrm{div}\mathbf{w}+\sum_{j=1}^N a_{sj}R_j^2\mathbf{w}). \tag{96}$$

Set

$$\begin{aligned}\mathbf{g}&=(\sum_{j=1}^N a_{1j}\tilde g_j(\Phi(x)),\dots,\sum_{j=1}^N a_{Nj}\tilde g_j(\Phi(x))),\\ \mathbf{h}&=(\sum_{j=1}^N a_{1j}\tilde h_j(\Phi(x)),\dots,\sum_{j=1}^N a_{Nj}\tilde h_j(\Phi(x))),\end{aligned}$$

and define remainder terms $\mathcal{R}^k$ by

$$\mathcal{R}^k\mathbf{w}=\Big(\sum_{j=1}^N a_{1j}R_j^k\mathbf{w},\dots,\sum_{j=1}^N a_{Nj}R_j^k\mathbf{w}\Big)\qquad(k=1,2).$$

Then, from (95) and (96) we have

$$\begin{aligned}\lambda\mathbf{w}-\mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{w})+(\beta-\alpha)\mathrm{div}\mathbf{w}\mathbb{I})-\mathcal{R}^1\mathbf{w}&=\mathbf{g}&&\text{in }\mathbb{R}^N_+,\\ (\alpha\mathbb{D}(\mathbf{w})+(\beta-\alpha)\mathrm{div}\mathbf{w}\mathbb{I})\mathbf{n}_0-\mathcal{R}^2\mathbf{w}&=\mathbf{h}&&\text{on }\partial\mathbb{R}^N_+.\end{aligned}\tag{97}$$

Let $\lambda_0>0$ and let $\mathcal{S}_{\mathbb{R}^N_+}(\lambda)$ be the solution operator given in Corollary 19. Recall that $\mathbf{w}=\mathcal{S}_{\mathbb{R}^N_+}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ is a unique solution of equations (40). Inserting this formula into equations (97), we have

$$\begin{aligned}\lambda\mathbf{w}-\mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{w})+(\beta-\alpha)\mathrm{div}\mathbf{w}\mathbb{I})-\mathcal{R}^1\mathbf{w}&=\mathbf{g}-\mathcal{R}^1\mathbf{w}&&\text{in }\mathbb{R}^N_+,\\ (\alpha\mathbb{D}(\mathbf{w})+(\beta-\alpha)\mathrm{div}\mathbf{w}\mathbb{I})\mathbf{n}_0-\mathcal{R}^2\mathbf{w}&=\mathbf{h}-\mathcal{R}^2\mathbf{w}&&\text{on }\partial\mathbb{R}^N_+.\end{aligned}\tag{98}$$

Define an operator $\mathcal{U}$ by

$$\mathcal{U}H=(\mathcal{R}^1\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H,\mathcal{R}^2\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H).$$

for $H\in B^\nu_{q,1}(\mathbb{R}^N_+)^{m(N)}$. Obvisouly,

$$(\mathbf{g}-\mathcal{R}^1\mathbf{w},\mathbf{h}-\mathcal{R}^2\mathbf{w})=(\mathbf{I}-\mathcal{U}\mathcal{O}_\lambda)(\mathbf{g},\mathbf{h}).\tag{99}$$

To estimate $\mathcal{U}$, we prepare some lemmas concerning the estimate of products in the Besov space.

Lemma 20. *Let $1<q<\infty$ and $-1+1/q<s<1/q$. Then, there exists a $\theta\in(0,1)$ such that for $u\in B^s_{q,1}(\mathbb{R}^N_+)$ and $v\in W^1_\infty(\mathbb{R}^N_+)$ such that $\mathrm{supp}\,v\subset B_d(x_0)$, there holds*

$$\|uv\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C_s\|u\|_{B^s_{q,1}(\mathbb{R}^N_+)}\|v\|^{1-\theta}_{L_\infty(\mathbb{R}^N_+)}\|v\|^\theta_{W^1_\infty(\mathbb{R}^N_+)}.$$

PROOF. We choose β as $N < \beta < \min(Nq, Nq')$. By Lemma 6,

$$\|uv\|_{B^s_{q,r}} \leq C\|u\|_{B^s_{q,r}}(\|v\|_{L_\infty} + \|v\|_{B^{N/\beta}_{\beta,\infty}\cap L_\infty}).$$

Since $N/\beta < 1$,, we know that $W^{N/\beta}_\beta = (L_\beta, W^1_\beta)_{[N/\beta]}$, where $(\cdot,\cdot)_{[\beta]}$ denotes the complex interpolation functor. Thus,

$$\|v\|_{B^{N/\beta}_{\beta,\infty}} \leq C\|v\|_{W^{N/\beta}_\beta} \leq C(\|v\|_{L_\beta} + \|v\|^{1-N/\beta}_{L_\beta}\|\nabla v\|^{N/\beta}_{L_q}).$$

If $\operatorname{supp} v \subset B_d(x_0)$, then

$$\|v\|_{B^{N/\beta}_{\beta,\infty}} \leq C|B_d(x_0)|^{1/q}(\|v\|_{L_\infty} + \|v\|^{1-N/\beta}_{L_\infty}\|\nabla v\|^{N/\beta}_{L_\infty}) \leq C|B_1|^{1/q}\|v\|^{1-N/\beta}_{L_\infty}\|v\|^{N/\beta}_{W^1_\infty}.$$

provided that $0 < d < 1$. Thus, setting $\theta = N/\beta$, we have the lemma. This completes the proof of Lemma 20. □

Continuation of the proof of Theorem 4.3. Since $\mathcal{S}_{\mathbb{R}^N_+}$ has (s, σ, q) properties, by (88), (89), (93) and Lemma 20, we see that for some $\omega \in (0, 1)$

$$\|\mathcal{O}_\lambda\mathcal{U}H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \leq C((M_1)^{1-\omega}D^\omega + D^{1-\omega}M_2^\omega|\lambda|^{-1/2})\|H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}.$$

Choosing M_1 so small that $(CM_1)^{1-\omega}D^\omega \leq 1/4$ and choosing $\lambda_1 \geq \lambda_0$ so large that $CD^{1-\omega}M_2^\omega\lambda_1^{-1/2} \leq 1/4$, we have

$$\|\mathcal{O}_\lambda\mathcal{U}H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \leq (1/2)\|H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)}. \tag{100}$$

Here, we choose $d > 0$ so small that $(CM_1)^{1-\omega}D^\omega \leq 1/4$ and fix such d. After this procedure, M_2 is fixed, and so we can choose λ_1 so large according to these fixed d and M_2.

Let us define $\mathcal{R}_\infty(\lambda)$ by setting

$$\mathcal{R}_\infty(\lambda) = \sum_{\ell=0}^\infty(\mathcal{U}\mathcal{O}_\lambda)^\ell.$$

By (100), we see that for $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}^\nu_{q,1}(\mathbb{R}^N_+)$ and $\lambda \in \Sigma_{\epsilon,\lambda_1}$,

$$\begin{aligned}
\|(\mathcal{U}\mathcal{O}_\lambda)^\ell(\mathbf{g}, \mathbf{h})\|_{\mathcal{H}^\nu_{q,1}(\mathbb{R}^N_+)} &\leq \max(1, \lambda_1^{-1/2})\|\mathcal{O}_\lambda(\mathcal{U}\mathcal{O}_\lambda)^\ell(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \\
&= \max(1, \lambda_1^{-1/2})\|(\mathcal{O}_\lambda\mathcal{U})^\ell\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \\
&\leq \max(1, \lambda_1^{-1/2})(1/2)^\ell\|\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \\
&\leq \max(1, \lambda_1^{-1/2})\max(1, |\lambda|^{1/2})(1/2)^\ell\|(\mathbf{g}, \mathbf{h})\|_{\mathcal{H}^\nu_{q,1}(\mathbb{R}^N_+)}.
\end{aligned}$$

Thus, $\mathcal{R}_\infty(\lambda) \in \mathcal{L}(\mathcal{H}^\nu_{q,1}(\mathbb{R}^N_+))$. If we define $\mathbf{v} \in B^{\nu+2}_{q,1}(\mathbb{R}^N_+)$ by

$$\mathbf{v} = \mathcal{S}_{\mathbb{R}^N_+}(\lambda)\mathcal{O}_\lambda\mathcal{R}_\infty(\lambda)(\mathbf{g},\mathbf{h})$$

then, in view of (99),

$$\mathcal{R}_\infty(\lambda)(\mathbf{g},\mathbf{h}) - \mathcal{U}\mathcal{O}_\lambda\mathcal{R}_\infty(\lambda)(\mathbf{g},\mathbf{h}) = (\mathbf{g},\mathbf{h}).$$

Thus, from (98), $\mathbf{v}$ satisfies equations:

$$\begin{aligned} \lambda\mathbf{v} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I}) - \mathcal{R}^1\mathbf{v} = \mathbf{g} \qquad & \text{in } \mathbb{R}^N_+, \\ (\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I})\mathbf{n}_0 - \mathcal{R}^2\mathbf{v} = \mathbf{h} \qquad & \text{on } \partial\mathbb{R}^N_+. \end{aligned} \tag{101}$$

Moreover, we observe that

$$\mathcal{O}_\lambda\mathcal{R}_\infty(\lambda) = \sum_{\ell=0}^\infty \mathcal{O}_\lambda(\mathcal{U}\mathcal{O}_\lambda)^\ell = (\sum_{\ell=0}^\infty(\mathcal{O}_\lambda\mathcal{U}))^\ell)\mathcal{O}_\lambda.$$

Let $\mathcal{Q}_\infty(\lambda)$ be an operator defined by

$$\mathcal{Q}_\infty(\lambda)H = \sum_{\ell=0}^\infty(\mathcal{O}_\lambda\mathcal{U})^\ell H,$$

and then by (100) we see that $\mathcal{Q}_\infty(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}(B^\nu_{q,1}(\mathbb{R}^N_+)^{m(N)}))$ and its operator norm does not exceed 2 for any $\lambda \in \Sigma_{\epsilon,\lambda_1}$. We define an operator $\mathcal{T}_{\Omega_+}(\lambda)$ by

$$\mathcal{T}_{\Omega_+}(\lambda)H = \mathcal{S}_{\mathbb{R}^N_+}(\lambda)\mathcal{Q}_\infty(\lambda)H.$$

Since $\mathcal{T}_{\Omega_+}(\lambda)\mathcal{O}_\lambda = \mathcal{T}(\lambda)\mathcal{O}_\lambda\mathcal{R}_\infty(\lambda)$, $\mathbf{v} = \mathcal{T}_{\Omega_+}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ is a solution of equations (101). Moreover, $\mathcal{S}_{\mathbb{R}^N_+}(\lambda)$ has (s,σ,q) properties, we see that

$$\begin{aligned} &\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{T}_{\Omega_+}(\lambda)H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} \leq C\|H\|_{B^\nu_{q,1}(\mathbb{R}^N_+)} && \text{for } H \in B^\nu_{q,1}(\mathbb{R}^N_+)^{m(N)}, \\ &\|(\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{T}_{\Omega_+}(\lambda)H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-\frac{\sigma}{2}}\|H\|_{B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)} && \text{for } H \in B^{s+\sigma}_{q,1}(\mathbb{R}^N_+)^{m(N)}, \\ &\|(1,\lambda^{-1/2}\bar\nabla)\mathcal{T}_{\Omega_+}H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\frac{\sigma}{2})}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} && \text{for } H \in B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)^{m(N)}. \end{aligned}$$

To estimate $\partial_\lambda\mathcal{T}_{\Omega_+}(\lambda)$, we write

$$\partial_\lambda\mathcal{T}_{\Omega_+}(\lambda)H = (\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda))\mathcal{Q}_\infty H + \mathcal{S}_{\mathbb{R}^N_+}(\lambda)(\partial_\lambda\mathcal{Q}_\infty)H.$$

Since $\mathcal{S}_{\mathbb{R}^N_+}(\lambda)$ has (s,σ,q) properties, we have

$$\begin{aligned} \|(\lambda,\lambda\bar\nabla,\bar\nabla^2)(\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda))\mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} &\leq C|\lambda|^{-1}\|\mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \\ &\leq C|\lambda|^{-1}\|H\|_{B^s_{q,1}(\mathbb{R}^N_+)}. \end{aligned}$$

Moreover,

$$\|(\lambda,\lambda\bar\nabla,\bar\nabla^2)(\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda))\mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\sigma/2)}\|\mathcal{Q}_\infty H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.$$

Since $-1+1/q < s-\sigma < 1/q$, we can show that

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \leq C\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}$$

and so, we may have

$$\|\mathcal{O}_\lambda\mathcal{U}H\|_{H^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \leq (1/2)\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \tag{102}$$

for $\lambda \in \Sigma_{\epsilon,\lambda_1}$. Thus, we have

$$\|\mathcal{Q}_\infty H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \leq C\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)},$$

which yields

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)(\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda))\mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.$$

To estimate $\partial_\lambda\mathcal{Q}_\infty$, we write

$$\partial_\lambda(\mathcal{O}_\lambda\mathcal{U})^\ell = \sum(\mathcal{O}_\lambda\mathcal{U})\cdots(\partial_\lambda\mathcal{O}_\lambda\mathcal{U})\cdots(\mathcal{O}_\lambda\mathcal{U}).$$

Since

$$\partial_\lambda(\mathcal{O}_\lambda\mathcal{U}) = (1/2)\lambda^{-1/2}R^2\mathcal{R}_{\mathbb{R}^N_+}(\lambda)H + \mathcal{O}_\lambda(R^1\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H, R^2\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H),$$

we have

$$\|\partial_\lambda(\mathcal{O}_\lambda\mathcal{U})H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|H\|_{B^s_{q,1}(\mathbb{R}^N_+)}$$

and so

$$\|\partial_\lambda(\mathcal{O}_\lambda\mathcal{U})^\ell\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C\ell(1/2)^{\ell-1}|\lambda|^{-1}\|H\|_{B^s_{q,1}(\mathbb{R}^N_+)},$$

which yields

$$\|\partial_\lambda\mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\sum_{\ell=1}^\infty \ell(1/2)^{\ell-1} \leq 4C|\lambda|^{-1}\|H\|_{B^s_{q,1}(\mathbb{R}^N_+)}.$$

Thus, we have

$$\|(\lambda, \lambda\bar\nabla, \bar\nabla^2)\mathcal{S}_{\mathbb{R}^N_+}(\lambda)\partial_\lambda\mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|H\|_{B^s_{q,1}(\mathbb{R}^N_+)}.$$

Moreover,

$$\begin{aligned}
&\|\partial_\lambda(\mathcal{O}_\lambda\mathcal{U}H)\|_{B^s_{q,1}(\mathbb{R}^N_+)} \\
&\quad \leq C(|\lambda|^{-1/2}\|\bar\nabla\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H\|_{B^s_{q,1}(\mathbb{R}^N_+)} + \|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{S}_{\mathbb{R}^N_+}(\lambda)H\|_{B^s_{q,1}(\mathbb{R}^N_+)}) \\
&\quad \leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.
\end{aligned}$$

Thus, by (100) and (102), we have

$$\begin{aligned}\|\partial_\lambda \mathcal{Q}_\infty H\|_{B^s_{q,1}(\mathbb{R}^N_+)} &\leq C\sum_{\ell=1}^\infty \ell(1/2)^{\ell-1}|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)} \\ &\leq 4C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)},\end{aligned}$$

which yields

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{S}_{\mathbb{R}^N_+}(\partial_\lambda \mathcal{Q}_\infty(\lambda)H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.$$

Summing up, we have

$$\begin{aligned}&\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}_{\Omega_+}(\lambda)H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-1}\|H\|_{B^s_{q,1}(\mathbb{R}^N_+)}, \\ &\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{T}_{\Omega_+}(\lambda)H\|_{B^s_{q,1}(\mathbb{R}^N_+)} \leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\mathbb{R}^N_+)}.\end{aligned} \tag{103}$$

Therefore, we see that $\mathcal{T}_{\Omega_+}(\lambda)$ has (s, σ, q) properties.

Finally, according to (98), for any $H = (H_1, H_2, H_3) \in B^\nu_{q,1}(\Omega_+)^{m(N)}$, we define $\mathcal{U}_{\Omega_+}(\lambda)$ by setting

$$\mathcal{U}_{\Omega_+}(\lambda)H = \mathcal{A}^\top(\mathcal{T}_{\Omega_+}(\lambda)(\mathcal{A}H_1, \mathcal{A}H_2, (\nabla\Phi)^\top\mathcal{A}H_3)\circ\Phi))\circ\Phi^{-1}.$$

Obviously, for any $(\tilde{\mathbf{u}}, \tilde{\mathbf{h}}) \in \mathcal{H}^\nu_{q,1}(\Omega_+)$, $\mathbf{u} = \mathcal{U}_{\Omega_+}(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})$ is a solution of equations (90). The uniqueness follows from the existence of solutions to the dual problem. Since $\mathcal{T}_{\Omega_+}(\lambda)$ has (s, σ, q) properties, so $\tilde{\mathcal{T}}_{\Omega_+}(\lambda)$ does. This completes the proof of Theorem 4.3. □

4.5. Spectral analysis of generalized Lamé equations in Ω, A proof of Theorem 3.1

In this subsection, we consider equations (5) and we prove Theorem 3.1. We only consider the case where Ω is an exterior domain and $\tilde\eta_0(x) \not\equiv 0$. Other cases can be treated in the same manner. Since we will use Lemma 3 below, we have to assume that $-\min(1-1/q, N/q) < s < 1/q$.

First, we consider the problem in $(B_R)^c$ with large $R > 0$. Let ψ and $\tilde\psi$ be two $C^\infty(\mathbb{R}^N)$ functions such that $\psi(x)$ equals to 1 for $|x| > 3$ and 0 for $|x| < 2$ and $\tilde\psi(x)$ equals to 1 for $|x| > 2$ and 0 for $|x| < 1$. Set $\psi_R(x) = \psi(x/R)$ and $\tilde\psi_R(x) = \tilde\psi(x/R)$. Notice that $\psi_R(x)\tilde\psi_R(x) = \psi_R(x)$. Let $\lambda_0 > 0$ and $\mathcal{S}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(B^\nu_{q,1}(\mathbb{R}^N), B^{\nu+2}_{q,1}(\mathbb{R}^N)^N))$ be the operator given in Theorem 4.1 and then $\mathbf{v}_R = \mathcal{S}(\rho_*\lambda)\tilde\psi_R\mathbf{g}$ satisfies equations

$$\rho_*\lambda\mathbf{v}_R - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}_R) + (\beta-\alpha)\mathrm{div}\mathbf{v}_R\mathbb{I}) = \psi_R\mathbf{g} \qquad \text{in } \mathbb{R}^N. \tag{104}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_0}$. Here, notice that $\rho_1 < \rho_* < \rho_2$. Let

$$A_R = \rho_* + \tilde\psi_R(x)(\eta_0(x) - \rho_*) = \rho_* + \tilde\psi_R(x)\tilde\eta_0(x).$$

We have

$$A_R\lambda\mathbf{v}_R - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}_R) + (\beta-\alpha)\mathrm{div}\mathbf{v}_R\mathbb{I}) = \tilde\psi_R\mathbf{g} - \mathcal{R}_R(\lambda)\tilde\psi_R\mathbf{g} \quad \text{in } \mathbb{R}^N,$$

where we have set

$$\mathcal{R}_R(\lambda)\mathbf{f} = -\tilde\psi_R(x)\tilde\eta_0(x)\lambda\mathcal{S}(\rho_*\lambda)\mathbf{f}.$$

By Lemma 3 and Theorem 4.1, we have

$$\|\mathcal{R}_R(\lambda)\mathbf{f}\|_{B^\nu_{q,1}(\mathbb{R}^N)} \leq C\|\tilde\psi_R\tilde\eta_0\|_{B^{N/q}_{q,1}(\mathbb{R}^N)}\|\mathbf{f}\|_{B^\nu_{q,1}(\mathbb{R}^N)}.$$

By Lemma 12 in [**31**], for any $\delta > 0$ there exists an $R_0 > 1$ such that $\|\tilde\psi_R\tilde\eta_0\|_{B^{N/q}_{q,1}(\Omega)} < \delta$ for any $R > R_0$, and so we have $\|\mathcal{R}_R(\lambda)\mathbf{f}\|_{B^\nu_{q,1}(\Omega)} \leq C\delta\|\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})\|_{B^\nu_{q,1}(\Omega)}$ for $\nu \in \{s-\sigma, s, s+\sigma\}$ and $R > R_0$. We choose $\delta > 0$ in such a way that $C\delta \leq 1/2$, we have

$$\|\mathcal{R}_R(\lambda)\mathbf{f}\|_{B^\nu_{q,1}(\Omega)} \leq (1/2)\|\mathbf{f}\|_{B^\nu_{q,1}(\Omega)}$$

for $\nu \in \{s-\sigma, s, s+\sigma\}$ and $R > R_0$. In the following this R is fixed. Thus, we can define

$$\mathcal{R}_{R,\infty}(\lambda) = (I - \mathcal{R}_R(\lambda))^{-1} = \sum_{\ell=0}^\infty \mathcal{R}_R(\lambda)^\ell.$$

Let $\mathcal{S}_R(\lambda)H = \mathcal{S}(\rho_*\lambda)\mathcal{R}_{R,\infty}(\lambda)\tilde\psi_R H_1$ for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $H \in B^\nu_{q,1}(\mathbb{R}^N_+)^{m(N)}$, and then from (104), $\mathbf{w}_R = \mathcal{S}_R(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ satisfies equations

$$A_R\lambda\mathbf{w}_R - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{w}_R) + (\beta-\alpha)\mathrm{div}\mathbf{w}_R\mathbb{I}) = \tilde\psi_R\mathbf{g} \quad \text{in } \mathbb{R}^N \tag{105}$$

for any $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $(\mathbf{g},\mathbf{h}) \in \mathcal{H}^\nu_{q,1}(\Omega)$. Moreover, by Theorem 4.1,

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{S}_R(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C\|H\|_{B^\nu_{q,1}(\Omega)} && \text{for } H \in B^\nu_{q,1}(\Omega)^{m(N)},\\
\|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{S}_R(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-\sigma/2}\|H\|_{B^{s+\sigma}_{q,1}(\Omega)} && \text{for } H \in B^{s+\sigma}_{q,1}(\Omega)^{m(N)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{S}_R(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} && \text{for } H \in B^s_{q,1}(\Omega)^{m(N)}
\end{aligned} \tag{106}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_0}$. Moreover, employing the similar argument to the poof of (103), we see that

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{S}_R(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C|\lambda|^{-1}\|H\|_{B^\nu_{q,1}(\Omega)} && \text{for } H \in B^\nu_{q,1}(\Omega)^{m(N)},\\
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{S}_R(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} && \text{for } H \in B^s_{q,1}(\Omega)^{m(N)}
\end{aligned} \tag{107}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$. Let $\mathbf{u}_R = \psi_R\mathcal{S}_R(\lambda)\tilde\psi_R\mathbf{g}$ and set

$$\begin{aligned}
U_R(\lambda)H = \psi_R&\mathrm{Div}\,(\alpha\mathbb{D}(\mathcal{S}_R(\lambda)\tilde\psi_R H_1) + (\beta-\alpha)\mathrm{div}(\mathcal{S}_R(\lambda)\tilde\psi_R H_1)\mathbb{I})\\
&- \mathrm{Div}\,(\alpha\mathbb{D}(\psi_R\mathcal{S}_R(\lambda)\tilde\psi_R H_1) + (\beta-\alpha)\mathrm{div}(\psi_R\mathcal{S}_R(\lambda)\tilde\psi_R H_1)\mathbb{I})
\end{aligned}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $H \in B^\nu_{q,1}(\Omega)^{m(N)}$. From (106) and Lemma 20, it follows that

$$\|U_R(\lambda)H\|_{B^\nu_{q,1}(\Omega)} \leq C_R|\lambda|^{-1/2}\|\tilde\psi_R H_1\|_{B^\nu_{q,1}(\Omega)} \tag{108}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}}$ and $H \in B^\nu_{q,1}(\Omega)^{m(N)}$. Moreover, by (105) $\mathbf{u}_R$ satisfies equations

$$\eta_0\lambda\mathbf{u}_R - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}_R) + (\beta-\alpha)\mathrm{div}\mathbf{u}_R\mathbb{I}) = \psi_R\mathbf{g} - U_R(\lambda)\mathcal{O}_s(\mathbf{g},\mathbf{h}) \quad \text{in } \mathbb{R}^N$$

for any $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $(\mathbf{g},\mathbf{h}) \in \mathcal{H}^\nu_{q,1}(\Omega)$.

Let $x_0 \in \Omega$ and let d_{x_0} be a positive number such that $B_{4d_{x_0}}(x_0) \subset \Omega$. Let φ and $\tilde\varphi$ be two $C^\infty_0(\mathbb{R}^N)$ functions such that φ equals 1 for $|x| < 1$ and 0 for $|x| > 2$ and $\tilde\varphi(x)$ equals 1 for $|x| < 2$ and 0 for $|x| > 3$. Set $\varphi_{x_0,d_{x_0}}(x) = \varphi((x-x_0)/d_{x_0})$ and $\tilde\varphi_{x_0,d_0}(x) = \tilde\varphi((x-x_0)/d_{x_0})$. Notice that $\varphi_{x_0,d_{x_0}}\tilde\varphi_{x_0,d_0} = \varphi_{x_0,d_{x_0}}$. Let $\mathcal{S}(\lambda)$ be the operator given in Theorem 4.1 again and set $\mathbf{v}_{x_0} = \mathcal{S}(\eta_0(x_0)\lambda)\tilde\varphi_{x_0,d_{x_0}}\mathbf{g}$ for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_0}$. Here, notice that $\rho_1 < \eta_0(x_0) < \rho_2$. Then, $\mathbf{v}_{x_0}$ satisfies equations:

$$\eta_0(x_0)\lambda\mathbf{v}_{x_0} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}_{x_0}) + (\beta-\alpha)\mathrm{div}\mathbf{v}_{x_0}\mathbb{I}) = \tilde\varphi_{x_0,d_{x_0}}\mathbf{g} \qquad \text{in } \mathbb{R}^N. \tag{109}$$

Let

$$A_{x_0} = \eta_0(x_0) + \tilde\varphi_{x_0,d_{x_0}}(x)(\eta_0(x) - \eta_0(x_0)).$$

We have

$$A_{x_0}\lambda\mathbf{v}_{x_0} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}_{x_0}) + (\beta-\alpha)\mathrm{div}\mathbf{v}_{x_0}\mathbb{I}) = \tilde\varphi_{x_0,d_{x_0}}\mathbf{g} - \mathcal{R}_{x_0}(\lambda)\tilde\varphi_{x_0,d_{x_0}}\mathbf{g} \quad \text{in } \mathbb{R}^N$$

where we have set

$$\mathcal{R}_{x_0}(\lambda)\mathbf{f} = -\tilde\varphi_{x_0,d_{x_0}}(x)(\eta_0(x) - \eta_0(x_0))\lambda\mathcal{S}(\lambda)\mathbf{f}.$$

By and Theorem 4.1, we have

$$\|\mathcal{R}_{x_0}(\lambda)\mathbf{f}\|_{B^\nu_{q,1}(\mathbb{R}^N)} \leq C\|\tilde\varphi_{x_0,d_{x_0}}(\eta_0(\cdot) - \eta_0(x_0))\|_{B^{N/q}_{q,1}(\mathbb{R}^N)}\|\mathbf{f}\|_{B^\nu_{q,1}(\mathbb{R}^N)}.$$

By Appendix in [**9**], for any $\delta > 0$ there exists a d_0 uniformly with respect to x_0 such that

$$\|\tilde\varphi_{x_0,d_{x_0}}(\eta_0(\cdot) - \eta_0(x_0))\|_{B^{N/q}_{q,1}(\mathbb{R}^N)} < \delta$$

provided $0 < d_{x_0} \leq d_0$, and so we have $\|\mathcal{R}_{x_0}(\lambda)\mathbf{f}\|_{B^\nu_{q,1}(\Omega)} \leq C\delta\|\mathbf{f}\|_{B^\nu_{q,1}(\Omega)}$ for $\nu \in \{s-\sigma, s, s+\sigma\}$. We choose $\delta > 0$ in such a way that $C\delta \leq 1/2$, we have

$$\|\mathcal{R}_{x_0}(\lambda)\mathbf{f}\|_{B^\nu_{q,1}(\Omega)} \leq (1/2)\|\mathbf{f}\|_{B^\nu_{q,1}(\Omega)}$$

for $\nu \in \{s-\sigma, s, s+\sigma\}$ and $0 < d_{x_0} < d_0$. In the following d_{x_0} is fixed. Thus, we can define $\mathcal{R}_{x_0,\infty}(\lambda) = (I - \mathcal{R}_{x_0}(\lambda))^{-1} = \sum_{\ell=0}^\infty \mathcal{R}_{x_0}(\lambda)^\ell$. Let $\mathcal{S}_{x_0}(\lambda)H = \mathcal{S}(\lambda)\mathcal{R}_{x_0,\infty}(\lambda)\tilde\varphi_{x_0,d_{x_0}}H_1$, then from (109), $\mathbf{w}_{x_0} = \mathcal{S}_{x_0}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ satisfies equations

$$A_{x_0}\lambda\mathbf{w}_{x_0} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{w}_{x_0}) + (\beta-\alpha)\mathrm{div}\mathbf{w}_{x_0}\mathbb{I}) = \tilde\varphi_{x_0,d_{x_0}}\mathbf{g} \quad \text{in } \mathbb{R}^N. \tag{110}$$

Moreover, by Theorem 4.1,

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{S}_{x_0}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C\|H\|_{B^\nu_{q,1}(\Omega)} && \text{for } H \in B^\nu_{q,1}(\Omega)^{m(N)},\\
\|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{S}_{x_0}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C|\lambda|^{-\sigma/2}\|H\|_{B^{s+\sigma}_{q,1}(\Omega)} && \text{for } H \in B^{s+\sigma}_{q,1}(\Omega)^{m(N)},\\
\|(1, \lambda^{-1/2}\bar\nabla)\mathcal{S}_{x_0}(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} && \text{for } H \in B^s_{q,1}(\Omega)^{m(N)}
\end{aligned} \tag{111}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$. Moreover, employing the similar argument to the proof of (103), we see that

$$\begin{aligned}
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{S}_{x_0}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C|\lambda|^{-1}\|H\|_{B^\nu_{q,1}(\Omega)} && \text{for } H \in B^\nu_{q,1}(\Omega)^{m(N)},\\
\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda\mathcal{S}_{x_0}(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} && \text{for } H \in B^{s-\sigma}_{q,1}(\Omega)^{m(N)}
\end{aligned} \tag{112}$$

for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$. Let $\mathbf{u}_{x_0} = \varphi_{x_0,d_{x_0}}\mathbf{v}_{x_0}$ and set

$$\begin{aligned}
U_{x_0}(\lambda)H = \varphi_{x_0,d_{x_0}}&\mathrm{Div}\,(\alpha\mathbb{D}(\mathcal{S}_{x_0}(\lambda)H) + (\beta-\alpha)\mathrm{div}(\mathcal{S}_{x_0}(\lambda)H)\mathbb{I})\\
&- \mathrm{Div}\,(\alpha\mathbb{D}(\varphi_{x_0,d_{x_0}}\mathcal{S}_{x_0}(\lambda)H) + (\beta-\alpha)\mathrm{div}(\varphi_{x_0,d_{x_0}}\mathcal{S}_{x_0}(\lambda)H)\mathbb{I}).
\end{aligned}$$

From (111) and Lemma 20, it follows that

$$\|U_{x_0}(\lambda)H\|_{B^\nu_{q,1}(\mathbb{R}^N)} \leq C_{x_0}|\lambda|^{-1/2}\|H\|_{B^\nu_{q,1}(\Omega)} \tag{113}$$

for any $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $H \in B^\nu_{q,1}(\Omega)$. Moreover, by (110) $\mathbf{u}_{x_0}$ satisfies equations

$$\eta_0\lambda\mathbf{u}_{x_0} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}_{x_0}) + (\beta-\alpha)\mathrm{div}\mathbf{u}_{x_0}\mathbb{I}) = \varphi_{x_0}\mathbf{g} - U_{x_0}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}) \quad \text{in } \mathbb{R}^N.$$

For $x_1 \in \partial\Omega$, let $d_{x_1} > 0$ be a small number and let Ω_+ a bent half-space given in Subsection 4.4 such that $B_{4d_{x_1}}(x_1) \cap \Omega \subset \Omega_+$. Let $\lambda_1 \geq 1$ and $\mathcal{S}_{\Omega_+} \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_1}, \mathcal{L}(B^\nu_{q,1}(\Omega_+)^{m(N)}, B^{\nu+2}_{q,1}(\Omega_+)^N))$ be the operator given in Theorem 4.3. Note that $|\eta_0(x_1)\lambda| \geq \lambda_1$ for $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$. Set $\mathbf{v}_{x_1} = \mathcal{S}_{\Omega_+}(\eta_0(x_1)\lambda)\mathcal{O}_\lambda\tilde\varphi_{x_0,d_{x_0}}(\mathbf{g},\mathbf{h})$, and then $\mathbf{v}_{x_1}$ satisfies equations:

$$\begin{aligned}
\eta_0(x_1)\lambda\mathbf{v}_{x_1} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}_{x_1}) + (\beta-\alpha)\mathrm{div}\mathbf{v}_{x_1}\mathbb{I}) &= \tilde\varphi_{x_1,d_{x_1}}\mathbf{g} && \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{v}_{x_1} + (\beta-\alpha)\mathrm{div}\mathbf{v}_{x_1}\mathbb{I})\mathbf{n} &= \tilde\varphi_{x_1,d_{x_1}}\mathbf{h} && \text{on } \partial\Omega.
\end{aligned}$$

Let

$$A_{x_1} = \eta_0(x_1) + \tilde\varphi_{x_1,d_{x_1}}(x)(\eta_0(x) - \eta_0(x_1)).$$

We have

$$\begin{aligned}
A_{x_1}\lambda\mathbf{v}_{x_1} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}_{x_1}) + (\beta-\alpha)\mathrm{div}\mathbf{v}_{x_1}\mathbb{I}) &= \tilde\varphi_{x_1,d_{x_1}}\mathbf{g} - \mathcal{R}^1_{x_1}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}) && \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{v}_{x_1} + (\beta-\alpha)\mathrm{div}\mathbf{v}_{x_1}\mathbb{I})\mathbf{n} &= \tilde\varphi_{x_1,d_{x_1}}\mathbf{h} && \text{on } \partial\Omega
\end{aligned}$$

for any $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $H \in B^\nu_{q,r}(\Omega)$, where we have set

$$\mathcal{R}^1_{x_1}(\lambda)H = -\tilde\varphi_{x_1,d_{x_1}}(x)(\eta_0(x)-\eta_0(x_1))\lambda\mathcal{S}_{\Omega_+}(\eta_0(x_1)\lambda)\tilde\varphi_{x_1,d_{x_1}}H.$$

By Lemma 3 and Theorem 4.3, we have

$$\|\mathcal{R}^1_{x_1}(\lambda)H\|_{B^\nu_{q,1}(\mathbb{R}^N)} \le C\|\tilde\varphi_{x_1,d_{x_1}}(\eta_0(\cdot)-\eta_0(x_1))\|_{B^{N/q}_{q,1}(\mathbb{R}^N)}\|H\|_{B^\nu_{q,1}(\mathbb{R}^N)}.$$

By Appendix in [**9**], for any $\delta > 0$ there exists a d_0 uniformly with respect to x_0 such that

$$\|\tilde\varphi_{x_1,d_{x_1}}(\eta_0(\cdot)-\eta_0(x_1))\|_{B^{N/q}_{q,1}(\mathbb{R}^N)} < \delta$$

provided $0 < d_{x_1} \le d_0$, and so we have $\|\mathcal{R}^1_{x_1}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} \le C\delta\|H\|_{B^\nu_{q,1}(\Omega)}$ for $\nu \in \{s-\sigma, s, s+\sigma\}$. We choose $\delta > 0$ in such a way that $C\delta \le 1/2$, we have

$$\|\mathcal{R}^1_{x_1}(\lambda)H\|_{B^\nu_{q,1}(\Omega_+)} \le (1/2)\|H\|_{B^\nu_{q,1}(\Omega_+)}$$

for $\nu \in \{s-\sigma, s, s+\sigma\}$ for $0 < d_{x_1} < d_0$. In the following d_{x_1} is fixed. Let $\mathcal{U}_{x_1}$ be an operator defined by $\mathcal{U}_{x_1}H = (\mathcal{R}^1_{x_1}(\lambda)H, 0)$ for $H = (H_1, H_2, H_3) \in \mathcal{H}^\nu_{q,1}(\Omega_+)$, and then

$$(\tilde\varphi_{x_1,d_{x_1}}\mathbf{g} - \mathcal{R}^1_{x_1}(\lambda)\tilde\varphi_{x_1,d_{x_1}}\mathbf{g}, \tilde\varphi_{x_1,d_{x_1}}\mathbf{h}) = (\mathbf{I} - \mathcal{U}_{x_1}\mathcal{O}_\lambda)\tilde\varphi_{x_1,d_{x_1}}(\mathbf{g},\mathbf{h}). \tag{114}$$

For $\lambda \in \Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$,

$$\|(\mathcal{O}_\lambda\mathcal{U}_{x_1})H\|_{B^\nu_{q,1}(\Omega_+)} \le \|\mathcal{R}_{x_1}(\lambda)H_1\|_{B^\nu_{q,1}(\Omega_+)} \le (1/2)\|H\|_{B^\nu_{q,1}(\Omega_+)}. \tag{115}$$

Thus,

$$\|\mathcal{O}_\lambda(\mathcal{U}_{x_1}\mathcal{O}_\lambda)^\ell H\|_{B^\nu_{q,1}(\Omega_+)} = \|(\mathcal{O}_\lambda\mathcal{U}_{x_1})^\ell\mathcal{O}_\lambda H\|_{B^\nu_{q,1}(\Omega_+)} \le (1/2)^\ell\|\mathcal{O}_\lambda H\|_{B^\nu_{q,1}(\Omega_+)}$$

and so $\mathcal{R}_{x_1,\infty}(\lambda) = \sum_{\ell=0}^\infty(\mathcal{U}\mathcal{O}_\lambda)^\ell = (\mathbf{I} - \mathcal{U}_{x_1}\mathcal{O}_\lambda)^{-1}$ can be defined. In fact,

$$\begin{aligned}\|\mathcal{O}_\lambda\mathcal{R}_{x_1,\infty}(\lambda)H\|_{B^\nu_{q,1}(\Omega_+)} &\le \sum_{\ell=0}^\infty\|(\mathcal{O}_\lambda\mathcal{U}_{x_1})^\ell\mathcal{O}_\lambda H\|_{B^\nu_{q,1}(\Omega_+)}\\ &\le \sum_{\ell=0}^\infty(1/2)^\ell\|\mathcal{O}_\lambda H\|_{B^\nu_{q,1}(\Omega_+)} = 2\|\mathcal{O}_\lambda H\|_{B^\nu_{q,1}(\Omega_+)}.\end{aligned}$$

If we define $\mathbf{w}_{x_1}$ by $\mathbf{w}_{x_1} = \mathcal{S}_{\Omega_+}(\eta_0(x_1)\lambda)\mathcal{O}_\lambda\mathcal{R}_{x_1,\infty}(\lambda)\tilde\varphi_{x_1,d_{x_1}}(\mathbf{g},\mathbf{h})$, then from (114) we see that $\mathbf{w}_{x_1}$ satisfies equations

$$\begin{aligned}A_{x_1}\lambda\mathbf{w}_{x_1} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{w}_{x_1}) + (\beta-\alpha)\mathrm{div}\mathbf{w}_{x_1}\mathbb{I}) &= \tilde\varphi_{x_1,d_{x_1}}\mathbf{g} && \text{in } \Omega,\\ (\alpha\mathbb{D}(\mathbf{w}_{x_1}) + (\beta-\alpha)\mathrm{div}\mathbf{w}_{x_1}\mathbb{I})\mathbf{n} &= \tilde\varphi_{x_1,d_{x_1}}\mathbf{h} && \text{on } \partial\Omega.\end{aligned} \tag{116}$$

In view of (115), we can define $\tilde{\mathcal{R}}_{x_1,\infty}(\lambda)H = \sum_{\ell=0}^\infty(\mathcal{O}_\lambda\mathcal{U})^\ell$, and then $\mathcal{O}_\lambda\tilde{\mathcal{R}}_{x_1,\infty} = \mathcal{R}_{x_1,\infty}\mathcal{O}_\lambda$ and the operator norm of $\tilde{\mathcal{R}}_{x_1,\infty}(\lambda)$ does not exceed 2. Thus, $\mathbf{w}_{x_1} =$

$\mathcal{U}_{x_1}(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})$ where $\mathcal{U}_{x_1}$ is defined by $\mathcal{U}_{x_1}(\lambda)H = \mathcal{S}_{\Omega_+}(\eta_0(x_1)\lambda)\tilde{\mathcal{R}}_{x_1,\infty}(\lambda)\varphi_{x_1,d_{x_1}}H$, and by Theorem 4.3,

$$\begin{aligned}
\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{U}_{x_1}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C\|H\|_{B^\nu_{q,1}(\Omega)} && \text{for } H\in B^\nu_{q,1}(\Omega)^{m(N)},\\
\|(\lambda^{1/2}\bar\nabla,\bar\nabla^2)\mathcal{U}_{x_1}(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-\sigma/2}\|H\|_{B^{s+\sigma}_{q,1}(\Omega)} && \text{for } H\in B^{s+\sigma}_{q,1}(\Omega)^{m(N)},\\
\|(1,\lambda^{-1/2}\bar\nabla)\mathcal{U}_{x_1}(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} && \text{for } H\in B^s_{q,1}(\Omega)^{m(N)}.
\end{aligned} \tag{117}$$

Moreover, employing the similar argument to the poof of (103), we see that

$$\begin{aligned}
\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\partial_\lambda\mathcal{U}_{x_1}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} &\leq C|\lambda|^{-1}\|H\|_{B^\nu_{q,1}(\Omega)} && \text{for } H\in B^\nu_{q,1}(\Omega)^{m(N)},\\
\|(\lambda,\lambda^{1/2}\bar\nabla,\bar\nabla^2)\partial_\lambda\mathcal{U}_{x_1}(\lambda)H\|_{B^s_{q,1}(\Omega)} &\leq C|\lambda|^{-(1-\sigma/2)}\|H\|_{B^{s-\sigma}_{q,1}(\Omega)} && \text{for } H\in B^s_{q,1}(\Omega)^{m(N)}
\end{aligned} \tag{118}$$

for any $\lambda\in\Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$. Let $\mathbf{u}_{x_1}=\varphi_{x_1,d_{x_1}}\mathbf{w}_{x_1}$ and set

$$\begin{aligned}
U^1_{x_1}(\lambda)H &= \varphi_{x_1,d_{x_1}}\mathrm{Div}\,(\alpha\mathbb{D}(\mathcal{U}_{x_1}(\lambda)H)+(\beta-\alpha)\mathrm{div}(\mathcal{U}_{x_1}(\lambda)H\mathbb{I})\\
&\quad-\mathrm{Div}\,(\alpha\mathbb{D}(\varphi_{x_1,d_{x_1}}\mathcal{U}_{x_1}(\lambda)H)+(\beta-\alpha)\mathrm{div}(\varphi_{x_1,d_{x_1}}\mathcal{U}_{x_1}(\lambda)H)\mathbb{I}),\\
U^2_{x_1}(\lambda)H &= \varphi_{x_1,d_{x_1}}(\alpha\mathbb{D}(\mathcal{U}_{x_1}(\lambda)H)+(\beta-\alpha)\mathrm{div}(\mathcal{U}_{x_1}(\lambda)H)\mathbb{I})\mathbf{n}\\
&\quad-(\alpha\mathbb{D}(\varphi_{x_1,d_{x_1}}\mathcal{U}_{x_1}(\lambda)H)+(\beta-\alpha)\mathrm{div}(\varphi_{x_1,d_{x_1}}\mathcal{U}_{x_1}(\lambda)H)\mathbb{I})\mathbf{n}.
\end{aligned}$$

From (117) and Lemma 20, it follows that

$$\|\mathcal{O}_\lambda(U^1_{x_1}(\lambda)H,U^2_{x_1}(\lambda)H)\|_{B^\nu_{q,1}(\Omega)}\leq C_{x_1}|\lambda|^{-1/2}\|H\|_{B^\nu_{q,1}(\Omega)} \tag{119}$$

for any $\lambda\in\Sigma_{\epsilon,\rho_1^{-1}\lambda_1}$ and $H\in B^\nu_{q,1}(\Omega)^{m(N)}$. Moreover, by (116) $\mathbf{u}_{x_1}$ satisfies equations

$$\begin{aligned}
\eta(x)\lambda\mathbf{u}_{x_1}-\mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}_{x_1})+(\beta-\alpha)\mathrm{div}\mathbf{u}_{x_1}\mathbb{I}) &= \varphi_{x_1,d_{x_1}}\mathbf{g}-U^1_{x_1}(\mathbf{g},\mathbf{h}) && \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{u}_{x_1})+(\beta-\alpha)\mathrm{div}\mathbf{u}_{x_1}\mathbb{I})\mathbf{n} &= \varphi_{x_1,d_{x_1}}\mathbf{h}-U^2_{x_1}(\mathbf{g},\mathbf{h}) && \text{on } \partial\Omega.
\end{aligned}$$

Now, we shall show the theorem. let $\overline{\Omega\cap B_{2R}}=\{x\in\Omega\cup\partial\Omega \mid |x|\leq 2R\}$. Notice that $\Omega\cup\partial\Omega=(B_{2R})^c\cup\overline{\Omega\cap B_{2R}}$. Since $\overline{\Omega\cap B_{2R}}$ is a compact set, there exist a finite set $\{x^0_j\}_{j=1}^{m_0}$ of points of Ω and a finite set $\{x^1_j\}_{j=1}^{m_1}$ of points of $\partial\Omega$ such that $\overline{\Omega}\subset(B_{2R})^c\cup(\bigcup_{j=1}^{m_0}B_{d_{x^0_j}/2}(x^0_j))\cup(\bigcup_{j=1}^{m_1}B_{d_{x^1_j}/2}(x^1_j))$. Let $\Phi(x)=\varphi_R(x)+(\sum_{j=1}^{m_0}\varphi_{x^0_j}(x))+(\sum_{j=1}^{m_1}\varphi_{x^1_j}(x))$. Obviously, $\Phi(x)\in C^\infty(\overline{\Omega})$ and $\Phi(x)\geq 1$ for $x\in\overline{\Omega}$. Thus, set $\omega_0(x)=\varphi_R(x)/\Phi(x)$, $\omega_j(x)=\varphi_{x^0_j}(x)/\Phi(x)$ ($j=1,\ldots,m_0$), and $\omega_{m_0+j}(x)=\varphi_{x^1_j}(x)/\Phi(x)$ ($j=1,\ldots,m_1$). Then, $\{\omega_j\}_{j=0}^{m_0+m_1}$ is a partition of unity on $\overline{\Omega}$. Let us define the parametrix operator $\mathcal{T}_\Omega(\lambda)$ and the remainder operator $\mathcal{U}^i(\lambda)$ by

$$\begin{aligned}
\mathcal{T}_\Omega(\lambda)H &= \omega_0\mathcal{S}_R(\lambda)H+\sum_{j=1}^{m_0}\omega_j\mathcal{S}_{x^0_j}(\lambda)H+\sum_{j=1}^{m_1}\omega_{m_0+j}\mathcal{U}_{x^1_j}(\lambda)H,\\
\mathcal{U}^1(\lambda)H &= U_R(\lambda)H+\sum_{j=1}^{m_0}U^1_{x^0_j}(\lambda)H+\sum_{j=1}^{m_1}U^1_{x^1_{m_0+j}}(\lambda)H,
\end{aligned}$$

$$\mathcal{U}^2(\lambda)H = \sum_{j=1}^{m_1} U^2_{x^1_{m_0+j}}(\lambda)H.$$

From (106), (107), (111), (112), (117), (118) and Lemma 20, it follows that $\mathcal{T}_\Omega(\lambda)$ has (s, σ, q) properties for $\lambda \in \Sigma_{\epsilon,\lambda_2}$. Moreover, we see that $\mathbf{v} = \mathcal{T}_\Omega(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})$, $\mathcal{U}^1(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})$, and $\mathcal{U}^2(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})$ satisfy equations:

$$\begin{aligned} \lambda\mathbf{v} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I}) &= \mathbf{g} - \mathcal{U}^1(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h}) && \text{in } \Omega, \\ (\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I})\mathbf{n} &= \mathbf{h} - \mathcal{U}^2(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h}) && \text{on } \partial\Omega. \end{aligned} \tag{120}$$

Let $\lambda_2 = \lambda_1\rho_1^{-1}$. Set $\mathcal{U}(\lambda) = (\mathcal{U}^1(\lambda), \mathcal{U}^2(\lambda))$. By (108), (113) and (119), we have

$$\|\mathcal{O}_\lambda\mathcal{U}(\lambda)H\|_{\mathcal{H}^\nu_{q,1}(\Omega)} \leq C|\lambda|^{-1/2}\|H\|_{B^\nu_{q,1}(\Omega)}$$

for $\lambda \in \Sigma_{\epsilon,\lambda_2}$, because the summation is finite number. Choosing $\lambda_3 \geq \lambda_2$ so large that $C\lambda_3^{-1/2} \leq 1/2$, we have

$$\|\mathcal{O}_\lambda\mathcal{U}(\lambda)H\|_{B^\nu_{q,1}(\Omega)} \leq (1/2)\|H\|_{B^\nu_{q,1}(\Omega)} \tag{121}$$

for $\lambda \in \Sigma_{\epsilon,\lambda_3}$. Let

$$\mathcal{W}(\lambda) = \sum_{\ell=0}^{\infty}(\mathcal{U}(\lambda)\mathcal{O}_\lambda)^\ell.$$

Noting that $\lambda_3^{-1/2} \leq 1$, we see that

$$\begin{aligned} \|\mathcal{W}(\lambda)(\mathbf{g}, \mathbf{h})\|_{\mathcal{H}^\nu_{q,1}(\Omega)} &\leq \|\mathcal{O}_\lambda\mathcal{W}(\lambda)(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\Omega)} \\ &\leq \sum_{\ell=0}^{\infty}\|(\mathcal{O}_\lambda\mathcal{U}(\lambda))^\ell\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\Omega)} \\ &\leq \sum_{\ell=0}^{\infty}(1/2)^\ell\|\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\Omega)} = 2\|\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})\|_{B^\nu_{q,1}(\Omega)} \\ &\leq 2\max(1, |\lambda|^{1/2})\|(\mathbf{g}, \mathbf{h})\|_{\mathcal{H}^\nu_{q,1}(\Omega)}. \end{aligned}$$

From (120) we see that $\mathbf{u} = \mathcal{T}_\Omega(\lambda)\mathcal{O}_\lambda\mathcal{W}(\lambda)(\mathbf{g}, \mathbf{h})$ satisfies equations:

$$\begin{aligned} \eta_0\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I}) &= \mathbf{g} && \text{in } \Omega, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n} &= \mathbf{h} && \text{on } \partial\Omega. \end{aligned}$$

Let $\tilde{\mathcal{W}}$ be an operator defined by $\tilde{\mathcal{W}} = \sum_{\ell=0}^\infty(\mathcal{O}_\lambda\mathcal{U}(\lambda))^\ell$, and then by (121), we see that the operator norm of $\tilde{\mathcal{W}}$ does not exceed 2 and $\mathcal{O}_\lambda\mathcal{W} = \tilde{\mathcal{W}}\mathcal{O}_\lambda$. Thus, defining $\mathcal{V}(\lambda)$ by $\mathcal{V}(\lambda) = \mathcal{T}_\Omega(\lambda)\tilde{\mathcal{W}}(\lambda)$, we see that $\mathbf{u} = \mathcal{V}(\lambda)\mathcal{O}_\lambda(\mathbf{g}, \mathbf{h})$ is a solution of equations (5). Moreover, since $\mathcal{T}_\Omega(\lambda)$ has (s, σ, q) properties, so $\mathcal{V}(\lambda)$ does. The uniqueness of solutions follows from the existence of solutions to the dual problem. This completes the proof of Theorem 3.1.

5. About generalized resolvent problems for the Stokes equations

In this section, we consider the generalized resolvent problem for Stokes equations with non-homogeneous free boundary conditions reading as

$$\begin{cases} \lambda\rho + \eta_0 \mathrm{div}\mathbf{u} = f & \text{in } \Omega, \\ \eta_0\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I}) = \mathbf{g} & \text{in } \Omega, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I})\mathbf{n} = \mathbf{h} & \text{on } \partial\Omega. \end{cases} \tag{122}$$

Recall that $\eta_0(x) = \rho_* + \tilde\eta_0(x)$, where ρ_* is a positive constant and $\tilde\eta_0 \in B^{N/q}_{q,1}(\Omega)$, that $P(s)$ is a C^∞ function defined for $s \in (0, \infty)$ such that $P'(s) > 0$, and that Assumption 10 holds. For any domain $D \subset \mathbb{R}^N$, let $\mathcal{J}^\nu_{q,1}(D) = B^{\nu+1}_{q,1}(D) \times B^\nu_{q,1}(D) \times B^{\nu+1}_{q,1}(D)$ and $\mathcal{D}^\nu_{q,1}(D) = B^{\nu+1}_{q,1}(D) \times B^{\nu+2}_{q,1}(D)$. Their norms $\|\cdot\|_{\mathcal{J}^\nu_{q,1}(D)}$ and $\|\cdot\|_{\mathcal{D}^\nu_{q,1}(D)}$ are defined by

$$\begin{aligned} \|(f,\mathbf{g},\mathbf{h})\|_{\mathcal{J}^\nu_{q,1}(D)} &= \|f\|_{B^{\nu+1}_{q,1}(D)} + \|\mathbf{g}\|_{B^\nu_{q,1}(D)} + \|\mathbf{h}\|_{B^{\nu+1}_{q,1}(D)}, \\ \|(\rho,\mathbf{u})\|_{\mathcal{D}^\nu_{q,1}(D)} &= \|\rho\|_{B^{\nu+1}_{q,1}(D)} + \|\mathbf{u}\|_{B^{\nu+2}_{q,1}(D)}. \end{aligned}$$

Moreover, for $(f, \mathbf{g}, \mathbf{h}) \in \mathcal{J}^\nu_{q,1}(D)$, we set

$$\|(f, \mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))\|_{B^{\nu+1,\nu}_{q,1}(D)} = \|f\|_{B^{\nu+1}_{q,1}(D)} + \|(\mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})\|_{B^\nu_{q,1}(D)}.$$

In what follows, we consider the two cases where $\rho_0 = \rho_*$ and $\tilde\eta_0 \not\equiv 0$. We shall show the following theorem.

Theorem 5.1. *Let $1 < q < \infty$. Assume that the following conditions* (1) *or* (2) *holds.*

(1) *If $\eta_0(x) = \rho_*$, then $-1 + 1/q < s < 1/q$.*

(2) *If $\tilde\eta_0(x) \not\equiv 0$ and $\tilde\eta_0(x) \in B^{s+1}_{q,1}(\Omega)$, then $-\min(1-1/q, N/q) < s < 1/q$.*

Then, there exists a large positive number $\lambda_4 \geq 1$ such that for any $\lambda \in \Sigma_{\epsilon,\lambda_4}$, $(f,\mathbf{g},\mathbf{h}) \in \mathcal{J}^s_{q,1}(\Omega)$, problem (122) *admits unique solutions $(\rho,\mathbf{u}) \in \mathcal{D}^s_{q,1}(\Omega)$ satisfying the estimate:*

$$\|\lambda(\rho,\mathbf{u})\|_{\mathcal{H}^s_{q,1}(\Omega)} + \|(\rho,\mathbf{u})\|_{\mathcal{D}^s_{q,1}(\Omega)} \leq C\|(f,\mathbf{g},\mathbf{h})\|_{\mathcal{J}^s_{q,1}(\Omega)}.$$

Moreover, there exist three operators $\mathcal{B}_v(\lambda)$, $\mathcal{C}_m(\lambda)$ and $\mathcal{C}_v(\lambda)$ with

$$\begin{aligned} &\mathcal{B}_v(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_4}, \mathcal{L}(B^s_{q,1}(\Omega)^{m(N)}, B^{s+2}_{q,1}(\Omega)^N)), \\ &\mathcal{C}_m(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_2}, \mathcal{L}(B^{s+1}_{q,1}(\Omega) \times B^s_{q,1}(\Omega)^{m(N)}, B^{s+1}_{q,1}(\Omega)), \\ &\mathcal{C}_v(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_2}, \mathcal{L}(B^{s+1}_{q,1}(\Omega) \times B^s_{q,1}(\Omega)^{m(N)}), B^{s+2}_{q,1}(\Omega)^N) \end{aligned}$$

such that $\mathcal{B}_v(\lambda)$ has (s,σ,q) properties and $\mathcal{C}_m(\lambda)$ and $\mathcal{C}_v(\lambda)$ have generalized resolvent properties for $X = B^{s+1}_{q,1}(\Omega) \times B^s_{q,1}(\Omega)^{m(N)}$ and for any $\lambda \in \Sigma_{\epsilon,\lambda_4}$ in the sense of Definition 14 and solutions ρ and $\mathbf{u}$ are represented by $\rho = \mathcal{C}_m(\lambda)(f, \mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))$ and $\mathbf{u} = \mathcal{B}_v(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}) + \mathcal{C}_v(\lambda)(f, \mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))$.

Proof. In what follows, we shall prove the theorem in the case where $\tilde\eta_0 \not\equiv 0$ only. In fact, in the case where $\eta_0 = \rho_*$ the theorem can be proved by using the similar argument. In (122), setting $\rho = \lambda^{-1}(f - \eta_0 \mathrm{div}\mathbf{u})$ and inserting this formula into the second equations, we have

$$\begin{aligned} \eta_0\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I}) & \\ = \mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f) & \quad \text{in } \Omega, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n} & \\ = \mathbf{h} + \lambda^{-1}P'(\eta_0)f\mathbf{n} & \quad \text{on } \partial\Omega. \end{aligned} \tag{123}$$

For a while, setting $\mathbf{G} = \mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f)$ and $\mathbf{H} = \mathbf{h} + \lambda^{-1}P'(\eta_0)f\mathbf{n}$, we shall solve equations:

$$\begin{aligned} \eta_0\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I}) = \mathbf{G} & \quad \text{in } \Omega, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n} = \mathbf{H} & \quad \text{on } \partial\Omega. \end{aligned} \tag{124}$$

Let $\mathcal{V}_\Omega(\lambda) \in \mathrm{Hol}\,(\Sigma_{\epsilon,\lambda_3}, \mathcal{L}(B^\nu_{q,1}(\Omega)^{m(N)}, B^{\nu+2}_{q,1}(\Omega)^N))$ be the solution operator of equations. Insert the formula $\mathbf{u} = \mathcal{V}_\Omega(\lambda)\mathcal{O}_\lambda(\mathbf{G},\mathbf{H})$ into (124) to obtain

$$\begin{aligned} \eta_0\lambda\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I}) & \\ = \mathbf{G} - \lambda^{-1}\nabla(P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}) & \quad \text{in } \Omega, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n} & \\ = \mathbf{H} + \lambda^{-1}(P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n} & \quad \text{on } \partial\Omega. \end{aligned}$$

We define an operator $\mathcal{P}(\lambda)$ by

$$\mathcal{P}(\lambda)H = (\nabla(P'(\eta_0)\eta_0\mathrm{div}\mathcal{V}_\Omega(\lambda)H), -P'(\eta_0)\eta_0\mathrm{div}(\mathcal{V}_\Omega(\lambda)H)\mathbf{n}).$$

Then, we have

$$(\mathbf{G},\mathbf{H}) - (\lambda^{-1}\nabla(P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}), -\lambda^{-1}(P'(\eta_0)\eta_0\mathrm{div}\mathbf{u}\mathbb{I})\mathbf{n}) = (\mathbf{I} - \lambda^{-1}\mathcal{P}\mathcal{O}_\lambda)(\mathbf{G},\mathbf{H}). \tag{125}$$

We will show that

$$\|\mathcal{O}_\lambda\mathcal{P}(\lambda)H\|_{B^s_{q,1}(\Omega)} \le C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)})\|H\|_{B^s_{q,1}(\Omega)}. \tag{126}$$

Here and in what follows, $C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)})$ denotes some constant depending on ρ_* and $\|\eta_0\|_{B^{s+1}_{q,1}(\Omega)}$ in the case where $\tilde\eta_0 \not\equiv 0$. If we consider the case where $\eta_0 = \rho_*$, $C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)})$ is replaced with simply a constant $C(\rho_*)$.

To prove (126), we shall use Lemma 7 and the fact that $B^{s+1}_{q,1}$ is a Banach algebra (cf. Lemma 5). In fact, noting that $N/q \le s+1$ by Lemma 3, we have

$$\begin{aligned} &\|uv\|_{B^{s+1}_{q,1}(\Omega)} \\ &\le \|(\nabla u)v\|_{B^s_{q,1}(\Omega)} + \|u(\nabla v)\|_{B^s_{q,1}(\Omega)} + \|uv\|_{B^s_{q,1}(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&\leq C(\|u\|_{B^{s+1}_{q,1}(\Omega)}\|v\|_{B^{N/q}_{q,1}(\Omega)} + \|u\|_{B^{N/q}_{q,1}(\Omega)}\|v\|_{B^{s+1}_{q,1}(\Omega)} + \|u\|_{B^{s}_{q,1}(\Omega)}\|v\|_{B^{N/q}_{q,1}(\Omega)}) \\
&\leq C\|u\|_{B^{s+1}_{q,1}(\Omega)}\|v\|_{B^{s+1}_{q,1}(\Omega)}. \qquad (127)
\end{aligned}
$$

To prove (126), recalling that $\eta_0 = \rho_* + \tilde\eta_0$, we write $P'(\eta_0)\eta_0 = P'(\rho_*)\rho_* + \mathcal{P}_1(\tilde\eta_0)$, where we have set

$$
\mathcal{P}_1(r) = P'(\rho_*)r + \int_0^1 P''(\rho_* + \theta r)\,d\theta r(\rho_* + r)
$$

with $r = \tilde\eta_0$. Note that $\mathcal{P}_1(0) = 0$ and $\rho_1 - \rho_* \leq \tilde\eta_0(x) \leq \rho_2 - \rho_*$ as follows from Assumption 10. By Lemma 7, we have

$$
\|\mathcal{P}_1(\tilde\eta_0)\|_{B^{s+1}_{q,1}(\Omega)} \leq C\|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)}.
$$

Thus,

$$
\begin{aligned}
&\|\nabla(P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H)\|_{B^s_{q,1}(\Omega)} \\
&\qquad \leq |P'(\rho_*)\rho_*|\|\nabla\mathrm{div}\mathcal{V}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)} + \|\mathcal{P}_1(\tilde\eta_0)\mathrm{div}\mathcal{V}_\Omega(\lambda)H\|_{B^{s+1}_{q,1}(\Omega)} \\
&\qquad \leq |P'(\rho_*)\rho_*|\|\mathcal{V}_\Omega(\lambda)H\|_{B^{s+2}_{q,1}(\Omega)} + C\|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)}\|\mathrm{div}\mathcal{V}_\Omega(\lambda)H\|_{B^{s+1}_{q,1}(\Omega)}.
\end{aligned}
$$

Using Theorem 3.1, we have

$$
\|\nabla(P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H)\|_{B^s_{q,1}(\Omega)} \leq C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)})\|H\|_{B^\nu_{q,1}(\Omega)}.
$$

Analogously,

$$
\begin{aligned}
&\|(\lambda^{1/2}, \nabla)(P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H)\mathbf{n}\|_{B^s_{q,1}(\Omega)} \\
&\quad \leq C(\|\nabla(P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H)\|_{B^s_{q,1}(\Omega)} + |\lambda|^{1/2}\|P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)}).
\end{aligned}
$$

Here and in the following, we may assume that $|\lambda| \geq \lambda_3 \geq 1$. We see that

$$
\|P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)} \leq (P'(\rho_*)\rho_* + C\|\mathcal{P}_1(\tilde\eta_0)\|_{B^{N/q}_{q,1}(\Omega)})\|\mathrm{div}\mathcal{V}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)}.
$$

Therefore, we have

$$
\begin{aligned}
&\|(\lambda^{1/2}, \nabla)(P'(\eta_0)\eta_0 \mathrm{div}\mathcal{V}_\Omega(\lambda)H)\mathbf{n}\|_{B^s_{q,1}(\Omega)} \\
&\quad \leq C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)})\|(\lambda^{1/2}\bar\nabla, \bar\nabla^2)\mathcal{V}_\Omega(\lambda)H\|_{B^s_{q,1}(\Omega)}.
\end{aligned}
$$

Combining these estimates with Theorem 3.1 yields (126).

Choosing $\lambda_4 \geq \lambda_3$ in such a way that $C(\rho_*, \|\tilde\eta_0\|_{B^{s+1}_{q,1}(\Omega)})\lambda_4^{-1} \leq 1/2$ in (126), we have

$$
\|\lambda^{-1}\mathcal{O}_\lambda\mathcal{P}(\lambda)H\|_{B^s_{q,1}(\Omega)} \leq (1/2)\|H\|_{B^s_{q,1}(\Omega)} \qquad (128)
$$

for any $\lambda \in \Sigma_{\epsilon,\lambda_4}$. Let us define $\mathcal{P}_\infty(\lambda)$ by

$$\mathcal{P}_\infty(\lambda) = \sum_{\ell=0}^\infty (\lambda^{-1}\mathcal{P}(\lambda)\mathcal{O}_\lambda)^\ell$$

for $\lambda \in \Sigma_{\epsilon,\lambda_4}$. In view of (128), we see that $\mathcal{P}_\infty(\lambda) \in \mathcal{L}(\mathcal{H}^\nu_{q,1}(\Omega))$. If we define $\mathbf{v} = \mathcal{V}_\Omega(\lambda)\mathcal{O}_\lambda\mathcal{P}_\infty(\lambda)(\mathbf{G},\mathbf{H})$, then by (125) $\mathbf{v}$ is a unique solution of the following equations:

$$\begin{aligned}
\eta_0\lambda\mathbf{v} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{v}\mathbb{I}) &= \mathbf{G} \qquad \text{in } \Omega,\\
(\alpha\mathbb{D}(\mathbf{v}) + (\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I} + \lambda^{-1}P'(\eta_0)\eta_0\mathrm{div}\mathbf{v}\mathbb{I})\mathbf{n} &= \mathbf{H} \qquad \text{on } \partial\Omega.
\end{aligned}$$

The uniqueness follows from the existence of solutions to the dual problem, which has essentially the same forms.

Let $\tilde{\mathcal{P}}_\infty(\lambda) = \sum_{\ell=0}^\infty(\lambda^{-1}\mathcal{O}_\lambda\mathcal{P}(\lambda))^\ell$, and then by (128), $\tilde{\mathcal{P}}_\infty(\lambda) \in \mathcal{L}(B^s_{q,1}(\Omega))$ and its operator norm does not exceed 2. Moreover, we have $\mathcal{O}_\lambda\mathcal{P}_\infty(\lambda) = \tilde{\mathcal{P}}_\infty(\lambda)\mathcal{O}_\lambda$. Let $\mathcal{W}^2_\Omega(\lambda)H = \mathcal{V}_\Omega(\lambda)\tilde{\mathcal{P}}_\infty H$, and then $\mathbf{v} = \mathcal{W}^2_\Omega(\lambda)\mathcal{O}_\lambda(\mathbf{G},\mathbf{H})$. Moreover, by Theorem 3.1 and the similar argument to the proof of (103), we see that $\mathcal{W}^2_\Omega(\lambda)$ has (s,σ,q) properties for any $\lambda \in \Sigma_{\epsilon,\lambda_4}$.

Setting $\mathbf{G} = \mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f)$ and $\mathbf{H} = \mathbf{h} + \lambda^{-1}P'(\eta_0)f\mathbf{n}$, we define $\mathbf{u}$ by

$$\mathbf{u} = \mathcal{W}^2_\Omega(\lambda)\mathcal{O}_\lambda(\mathbf{g} - \lambda^{-1}\nabla(P'(\eta_0)f), \mathbf{h} + \lambda^{-1}P'(\eta_0)f\mathbf{n}). \tag{129}$$

Obvisouly, $\mathbf{u}$ is a solution of equations (123) and satisfies the estimate:

$$\|\lambda\mathbf{u}\|_{B^s_{q,1}(\Omega)} + \|\mathbf{u}\|_{B^{s+2}_{q,1}(\Omega)} \le C(\rho_*, \|\tilde{\eta}_0\|_{B^{s+1}_{q,1}(\Omega)})(\|f\|_{B^{s+1}_{q,1}(\Omega)} + \|\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})\|_{B^s_{q,1}(\Omega)}).$$

And therefore, setting $\rho = \lambda^{-1}(f - \eta_0\mathrm{div}\mathbf{u})$, we see that ρ and $\mathbf{u}$ are solutions of equations (122), and satisfy the estimate:

$$\begin{aligned}
&\|(\lambda\rho,\rho)\|_{B^{s+1}_{q,1}(\Omega)} + \|(\lambda,\bar{\nabla}^2)\mathbf{u}\|_{B^s_{q,1}(\Omega)}\\
&\quad\le C(\rho_*, \|\tilde{\eta}_0\|_{B^{s+1}_{q,1}(\Omega)})(\|f\|_{B^{s+1}_{q,1}(\Omega)} + \|\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})\|_{B^s_{q,1}(\Omega)}).
\end{aligned}$$

The uniqueness of solutions to equations (122) follows from the uniqueness of solutions of equations (124).

In view of (129), if we define an operator $\mathcal{Z}_\Omega(\lambda)$ by

$$\mathcal{Z}_\Omega(\lambda)(f,H) = \mathcal{W}^2_\Omega(\lambda)H + \lambda^{-1}\mathcal{W}^2_\Omega(\lambda)\mathcal{O}_\lambda(-\nabla(P'(\eta_0)f), P'(\eta_0)f\mathbf{n}), \tag{130}$$

then, we have $\mathbf{u} = \mathcal{Z}_\Omega(\lambda)(f, \mathcal{O}(\mathbf{g},\mathbf{h}))$. Therefore, we define an operator $\mathcal{C}_m(\lambda)$ by

$$\mathcal{C}_m(\lambda)(f,H) = \lambda^{-1}f - \lambda^{-1}\eta_0(x)\mathrm{div}(\mathcal{Z}_\Omega(\lambda)(f,H)).$$

Obviously, $\rho = \mathcal{C}_m(\lambda)(f, \mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))$.

To define $\mathcal{B}_m(\lambda)$ and $\mathcal{C}_m(\lambda)$, we observe that $\sum_{\ell=1}^\infty(\lambda^{-1}\mathcal{O}_\lambda\mathcal{P}(\lambda))^\ell = \lambda^{-1}\mathcal{O}_\lambda\mathcal{P}(\lambda)\mathcal{Q}_\infty(\lambda)$, where we have set $\mathcal{Q}_\infty(\lambda) = \sum_{\ell=0.}^\infty(\lambda^{-1}\mathcal{O}_\lambda\mathcal{P}(\lambda))^\ell$. By (128), $\mathcal{Q}_\infty$ is well-defined and $\|\mathcal{Q}_\infty(\lambda)H\|_{B^s_{q,1}(\Omega)} \le C\|H\|_{B^s_{q,1}(\Omega)}$ for any $\lambda \in \Sigma_{\epsilon,\lambda_4}$. Moreover, $\tilde{\mathcal{P}}_\infty(\lambda) = \mathbf{I} + \lambda^{-1}\mathcal{Q}_\infty(\lambda)$. Thus, in view of (130), we define $\mathcal{B}_v(\lambda)$ and $\mathcal{C}_v(\lambda)$ by

$$\mathcal{B}_v(\lambda)H = \mathcal{V}_\Omega(\lambda)H,$$
$$\mathcal{C}_v(\lambda)(f, H) = \lambda^{-1}\mathcal{V}_\Omega(\lambda)\mathcal{Q}_\infty(\lambda)H + \lambda^{-1}\mathcal{W}^2_\Omega(\lambda)\mathcal{O}_\lambda(-\nabla(P'(\eta_0)f), P'(\eta_0)f\mathbf{n}).$$

Using Theorem 3.1, we see that $\mathcal{B}_v(\lambda)$ has (s, σ, q) properties. Using Theorem 3.1, the similar argument to the proof of (103), and (127), we see that

$$\|(\lambda, \lambda^{1/2}\bar\nabla, \bar\nabla^2)\partial_\lambda^\ell \mathcal{D}(\lambda)(f, H)\|_{B^s_{q,1}(\Omega)} \leq C|\lambda|^{-1-\ell}\|(f, H)\|_{B^s_{q,1}(\Omega)\times\mathcal{H}^s_{q,1}(\Omega)} \quad (\ell = 0, 1)$$

for $\mathcal{D} \in \{\mathcal{C}_m, \mathcal{C}_v\}$. Namely, $\mathcal{C}_m(\lambda)$ and $\mathcal{C}_v(\lambda)$ have generalized resolvent properties with $X = B^{s+1}_{q,1}(\Omega) \times \mathcal{H}^s_{q,1}(\Omega)$ in the sense of Definition 14. This completes the proof of Theorem 5.1. □

6. L_1 maximal regularity, A proof of Theorem 1.1.

In this section, we consider the following evolution equations (1) and we shall prove Theorem 1.1. To prove the theorem, problem (1) is divided into the following two equations:

$$\begin{cases} \partial_t\rho + \eta_0 \mathrm{div}\mathbf{u} = f & \text{in } \Omega \times \mathbb{R}, \\ \eta_0\partial_t\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I}) = \mathbf{g} & \text{in } \Omega \times \mathbb{R}, \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I})\mathbf{n} = \mathbf{h} & \text{on } \partial\Omega \times \mathbb{R}; \end{cases} \tag{131}$$

$$\begin{cases} \partial_t\rho + \eta_0 \mathrm{div}\mathbf{u} = 0 & \text{in } \Omega \times (0,\infty), \\ \eta_0\partial_t\mathbf{u} - \mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I}) = 0 & \text{in } \Omega \times (0,\infty), \\ (\alpha\mathbb{D}(\mathbf{u}) + (\beta-\alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I})\mathbf{n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{132}$$

First, we consider equations (131). We shall prove the following theorem.

THEOREM 6.1. *Let $1 < q < \infty$. Assume that the following conditions* (1) *or* (2) *holds.*

(1) *If $\eta_0(x) = \rho_*$, then $-1 + 1/q < s < 1/q$.*

(2) *If $\tilde\eta_0(x) \not\equiv 0$ and $\tilde\eta_0(x) \in B^{s+1}_{q,1}(\Omega)$, then $-\min(1 - 1/q, N/q) < s < 1/q$.*

Let λ_4 be the positive number given in Theorem 5.1 and $\gamma \geq \lambda_4$. Then, for any right members f, $\mathbf{g}$ and $\mathbf{h}$ with

$$e^{-\gamma t}f \in L_1(\mathbb{R}, B^{s+1}_{q,1}(\Omega)), \quad e^{-\gamma t}\mathbf{g} \in L_1(\mathbb{R}, B^s_{q,1}(\Omega)^N),$$
$$e^{-\gamma t}\mathbf{h} \in L_1(\mathbb{R}, B^{s+1}_{q,1}(\Omega)^N) \cap W^{1/2}_1(\mathbb{R}, B^s_{q,1}(\Omega)^N)$$

for some $\gamma \geq \lambda_5$, problem (131) *admits unique solutions ρ and $\mathbf{u}$ with*

$$e^{-\gamma t}\rho \in W^1_1(\mathbb{R}, B^{s+1}_{q,1}(\Omega)), \quad e^{-\gamma t}\mathbf{u} \in W^1_1(\mathbb{R}, B^s_{q,1}(\Omega)^N) \cap L_1(\mathbb{R}, B^{s+2}_{q,1}(\Omega)^N)$$

possessing the estimate:

$$\begin{aligned}&\|e^{-\gamma t}(\partial_t\rho,\rho)\|_{L_1(\mathbb{R},B^{s+1}_{q,1}(\Omega))}+\|e^{-\gamma t}\partial_t\mathbf{u}\|_{L_1(\mathbb{R},B^{s}_{q,1}(\Omega))}+\|e^{-\gamma t}\mathbf{u}\|_{L_1(\mathbb{R},B^{s+2}_{q,1}(\Omega))}\\&\leq C\|e^{-\gamma t}(\bar\nabla f,\mathbf{g},\Lambda^{1/2}\mathbf{h},\nabla\mathbf{h})\|_{L_1(\mathbb{R},B^{s}_{q,1}(\Omega))}.\end{aligned}$$

Here, the constant C depends on λ_4 but is independent of γ when $\gamma \geq \lambda_4$.

Proof. First, we consider equations (131). Applying the Laplace transform to equations (131), we have

$$\begin{aligned}\lambda u+\eta_0\mathrm{div}\hat{\mathbf{v}}&=\hat f &&\text{in }\Omega,\\ \eta_0\lambda\mathbf{v}-\mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{v})+(\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I}-P'(\eta_0)u\mathbb{I})&=\hat{\mathbf{g}}&&\text{in }\Omega,\\ (\alpha\mathbb{D}(\mathbf{v})+(\beta-\alpha)\mathrm{div}\mathbf{v}\mathbb{I}-P'(\eta_0)u\mathbb{I})\mathbf{n}&=\hat{\mathbf{h}}&&\text{on }\partial\Omega,\end{aligned}$$

where $\hat f=\mathcal{L}[f]$ and $\hat{\mathbf{g}}=\mathcal{L}[\mathbf{g}]$. From Theorem 5.1, it follows that $u=\mathcal{C}_m(\lambda)(f,\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))$ and $\mathbf{v}=\mathcal{B}_v(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})+\mathcal{C}_v(\lambda)(f,\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))$. We define ρ and $\mathbf{u}$ by

$$\begin{aligned}\rho&=\mathcal{L}^{-1}[u]=\mathcal{L}^{-1}[\mathcal{C}_m(\lambda)(f,\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))],\\ \mathbf{u}&=\mathcal{L}^{-1}[\mathbf{v}]=\mathcal{L}[\mathcal{B}_v(\lambda)\mathcal{O}_\lambda(\mathbf{g},\mathbf{h})+\mathcal{C}_v(\lambda)(f,\mathcal{O}_\lambda(\mathbf{g},\mathbf{h}))].\end{aligned}$$

Then, ρ and $\mathbf{u}$ are solutions of equations (131). Moreover, by Propositions 13 and 15, we have

$$\begin{aligned}&\int_{-\infty}^\infty e^{-\gamma t}\|\rho(\cdot,t)\|_{B^{s+1}_{q,1}(\Omega)}\,dt\\&\quad\leq C\int_{-\infty}^\infty e^{-\gamma t}(\|f(\cdot,t)\|_{B^{s}_{q,1}(\Omega)}+\|(\mathbf{g},\Lambda^{1/2}\mathbf{h},\nabla\mathbf{h})(t)\|_{B^{s}_{q,1}(\Omega)})\,dt,\\&\int_{-\infty}^\infty e^{-\gamma t}\|\mathbf{u}(\cdot,t)\|_{B^{s+2}_{q,1}(\Omega)}\,dt\\&\quad\leq C\int_{-\infty}^\infty e^{-\gamma t}(\|f(\cdot,t)\|_{B^{s}_{q,1}(\Omega)}+\|(\mathbf{g},\Lambda^{1/2}\mathbf{h},\nabla\mathbf{h})(t)\|_{B^{s}_{q,1}(\Omega)})\,dt.\end{aligned}\tag{133}$$

For the estimate of the time derivatives, we use equations, and then writing $\partial_t\rho(t)=-\rho_0\mathrm{div}\mathbf{u}+f$, we have

$$\begin{aligned}&\int_{-\infty}^\infty e^{-\gamma t}\|\partial_t\rho(t)\|_{B^{s+1}_{q,1}(\Omega)}\,dt\\&\leq\int_{-\infty}^\infty e^{-\gamma t}(\|f(t)\|_{B^{s+1}_{q,1}(\Omega)}+C(\rho_*,\|\tilde\eta_0\|_{B^{N/q+1}_{q,1}(\Omega)})\|\mathbf{u}\|_{B^{s+2}_{q,1}(\Omega)})\,dt\\&\leq C(\rho_*,\|\tilde\eta_0\|_{B^{N/q+1}_{q,1}(\Omega)})\int_{-\infty}^\infty e^{-\gamma t}(\|f(t)\|_{B^{s}_{q,1}(\Omega)}+\|(\mathbf{g},\Lambda^{1/2}\mathbf{h},\nabla\mathbf{h})(t)\|_{B^{s}_{q,1}(\Omega)})\,dt.\end{aligned}$$

To estimate $\partial_t\mathbf{u}$, we have to investigate the multiplication $\eta_0(x)^{-1}$. From (2) we have the following lemma:

Lemma 21. *Let $1<q<\infty$ and $-\min(1-1/q,N/q)<s<1/q$. Let $\eta_0(x)=\rho_*+\tilde\eta_0(x)$ such that $\tilde\eta_0(x)\in B^{N/q}_{q,1}(\Omega)$ and $\eta_0(x)$ satisfies Assumption 10. Then, for any*

$u \in B^s_{q,1}(\Omega)$, there holds

$$\|u\eta_0^{-1}\|_{B^s_{q,1}(\Omega)} \leq \rho_*^{-1}\|u\|_{B^s_{q,1}(\Omega)} + C\|\tilde\eta_0\|_{B^{N/q}_{q,1}(\Omega)}\|u\|_{B^s_{q,1}(\Omega)}$$

for some constant $C > 0$ depending on ρ_*, ρ_1 and ρ_2.

Proof. Note that $\eta_0(x)^{-1} = \rho_*^{-1} - \tilde\eta_0(x)(\rho_*\eta_0(x))^{-1}$. If we define $F(t)$ by $F(t) = t(\rho_*(\rho_* + t))^{-1}$, then $F(\tilde\eta_0(x)) = \tilde\eta_0(x)(\rho_*\eta_0(x))^{-1}$. From (2), $\rho_1 - \rho_* < \tilde\eta_0(x) < \rho_2 - \rho_*$ for any $x \in \Omega$. Set $I = (\rho_1 - \rho_*, \rho_2 - \rho_*)$ and let $\tilde s$ is the smallest integer such that $\tilde s \geq N/q$. Noticing that $0 \in I$ and using Lemma 7 we have

$$\|F(\tilde\eta_0)\|_{B^{N/q}_{q,1}(\Omega)} \leq C(1 + \|\tilde\eta_0\|_{L_\infty(\Omega)})^{\tilde s}\|F'\|_{BC^{\tilde s}(I,\mathbb{R})}\|\tilde\eta\|_{B^{N/q}_{q,1}(\Omega)}.$$

Notice that $\|\tilde\eta_0\|_{L_\infty(\Omega)} \leq \max(\rho_* - \rho_1, \rho_2 - \rho_*)$. Thus, using (13), we have

$$\|u\eta_0^{-1}\|_{B^s_{q,1}(\Omega)} \leq \rho_*^{-1}\|u\|_{B^s_{q,1}(\Omega)} + C\|\tilde\eta_0\|_{B^{N/q}_{q,1}(\Omega)}\|u\|_{B^s_{q,1}(\Omega)}.$$

This completes the proof of Lemma 21. □

From equations (132), we write

$$\partial_t\mathbf{u} = \eta_0^{-1}(\mathrm{Div}\,(\alpha\mathbb{D}(\mathbf{u}) + (\beta - \alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I}) + \eta_0^{-1}\mathbf{g}.$$

Using (133), we have

$$\begin{aligned}
&\int_{-\infty}^\infty e^{-\gamma t}\|\partial_t\mathbf{u}(t)\|_{B^s_{q,1}(\Omega)}\,dt \\
&\leq C(\rho_*, \|\tilde\eta_0\|_{B^{N/q+1}_{q,1}(\Omega)})\int_{-\infty}^\infty e^{-\gamma t}(\|\mathbf{u}(t)\|_{B^{s+2}_{q,1}(\Omega)} + \|\rho(t)\|_{B^{s+1}_{q,1}(\Omega)})\,dt \\
&\leq C(\rho_*, \|\tilde\eta_0\|_{B^{N/q+1}_{q,1}(\Omega)})\int_{-\infty}^\infty e^{-\gamma t}(\|f(\cdot, t)\|_{B^{s+1}_{q,1}(\Omega)} + \|(\mathbf{g}, \Lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})(t)\|_{B^s_{q,1}(\Omega)})\,dt.
\end{aligned}$$

This completes the proof of Theorem 6.1. □

We now consider equations (132). We shall show the generation of C^0 analytic semigroup $\{T(t)\}_{t\geq 0}$ defined on $\mathcal{H}^s_{q,1}(\Omega)$. To formulate equations (132) in the semigroup setting, we define a set $\mathcal{D}^\nu_{q,1}(\Omega)$ by setting

$$\mathcal{D}^\nu_{q,1} = \{(\rho, \mathbf{u}) \in B^{\nu+1}_{q,1}(\Omega) \times B^{\nu+2}_{q,1}(\Omega) \mid (\alpha\mathbb{D}(\mathbf{u}) + (\beta - \alpha)\mathrm{div}\mathbf{u}\mathbb{I} - P'(\eta_0)\rho\mathbb{I})\mathbf{n}|_{\partial\Omega} = 0\}.$$

Let $\mathcal{A}_m$, $\mathbf{A}_v$ and $\mathcal{A}$ be operators defined by

$$\begin{aligned}
&\mathcal{A}_m(\rho, \mathbf{u}) = \eta_0\mathrm{div}\mathbf{u}, \quad \mathcal{A}_v(\rho, \mathbf{u}) = \eta_0^{-1}(-\alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\mathbf{u} + \nabla(P'(\eta_0)\rho)), \\
&\mathcal{A}(\rho, \mathbf{u}) = (\mathcal{A}_m(\rho, \mathbf{u}), \mathcal{A}_v(\rho, \mathbf{u}))
\end{aligned}$$

for $(\rho, \mathbf{u}) \in \mathcal{D}^\nu_{q,1}$. Then, problem (132) is written as

$$(\partial_t + \mathcal{A})(\rho, \mathbf{u}) = (0, 0) \quad \text{for } t > 0, \quad (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) \quad \text{in } \Omega.$$

From Theorem 5.1, $\Sigma_{\epsilon,\lambda_4}$ is contained in a resolvent set of the operator $\mathcal{A}$ and for $\lambda \in \Sigma_{\epsilon,\lambda_4}$, we have

$$\|\lambda(\lambda\mathbf{I}+\mathcal{A})^{-1}(f,\mathbf{g})\|_{\mathcal{H}^s_{q,1}(\Omega)} \le C\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,1}(\Omega)}.$$

Therefore, there exists a C_0 analytic semigroup $\{T(t)\}_{t\ge0}$ associated with equations (132). Moreover, we have the L_1 maximal regularity of $\{T(t)\}_{t\ge0}$ as follows.

Theorem 6.2. *Let $1<q<\infty$. Assume that the following conditions* (1) *or* (2) *holds.*

(1) *If $\eta_0(x)=\rho_*$, then $-1+1/q<s<1/q$.*

(2) *If $\tilde\eta_0(x)\not\equiv 0$ and $\tilde\eta_0(x)\in B^{s+1}_{p,1}(\Omega)$, then $-\min(1-1/q,N/q)<s<1/q$.*

Let $T(t)(\rho_0,\mathbf{u}_0)=(T_m(t)(\rho_0,\mathbf{u}_0),T_v(t)(\rho_0,\mathbf{u}_0))$, that is $T_m(t)$ and $T_v(t)$ are the mass density part and the velocity part, respectively. Then, there holds

$$\begin{aligned}\int_0^\infty e^{-\gamma t}(\|(\partial_t,\bar\nabla^2)T_v(t)(f,\mathbf{g})\|_{B^s_{q,1}(\Omega)} + \|(1,\partial_t)T_m(t)(f,\mathbf{g})\|_{B^{s+1}_{q,1}(\Omega)})\,dt \\ \le C\|(f,\mathbf{g})\|_{\mathcal{H}^s_{q,1}(\Omega)}.\end{aligned} \tag{134}$$

Proof. In view of Theorem 5.1, we can write

$$(\lambda\mathbf{I}+\mathcal{A})^{-1}(f,\mathbf{g}) = (\mathcal{C}_m(\lambda)(f,\mathcal{O}_\lambda(\mathbf{g},0)),\mathcal{B}_v(\lambda)\mathcal{O}_\lambda(\mathbf{g},0)+\mathcal{C}_v(\lambda)(f,\mathcal{O}_\lambda(\mathbf{g},0)).$$

As is known well, $T(t)(f,\mathbf{g})=\mathcal{L}^{-1}[(\lambda\mathbf{I}+\mathcal{A})^{-1}(f,\mathbf{g})]$. By Proposition 13 and 15 we see that (134) hold. This completes the proof of Theorem 6.2. □

Let ρ and $\mathbf{u}$ be solutions given in Theorem 6.1. By the time trace theorem, we see that

$$\begin{aligned}\|(\rho,\mathbf{u})|_{t=0}\|_{B^{s+1}_{q,1}(\Omega)\times B^s_{q,1}(\Omega)} &\le C(\|e^{-\gamma t}(\partial_t\rho,\rho)\|_{L_1((0,\infty),B^{s+1}_{q,1}(\Omega))} \\ &\quad + \|e^{-\gamma t}\mathbf{u}\|_{L_1((0,\infty),B^{s+2}_{q,1}(\Omega))} + \|e^{-\gamma t}\partial_t\mathbf{u}\|_{L_1((0,\infty),B^s_{q,1}(\Omega))}).\end{aligned}$$

Thus, combining Theorems 6.1 and 6.2, we can prove the existence part of Theorem 1.1. The uniqueness of solutions to equations (1) can be proved using Duhamel's principle for the analytic semigroup $\{T(t)\}_{t\ge0}$. This completes the proof of Theorem 1.1.

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