

Values and derivative values at nonpositive integers of generalized multiple Hurwitz zeta functions

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Abstract. We establish the meromorphic continuation of certain multiple zeta functions of generalized Hurwitz type. From this meromorphic continuation, we obtain explicit formulae for their (derivative) values at nonpositive integers along a given direction. As an application, we provide explicit formulae for special values of Witten zeta functions of Lie groups of rank 2. Furthermore, by employing a Meinardus-type theorem, we investigate the asymptotic behavior of the number of n -dimensional representations of the exceptional Lie group G_2 .

1. Introduction

Consider positive integers $r, m \in \mathbb{N}$, a tuple $\mathbf{x} = (x_1, \dots, x_r)$ and a matrix $\mathbf{c} = (c_{q,p})_{1 \leq q \leq m, 1 \leq p \leq r}$ such that $x_1, \dots, x_r > 0$ and $\Re(c_{1,1}), \dots, \Re(c_{m,r}) > 0$. We define the following Dirichlet series

$$\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s}) := \sum_{n_1, \dots, n_r \geq 0} \prod_{i=1}^r (n_i + x_i)^{-s_i} \prod_{j=1}^m \left(\sum_{i=1}^r c_{j,i} (n_i + x_i) \right)^{-s_{r+j}} \quad (1)$$

which converges absolutely for $\Re(s_1), \dots, \Re(s_{r+m}) > 1$. This series is a natural generalization of both Shintani zeta functions and multiple zeta functions of generalized Mordell–Tornheim type, and may be regarded as a multiple zeta function of generalized Hurwitz type. The analytic properties of various classes of multiple zeta functions have been studied by many, including H. Mellin, P. Cassou-Noguès, P. Sargos, B. Lichtin, M. Kaneko, K. Matsumoto, J. Zhao, D. Essouabri, and Y. Komori. In [Kom10], Komori studied the analytic continuation and the singularities of multiple zeta functions of generalized Hurwitz–Lerch type. He proved that these functions admit a meromorphic continuation to the whole complex space and that nonpositive integers are points of indeterminacy. In particular, the values at nonpositive integers of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ are not well-defined. Values at nonpositive integers can nevertheless be assigned by taking successive limits, as introduced in [AET01]. One can also consider directional values, which consist of approaching a tuple $-\mathbf{N} = (-N_1, \dots, -N_{r+m}) \in \mathbb{Z}_{\leq 0}^{r+m}$ along a direction $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{r+m}) \in \mathbb{C}^{r+m}$. Komori proved in [Kom10] that these directional values are well-defined for a large class of zeta functions, assuming certain conditions on the direction $\boldsymbol{\theta}$. He also gave a non-explicit formula for directional values at nonpositive integers involving generalized Bernoulli numbers.

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The *Witten zeta function* of a compact connected semisimple Lie group G corresponds to the Dirichlet series

$$\zeta_G(s) = \sum_{\varphi} (\dim \varphi)^{-s} \quad (\Re(s) \gg 1),$$

where we sum over all (isomorphism classes of) finite-dimensional irreducible representations of G^1 . Using the Weyl dimension formula, $\zeta_G(s)$ can be expressed as an explicit series involving linear forms in the denominator. A first example is the single-variable Mordell–Tornheim zeta function $\zeta_{\mathrm{SU}_3}(s) = \sum_{n_1, n_2 \geq 1} \frac{2^s}{[n_1 n_2 (n_1 + n_2)]^s}$. The other Witten zeta functions of simple Lie groups of rank 2 are

$$\zeta_{\mathrm{PU}_3}(s) = \sum_{n_1, n_2 \geq 1, n_1 \equiv n_2 \pmod{3}} \frac{2^s}{[n_1 n_2 (n_1 + n_2)]^s}, \quad (2)$$

$$\zeta_{\mathrm{Spin}_5}(s) = \sum_{n_1, n_2 \geq 1} \frac{6^s}{[n_1 n_2 (n_1 + n_2)(n_1 + 2n_2)]^s}, \quad (3)$$

$$\zeta_{\mathrm{SO}_5}(s) = \sum_{n_1 \geq 0, n_2 \geq 1} \frac{6^s}{[(2n_1 + 1)n_2(2n_1 + n_2 + 1)(2n_1 + 2n_2 + 1)]^s}, \quad (4)$$

$$\zeta_{\mathrm{G}_2}(s) = \sum_{n_1, n_2 \geq 1} \frac{120^s}{[n_1 n_2 (n_1 + n_2)(n_1 + 2n_2)(n_1 + 3n_2)(2n_1 + 3n_2)]^s}. \quad (5)$$

In [KMT23, Chap. 15], these zeta functions and their multiple variable generalization called zeta functions of weight lattices of Lie groups have been introduced and studied. They constitute a key motivation for investigating directional values of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$. We point out that Witten zeta functions of rank 3 and higher are not of the form of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ as defined in (1), because we assume all the coefficients $c_{q,p}$ to be nonzero. This condition is crucial for the techniques used in the meromorphic continuation of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$, and we refer to Remark 10 for more details on the matter.

Recently, the analytic properties of Witten zeta functions have been studied for their connection with asymptotic formulae for some generalized partition sequence. We denote by $r_G(n)$ (resp. $f_G(n)$) the number of isomorphism classes of n -dimensional representations (resp. irreducible representations) of a Lie group G . Such a sequence can be seen as a generalization of partition numbers, because the generating series of $r_G(n)$ corresponds to an infinite product. Indeed, we have that

$$\sum_{n=0}^{+\infty} r_G(n) q^n = \prod_{n=1}^{+\infty} \frac{1}{(1 - q^n)^{f_G(n)}}. \quad (6)$$

Romik [Rom17] and Bridges, Brindle, Bringmann, and Franke [BBBF24] proved an asymptotic formula for $r_{\mathrm{SU}_3}(n)$ and $r_{\mathrm{Spin}_5}(n)$ respectively. Both formulae feature residues at positive poles of $\zeta_{\mathrm{SU}_3}(s)$ (resp. $\zeta_{\mathrm{Spin}_5}(s)$), and special values $\zeta_{\mathrm{SU}_3}(0)$, $\zeta'_{\mathrm{SU}_3}(0)$ (resp. $\zeta_{\mathrm{Spin}_5}(0)$, $\zeta'_{\mathrm{Spin}_5}(0)$). All these residues and values are now known, except for the special

¹Witten zeta functions are also defined for Lie algebras. When G is simply connected, the Witten zeta function of G coincides with the Witten zeta function of the Lie algebra of G .

value $\zeta'_{\text{Spin}_5}(0)$. We provide a formula for this value in Theorem B, which now makes the asymptotic formula of $r_{\text{Spin}_5}(n)$ fully explicit.

Main results. Consider a tuple $-\mathbf{N} = (-N_1, \dots, -N_{r+m}) \in \mathbb{Z}_{\leq 0}^{r+m}$, and a direction $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{r+m}) \in \mathbb{C}^{r+m}$ satisfying the *non-vanishing condition*

$$\forall \mathcal{A} \subseteq \llbracket 1, r \rrbracket, \quad \sum_{a \in \mathcal{A}} \theta_a + \sum_{q=1}^m \theta_{r+q} \neq 0. \quad (7)$$

We define the *directional values* and *directional derivative values* of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ as the two limits

$$\begin{aligned} \zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N}) &:= \lim_{\boldsymbol{\theta}} \zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}), \\ \zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N}) &:= \lim_{\boldsymbol{\theta}} \partial_z (\zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta})). \end{aligned}$$

Theorem 6 establishes an explicit meromorphic continuation to the whole complex space of the function $\mathbf{s} \mapsto \zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$. We also show that the single-variable function $z \mapsto \zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta})$ is regular at $z = 0$, which justifies the existence of the two limits above. We then derive the following theorem on those two directional limits.

THEOREM A. *For any tuple $-\mathbf{N} = (-N_1, \dots, -N_{r+m}) \in \mathbb{Z}_{\leq 0}^{r+m}$, and any direction $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{r+m}) \in \mathbb{C}^{r+m}$ satisfying the non-vanishing condition, we have that*

- i) the directional value $\zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N})$ can be written down explicitly in terms of Bernoulli polynomials $B_k(x_i)$ ($0 \leq k, 1 \leq i \leq r$);*
- ii) the directional derivative value $\zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N})$ can be written down explicitly in terms of Bernoulli polynomials $B_k(x_i)$ ($0 \leq k, 1 \leq i \leq r$), and of derivative values of Hurwitz zeta function and generalized Barnes zeta functions at nonpositive integers.*

The explicit formulae for the directional (derivative) values are contained in Theorems 20 and 21. From both of them, we obtain explicit formulae for special values of Witten zeta functions of rank 2.

THEOREM B. *We have*

G	$\zeta_G(0)$	$\zeta_G(-1)$	$\zeta_G(-3)$	$\zeta_G(-5)$	$\zeta'_G(0)$
PU_3	$\frac{1}{3}$	0	0	0	$\ln(2\pi) + \frac{1}{3} \ln 2$
Spin_5	$\frac{3}{8}$	$-\frac{11}{26880}$	$-\frac{509}{37847040}$	$-\frac{3553343203}{24624192552960}$	$\frac{3}{2} \ln(2\pi) - \frac{1}{4} \ln 2 + \frac{3}{8} \ln(3!)$
SO_5	0	$\frac{11}{8960}$	$\frac{1527}{1802240}$	$\frac{110153639293}{746187653120}$	$\frac{3}{4} \ln 2$
G_2	$\frac{5}{12}$	$\frac{6641}{62705664}$	$\frac{12522872818983257}{109242202556140093440}$	$\frac{10901501636536860814654034179423}{18474145623495767864012640878592}$	$\frac{5}{2} \ln(2\pi) - \frac{1}{2} \ln(3!) + \frac{5}{12} \ln(5!)$

Table 1. Special values of Witten zeta functions of rank 2

We checked that all of these Witten zeta functions vanish at small negative even integers (up to $s = -10$). This fact was predicted in [KO13] by the Kurokawa–Ochiai conjecture when $s = -2$. Au recently showed in [Au24] a lower bound on the vanishing order at negative integers of $\zeta_G(s)$ when G is simply connected, which implies in particular the vanishing of $\zeta_{\mathrm{Spin}_5}(s)$ and $\zeta_{\mathrm{G}_2}(s)$ at negative even integers. It is worth noting that the formulae for the special values of $\zeta_{\mathrm{Spin}_5}(s)$ and $\zeta_{\mathrm{G}_2}(s)$ were rediscovered in [Au25].

Applying a Meinardus-type theorem, we establish an almost-explicit asymptotic formula for the number of n -dimensional representations of the exceptional Lie group G_2 , denoted by $r_{\mathrm{G}_2}(n)$.

THEOREM C. *For $N \geq 1$, there exist explicit constants $\tilde{B}_2, \dots, \tilde{B}_{N+1}$ such that*

$$r_{\mathrm{G}_2}(n) \underset{n \rightarrow +\infty}{=} \frac{C}{n^{\frac{9}{16}}} \exp\left(A_1 n^{\frac{1}{4}} + A_2 n^{\frac{3}{20}} + A_3 n^{\frac{1}{20}}\right) \left(1 + \sum_{j=2}^{N+1} \frac{\tilde{B}_j}{n^{\frac{j-1}{20}}} + O\left(n^{-\frac{N+1}{20}}\right)\right),$$

where A_1, A_2, A_3, C are defined in §7.2.

The coefficients in this formula depend on special values of $\zeta_{\mathrm{G}_2}(s)$, and on its residues at positive poles. We point out that these residues were expressed using integrals difficult to evaluate in [Rut24], and will not be computed in this paper. Recently, Au obtained explicit formulae for the residues at positive poles of $\zeta_{\mathrm{G}_2}(s)$ in [Au25], thus the asymptotic formula for $r_{\mathrm{G}_2}(n)$ is now fully explicit.

Paper’s layout. In §2, we fix some notation and introduce some functions used throughout the text. In §3, we show the meromorphic continuation of $\zeta(\mathbf{c}, \mathbf{x}, s)$ using Crandall’s expansion techniques [Cra12], and following ideas of [BD18] and [Ono21]. In §4, we compute the directional values $\zeta(\mathbf{c}, \mathbf{x}, -\frac{N}{\theta})$ and $\zeta'(\mathbf{c}, \mathbf{x}, -\frac{N}{\theta})$. Both formulae feature several parameter-dependent integrals studied in §5, and the values of $\zeta'(\mathbf{c}, \mathbf{x}, -\frac{N}{\theta})$ depend on derivative values of both Hurwitz and Barnes zeta functions at nonpositive integers, studied in §6. In §6, we also compute the special values of Witten zeta functions of rank 2. In §7, we review a generalized Meinardus theorem proved in [BBBF24] and use it to derive an asymptotic formula for the number of representations $r_{\mathrm{G}_2}(n)$. Finally, in §8, we compute some coefficients featured in the formula of $\zeta'(\mathbf{c}, \mathbf{x}, -\frac{N}{\theta})$.

2. Notation

For all $s \in \mathbb{C}$, we write $s = \sigma + i\tau$, with $\sigma, \tau \in \mathbb{R}$. For any $x \in \mathbb{R}$, we set $H_x := \{s \in \mathbb{C} \mid \sigma > x\}$, and let \bar{H}_x denote its closure in \mathbb{C} . We denote by $D_a(r)$ the open disk centered at $a \in \mathbb{C}$ with radius $r > 0$, and let $\bar{D}_a(r)$ denote its closure in \mathbb{C} . We write $\llbracket k, n \rrbracket := \{k, \dots, n\}$ for $k, n \in \mathbb{N}$. Throughout the paper, we assume that a product over the empty set is 1, and a sum over the empty set is 0.

For any finite sets $\mathcal{B} \subseteq \mathcal{A}$ and any complex tuple $\mathbf{z} = (z_a)_{a \in \mathcal{A}}$, we write $|\mathbf{z}|_{\mathcal{B}} := \sum_{b \in \mathcal{B}} z_b$. If $\mathcal{B} = \mathcal{A}$, we simply write $|\mathbf{z}| := |\mathbf{z}|_{\mathcal{A}}$. Note that if $\mathcal{A} = \emptyset$, then \mathbf{z} corresponds to the empty tuple \emptyset , thus by convention $|\mathbf{z}| = 0$.

For any finite sets \mathcal{A}, \mathcal{B} and any integer matrix $\mathbf{u} = (u_{a,b})_{(a,b) \in \mathcal{A} \times \mathcal{B}}$, we set $\mathbf{u}_{\bullet b} :=$

$(u_{a,b})_{a \in \mathcal{A}}$ for all $b \in \mathcal{B}$, and $\mathbf{u}_{a \bullet} := (u_{a,b})_{b \in \mathcal{B}}$ for all $a \in \mathcal{A}$.

To simplify the notation of certain finite sums in the paper, we introduce two special sum notations. Let $m \in \mathbb{Z}_{\geq 1}$, $k \in \mathbb{Z}_{\geq 0}$, $j \neq \ell \in \llbracket 1, m \rrbracket$ be integers, and let $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$ be a tuple. The sum $\sum_{\substack{\hat{j}, \ell \\ u_{1,1} + \dots + u_{1,k} = n_1 \\ \dots \\ u_{m,1} + \dots + u_{m,k} = n_m}}^{\hat{j}, \ell}$ denotes the finite sum over all matrices $\mathbf{u} = (u_{q,p})_{\substack{q \in \llbracket 1, m \rrbracket \setminus \{j, \ell\} \\ p \in \llbracket 1, k \rrbracket}}$ with nonnegative integer coefficients such that $u_{q,1} + \dots + u_{q,k} = n_q$ ($1 \leq q \leq m$, $q \neq j, \ell$). Similarly, the sum $\sum_{\substack{\hat{j} \\ u_{1,1} + \dots + u_{1,k} = n_1 \\ \dots \\ u_{m,1} + \dots + u_{m,k} = n_m}}^{\hat{j}}$ denotes the finite sum over all matrices $\mathbf{u} = (u_{q,p})_{\substack{q \in \llbracket 1, m \rrbracket \setminus \{j\} \\ p \in \llbracket 1, k \rrbracket}}$ with nonnegative integer coefficients such that $u_{q,1} + \dots + u_{q,k} = n_q$ ($1 \leq q \leq m$, $q \neq j$).

REMARK 1. i) If $k = 0$ or $m = 1, 2$ (resp. $k = 0$ or $m = 1$), then the only admissible \mathbf{u} in the first sum (resp. the second sum) corresponds to the empty tuple \emptyset .

ii) If there exists an integer $q \neq j, \ell$ (resp. $q \neq j$) such that $n_q < 0$, then the first sum (resp. the second sum) is 0 by convention.

We now introduce some terms and functions that will appear in the main results. We denote the n -th harmonic number by $h_n := \sum_{k=1}^n \frac{1}{k}$. We denote the binomial and multinomial coefficients by

$$\binom{s}{n} := \frac{s(s-1) \cdots (s-n+1)}{n!} \quad (s \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0})$$

$$\binom{n}{\mathbf{k}} := \frac{n!}{k_1! \cdots k_\ell!} \quad (n \in \mathbb{Z}_{\geq 0}, \mathbf{k} = (k_1, \dots, k_\ell) \in \mathbb{Z}_{\geq 0}^\ell \text{ s.t. } \sum_{j=1}^{\ell} k_j = n).$$

We assume that $h_0 = 0$, and if $\ell = 0$ then \mathbf{k} is the empty tuple \emptyset , and $\binom{0}{\emptyset} = 1$.

DEFINITION 2. The *Lerch zeta function* $\phi(z, s, x)$ is the meromorphic continuation in s to the whole complex plane \mathbb{C} of the Dirichlet series $\sum_{n=0}^{+\infty} \frac{z^n}{(n+x)^s}$, ($|z| \leq 1, \sigma > 1, \Re(x) > 0$). We denote by $\zeta(s, x) := \phi(1, s, x)$ the *Hurwitz zeta function*.

DEFINITION 3. Let $\mathbf{c} = (c_1, \dots, c_r) \in H_0^r$, $\mathbf{x} = (x_1, \dots, x_r) \in H_0^r$, and $\mathbf{M} = (M_1, \dots, M_r) \in \mathbb{Z}_{\geq 0}^r$. The *generalized Barnes zeta function* is defined as the meromorphic continuation to \mathbb{C} of the Dirichlet series

$$\zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}) := \sum_{n_1, \dots, n_r \geq 0} \frac{(n_1 + x_1)^{M_1} \cdots (n_r + x_r)^{M_r}}{(c_1(n_1 + x_1) + \cdots + c_r(n_r + x_r))^s}.$$

Its derivative with respect to s is denoted by $\zeta'_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}) := \partial_s \zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c})$. By setting $\mathbf{M} = \mathbf{0} := (0, \dots, 0)$, we get the classical Barnes zeta function

$$\zeta_B(s, \mathbf{0}, \mathbf{x} | \mathbf{c}) = \sum_{n_1, \dots, n_r \geq 0} \frac{1}{(c_1(n_1 + x_1) + \dots + c_r(n_r + x_r))^s}.$$

DEFINITION 4. We denote the *Euler Gamma function* by Γ , and the *Euler–Mascheroni constant* by $\gamma := -\Gamma'(1) = 0.577215\dots$ We define the *upper* (resp. *lower*) *incomplete gamma function* by

$$\begin{aligned} \Gamma(s, t, z) &:= \int_t^{+\infty} e^{-zy} y^{s-1} dy && (s \in \mathbb{C}, t > 0, \Re(z) > 0), \\ \gamma(s, t, z) &:= \int_0^t e^{-zy} y^{s-1} dy && (\sigma > 0, t > 0, \Re(z) > 0). \end{aligned}$$

REMARK 5. If $z = 1$, we recover the classical incomplete gamma functions used in the literature (see [EMOT81, Chap. IX]). We see that, for all $t > 0$, the functions $(s, z) \in H_0 \times H_0 \mapsto \gamma(s, t, z)$ and $(s, z) \in \mathbb{C} \times H_0 \mapsto \Gamma(s, t, z)$ are holomorphic, and we have the identity

$$\Gamma(s)z^{-s} = \Gamma(s, t, z) + \gamma(s, t, z) \quad (\sigma > 0, t > 0, \Re(z) > 0). \quad (8)$$

3. Meromorphic continuation of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$

For this section and the following section, we consider the linear forms $l_q(\mathbf{y}) := c_{q,1}y_1 + \dots + c_{q,r}y_r$ ($1 \leq q \leq m$), assuming $\Re(c_{q,p}) > 0$ for all q, p . Our goal is to prove an explicit meromorphic continuation of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ in terms of the Hurwitz zeta function, the Euler gamma function, and certain parameter-dependent integrals which will later be evaluated at special values.

THEOREM 6. *The function $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ has an explicit meromorphic continuation with respect to $\mathbf{s} = (s_1, \dots, s_{r+m})$ to the whole complex space given by*

$$\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s}) = H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) + F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) \quad (t > 0 \text{ small enough}),$$

where the function $F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ is meromorphic (resp. $H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ is holomorphic) and satisfies (35) (resp. (15)). Moreover, its singularities belong to the set $\mathcal{S}_{r,m} \subset \mathbb{C}^{r+m}$ defined as a union of all the hyperplanes

$$\begin{aligned} \mathcal{H}_{p,n}^{(1)} &:= \{\mathbf{s} \in \mathbb{C}^{r+m} \mid s_p = n\} && (1 \leq p \leq r, n \in \mathbb{N}), \\ \mathcal{H}_{\mathcal{A},n}^{(2)} &:= \left\{ \mathbf{s} \in \mathbb{C}^{r+m} \mid s_{r+1} + \dots + s_{r+m} + \sum_{a \in \mathcal{A}} s_a = n \right\} && (\mathcal{A} \subseteq \llbracket 1, r \rrbracket, n \in \mathbb{Z}_{\leq r}). \end{aligned}$$

Furthermore, if $\mathbf{s} \in \mathcal{S}_{r,m} \setminus \left(\bigcup_{\substack{1 \leq p \leq r \\ 1 \leq n}} \mathcal{H}_{p,n}^{(1)} \right)$ is a singularity, then its multiplicity is 1.

3.1. Erdélyi's formula

Erdélyi's formula [EMOT81, §1.11], as stated in the following lemma, is crucial for rewriting integrands involving the Lerch zeta function, which appear in the next subsection.

LEMMA 7 ([Ono21, Lemma 2.2]). *Let $u \in \mathbb{C} \setminus (-\infty, 0]$. We have*

$$e^{-xu} \phi(e^{-u}, s, x) = \Gamma(1-s)u^{s-1} + \sum_{k=0}^{+\infty} \frac{(-u)^k}{k!} \zeta(s-k, x) \quad (|u| < 2\pi, s \in \mathbb{C} \setminus \mathbb{N}, x > 0).$$

To prove that the above series is absolutely convergent, Onodera established a bound using the Hurwitz formula. This bound will also be of use later in the paper.

LEMMA 8 ([Ono21, Lemma 2.1]). *For any $\delta \in \mathbb{R}_{>0}$, $R \in \mathbb{N}$, we set $\overline{D}(\delta, R) := \{s \in \mathbb{C} \mid \min_{n \in \mathbb{N}} |s-n| \geq \delta, |s| \leq R\}$. We then have*

$$|\partial_s^n \zeta(s-k, x)| \ll_{R, \varepsilon, \delta} k!(k+1)^{R+\varepsilon} (2\pi)^{-k} \quad (k, n \in \mathbb{Z}_{\geq 0}, s \in \overline{D}(\delta, R), \varepsilon > 0).$$

3.2. Crandall's expansion

Let $\mathbf{s} = (s_1, \dots, s_{r+m}) \in \mathbb{C}^{r+m}$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$. For all tuples $\mathbf{n} \in \mathbb{Z}_{\geq 0}^r$ we have $\Gamma(s_{r+q})(l_q(\mathbf{n} + \mathbf{x}))^{-s_{r+q}} = \int_0^{+\infty} e^{-l_q(\mathbf{n} + \mathbf{x})y} y^{s_{r+q}-1} dy$, thus we can write

$$\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s}) = \sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{p=1}^r (n_p + x_p)^{s_p}} \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{q=1}^m \frac{e^{-l_q(\mathbf{n} + \mathbf{x})y_q} y_q^{s_{r+q}-1}}{\prod_{q=1}^m \Gamma(s_{r+q})} dy_1 \cdots dy_m.$$

Next, we split the range of each integral such that $[0, +\infty) = [0, t] \cup (t, +\infty)$ for any given $t > 0$. By using (8), one can then prove that the general term of the Dirichlet series associated with $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ satisfies the following identity

$$\begin{aligned} & \frac{1}{\prod_{p=1}^r (n_p + x_p)^{s_p} \prod_{q=1}^m l_q(\mathbf{n} + \mathbf{x})^{s_{r+q}}} \\ &= \frac{1}{\prod_{p=1}^r (n_p + x_p)^{s_p}} \int_0^t \cdots \int_0^t \prod_{q=1}^m \frac{e^{-l_q(\mathbf{n} + \mathbf{x})y_q} y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} dy_1 \cdots dy_m \\ &+ \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, m \rrbracket \\ \mathcal{C} \subseteq \mathcal{A}^c}} \frac{(-1)^{|\mathcal{A}^c \setminus \mathcal{C}|}}{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q})} \prod_{p=1}^r (n_p + x_p)^{-s_p} \\ & \quad \times \prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q}, t, l_q(\mathbf{n} + \mathbf{x})) \prod_{q \in \mathcal{C}} l_q(\mathbf{n} + \mathbf{x})^{-s_{r+q}}. \end{aligned} \tag{9}$$

On the right-hand side, we obtain two types of terms. The first one depends on an integral over $[0, t]^m$, and the second term is a finite sum of a non-empty product involving upper incomplete gamma functions and linear forms. We now consider the sum over $n_1, \dots, n_r \geq 0$ for each of these terms, and study the nature of each corresponding series.

PROPOSITION 9. *Let $t > 0$, $\mathcal{C} \subsetneq \llbracket 1, m \rrbracket$, and let $K \subset \mathbb{C}^{r+m}$ be a compact set. Then the series*

$$\sum_{n_1, \dots, n_r \geq 0} \frac{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q}, t, l_q(\mathbf{n} + \mathbf{x}))}{\prod_{p=1}^r (n_p + x_p)^{s_p} \prod_{q \in \mathcal{C}} l_q(\mathbf{n} + \mathbf{x})^{s_{r+q}}} \tag{10}$$

is normally convergent on K in the variables (s_1, \dots, s_{r+m}) .

PROOF. We proceed by bounding the general term of the series above. Let $s \in \mathbb{C}$, $t > 0$, $z \in H_0$. By a change of variables, we have $\Gamma(s, t, z) = 2^s \int_{t/2}^{+\infty} e^{-2zy} y^{s-1} dy$. This yields the following bound for the upper incomplete gamma function

$$|\Gamma(s, t, z)| \leq 2^\sigma e^{-\Re(z)t/2} \Gamma(\sigma, t/2, \Re(z)).$$

By the previous inequality, we find

$$\begin{aligned} & \left| \prod_{p=1}^r (n_p + x_p)^{-s_p} \prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q}, t, l_q(\mathbf{n} + \mathbf{x})) \prod_{q \in \mathcal{C}} l_q(\mathbf{n} + \mathbf{x})^{-s_{r+q}} \right| \\ & \leq \frac{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \left(2^{\sigma_{r+q}} e^{-\frac{\Re(l_q(\mathbf{n} + \mathbf{x}))t}{2}} \Gamma(\sigma_{r+q}, t/2, \Re(l_q(\mathbf{n} + \mathbf{x}))) \right)}{\prod_{p=1}^r (n_p + x_p)^{\sigma_p} \prod_{q \in \mathcal{C}} |l_q(\mathbf{n} + \mathbf{x})|^{\sigma_{r+q}}}. \end{aligned}$$

Note that $x \in \mathbb{R}_{>0} \mapsto \Gamma(\sigma, t/2, x)$ is increasing for any $\sigma > 0$, $t > 0$. Since $\Re(l_q(\mathbf{n} + \mathbf{x})) \geq c$ ($1 \leq q \leq m$) where $c := \min_{1 \leq q \leq m, 1 \leq p \leq r} (\Re(c_{q,p} x_p)) > 0$, we find

$$\Gamma(\sigma_{r+q}, t/2, \Re(l_q(\mathbf{n} + \mathbf{x}))) \leq \Gamma(\sigma_{r+q}, t/2, c).$$

Since K is compact, we have

$$|2^{\sigma_{r+q}}| \ll_K 1, \quad \Gamma(\sigma_{r+q}, t/2, c) \ll_K 1.$$

Since the set $\llbracket 1, m \rrbracket \setminus \mathcal{C}$ is non-empty, and since $\Re(c_{q,p}) > 0$ ($1 \leq q \leq m$, $1 \leq p \leq r$), there exist $d_1, \dots, d_r > 0$ such that

$$\begin{aligned} & \left| \prod_{p=1}^r (n_p + x_p)^{-s_p} \prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q}, t, l_q(\mathbf{n} + \mathbf{x})) \prod_{q \in \mathcal{C}} l_q(\mathbf{n} + \mathbf{x})^{-s_{r+q}} \right| \\ & \ll_K \frac{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} e^{-\frac{\Re(l_q(\mathbf{n} + \mathbf{x}))t}{2}}}{\prod_{p=1}^r (n_p + x_p)^{\sigma_p} \prod_{q \in \mathcal{C}} |l_q(\mathbf{n} + \mathbf{x})|^{\sigma_{r+q}}} = O(e^{-d_1(n_1 + x_1) - \dots - d_r(n_r + x_r)}). \end{aligned}$$

Therefore, the series (10) is normally convergent in the variables (s_1, \dots, s_{r+m}) on K . \square

REMARK 10. The series (10) may diverge if we no longer assume $\Re(c_{j,i}) > 0$ for all $1 \leq i \leq r$, $1 \leq j \leq m$.

PROPOSITION 11. *The following series is normally convergent in the variables s_1, \dots, s_{r+m} on any given compact set $K \subset H_1^{r+m}$,*

$$\sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{p=1}^r (n_p + x_p)^{s_p}} \int_0^t \dots \int_0^t \prod_{q=1}^m \left(e^{-l_q(\mathbf{n} + \mathbf{x}) y_q} y_q^{s_{r+q} - 1} \right) dy_1 \dots dy_m.$$

PROOF. It follows from bounding the exponential term in the integrand by 1. \square

From the previous proposition, we can perform a series-integral swap

$$\begin{aligned} & \sum_{n_1, \dots, n_r \geq 0} \frac{1}{\prod_{p=1}^r (n_p + x_p)^{s_p}} \int_0^t \cdots \int_0^t \prod_{q=1}^m \left(e^{-l_q(\mathbf{n} + \mathbf{x}) y_q} y_q^{s_{r+q}-1} \right) dy_1 \cdots dy_m \\ &= \int_0^t \cdots \int_0^t \prod_{p=1}^r \left(e^{-l_p^*(\mathbf{y}) x_p} \phi(e^{-l_p^*(\mathbf{y})}, s_p, x_p) \right) \prod_{q=1}^m y_q^{s_{r+q}-1} dy_1 \cdots dy_m, \end{aligned} \quad (11)$$

where $\sigma_1, \dots, \sigma_{r+m} > 1$ and

$$l_p^*(\mathbf{y}) := c_{1,p} y_1 + \cdots + c_{m,p} y_m \quad (1 \leq p \leq r). \quad (12)$$

By (9), by (11), by Proposition 11, and by Proposition 9, we find the following expansion

$$\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s}) = F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) + H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) \quad (\sigma_1, \dots, \sigma_{r+m} > 1, t > 0), \quad (13)$$

where

$$F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) := \int_0^t \cdots \int_0^t \prod_{p=1}^r \left(e^{-l_p^*(\mathbf{y}) x_p} \phi(e^{-l_p^*(\mathbf{y})}, s_p, x_p) \right) \prod_{q=1}^m \frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} dy_1 \cdots dy_m \quad (14)$$

$$\begin{aligned} H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) := & \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, m \rrbracket \\ \mathcal{C} \subseteq \mathcal{A}^c}} \frac{(-1)^{|\mathcal{A}^c \setminus \mathcal{C}|}}{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q})} \\ & \times \sum_{n_1, \dots, n_r \geq 0} \frac{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(s_{r+q}, t, l_q(\mathbf{n} + \mathbf{x}))}{\prod_{p=1}^r (n_p + x_p)^{s_p} \prod_{q \in \mathcal{C}} l_q(\mathbf{n} + \mathbf{x})^{s_{r+q}}}. \end{aligned} \quad (15)$$

Such an equality is an analogue of Crandall's expansion of the Mordell–Tornheim zeta function (see [BD18]). By Proposition 9, we get that:

COROLLARY 12. *Let $t > 0$. Then $\mathbf{s} \mapsto H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ is entire and vanishes at all nonpositive integers $-\mathbf{N} \in \mathbb{Z}_{\leq 0}^{r+m}$.*

By the previous corollary, $H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ does not contribute to the directional values of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ at nonpositive integers. Thus, only $F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ determines those values, which we ultimately compute using Erdélyi's formula (see Lemma 7).

PROPOSITION 13. *Let $\mathbf{s} = (s_1, \dots, s_{r+m}) \in (\mathbb{C} \setminus \mathbb{N})^{r+m}$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$. There exists $t_0 > 0$ such that, for all $t \in (0, t_0)$, we have*

$$\begin{aligned} F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) = & \sum_{\substack{\mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ k_1, \dots, k_\alpha \geq 0}} (-1)^{|\mathbf{k}|} \prod_{p=1}^\beta \Gamma(1 - s_{b_p}) \prod_{p=1}^\alpha \frac{\zeta(s_{a_p} - k_p, x_{a_p})}{k_p!} \\ & \times \int_0^t \cdots \int_0^t \prod_{q=1}^m \frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} \prod_{p=1}^\beta l_{b_p}^*(\mathbf{y})^{s_{b_p}-1} \prod_{p=1}^\alpha l_{a_p}^*(\mathbf{y})^{k_p} dy_1 \cdots dy_m. \end{aligned} \quad (16)$$

Recall that we write $\mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$.

Note that \mathcal{A} or \mathcal{B} can be empty, thus α or β can be 0. In such a case, we understand that the product over an empty set is 1. Also, if $\alpha = 0$, then the tuple $\mathbf{k} = (k_1, \dots, k_\alpha)$ corresponds to the empty tuple \emptyset , thus $|\mathbf{k}| = 0$.

PROOF. Let

$$\overline{D}_{r,m}(\delta, R) := \left\{ \mathbf{s} \in \overline{D}_0(R)^{r+m} \mid \min_{n \in \mathbb{Z}} |s_p - n| \geq \delta \ (1 \leq p \leq r) \right\} \quad (17)$$

where $0 < \delta < 1$ and $R > 0$ are both arbitrary. Applying Erdélyi's formula on the right-hand side of (14), we will prove that (16) holds for all $\mathbf{s} \in \overline{D}_{r,m}(\delta, R)$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$.

Let $\mathbf{s} \in \overline{D}_{r,m}(\delta, R)$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$. In particular, we have $s_p \in \mathbb{C} \setminus \mathbb{N}$ ($1 \leq p \leq r$). Note also that

$$\forall \mathbf{y} = (y_1, \dots, y_m) \in [0, t]^m, \quad |l_p^*(\mathbf{y})| \ll t \quad (1 \leq p \leq r). \quad (18)$$

By Erdélyi's formula we obtain the following for $t > 0$ sufficiently small

$$e^{-l_p^*(\mathbf{y})x_p} \phi(e^{-l_p^*(\mathbf{y})}, s_p, x_p) = \Gamma(1 - s_p) l_p^*(\mathbf{y})^{s_p-1} + \sum_{k_p=0}^{+\infty} \frac{(-l_p^*(\mathbf{y}))^{k_p}}{k_p!} \zeta(s_p - k_p, x_p), \quad (19)$$

with $1 \leq p \leq r$ and $\mathbf{y} \in [0, t]^m$. We can then substitute the product $\prod_{p=1}^r \left(e^{-l_p^*(\mathbf{y})x_p} \phi(e^{-l_p^*(\mathbf{y})}, s_p, x_p) \right)$ in the integrand of $F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ using the right-hand side of (19), and we will then perform another series-integral inversion. In order to justify such inversion, we first need to bound each term on the right-hand side of (19) for all $\mathbf{s} \in \overline{D}_{r,m}(\delta, R)$.

We first recall the following standard results:

- i) The function Γ is meromorphic on \mathbb{C} with simple poles at nonpositive integers, and the function $1/\Gamma$ is entire.
- ii) The Hurwitz zeta function is meromorphic on \mathbb{C} with a simple pole at $s = 1$.
- iii) As $s_p \mapsto \Gamma(1 - s_p)$ is holomorphic on the compact set $\overline{D}_{r,m}(\delta, R)$, it follows that

$$|\Gamma(1 - s_p)| \underset{R, \delta}{\ll} 1 \quad (\mathbf{s} = (s_1, \dots, s_{r+m}) \in \overline{D}_{r,m}(\delta, R)). \quad (20)$$

- iv) Let $\varepsilon > 0$. By Lemma 8 we have

$$\left| \frac{\zeta(s_p - k_p, x_p)}{k_p!} \right| \underset{R, \varepsilon, \delta}{\ll} (2\pi)^{-k_p} (k_p + 1)^{R+\varepsilon} \quad (\mathbf{s} \in \overline{D}_{r,m}(\delta, R), k_p \in \mathbb{Z}_{\geq 0}).$$

Moreover, the series $\sum_{k_p=0}^{+\infty} (k_p + 1)^{R+\varepsilon} \left(\frac{t}{2\pi}\right)^{k_p}$ converges for all $t > 0$ sufficiently small.

Using iv) together with bounds (18) and (20), we can perform a series-integral inversion

$$\begin{aligned}
 & \int_0^t \cdots \int_0^t \prod_{p=1}^r \left(\Gamma(1 - s_p) l_p^*(\mathbf{y})^{s_p-1} + \sum_{k_p=0}^{+\infty} \frac{(-l_p^*(\mathbf{y}))^{k_p}}{k_p!} \zeta(s_p - k_p, x_p) \right) \prod_{q=1}^m (y_q^{s_{r+q}-1} dy_q) \\
 &= \sum_{\substack{\mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ k_1, \dots, k_\alpha \geq 0}} (-1)^{|\mathbf{k}|} \prod_{p=1}^{\beta} \Gamma(1 - s_{b_p}) \prod_{p=1}^{\alpha} \frac{\zeta(s_{a_p} - k_p, x_{a_p})}{k_p!} \\
 & \quad \times \int_0^t \cdots \int_0^t \prod_{q=1}^m y_q^{s_{r+q}-1} \prod_{p=1}^{\beta} l_{b_p}^*(\mathbf{y})^{s_{b_p}-1} \prod_{p=1}^{\alpha} l_{a_p}^*(\mathbf{y})^{k_p} dy_1 \cdots dy_m.
 \end{aligned} \tag{21}$$

Dividing both sides of (21) by $\prod_{q=1}^m \Gamma(s_{r+q})$, we obtain (16). \square

We now want to extend (16) to the whole complex space. One major obstruction is that the integral on the right-hand side of (16) is not convergent when we no longer assume $\sigma_1, \dots, \sigma_{r+m} > 1$, due to a singularity at the origin of the range of the integral. For this reason, we shall perform a suitable change of variables in order to perform an analytic continuation of this parameter-dependent integral.

3.3. Analytic continuation of a parameter-dependent integral

In this subsection, we fix $\mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$, and a tuple $\mathbf{k} = (k_1, \dots, k_\alpha) \in \mathbb{Z}_{\geq 0}^\alpha$. We set

$$V_j(t) := \{\mathbf{y} \in [0, t]^m \mid y_j \geq y_q \ (1 \leq q \neq j \leq m)\} \quad (1 \leq j \leq m).$$

The range of the integral on the right-hand side of (16) can be decomposed as $[0, t]^m = \cup_{j=1}^m V_j(t)$ with pairwise intersections of measure zero. Let $\mathbf{s} = (s_1, \dots, s_{r+m}) \in \mathbb{C}^{r+m}$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$. By the additivity property of the integral, and using the change of variables

$$\varphi_j : \begin{cases} [0, 1]^{j-1} \times [0, t] \times [0, 1]^{m-j} \rightarrow V_j(t) \\ \mathbf{y} = (y_1, \dots, y_m) \mapsto (y_1 y_j, \dots, y_j, \dots, y_m y_j) \end{cases} \quad (1 \leq j \leq m)$$

we obtain

$$\begin{aligned}
 & \int_0^t \cdots \int_0^t \prod_{q=1}^m \frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} \prod_{p=1}^{\beta} l_{b_p}^*(\mathbf{y})^{s_{b_p}-1} \prod_{p=1}^{\alpha} l_{a_p}^*(\mathbf{y})^{k_p} dy_1 \cdots dy_m \\
 &= \sum_{j=1}^m \frac{t^{|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}}{\Gamma(s_{r+j}) (|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, \mathbf{s})
 \end{aligned} \tag{22}$$

where l_1^*, \dots, l_r^* are linear forms defined in (12), and where we have set

$$f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, \mathbf{s}) := \int_0^1 \cdots \int_0^1 \prod_{\substack{q=1 \\ q \neq j}}^m \frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} \prod_{p=1}^{\beta} l_{b_p}^*(\widehat{\mathbf{y}}^j)^{s_{b_p}-1} \prod_{p=1}^{\alpha} l_{a_p}^*(\widehat{\mathbf{y}}^j)^{k_p} dy_1 \cdots \widehat{dy_j} \cdots dy_m \tag{23}$$

with $\widehat{\mathbf{y}}^j := (y_1, \dots, y_{j-1}, 1, y_{j+1}, \dots, y_m)$, and $dy_1 \cdots \widehat{dy_j} \cdots dy_m := dy_1 \cdots dy_{j-1} dy_{j+1} \cdots dy_m$. It is clear that the function $\mathbf{s} \mapsto f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s})$ is holomorphic over H_1^{r+m} .

In the next proposition, we establish a holomorphic continuation for the function $\mathbf{s} \mapsto f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s})$ involving a normally convergent series with respect to the variables s_1, \dots, s_{r+m} . We also provide a bound for this function, which will allow us to prove that the right-hand side of (16) converges for all $\mathbf{s} \in \mathbb{C}^{r+m}$, except for some tuples \mathbf{s} contained in a union of hyperplanes.

PROPOSITION 14. *The function $\mathbf{s} \mapsto f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s})$ has a holomorphic continuation on \mathbb{C}^{r+m} given by*

$$\begin{aligned} f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s}) &= \sum_{\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}} \sum_{\substack{u_{1,1}, \dots, u_{\alpha', \beta} \geq 0 \\ v_{1,1} + \dots + v_{m,1} = k_1 \\ \dots \\ v_{1,\alpha} + \dots + v_{m,\alpha} = k_\alpha}} \frac{\varepsilon^{\sum_{q=1}^{\alpha'} (s_{r+a'_q} + |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}|)}}{\prod_{q=1}^{\alpha'} (\Gamma(s_{r+a'_q})(s_{r+a'_q} + |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}|))} \\ &\times \prod_{p=1}^{\alpha} \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m c_{q,a_p}^{v_{q,p}} \right) \prod_{p=1}^{\beta} \left(\binom{s_{b_p} - 1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \prod_{q=1}^{\alpha'} c_{a'_q, b_p}^{u_{q,p}} \right) \\ &\times \int_{\varepsilon}^1 \cdots \int_{\varepsilon}^1 \prod_{q=1}^{\beta'} \frac{y_q^{s_{r+b'_q} - 1 + |\mathbf{v}_{b'_q\bullet}|}}{\Gamma(s_{r+b'_q})} \prod_{p=1}^{\beta} \left(c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q, b_p} y_q \right)^{s_{a_p} - 1 - |\mathbf{u}_{\bullet p}|} dy_1 \cdots dy_{\beta'} \end{aligned} \quad (24)$$

where $\varepsilon \in (0, 1)$ is sufficiently small, and where we write $\mathcal{A}' = (a'_1, \dots, a'_{\alpha'}) \subseteq \llbracket 1, m \rrbracket \setminus \{j\}$ and $\mathcal{B}' = (b'_1, \dots, b'_{\beta'}) := \llbracket 1, m \rrbracket \setminus (\mathcal{A}' \cup \{j\})$. Moreover, for any $R > 0$, we have

$$|f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s})| \ll_{R,\varepsilon} (m \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}|)^{|\mathbf{k}|} \quad (|s_1|, \dots, |s_{r+m}| \leq R). \quad (25)$$

Before proving the proposition, we need the following elementary uniform bounds.

LEMMA 15. *For all $R > 0$, we have*

$$\left| \frac{1}{\Gamma(s)} \right| \ll_R 1, \quad \left| \frac{1}{\Gamma(s)(s+n)} \right| \ll_R 1 \quad (n \in \mathbb{Z}_{\geq 0}, s \in \overline{D}_0(R)).$$

PROOF OF PROPOSITION 14. Firstly, we prove that the series on the right-hand side of (24) is normally convergent on any given compact subset of \mathbb{C}^{r+m} (in the variables \mathbf{s}) by establishing bounds for each term of this series. Secondly, we show that (24) holds for all $s_1, \dots, s_{r+m} \in \mathbb{C}$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$, which will conclude the proof.

i) We begin by bounding the general term, denoted $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\mathbf{s})$, of the series on the right-hand side of (24), on the compact set $K := \{\mathbf{s} \in \mathbb{C}^{r+m} \mid \forall k \in \llbracket 1, r+m \rrbracket, s_k \in \overline{D}_0(R)\}$ with $R \in \mathbb{N}$. Let $u_{1,1}, \dots, u_{\alpha', \beta} \geq 0$ and $v_{1,1}, \dots, v_{m,\alpha} \geq 0$ be nonnegative integers such that $v_{1,p} + \dots + v_{m,p} = k_p$ ($1 \leq p \leq \alpha$). For all $\mathbf{s} \in K$, we have:

- From Lemma 15, we obtain

$$\left| \frac{1}{\Gamma(s_{r+q})(s_{r+q} + n)} \right| \ll_{\bar{R}} 1 \quad (1 \leq q \leq m, n \in \mathbb{Z}_{\geq 0}), \quad (26)$$

$$\left| \frac{1}{\Gamma(s_{r+q})} \right| \ll_{\bar{R}} 1 \quad (1 \leq q \leq m). \quad (27)$$

- By compactness, it is clear that

$$\left| x^{s_{r+q}-1+|\mathbf{v}_{q\bullet}|} \right| \ll_{\bar{R}, \varepsilon} 1 \quad (1 \leq q \leq m, x \in [\varepsilon, 1]), \quad (28)$$

$$\left| \varepsilon^{\sum_{q=1}^{\alpha'} (s_{r+a'_q} + |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}|)} \right| \ll_{\bar{R}, \varepsilon} \varepsilon^{\sum_{q=1}^{\alpha'} |\mathbf{u}_{q\bullet}|}. \quad (29)$$

- By setting $L := \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}|$, we clearly have that

$$\left| \binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m c_{q, a'_p}^{v_{q,p}} \right| \leq \binom{k_p}{\mathbf{v}_{\bullet p}} L^{k_p} \quad (1 \leq p \leq \alpha). \quad (30)$$

- Using the bound

$$\left| \binom{z-1}{k} \right| \leq \binom{R+k}{k} \quad (k \in \mathbb{Z}_{\geq 0}, z \in \bar{D}_0(R))$$

we obtain

$$\left| \prod_{p=1}^{\beta} \left(\binom{s_{b_p}-1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \prod_{q=1}^{\alpha'} c_{q, a'_p}^{u_{q,p}} \right) \right| \ll_{\bar{R}, \varepsilon} L^{|\mathbf{u}|} \prod_{p=1}^{\beta} \left(\binom{R+|\mathbf{u}_{\bullet p}|}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \right). \quad (31)$$

- Let $1 \leq p \leq \beta$ and $(y_1, \dots, y_{\beta'}) \in [\varepsilon, 1]^{\beta'}$. It is clear that

$$\left| \left(c_{j, b_p} + \sum_{q=1}^{\beta'} c_{b'_q, b_p} y_q \right)^{s_{b_p}-1-|\mathbf{u}_{\bullet p}|} \right| \leq \left| c_{j, b_p} + \sum_{q=1}^{\beta'} c_{b'_q, b_p} y_q \right|^{\sigma_{b_p}-1-|\mathbf{u}_{\bullet p}|} e^{R \frac{\pi}{2}}.$$

Note that we have the following bound:

$$l \leq \left(\Re(c_{j, b_p}) + \sum_{q=1}^{\beta'} \Re(c_{b'_q, b_p}) \varepsilon \right) \leq \left| c_{j, b_p} + \sum_{q=1}^{\beta'} c_{b'_q, b_p} y_q \right| \leq \sum_{q=1}^m |c_{q, b_p}| \leq mL$$

with $L := \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}| > 0$, $l := \min_{1 \leq p \leq r, 1 \leq q \leq m} \Re(c_{q,p}) > 0$. Thus

$$\left| c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right|^{\sigma_{b_p}-1-|\mathbf{u}_{\bullet p}|} \leq \begin{cases} (mL)^{\sigma_{b_p}-|\mathbf{u}_{\bullet p}|-1} & \text{if } \left| c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right| \geq 1 \\ & \text{and } \sigma_{b_p} - |\mathbf{u}_{\bullet p}| \geq 1, \\ 1 & \text{if } \left| c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right| \geq 1 \\ & \text{and } \sigma_{b_p} - |\mathbf{u}_{\bullet p}| < 1, \\ 1 & \text{if } \left| c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right| < 1 \\ & \text{and } \sigma_{b_p} - |\mathbf{u}_{\bullet p}| \geq 1, \\ K^{\sigma_{b_p}-|\mathbf{u}_{\bullet p}|-1} & \text{else.} \end{cases}$$

By compactness, we also have:

$$(mL)^{\sigma_{b_p}-|\mathbf{u}_{\bullet p}|-1} \ll_R (mL)^{-|\mathbf{u}_{\bullet p}|}, \quad l^{\sigma_{b_p}-|\mathbf{u}_{\bullet p}|-1} \ll_R l^{-|\mathbf{u}_{\bullet p}|}.$$

Finally, we find that

$$\left| \left(c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right)^{\sigma_{b_p}-1-|\mathbf{u}_{\bullet p}|} \right| \ll_R \min(l, mL, 1)^{-|\mathbf{u}_{\bullet p}|}. \quad (32)$$

From (28) and (32), we get:

$$\left| \int_{\varepsilon}^1 \cdots \int_{\varepsilon}^1 \prod_{q=1}^{\beta'} \frac{y_q^{s_r+b'_q-1+|\mathbf{v}_{b'_q \bullet}|}}{\Gamma(s_r+b'_q)} \prod_{p=1}^{\beta} \left(c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right)^{s_{a_p}-1-|\mathbf{u}_{\bullet p}|} dy_1 \cdots dy_{\beta'} \right| \ll_{R,\varepsilon} \min(l, mL, 1)^{-|\mathbf{u}_{\bullet p}|}.$$

From the above inequality and from (26), (27), (29), (30), (31), we find that

$$|T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\mathbf{s})| \ll_{R,\varepsilon} \left(\frac{L\varepsilon}{\min(l, mL, 1)} \right)^{|\mathbf{u}|} \prod_{p=1}^{\alpha} \left(\binom{k_p}{\mathbf{v}_{\bullet p}} L^{k_p} \right) \prod_{p=1}^{\beta} \left(\binom{R+|\mathbf{u}_{\bullet p}|}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \right).$$

By the multinomial theorem, we have $\sum_{\substack{v_{1,1}+\dots+v_{m,1}=k_1 \\ \dots \\ v_{1,\alpha}+\dots+v_{m,\alpha}=k_\alpha}} \prod_{p=1}^{\alpha} \left(\binom{k_p}{\mathbf{v}_{\bullet p}} L^{k_p} \right) = (mL)^{|\mathbf{k}|}$. It is

straightforward to check that the series

$$(mL)^{|\mathbf{k}|} \sum_{u_{1,1}, \dots, u_{\alpha', \beta} \geq 0} \left(\frac{L\varepsilon}{\min(l, mL, 1)} \right)^{|\mathbf{u}|} \prod_{p=1}^{\beta} \left(\binom{R+|\mathbf{u}_{\bullet p}|}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \right)$$

converges absolutely for all $0 < \varepsilon < \frac{\min(l, mL, 1)}{L}$. Therefore, the series on the right-hand side of (16) is normally convergent on the compact set K , for $\varepsilon > 0$ sufficiently small.

ii) Let $\mathbf{s} = (s_1, \dots, s_{r+m}) \in \mathbb{C}^{r+m}$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$. We now want to prove that (16) holds for such \mathbf{s} . Let $\mathcal{A}' = (a'_{\alpha'}, \dots, a'_{\alpha'}) \subseteq \llbracket 1, m \rrbracket \setminus \{j\}$ and $\mathcal{B}' =$

$(b'_1, \dots, b'_{\beta'}) := \llbracket 1, m \rrbracket \setminus (\mathcal{A}' \cup \{j\})$. We set

$$V_{\mathcal{A}', j}(\varepsilon) := \left\{ (y_1, \dots, \widehat{y}_j, \dots, y_m) \in [0, 1]^{m-1} \mid \begin{array}{ll} y_{\alpha'_q} \leq \varepsilon & (1 \leq q \leq \alpha') \\ y_{\beta'_q} \geq \varepsilon & (1 \leq q \leq \beta') \end{array} \right\} \quad (0 < \varepsilon < 1).$$

We can then partition the following integration range:

$$\{(y_1, \dots, \widehat{y}_j, \dots, y_m) \mid \forall q \neq j, y_q \in [0, 1]\} = \bigcup_{\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}} V_{\mathcal{A}', j}(\varepsilon),$$

with pairwise intersections of measure zero. By the additivity property of the integral, we get

$$\begin{aligned} & f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, \mathbf{s}) \\ &= \sum_{\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}} \int_{V_{\mathcal{A}', j}(\varepsilon)} \prod_{\substack{q=1 \\ q \neq j}}^m \frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} \prod_{p=1}^{\beta} l_{b_p}^* (\widehat{\mathbf{y}}^j)^{s_{b_p}-1} \prod_{p=1}^{\alpha} l_{a_p}^* (\widehat{\mathbf{y}}^j)^{k_p} dy_1 \cdots \widehat{dy}_j \cdots dy_m. \end{aligned} \quad (33)$$

We now aim to expand the integrand of each integral into power series around the origin. Note that these power series converge uniformly on a sufficiently small compact disk. Therefore we are able to perform a series-integral inversion.

Let $\mathcal{A}' = (a'_1, \dots, a'_{\alpha'}) \subseteq \llbracket 1, m \rrbracket \setminus \{j\}$ and $\varepsilon \in (0, \frac{1}{m \max_{1 \leq q \leq m, 1 \leq p \leq r} |c_{q,p}|})$. By Taylor's expansion

$$(1 + X_1 + \cdots + X_d)^{-s} = \sum_{k_1, \dots, k_d \geq 0} \binom{-s}{|\mathbf{k}|} \binom{|\mathbf{k}|}{\mathbf{k}} X_1^{k_1} \cdots X_d^{k_d} \quad (|X_1| + \cdots + |X_d| < 1),$$

we obtain the following expansion, uniformly in $\widehat{\mathbf{y}}^j \in V_{\mathcal{A}', j}(\varepsilon)$,

$$\begin{aligned} & l_{b_p}^* (\widehat{\mathbf{y}}^j)^{s_{b_p}-1} \\ &= \sum_{u_{p,1}, \dots, u_{p,\alpha'} \geq 0} \binom{s_{b_p}-1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \left(c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q, b_p} y_q \right)^{s_{b_p}-1-|\mathbf{u}_{\bullet p}|} \prod_{q=1}^{\alpha'} \left(c_{a'_q, b_p}^{u_{q,p}} y_q^{u_{q,p}} \right). \end{aligned}$$

Let $p \in \llbracket 1, \alpha \rrbracket$ and $k_p \in \mathbb{Z}_{\geq 0}$. By Newton's multinomial, we have

$$l_{a_p}^* (\widehat{\mathbf{y}}^j)^{k_p} = \sum_{\substack{v_{1,1} + \cdots + v_{m,1} = k_1 \\ \vdots \\ v_{1,\alpha} + \cdots + v_{m,\alpha} = k_\alpha}} \binom{k_p}{\mathbf{v}_{\bullet p}} c_{j,p}^{v_{j,p}} \prod_{\substack{q=1 \\ q \neq j}}^m c_{q,p}^{v_{q,p}} y_q^{v_{q,p}}.$$

Plugging these last two expressions into the integrand on the right-hand side of (33), we obtain

$$\begin{aligned}
& \int_{V_{\mathcal{A}',j}(\varepsilon)} \prod_{\substack{q=1 \\ q \neq j}}^m \frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} \prod_{p=1}^{\beta} l_{b_p}^* (\widehat{\mathbf{y}}^j)^{s_{b_p}-1} \prod_{p=1}^{\alpha} l_{a_p}^* (\widehat{\mathbf{y}}^j)^{k_p} dy_1 \cdots \widehat{dy}_j \cdots dy_m \\
&= \sum_{\substack{u_{1,1}, \dots, u_{\alpha',\beta} \geq 0 \\ v_{1,1} + \dots + v_{m,1} = k_1 \\ \dots \\ v_{1,\alpha} + \dots + v_{m,\alpha} = k_\alpha}} \prod_{p=1}^{\alpha} \binom{k_p}{\mathbf{v}_{\bullet,p}} c_{j,a_p}^{v_{j,p}} \prod_{p=1}^{\beta} \binom{s_{b_p}-1}{|\mathbf{u}_{\bullet,p}|} \binom{|\mathbf{u}_{\bullet,p}|}{\mathbf{u}_{\bullet,p}} \\
&\quad \times \int_{V_{\mathcal{A}',j}(\varepsilon)} \prod_{p=1}^{\beta} \left(\left(c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right)^{s_{b_p}-1-|\mathbf{u}_{\bullet,p}|} \prod_{q=1}^{\alpha'} \left(c_{a'_q,b_p}^{u_{q,p}} y_q^{u_{q,p}} \right) \right) \\
&\quad \quad \quad \times \prod_{\substack{q=1 \\ q \neq j}}^m \left(\frac{y_q^{s_{r+q}-1}}{\Gamma(s_{r+q})} \prod_{p=1}^{\alpha} \left(c_{q,a_p}^{v_{q,p}} y_q^{v_{q,p}} \right) \right) dy_1 \cdots \widehat{dy}_j \cdots dy_m.
\end{aligned} \tag{34}$$

Using Fubini's theorem, we then obtain that the integral inside that last expression is equal to

$$\begin{aligned}
& \prod_{q=1}^{\alpha'} \frac{\varepsilon^{s_{r+a'_q} + |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}|}}{\Gamma(s_{r+a'_q})(s_{r+a'_q} + |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}|)} \prod_{q=1}^{\alpha'} \left(\prod_{p=1}^{\beta} c_{a'_q,b_p}^{u_{q,p}} \right) \prod_{\substack{q=1 \\ q \neq j}}^m \left(\prod_{p=1}^{\alpha} c_{q,a_p}^{v_{q,p}} \right) \\
&\quad \times \int_{\varepsilon}^1 \cdots \int_{\varepsilon}^1 \prod_{q=1}^{\beta'} \frac{y_q^{s_{r+b'_q}-1 + |\mathbf{v}_{b'_q\bullet}|}}{\Gamma(s_{r+b'_q})} \prod_{p=1}^{\beta} \left(c_{j,b_p} + \sum_{q=1}^{\beta'} c_{b'_q,b_p} y_q \right)^{s_{b_p}-1-|\mathbf{u}_{\bullet,p}|} dy_1 \cdots dy_{\beta'}.
\end{aligned}$$

Plugging the previous equality into (34), and using (33), we get (24). \square

REMARK 16. We proved that the series (with respect to (\mathbf{u}, \mathbf{v})) on the right-hand side of (24) is normally convergent on any compact subset of \mathbb{C}^{r+m} in the variables s_1, \dots, s_{r+m} . Therefore, one can compute the limit of $f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s})$ at $\mathbf{s} = (-N_1, \dots, -N_{r+m})$ by interchanging the limit and the series on the right-hand side of (24). One can also compute the directional derivative values of $f_{j,\mathcal{A},\mathbf{k}}$ by taking the directional derivative values of the general term of the series on the right-hand side of (24).

3.4. Meromorphic continuation of $F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$

We established that $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s}) = F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) + H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ ($t > 0, \sigma_1, \dots, \sigma_{r+m} > 1$), where $\mathbf{s} \mapsto H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ is entire and vanishes when $\mathbf{s} \in \mathbb{Z}_{\leq 0}^{r+m}$. We also showed in (16) an explicit formula for $F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$, where $\sigma_1, \dots, \sigma_{r+m} > 1$. Here, we extend this formula to \mathbb{C}^{r+m} , which introduces singularities along a union of hyperplanes.

PROPOSITION 17. *Let $\mathbf{s} = (s_1, \dots, s_{r+m}) \in \mathbb{C}^{r+m} \setminus \mathcal{S}_{r,m}$. There exists $t_0 > 0$ such that for all $t \in (0, t_0)$, we have*

$$\begin{aligned}
 F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) &= \sum_{\substack{\mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \prod_{p=1}^{\beta} \Gamma(1 - s_{b_p}) \sum_{k_1, \dots, k_\alpha \geq 0} (-1)^{|\mathbf{k}|} f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, \mathbf{s}) \\
 &\quad \times \frac{t^{|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}}{\Gamma(s_{r+j})(|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \prod_{p=1}^{\alpha} \frac{\zeta(s_{a_p} - k_p, x_{a_p})}{k_p!}
 \end{aligned} \tag{35}$$

where we recall that $\mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$. Moreover, the singularities of $\mathbf{s} \mapsto F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t)$ belong to $\mathcal{S}_{r,m}$.

PROOF. By plugging (22) into (16), we obtain that (35) holds for all $\mathbf{s} \in (\mathbb{C} \setminus \mathbb{N})^{r+m}$ such that $\sigma_1, \dots, \sigma_{r+m} > 1$. We now prove that the series on the right-hand side of (35) is normally convergent on a suitable compact set that does not intersect the set $\mathcal{S}_{r,m}$, and we prove such a result by using bounds obtained for the Hurwitz zeta function and for $f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, \mathbf{s})$. We need to choose a suitable compact set by accounting for the possible singularities arising from the right-hand side of (35).

Let $R > 0$ and $\delta \in (0, 1)$. We set

$$K_{r,m}(\delta, R) := \left\{ \mathbf{s} \in \overline{D}_0(R)^{r+m} \mid \begin{array}{l} \min_{n \in \mathbb{N}, 1 \leq p \leq r} |s_p - n| \geq \delta \\ \min_{n \in \mathbb{Z}_{\leq m}, \mathcal{A} \subseteq \llbracket 1, r \rrbracket} \left| \sum_{a \in \mathcal{A}} s_a + \sum_{q=1}^m s_{r+q-n} \right| \geq \delta \end{array} \right\}. \tag{36}$$

Note that the set $K_{r,m}(\delta, R)$ contains the compact set $D_{r,m}(\delta, R)$ as defined in (17). Observe that each term of the series on the right-hand side of (35) is holomorphic on $K_{r,m}(\delta, R)$. Let $\mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$, $j \in \llbracket 1, m \rrbracket$. For all $\mathbf{k} = (k_1, \dots, k_\alpha) \in \mathbb{Z}_{\geq 0}^\alpha$, we have:

- If $k_1 + \dots + k_\alpha \geq \beta$, then by Lemma 15 we find that

$$\left| \frac{1}{\Gamma(s_{r+j})(|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \right| \ll_{R, \delta} 1 \quad (\mathbf{s} \in K_{r,m}(\delta, R)).$$

- If $k_1 + \dots + k_\alpha < \beta$, we know that the function

$$\mathbf{s} \in K_{r,m}(\delta, R) \mapsto \frac{1}{\Gamma(s_{r+j})(|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})}$$

is entire. By compactness, we have

$$\left| \frac{1}{\Gamma(s_{r+j})(|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \right| \ll_{R, \delta, \mathbf{k}} 1 \quad (\mathbf{s} \in K_{r,m}(\delta, R)).$$

Since there is only a finite number of nonnegative tuples \mathbf{k} satisfying $k_1 + \dots + k_\alpha \leq \beta$, we get that

$$\left| \frac{1}{\Gamma(s_{r+j})(|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \right| \ll_{R, \delta} 1 \quad (\mathbf{s} \in K_{r,m}(\delta, R), \mathbf{k} \in \mathbb{Z}_{\geq 0}^\alpha). \tag{37}$$

We see that the function $\mathbf{s} \mapsto \prod_{p=1}^{\beta} \Gamma(1 - s_{b_p})$ is entire on $K_{r,m}(\delta, R)$. By compactness we have

$$\left| \prod_{p=1}^{\beta} \Gamma(1 - s_{b_p}) \right| \ll_{R,\delta} 1 \quad (\mathbf{s} \in K_{r,m}(\delta, R)). \quad (38)$$

One can also prove that

$$\left| t^{|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B}\cup[r+1, r+m]}} \right| \ll_{R,\delta} t^{|\mathbf{k}|} \quad (\mathbf{s} \in K_{r,m}(\delta, R), \mathbf{k} \in \mathbb{Z}_{\geq 0}^{\alpha}). \quad (39)$$

By (37), (38), (25), (39), and using Lemma 8, we obtain that

$$\begin{aligned} & \left| f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s}) \frac{t^{|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B}\cup[r+1, r+m]}}}{\Gamma(s_{r+j})(|\mathbf{k}| - \beta + |\mathbf{s}|_{\mathcal{B}\cup[r+1, r+m]})} \prod_{p=1}^{\beta} \Gamma(1 - s_{b_p}) \prod_{p=1}^{\alpha} \frac{\zeta(s_{a_p} - k_p, x_{a_p})}{k_p!} \right| \\ & \ll_{R,\varepsilon,\delta} \left(\frac{m \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}|}{2\pi} t \right)^{|\mathbf{k}|} \prod_{p=1}^{\alpha} (k_p + 1)^{R+\varepsilon} \quad (\varepsilon > 0). \end{aligned} \quad (40)$$

Note that the series $\sum_{k_1, \dots, k_{\alpha} \geq 0} \left(\frac{m \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}|}{2\pi} t \right)^{|\mathbf{k}|} \prod_{p=1}^{\alpha} (k_p + 1)^{R+\varepsilon}$ converges for all $0 < t < \frac{2\pi}{m \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}|} =: t_0$. Combining the previous fact with the bound (40), we conclude that the series on the right-hand side of (35) is normally convergent on $K_{r,m}(\delta, R)$ in the variables $\mathbf{s} = (s_1, \dots, s_{r+m})$, for any $t \in (0, t_0)$. \square

Combining the expansion (13) with the convergence result shown above, we obtain a meromorphic continuation for $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$ to the whole complex space \mathbb{C}^{r+m} .

COROLLARY 18. *Let $t > 0$ sufficiently small. We have*

$$\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s}) = F(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) + H(\mathbf{c}, \mathbf{x}, \mathbf{s}, t) \quad (\mathbf{s} \in \mathbb{C}^{r+m} \setminus \mathcal{S}_{r,m}).$$

From Corollary 18, Proposition 17, and Corollary 12, we obtain Theorem 6.

3.5. Regularity of $F(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t)$ at $z = 0$

Let $\mathbf{N} \in \mathbb{Z}_{\geq 0}^{r+m}$, and let $\boldsymbol{\theta} \in \mathbb{C}^{r+m}$ such that (7) holds. We prove in this subsection that the single-variable function $z \mapsto \zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta})$ is meromorphic on \mathbb{C} and regular at $z = 0$. Thanks to Theorem 6 and Corollary 12, it suffices to show that $z \mapsto F(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t)$ is regular at $z = 0$. To that end, we will show that (35) holds around $z = 0$ after setting $\mathbf{s} = -\mathbf{N} + z\boldsymbol{\theta}$.

PROPOSITION 19. *Let $t, \eta > 0$ sufficiently small. For all $z \in D_0(\eta)$, we have*

$$\begin{aligned} F(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t) &= \sum_{\substack{\mathcal{A} \subseteq [1, r] \\ 1 \leq j \leq m}} \prod_{p=1}^{\beta} \Gamma(1 + N_{b_p} - z\theta_{b_p}) \sum_{k_1, \dots, k_{\alpha} \geq 0} (-1)^{|\mathbf{k}|} f_{j,\mathcal{A},\mathbf{k}}(-\mathbf{N} + z\boldsymbol{\theta}) \\ &\quad \times \frac{t^{|\mathbf{k}| - \beta - |\mathbf{N}|_{\mathcal{B}\cup[r+1, r+m]} + z|\boldsymbol{\theta}|_{\mathcal{B}\cup[r+1, r+m]}}}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{\mathcal{B}\cup[r+1, r+m]} + z|\boldsymbol{\theta}|_{\mathcal{B}\cup[r+1, r+m]})} \\ &\quad \times \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} + z\theta_{a_p} - k_p, x_{a_p})}{k_p!}. \end{aligned} \quad (41)$$

PROOF. Let $t > 0$ be sufficiently small. By definition of $\mathcal{S}_{r,m}$ in Theorem 6, we have

$$-\mathbf{N} + z\boldsymbol{\theta} \in \mathcal{S}_{r,m} \iff z \in \left(\bigcup_{\mathcal{A} \subseteq \llbracket 1, r \rrbracket} \frac{1}{|\boldsymbol{\theta}|_{|\mathcal{A} \cup \llbracket r+1, r+m \rrbracket}}} \mathbb{Z}_{\leq |\mathbf{N}|_{|\mathcal{A} \cup \llbracket r+1, r+m \rrbracket}}} \right) \cup \left(\bigcup_{\substack{p \in \llbracket 1, r \rrbracket \\ \text{s.t. } \theta_p \neq 0}} \frac{1}{\theta_p} \mathbb{N} \right).$$

We take a radius η so that the function $z \in \overline{D}_0(\eta) \mapsto F(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t)$ has at most one singularity in the disk, at $z = 0$. Let $\eta > 0$ and $R \in \mathbb{N}$ such that

$$\eta := \frac{1}{2} \min \left(\min_{p \in \llbracket 1, r \rrbracket} \min_{\text{s.t. } \theta_p \neq 0} \left| \frac{1}{\theta_p} \right|, \min_{\mathcal{A} \subseteq \llbracket 1, r \rrbracket} \left| \frac{1}{\sum_{a \in \mathcal{A}} \theta_a + \sum_{q=1}^m \theta_{r+q}} \right| \right)$$

$$R := \left\lceil \max_{1 \leq k \leq r+m} (N_k + |\theta_k \eta|) \right\rceil \in \mathbb{N}.$$

Such an η exists because of the non-vanishing condition (7) satisfied by the direction $\boldsymbol{\theta}$. We clearly have

$$\forall z \in \overline{D}_0(\eta) \setminus \{0\}, -\mathbf{N} + z\boldsymbol{\theta} \notin \mathcal{S}_{r,m},$$

and with our choice for η and R , we also have

$$\{-\mathbf{N} + z\boldsymbol{\theta} \mid z \in \overline{D}_0(\eta)\} \subseteq \overline{D}_0(R).$$

Therefore, for $\delta > 0$ sufficiently small, the compact set

$$K(\eta, \delta, R) := \{-\mathbf{N} + z\boldsymbol{\theta} \mid z \in \overline{D}_0(\eta)\} \cap K_{r,m}(\delta, R)$$

contains a non-empty open connected subset of the compact set $K_{r,m}(\delta, R)$ defined in (36). We observe that (41) holds for all $z \in \overline{D}_0(\eta)$ such that $-\mathbf{N} + z\boldsymbol{\theta} \in K_{r,m}(\delta, R)$. Therefore, it remains to show that the series on the right-hand side of (41) is normally convergent on $\overline{D}_0(\eta)$. We will then bound the general term of this series, and we will use the same bounds as those used in the proof of Proposition 17.

Let $\mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$, $\mathbf{k} = (k_1, \dots, k_\alpha) \in \mathbb{Z}_{\geq 0}^\alpha$, $1 \leq j \leq m$. By the definition of η , no singularity arises from the terms $\zeta(-N_{a_p} + z\theta_{a_p} - k_p, x_{a_p})$ ($1 \leq p \leq \alpha$) for $z \in \overline{D}_0(\eta)$. Thanks to (25), Lemma 15, and Lemma 8, we obtain:

$$\begin{aligned} |f_{j,\mathcal{A},\mathbf{k}}(-\mathbf{N} + z\boldsymbol{\theta})| &\llcorner_{\eta} \left(m \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}| \right)^{|\mathbf{k}|} \\ &\left| \frac{t^{|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} + z^{|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} + z^{|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}})} \right| \llcorner_{\eta} t^{|\mathbf{k}|} \\ &\left| \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} + z\theta_{a_p} - k_p, x_{a_p})}{k_p!} \right| \llcorner_{\varepsilon, \eta} (2\pi)^{-|\mathbf{k}|} \prod_{p=1}^{\alpha} (k_p + 1)^{\eta + \varepsilon} \quad (\varepsilon > 0) \end{aligned}$$

uniformly for $z \in \overline{D}_0(\eta)$. By compactness, we also have

$$\left| \prod_{p=1}^{\beta} \Gamma(1 + N_{b_p} - z\theta_{b_p}) \right| \ll_{\eta} 1 \quad (z \in \overline{D}_0(\eta)).$$

Therefore, the general term of the series on the right-hand side of (41) is bounded by $(\frac{m \max_{1 \leq p \leq r, 1 \leq q \leq m} |c_{q,p}| t}{2\pi})^{|k|} \prod_{p=1}^{\alpha} (k_p + 1)^{\eta + \varepsilon}$. Assuming that $t > 0$ is sufficiently small, we get that the series on the right-hand side of (41) converges normally on $\overline{D}_0(\eta)$. \square

4. Computation of $\zeta(\mathbf{c}, \mathbf{x}, -N)$ and $\zeta'(\mathbf{c}, \mathbf{x}, -N)$

In this section, we state in Theorems 20 and 21 explicit formulae for the directional (derivative) values of $\zeta(\mathbf{c}, \mathbf{x}, \mathbf{s})$. The proof of both theorems relies on Crandall's expansion established in the previous section, as well as on the meromorphic continuations of the functions F and H . Those two explicit formulae clearly imply Theorem A. Note that both formulae involve some coefficients $C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, \mathbf{N})$ and $C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, \mathbf{N}_{\theta})$, and are computed in Proposition 27. They are first defined in terms of directional (derivative) values of the parameter-dependent integral $f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, \mathbf{s})$ given in (23).

THEOREM 20. *We have*

$$\begin{aligned} \zeta(\mathbf{c}, \mathbf{x}, -N) &= \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \frac{(-1)^{|N|_{\mathcal{B}} + \beta} \theta_{r+j}}{|\theta|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} \prod_{p=1}^{\beta} N_{b_p}! \\ &\quad \times \sum_{k_1 + \dots + k_{\alpha} = \beta + |N|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, \mathbf{N}) \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \end{aligned}$$

where we write $\mathcal{A} = \{a_1, \dots, a_{\alpha}\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_{\beta}\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$, and

$$C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, \mathbf{N}) := (-1)^{N_{r+1} + \dots + \widehat{N_{r+j}} + \dots + N_{r+m}} N_{r+j}! f_{j,\mathcal{A},\mathbf{k}}(-N). \quad (42)$$

THEOREM 21. *We have*

$$\begin{aligned} \zeta'(\mathbf{c}, \mathbf{x}, -N) &= \sum_{j=1}^m \theta_{r+j} \sum_{\substack{\widehat{j} \\ u_{1,1} + \dots + u_{1,r} = N_{r+1} \\ \dots \\ u_{m,1} + \dots + u_{m,r} = N_{r+m}}} \prod_{\substack{q=1 \\ q \neq j}}^m \left(\binom{N_{r+q}}{\mathbf{u}_{q,\bullet}} \prod_{p=1}^r c_{q,p}^{u_{q,p}} \right) \quad (43) \\ &\quad \times \left[\zeta'_B(-N_{r+j}, \mathbf{M}(N, j, \mathbf{u}), \mathbf{x} | \mathbf{c}_{j,\bullet}) - (\gamma - h_{N_{r+j}}) N_{r+j}! \sum_{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket} \prod_{p=1}^{\beta} \frac{(N_{b_p} + |\mathbf{u}_{\bullet, b_p}|)!}{(-c_{j,b_p})^{N_{b_p} + |\mathbf{u}_{\bullet, b_p}| + 1}} \right. \\ &\quad \left. \times \sum_{k_1 + \dots + k_{\alpha} = N_{r+j} + \sum_{p=1}^{\beta} (N_{b_p} + |\mathbf{u}_{\bullet, b_p}| + 1)} \prod_{p=1}^{\alpha} \frac{c_{j,a_p}^{k_p} \zeta(-N_{a_p} - |\mathbf{u}_{\bullet, a_p}| - k_p, x_{a_p})}{k_p!} \right] \\ &\quad + \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \frac{(-1)^{|N|_{\mathcal{B}} + \beta} \theta_{r+j}}{|\theta|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} \prod_{p=1}^{\beta} N_{b_p}! \sum_{\substack{k_1 + \dots + k_{\alpha} \\ = \beta + |N|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} \sum_{\substack{i=1 \\ p \neq i}}^{\alpha} \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \end{aligned}$$

$$\times \left(\frac{1}{\alpha} C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, \mathbf{N}_{\boldsymbol{\theta}}) \frac{\zeta(-N_{a_i} - k_i, x_{a_i})}{k_i!} + \theta_{a_i} C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, \mathbf{N}) \frac{\zeta'(-N_{a_i} - k_i, x_{a_i})}{k_i!} \right)$$

where we have set

$$\mathbf{M}(\mathbf{N}, j, \mathbf{u}) := (N_p + |\mathbf{u}_{\bullet p}|)_{p \in \llbracket 1, r \rrbracket} \quad (1 \leq j \leq m, \mathbf{u} = (u_{q,p})_{\substack{q \in \llbracket 1, m \rrbracket \setminus \{j\} \\ p \in \llbracket 1, r \rrbracket}}), \quad (44)$$

$$C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, \mathbf{N}_{\boldsymbol{\theta}}) := N_{r+j}! \prod_{\substack{q=1 \\ q \neq j}}^m (-1)^{N_{r+q}} \quad (45)$$

$$\times \left[f'_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -\mathbf{N}_{\boldsymbol{\theta}}) + f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -\mathbf{N}) \left(\theta_{r+j}(h_{N_{r+j}} - \gamma) + \sum_{p=1}^{\beta} \theta_{b_p}(h_{N_{b_p}} - \gamma) \right) \right].$$

Note that both coefficients $C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, \mathbf{N})$ and $C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, \mathbf{N}_{\boldsymbol{\theta}})$ are computed in Proposition 27. To prove the theorems, we first notice that by Corollary 12 and 18, we have

$$\zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}}) = F(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t), \quad (46)$$

$$\zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}}) = F'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) + H'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t), \quad (47)$$

for $t > 0$ sufficiently small, where we have set

$$F(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}}, t) := \lim_{z \rightarrow 0} F(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t),$$

$$F'(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}}, t) := \lim_{z \rightarrow 0} \partial_z (F(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t)),$$

$$H'(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}}, t) := \lim_{z \rightarrow 0} \partial_z (H(\mathbf{c}, \mathbf{x}, -\mathbf{N} + z\boldsymbol{\theta}, t)).$$

In the following subsections, we compute the three directional limits to prove Theorems 20 and 21.

4.1. Lemmata

We state two key lemmata to compute the two special values $\zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}})$ and $\zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N}_{\boldsymbol{\theta}})$.

LEMMA 22. *Let $a, b \in \mathbb{C} \setminus \{0\}$, $N \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. Then we have*

$$\frac{1}{\Gamma(az - N)(bz + n)} \underset{z \rightarrow 0}{=} \begin{cases} \frac{(-1)^N a N!}{b} + \frac{(-1)^N a^2 N!}{b} (\gamma - h_N) z + O(z^2) & \text{if } n = 0 \\ \frac{(-1)^N a N!}{n} z + O(z^2) & \text{if } n \neq 0 \end{cases}$$

where h_N is the N -th harmonic number. Moreover, we have $(1/\Gamma)'(-N) = (-1)^N N!$.

PROOF. It follows from applying Gamma's functional relation, and using the first-order Taylor expansion of $1/\Gamma$ at nonpositive integers. \square

LEMMA 23. *Let $(n_1, \dots, n_\alpha) \in \mathbb{Z}_{\geq 0}^\alpha$. We have*

$$\lim_{z \rightarrow 0} \partial_z \left(\prod_{i=1}^{\alpha} \binom{-N_i - 1 + z\theta_i}{n_i} \right) = \left(\sum_{i=1}^{\alpha} \theta_i (h_{N_i} - h_{N_i + n_i}) \right) \prod_{i=1}^{\alpha} \binom{-N_i - 1}{n_i}.$$

PROOF. It follows from Leibniz' rule and from

$$\begin{aligned} \lim_{z \rightarrow 0} \partial_z \left(\binom{-N_i - 1 + z\theta_i}{n_i} \right) &= \theta_i \left(\sum_{j=0}^{n_i-1} \frac{(-N_i - 1) \cdots \widehat{(-N_i - 1 - j)} \cdots (-N_i - n_i)}{n_i!} \right) \\ &= \theta_i \binom{-N_i - 1}{n_i} \sum_{j=0}^{n_i-1} \frac{1}{-N_i - 1 - j}, \end{aligned}$$

where $\widehat{(-N_i - 1 - j)}$ indicates that the factor $(-N_i - 1 - j)$ is omitted from the product. Note that the last sum yields a difference between harmonic numbers. \square

4.2. Proof of Theorem 20

Using (46) and Proposition 19, we obtain

$$\begin{aligned} \zeta(\mathbf{c}, \mathbf{x}, -\mathbf{N})_{\boldsymbol{\theta}} &= \sum_{\substack{\mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \prod_{p=1}^{\beta} N_{b_p}! \sum_{k_1, \dots, k_{\alpha} \geq 0} (-1)^{|\mathbf{k}|} f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \\ &\quad \times \lim_{z \rightarrow 0} \frac{t^{|\mathbf{k}| - \beta - |\mathbf{N}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket} + z|\boldsymbol{\theta}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket} + z|\boldsymbol{\theta}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})}. \end{aligned} \quad (48)$$

Let $\mathcal{A} = \{a_1, \dots, a_{\alpha}\} \subseteq \llbracket 1, r \rrbracket$, $j \in \llbracket 1, m \rrbracket$, $\mathbf{k} = (k_1, \dots, k_{\alpha}) \in \mathbb{Z}_{\geq 0}^{\alpha}$. By Lemma 22 we find that

$$\begin{aligned} &\lim_{z \rightarrow 0} \frac{1}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket} + z|\boldsymbol{\theta}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \\ &= \begin{cases} 0 & \text{if } |\mathbf{k}| \neq \beta + |\mathbf{N}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket} \\ \frac{(-1)^{N_{r+j} + \theta_{r+j} N_{r+j}}}{|\boldsymbol{\theta}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} & \text{otherwise.} \end{cases} \end{aligned} \quad (49)$$

In particular, if $\mathcal{A} = \emptyset$, then \mathbf{k} corresponds to the empty tuple \emptyset , and so $|\mathbf{k}| = 0$. Since $0 < \beta = m$, we conclude that the limit on the left-hand side of (49) vanishes when $\mathcal{A} = \emptyset$.

Using (49) and splitting the sum on the right-hand side of (48) as follows

$$\sum_{k_1, \dots, k_{\alpha} \geq 0} = \sum_{k_1 + \dots + k_{\alpha} = \beta + |\mathbf{N}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} + \sum_{k_1 + \dots + k_{\alpha} \neq \beta + |\mathbf{N}|_{\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}},$$

we obtain Theorem 20. \square

4.3. Computation of $F'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t)_{\boldsymbol{\theta}}$

One can evaluate at $z = 0$ the derivative with respect to z termwise, via (41).

PROPOSITION 24. *There exists a function $F^*(\mathbf{c}, \mathbf{x}, -\mathbf{N})_{\boldsymbol{\theta}}(t) = \mu \ln t + \sum_{n=\ell}^{+\infty} \lambda_n t^n$ without constant term (i.e. $\lambda_0 = 0$), defined for all $0 < t \ll 1$ where $\ell \in \mathbb{Z}_{\leq 0}$, such that:*

$$\begin{aligned}
 F'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) &= F^*(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) \\
 &+ \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \frac{(-1)^{|\mathbf{N}|_{\mathcal{B} + \beta} \theta_{r+j}}}{|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} \prod_{p=1}^{\beta} N_{b_p}! \sum_{\substack{k_1 + \dots + k_{\alpha} \\ = \beta + |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} \sum_{\substack{i=1 \\ p \neq i}}^{\alpha} \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \\
 &\times \left(\frac{1}{\alpha} C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{N}, \boldsymbol{\theta}) \frac{\zeta(-N_{a_i} - k_i, x_{a_i})}{k_i!} + \theta_{a_i} C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{N}) \frac{\zeta'(-N_{a_i} - k_i, x_{a_i})}{k_i!} \right)
 \end{aligned} \tag{50}$$

where $C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{N})$ and $C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{N}, \boldsymbol{\theta})$ are defined in (42) and (45).

PROOF. Differentiating (41) with respect to z and taking the limit at $z = 0$, we obtain

$$\begin{aligned}
 F'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) &= \sum_{\substack{\mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \prod_{p=1}^{\beta} N_{b_p}! \sum_{k_1, \dots, k_{\alpha} \geq 0} (-1)^{|\mathbf{k}| t^{|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}}} \left(T_{j, \mathcal{A}, \mathbf{k}}^{(1)} + T_{j, \mathcal{A}, \mathbf{k}}^{(2)} + T_{j, \mathcal{A}, \mathbf{k}}^{(3)} \ln t \right),
 \end{aligned} \tag{51}$$

with

$$\begin{aligned}
 T_{j, \mathcal{A}, \mathbf{k}}^{(1)} &= \lim_{z \rightarrow 0} \left(\frac{1}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} + z|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \right) \\
 &\times \sum_{i=1}^{\alpha} \left[\frac{\zeta(-N_{a_i} - k_i, x_{a_i})}{k_i!} \left(f'_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}, \boldsymbol{\theta}) + \sum_{p=1}^{\beta} \theta_{b_p} (h_{N_{b_p}} - \gamma) f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \right) \right. \\
 &\quad \left. + \theta_{a_i} \frac{\zeta'(-N_{a_i} - k_i, x_{a_i})}{k_i!} f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \right] \prod_{\substack{p=1 \\ p \neq i}}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \\
 T_{j, \mathcal{A}, \mathbf{k}}^{(2)} &= \lim_{z \rightarrow 0} \partial_z \left(\frac{1}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} + z|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \right) \\
 &\quad \times f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \\
 T_{j, \mathcal{A}, \mathbf{k}}^{(3)} &= \lim_{z \rightarrow 0} \left(\frac{1}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} + z|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})} \right) \\
 &\quad \times |\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!}.
 \end{aligned}$$

It remains to compute the limits for each term. Let us fix $\mathcal{A} = \{a_1, \dots, a_{\alpha}\} \subseteq \llbracket 1, r \rrbracket$, $j \in \llbracket 1, m \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_{\beta}\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$, and $\mathbf{k} = (k_1, \dots, k_{\alpha})$. Thanks to Lemma 22, we have

$$\frac{1}{\Gamma(-N_{r+j} + z\theta_{r+j})(|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} + z|\boldsymbol{\theta}|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket})}$$

$$\stackrel{=}{z \rightarrow 0} \begin{cases} \frac{(-1)^{N_{r+j}} \theta_{r+j} N_{r+j}!}{|\mathbf{k}|^{-\beta} - |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}} z + O(z^2) & \text{if } |\mathbf{k}| \neq \beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}, \\ \frac{(-1)^{N_{r+j}} \theta_{r+j} N_{r+j}!}{|\boldsymbol{\theta}|_{|\mathcal{B} \cup [r+1, r+m]}} + \frac{(-1)^{N_{r+j}} \theta_{r+j}^2 N_{r+j}!}{|\boldsymbol{\theta}|_{|\mathcal{B} \cup [r+1, r+m]}} (\gamma - h_{N_{r+j}}) z + O(z^2) & \text{else.} \end{cases}$$

Therefore, many terms actually vanish depending on an equality satisfied by $|\mathbf{k}|$.

i) If $|\mathbf{k}| \neq \beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}$ we find that $T_{j, \mathcal{A}, \mathbf{k}}^{(1)} = T_{j, \mathcal{A}, \mathbf{k}}^{(3)} = 0$. In particular, if $\mathcal{A} = \emptyset$ then \mathbf{k} corresponds to the empty tuple \emptyset , thus $|\mathbf{k}| = 0 < \beta = m$ and then $T_{j, \emptyset, \mathbf{k}}^{(1)} = T_{j, \emptyset, \mathbf{k}}^{(3)} = 0$.

ii) If $|\mathbf{k}| = \beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}$, we find

$$\begin{aligned} T_{j, \mathcal{A}, \mathbf{k}}^{(1)} &= \frac{(-1)^{N_{r+j}} \theta_{r+j} N_{r+j}!}{|\boldsymbol{\theta}|_{|\mathcal{B} \cup [r+1, r+m]}} \tag{52} \\ &\times \sum_{i=1}^{\alpha} \left[\left(f'_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\frac{\mathbf{N}}{\boldsymbol{\theta}}) + f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \sum_{p=1}^{\beta} \theta_{b_p} (h_{N_{b_p}} - \gamma) \right) \frac{\zeta(-N_{a_i} - k_i, x_{a_i})}{k_i!} \right. \\ &\quad \left. + \theta_{a_i} f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \frac{\zeta'(-N_{a_i} - k_i, x_{a_i})}{k_i!} \right] \prod_{\substack{p=1 \\ p \neq i}}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!} \\ T_{j, \mathcal{A}, \mathbf{k}}^{(2)} &= \frac{(-1)^{N_{r+j}} \theta_{r+j}^2 N_{r+j}!}{|\boldsymbol{\theta}|_{|\mathcal{B} \cup [r+1, r+m]}} (\gamma - h_{N_{r+j}}) f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N}) \prod_{p=1}^{\alpha} \frac{\zeta(-N_{a_p} - k_p, x_{a_p})}{k_p!}. \tag{53} \end{aligned}$$

Using i) and ii), and splitting the series $\sum_{k_1, \dots, k_{\alpha} \geq 0}$ in (51), we obtain

$$\begin{aligned} F'(\mathbf{c}, \mathbf{x}, -\frac{\mathbf{N}}{\boldsymbol{\theta}}, t) &= F^*(\mathbf{c}, \mathbf{x}, -\frac{\mathbf{N}}{\boldsymbol{\theta}}, t) + \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq [1, r] \\ 1 \leq j \leq m}} (-1)^{\beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}} \prod_{p=1}^{\beta} N_{b_p}! \tag{54} \\ &\quad \times \sum_{k_1 + \dots + k_{\alpha} = \beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}} \left(T_{j, \mathcal{A}, \mathbf{k}}^{(1)} + T_{j, \mathcal{A}, \mathbf{k}}^{(2)} \right) \end{aligned}$$

where

$$\begin{aligned} F^*(\mathbf{c}, \mathbf{x}, -\frac{\mathbf{N}}{\boldsymbol{\theta}}, t) &:= \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq [1, r] \\ 1 \leq j \leq m}} (-1)^{\beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}} \prod_{p=1}^{\beta} N_{b_p}! \\ &\quad \times \sum_{k_1 + \dots + k_{\alpha} = \beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}} T_{j, \mathcal{A}, \mathbf{k}}^{(3)} \ln t \\ &\quad + \sum_{\substack{\mathcal{A} \subseteq [1, r] \\ 1 \leq j \leq m}} \prod_{p=1}^{\beta} N_{b_p}! \sum_{k_1 + \dots + k_{\alpha} \neq \beta + |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}} (-1)^{|\mathbf{k}|} T_{j, \mathcal{A}, \mathbf{k}}^{(2)} t^{|\mathbf{k}| - \beta - |\mathbf{N}|_{|\mathcal{B} \cup [r+1, r+m]}}. \end{aligned}$$

By plugging formulae (52) and (53) into (54), we notice that the coefficients $C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{N})$

and $C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, \mathbf{N})$ defined in (42) and (45) both appear in the summand, and we finally obtain (50). \square

4.4. Computation of $H'(\mathbf{c}, \mathbf{x}, -N, t)$

By (15), we have

$$H(\mathbf{c}, \mathbf{x}, -N + z\boldsymbol{\theta}, t) = \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, m \rrbracket \\ \mathcal{C} \subseteq \mathcal{A}^c}} \frac{(-1)^{|\mathcal{A}^c \setminus \mathcal{C}|}}{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(-N_{r+q} + z\theta_{r+q})} \quad (55)$$

$$\times \sum_{n_1, \dots, n_r \geq 0} \frac{\prod_{q \in \llbracket 1, m \rrbracket \setminus \mathcal{C}} \Gamma(-N_{r+q} + z\theta_{r+q}, t, l_q(\mathbf{n} + \mathbf{x}))}{\prod_{p=1}^r (n_p + x_p)^{-N_p + z\theta_p} \prod_{q \in \mathcal{C}} l_q(\mathbf{n} + \mathbf{x})^{-N_{r+q} + z\theta_{r+q}}}$$

for any $t > 0$ and any $z \in \mathbb{C}$. Thanks to Proposition 9, we know that all the series above are normally convergent with respect to z over any compact subset of \mathbb{C} . In particular, we can differentiate the right-hand side of (55) termwise with respect to z . By taking the limit at $z = 0$ of such derivatives, many terms vanish because $1/\Gamma$ has a zero of order 1 at nonpositive integers. Therefore, we obtain the following relation

$$H'(\mathbf{c}, \mathbf{x}, -N, t) = \sum_{j=1}^m \lim_{z \rightarrow 0} \partial_z \left(\frac{1}{\Gamma(-N_{r+j} + z\theta_{r+j})} \right)$$

$$\times \sum_{n_1, \dots, n_r \geq 0} \Gamma(-N_{r+j}, t, l_j(\mathbf{n} + \mathbf{x})) \prod_{p=1}^r (n_p + x_p)^{N_p} \prod_{\substack{q=1 \\ q \neq j}}^m (l_q(\mathbf{n} + \mathbf{x}))^{N_{r+q}}.$$

By Lemma 22 and by the multinomial theorem, we get

$$H'(\mathbf{c}, \mathbf{x}, -N, t) = \sum_{j=1}^m \theta_{r+j} \sum_{\substack{\widehat{j} \\ u_{1,1} + \dots + u_{1,r} = N_{r+1} \\ \dots \\ u_{m,1} + \dots + u_{m,r} = N_{r+m}}} \prod_{\substack{q=1 \\ q \neq j}}^m \left(\binom{N_{r+q}}{\mathbf{u}_{q,\bullet}} \prod_{p=1}^r c_{q,p}^{u_{q,p}} \right) \quad (56)$$

$$\times (-1)^{N_{r+j}} N_{r+j}! \sum_{n_1, \dots, n_r \geq 0} \Gamma(-N_{r+j}, t, l_j(\mathbf{n} + \mathbf{x})) \prod_{p=1}^r (n_p + x_p)^{N_p + |\mathbf{u}_{\bullet,p}|}.$$

We observe that this last expression is not very explicit due to the presence of the series on the right-hand side of (56). In §4.4.1, we will show via another Crandall's expansion that this series can be written as a derivative value of a generalized Barnes zeta function, plus an explicit constant term.

4.4.1. Crandall's expansion for generalized Barnes zeta function

We fix $\mathbf{M} = (M_1, \dots, M_r) \in \mathbb{Z}_{\geq 0}^r$, $j \in \llbracket 1, m \rrbracket$. By applying Corollary 18, Proposition 17 and (15) with suitable data for the generalized Barnes zeta function

$$\zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}_j \bullet) = \sum_{n_1, \dots, n_r \geq 0} \frac{(n_1 + x_1)^{M_1} \dots (n_r + x_r)^{M_r}}{(c_{j,1}(n_1 + x_1) + \dots + c_{j,r}(n_r + x_r))^s} \quad (\sigma \gg 1),$$

we obtain the following Crandall expansion for the generalized Barnes zeta function $\zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}_{j\bullet})$:

$$\zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}_{j\bullet}) = H_{\mathbf{M}}(\mathbf{c}_{j\bullet}, \mathbf{x}, s, t) + F_{\mathbf{M}}(\mathbf{c}_{j\bullet}, \mathbf{x}, s, t), \quad (57)$$

where $t > 0$ is sufficiently small, and

$$H_{\mathbf{M}}(\mathbf{c}_{j\bullet}, \mathbf{x}, s, t) := \frac{1}{\Gamma(s)} \sum_{n_1, \dots, n_r \geq 0} \Gamma(s, t, l_j(\mathbf{n} + \mathbf{x})) \prod_{p=1}^r (n_p + x_p)^{M_p} \quad (s \in \mathbb{C}), \quad (58)$$

$$F_{\mathbf{M}}(\mathbf{c}_{j\bullet}, \mathbf{x}, s, t) := \sum_{\mathcal{A} \subseteq \llbracket 1, r \rrbracket} \prod_{p=1}^{\beta} \frac{M_{b_p}!}{c_{j, b_p}^{M_{b_p}+1}} \sum_{k_1, \dots, k_{\alpha} \geq 0} (-1)^{|\mathbf{k}|} \prod_{p=1}^{\alpha} \frac{c_{j, a_p}^{k_p} \zeta(-M_{a_p} - k_p, x_{a_p})}{k_p!} \quad (59)$$

$$\times \frac{t^{s - |\mathbf{M}|_{\mathcal{B}} - \beta + |\mathbf{k}|}}{\Gamma(s)(s - |\mathbf{M}|_{\mathcal{B}} - \beta + |\mathbf{k}|)} \quad (s \neq 1, \dots, |\mathbf{M}| + r).$$

We denote by $H'_{\mathbf{M}}$ and $F'_{\mathbf{M}}$ the derivatives with respect to s of the two functions defined above. We now give an expression for both of these functions at nonpositive integers.

PROPOSITION 25. *The function $s \mapsto \zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}_{j\bullet})$ is meromorphic over \mathbb{C} , and is regular at nonpositive integers. Moreover, there exists a function with no constant term $F_{\mathbf{M}}^*(\mathbf{c}_{j\bullet}, \mathbf{x}, -N_{r+j}, t) = \mu \ln t + \sum_{n=\ell}^{+\infty} \lambda_n t^n$ defined for all $0 < t \ll 1$ where $\ell \in \mathbb{Z}_{\leq 0}$, such that*

$$F'_{\mathbf{M}}(\mathbf{c}_{j\bullet}, \mathbf{x}, -N_{r+j}, t) = F_{\mathbf{M}}^*(\mathbf{c}_{j\bullet}, \mathbf{x}, -N_{r+j}, t) + N_{r+j}!(\gamma - h_{N_{r+j}}) \quad (60)$$

$$\times \sum_{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket} \prod_{p=1}^{\beta} \frac{M_{b_p}!}{(-c_{j, b_p})^{M_{b_p}+1}} \sum_{k_1 + \dots + k_{\alpha} = N_{r+j} + |\mathbf{M}|_{\mathcal{B}} + \beta} \prod_{p=1}^{\alpha} \frac{c_{j, a_p}^{k_p} \zeta(-M_{a_p} - k_p, x_{a_p})}{k_p!},$$

$$H'_{\mathbf{M}}(\mathbf{c}_{j\bullet}, \mathbf{x}, -N_{r+j}, t) = (-1)^{N_{r+j}} N_{r+j}! \sum_{n_1, \dots, n_r \geq 0} \Gamma(-N_{r+j}, t, l_j(\mathbf{n} + \mathbf{x})) \prod_{p=1}^r (n_p + x_p)^{M_p}. \quad (61)$$

PROOF. By applying Theorem 6 to ζ_B , we know that the function $s \mapsto \zeta_B(s, \mathbf{M}, \mathbf{x} | \mathbf{c}_{j\bullet})$ has a meromorphic continuation to the whole complex plane, given by Crandall's expansion written in (57). Differentiating (58) with respect to s , and using Lemma 22, we obtain (61). To obtain (60), we take the derivative of (59), and the computations are similar to the ones in the proof of Proposition 24. \square

4.4.2. Explicit formula for $H'_{\theta}(\mathbf{c}, \mathbf{x}, -N, t)$

Via Proposition 25 we eliminate the non-explicit series on the right-hand side of (56) by first writing

$$H'_{\theta}(\mathbf{c}, \mathbf{x}, -N, t) = \sum_{j=1}^m \theta_{r+j} \sum_{\substack{\widehat{j} \\ u_{1,1} + \dots + u_{1,r} = N_{r+1} \\ \dots \\ u_{m,1} + \dots + u_{m,r} = N_{r+m}}} \prod_{\substack{q=1 \\ q \neq j}}^m \left(\binom{N_{r+q}}{\mathbf{u}_{q\bullet}} \prod_{p=1}^r c_{q,p}^{u_{q,p}} \right) \quad (62)$$

$$\times \left(\zeta'_B(-N_{r+j}, \mathbf{M}(\mathbf{N}, j, \mathbf{u}), \mathbf{x} | \mathbf{c}) - F'_{\mathbf{M}(\mathbf{N}, j, \mathbf{u})}(-N_{r+j}, \mathbf{c}_{j\bullet}, \mathbf{x}, t) \right)$$

where $\mathbf{M}(\mathbf{N}, j, \mathbf{u})$ is the tuple defined in (44). By plugging (60) into (62), we obtain a function with no constant term $H^*_{\mathbf{M}(\mathbf{N}, j, \mathbf{u})}(\mathbf{c}_{j\bullet}, \mathbf{x}, -N_{r+j}, t) = \mu \ln t + \sum_{n=\ell}^{+\infty} \lambda_n t^n$ defined for all $0 < t \ll 1$ where $\ell \in \mathbb{Z}_{\leq 0}$, such that

$$\begin{aligned} H'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) &= H^*(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) \\ &+ \sum_{j=1}^m \theta_{r+j} \sum_{\substack{\hat{j} \\ u_{1,1}+\dots+u_{1,r}=N_{r+1} \\ \dots \\ u_{m,1}+\dots+u_{m,r}=N_{r+m}}} \prod_{\substack{q=1 \\ q \neq j}}^m \left(\binom{N_{r+q}}{\mathbf{u}_{q\bullet}} \prod_{p=1}^r c_{q,p}^{u_{q,p}} \right) \left[\zeta'_B(-N_{r+j}, \mathbf{M}(\mathbf{N}, j, \mathbf{u}), \mathbf{x} | \mathbf{c}_{j\bullet}) \right. \\ &- (\gamma - h_{N_{r+j}}) N_{r+j}! \sum_{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket} \prod_{p=1}^{\beta} \frac{(N_{b_p} + |\mathbf{u}_{\bullet b_p}|)!}{(-c_{j,b_p})^{N_{b_p} + |\mathbf{u}_{\bullet b_p}| + 1}} \\ &\times \sum_{k_1 + \dots + k_\alpha = N_{r+j} + \sum_{p=1}^{\beta} (N_{b_p} + |\mathbf{u}_{\bullet b_p}| + 1)} \prod_{p=1}^{\alpha} \frac{c_{j,a_p}^{k_p} \zeta(-N_{a_p} - |\mathbf{u}_{\bullet a_p}| - k_p, x_{a_p})}{k_p!} \left. \right]. \end{aligned} \quad (63)$$

4.5. Proof of Theorem 21

By (47), recall that we have $\zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N}) = F'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) + H'(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t)$ for $t > 0$ sufficiently small. A crucial step for the proof of Theorem 21 relies on the linear independence of $(\ln t, (t^n)_{n \geq \ell})$.

LEMMA 26. *Let $\ell \in \mathbb{Z}_{\leq 0}$, and let $(\lambda_n)_{n \in \mathbb{Z}_{\geq \ell}}$ and $(\mu_n)_{n \in \mathbb{Z}_{\geq \ell}}$ be two sequences of complex numbers such that the power series $\sum_{n=0}^{+\infty} \lambda_n y^n$ and $\sum_{n=0}^{+\infty} \mu_n y^n$ have a positive radius of convergence R . Let us also assume that*

$$\forall y \in (0, R), \quad \sum_{n=\ell}^{+\infty} \lambda_n y^n + \sum_{n=\ell}^{+\infty} \mu_n y^n \ln y = 0.$$

Then for all $n \in \mathbb{Z}_{\geq \ell}$, we have $\lambda_n = 0$ and $\mu_n = 0$.

PROOF. If all μ_n vanish (resp. all λ_n vanish) for all $n \geq \ell$, the result is trivial. Otherwise, let us assume that there exist two integers $i, j \geq \ell$ such that $\lambda_i \neq 0$ and $\mu_j \neq 0$. Suppose $i = \min\{n \mid \lambda_n \neq 0\}$ and $j = \min\{n \mid \mu_n \neq 0\}$, then $\lambda_i y^i + \mu_j y^j \ln y + o(y^j \ln y) + o(y^i) \underset{y \rightarrow 0}{=} 0$, thus $\mu_j y^{j-i} \ln y \underset{y \rightarrow 0}{\sim} \lambda_i$, hence the contradiction. \square

By adding (50) and (63), we obtain that $\zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N})$ is equal to a constant term (with respect to t) that corresponds to the right-hand side of (43), plus a function in the variable t with no constant term $H^*(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) + F^*(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t)$. This equality holds for all $t > 0$ sufficiently small, yet $\zeta'(\mathbf{c}, \mathbf{x}, -\mathbf{N})$ does not depend on t , therefore by Lemma 26 we obtain for all $t > 0$ sufficiently small,

$$H^*(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) + F^*(\mathbf{c}, \mathbf{x}, -\mathbf{N}, t) = 0.$$

□

5. Computations of $C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, N)$ and $C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, N_\theta)$

Throughout this section, we fix a set $\emptyset \neq \mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, a tuple $\mathbf{k} = (k_1, \dots, k_\alpha) \in \mathbb{Z}_{\geq 0}^\alpha$, and for convenience we also set $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$. We have

PROPOSITION 27. *Let $\emptyset \neq \mathcal{A} = \{a_1, \dots, a_\alpha\} \subseteq \llbracket 1, r \rrbracket$, $\mathcal{B} = \{b_1, \dots, b_\beta\} := \llbracket 1, r \rrbracket \setminus \mathcal{A}$, $j \in \llbracket 1, m \rrbracket$, and $\mathbf{k} = (k_1, \dots, k_\alpha) \in \mathbb{Z}_{\geq 0}^\alpha$. We have*

$$C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, N) = \prod_{q=1}^m N_{r+q}! \sum_{\substack{v_{1,1}+\dots+v_{m,1}=k_1 \\ \dots \\ v_{1,\alpha}+\dots+v_{m,\alpha}=k_\alpha}} \sum_{\substack{u_{1,1}+\dots+u_{1,\beta}=N_{r+1}-|\mathbf{v}_{1\bullet}| \\ \dots \\ u_{m,1}+\dots+u_{m,\beta}=N_{r+m}-|\mathbf{v}_{m\bullet}|}} \sum^{\widehat{j}} \quad (64)$$

$$\times \prod_{p=1}^\alpha \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m C_{q,a_p}^{v_{q,p}} \right) \prod_{p=1}^\beta \left(\binom{-N_{b_p}-1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} C_{j,b_p}^{-N_{b_p}-1-|\mathbf{u}_{\bullet p}|} \prod_{\substack{q=1 \\ q \neq j}}^m C_{q,b_p}^{u_{q,p}} \right)$$

$$C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, N_\theta) = \prod_{q=1}^m N_{r+q}! \sum_{\substack{v_{1,1}+\dots+v_{m,1}=k_1 \\ \dots \\ v_{1,\alpha}+\dots+v_{m,\alpha}=k_\alpha}} \sum_{\substack{u_{1,1}+\dots+u_{1,\beta}=N_{r+1}-|\mathbf{v}_{1\bullet}| \\ \dots \\ u_{m,1}+\dots+u_{m,\beta}=N_{r+m}-|\mathbf{v}_{m\bullet}|}} \sum^{\widehat{j}} \quad (65)$$

$$\left(\sum_{p=1}^\beta \theta_{b_p} (\gamma - h_{N_{b_p}+|\mathbf{u}_{\bullet p}|} + \ln C_{j,b_p}) + \sum_{q=1}^m \theta_{r+q} (\gamma - h_{N_{r+q}}) \right)$$

$$\times \prod_{p=1}^\alpha \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m C_{q,a_p}^{v_{q,p}} \right) \prod_{p=1}^\beta \left(\binom{-N_{b_p}-1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} C_{j,b_p}^{-N_{b_p}-1-|\mathbf{u}_{\bullet p}|} \prod_{\substack{q=1 \\ q \neq j}}^m C_{q,b_p}^{u_{q,p}} \right)$$

$$+ \prod_{q=1}^m N_{r+q}! \sum_{\substack{\ell=1 \\ \ell \neq j}}^m \theta_{r+\ell} \sum_{\substack{v_{1,1}+\dots+v_{m,1}=k_1 \\ \dots \\ v_{1,\alpha}+\dots+v_{m,\alpha}=k_\alpha}} \sum_{\substack{u_{1,1}+\dots+u_{1,\beta}=N_{r+1}-|\mathbf{v}_{1\bullet}| \\ \dots \\ u_{m,1}+\dots+u_{m,\beta}=N_{r+m}-|\mathbf{v}_{m\bullet}|}} \sum^{\widehat{j,\ell}} \prod_{p=1}^\alpha \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m C_{q,a_p}^{v_{q,p}} \right)$$

$$\times \prod_{p=1}^\beta \left(\binom{-N_{b_p}-1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \prod_{\substack{q=1 \\ q \neq j,\ell}}^m C_{q,b_p}^{u_{q,p}} \right) W_{\mathcal{B},j,\ell,\mathbf{n}(N,\mathbf{u}),N_{\ell+r}-|\mathbf{v}_{\ell\bullet}|}(\mathbf{c})$$

where $\mathbf{n}(N, \mathbf{u}) := (N_{b_1} + |\mathbf{u}_{\bullet 1}|, \dots, N_{b_\beta} + |\mathbf{u}_{\bullet \beta}|)$, and W is the constant defined in (76).

We recall that $C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, N)$ (resp. $C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, N_\theta)$) is defined in (42) (resp. (45)) by the value $f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -N)$ (resp. by the values $f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -N)$ and $f'_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -N)$). Here, we will derive from (24) two explicit expressions of $f_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -N)$ and $f'_{j,\mathcal{A},\mathbf{k}}(\mathbf{c}, -N)$, which yield explicit formulae for the coefficients $C_{j,\mathcal{A},\mathbf{k}}^0(\mathbf{c}, N)$ and $C_{j,\mathcal{A},\mathbf{k}}^1(\mathbf{c}, N_\theta)$.

NOTATION. Let $\mathcal{A}' = \{a'_1, \dots, a'_{\alpha'}\} \subseteq \llbracket 1, m \rrbracket \setminus \{j\}$, $\mathcal{B}' = \{b'_1, \dots, b'_{\beta'}\} := \llbracket 1, m \rrbracket \setminus (\mathcal{A}' \cup \{j\})$. We consider two matrices $\mathbf{u} = (u_{q,p})_{1 \leq q \leq \alpha', 1 \leq p \leq \beta}$, $\mathbf{v} = (v_{q,p})_{1 \leq q \leq m, 1 \leq p \leq \alpha}$ with nonpositive integer coefficients. We denote

$$T_{j, \mathcal{A}, \mathbf{k}, \mathcal{A}', \mathbf{u}, \mathbf{v}}(\mathbf{c}, \mathbf{s}, \varepsilon) := \frac{1}{\prod_{\substack{q=1 \\ q \neq j}}^m \Gamma(s_{r+q})} \frac{\varepsilon^{\sum_{q=1}^{\alpha'} (s_{r+a'_q} + |\mathbf{u}_{q \bullet}| + |\mathbf{v}_{a'_q \bullet}|)}}{\prod_{q=1}^{\alpha'} (s_{r+a'_q} + |\mathbf{u}_{q \bullet}| + |\mathbf{v}_{a'_q \bullet}|)} \quad (66)$$

$$\times \int_{\varepsilon}^1 \cdots \int_{\varepsilon}^1 \prod_{p=1}^{\beta} \left(c_{j, b_p} + \sum_{q=1}^{\beta'} c_{b'_q, b_p} y_q \right)^{s_{a_p} - 1 - |\mathbf{u}_{\bullet p}|} \prod_{q=1}^{\beta'} y_q^{s_{r+b'_q} - 1 + |\mathbf{v}_{b'_q \bullet}|} dy_1 \cdots dy_{\beta'}$$

$$\times \prod_{p=1}^{\alpha} \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m c_{q, a_p}^{v_{q, a_p}} \right) \prod_{p=1}^{\beta} \left(\binom{s_{b_p} - 1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \prod_{q=1}^{\alpha'} c_{a'_q, b_p}^{u_{q, b_p}} \right).$$

For convenience, since we already fixed the data \mathbf{c} , \mathcal{A} , j and \mathbf{k} for this section, we simply write $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\mathbf{s}, \varepsilon) := T_{j, \mathcal{A}, \mathbf{k}, \mathcal{A}', \mathbf{u}, \mathbf{v}}(\mathbf{c}, \mathbf{s}, \varepsilon)$.

Recall that we proved in Proposition 14 that the series on the right-hand side of (24) is normally convergent on any compact subset of \mathbb{C}^{r+m} in the variables (s_1, \dots, s_{r+m}) . Therefore, by exchanging the limits and series, and by plugging the resulting formulae for $f_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\mathbf{N})$ and $f'_{j, \mathcal{A}, \mathbf{k}}(\mathbf{c}, -\frac{\mathbf{N}}{\theta})$ into (42) and (45), we find

$$C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{N}) = (-1)^{N_{r+1} + \cdots + \widehat{N_{r+j}} + \cdots + N_{r+m}} N_{r+j}! \sum_{\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}} \sum_{\substack{u_{1,1}, \dots, u_{\alpha', \beta} \geq 0 \\ v_{1,1} + \cdots + v_{m,1} = k_1 \\ \vdots \\ v_{1,\alpha} + \cdots + v_{m,\alpha} = k_\alpha}} T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon) \quad (67)$$

$$C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \frac{\mathbf{N}}{\theta}) = C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{N}) \left(\theta_{r+j} (h_{N_{r+j}} - \gamma) + \sum_{p=1}^{\beta} \theta_{b_p} (h_{N_{b_p}} - \gamma) \right) \quad (68)$$

$$+ (-1)^{N_{r+1} + \cdots + \widehat{N_{r+j}} + \cdots + N_{r+m}} N_{r+j}! \sum_{\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}} \sum_{\substack{u_{1,1}, \dots, u_{\alpha', \beta} \geq 0 \\ v_{1,1} + \cdots + v_{m,1} = k_1 \\ \vdots \\ v_{1,\alpha} + \cdots + v_{m,\alpha} = k_\alpha}} T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\frac{\mathbf{N}}{\theta}, \varepsilon)$$

where we set $T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\frac{\mathbf{N}}{\theta}, \varepsilon) = \lim_{z \rightarrow 0} \partial_z (T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N} + z\theta, \varepsilon))$. We observe that both values $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon)$ and $T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\frac{\mathbf{N}}{\theta}, \varepsilon)$ vanish for most of the matrices \mathbf{u} and \mathbf{v} .

LEMMA 28. *Let $\varepsilon \in (0, 1)$, we have*

$$T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N} + z\theta, \varepsilon)$$

$$\stackrel{z \rightarrow 0}{=} \begin{cases} O(1) & \text{if } \mathcal{A}' = \llbracket 1, m \rrbracket \setminus \{j\}, \text{ and } \forall q, |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}| = N_{r+a'_q}, \\ O(z) & \text{if } \mathcal{A}' = \llbracket 1, m \rrbracket \setminus \{j\}, \text{ and } \exists \ell \text{ s.t.} \\ & |\mathbf{u}_{\ell\bullet}| + |\mathbf{v}_{a'_\ell\bullet}| \neq N_{r+a'_\ell} \text{ and } \forall q \neq \ell, |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}| = N_{r+a'_q}, \\ O(z) & \text{if } \mathcal{A}' = \llbracket 1, m \rrbracket \setminus \{j, \ell\}, \text{ and } \forall q, |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}| = N_{r+a'_q}, \\ O(z^2) & \text{else.} \end{cases}$$

PROOF. Observe that the second and third lines on the right-hand side of (66) define entire functions in the variables (s_1, \dots, s_{r+m}) . By substituting $\mathbf{s} = -\mathbf{N} + z\boldsymbol{\theta}$ in (66), and recalling that the function $1/\Gamma$ has a zero of order 1 at nonpositive integers, we find that the function $z \mapsto T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N} + z\boldsymbol{\theta}, \varepsilon)$ has a zero at $z = 0$ of order at least

$$m - 1 - \left| \{q \in \llbracket 1, \alpha' \rrbracket \mid |\mathbf{u}_{q\bullet}| + |\mathbf{v}_{a'_q\bullet}| = N_{r+a'_q}\} \right|.$$

□

Lemma 28 indicates that there is only one situation to check when looking at the directional values of $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}$, and only three situations to check when studying its directional derivative values. In particular we will study the function $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}$ when $\mathcal{A}' = \llbracket 1, m \rrbracket \setminus \{j\}$, and $\mathcal{A}' = \llbracket 1, m \rrbracket \setminus \{j, \ell\}$, and we wish to simplify the notations used for the matrix $\mathbf{u} = (u_{q,p})_{1 \leq q \leq \alpha', 1 \leq p \leq \beta}$ for the rest of the section. In the first case $\mathcal{A}' = \{a'_1, \dots, a'_{m-1}\}$ (i.e. $\alpha' = m - 1$), we replace \mathbf{u} by $\tilde{\mathbf{u}} = (\tilde{u}_{q,p})_{1 \leq q \neq j \leq m, 1 \leq p \leq \beta}$ where $\tilde{u}_{a'_q,p} = u_{q,p}$ for all integers q, p . Similarly, in the second case $\mathcal{A}' = \{a'_1, \dots, a'_{m-2}\}$ (i.e. $\alpha' = m - 2$), we replace \mathbf{u} by $\tilde{\mathbf{u}} = (\tilde{u}_{q,p})_{1 \leq q \neq j, \ell \leq m, 1 \leq p \leq \beta}$ where $\tilde{u}_{a'_q,p} = u_{q,p}$ for all integers q, p .

5.1. Computation of $C_{j, \mathcal{A}, k}^0(\mathbf{c}, \mathbf{N})$

Considering (67), it is enough to study the value $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon)$ to obtain (64). By taking $\mathbf{s} = (-N_1, \dots, -N_{r+m})$ in (66), we obtain two possible types of values for $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon)$, depending on whether the matrices \mathbf{v} and \mathbf{u} satisfy certain conditions on their components.

PROPOSITION 29. *Let $\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}$.*

i) *If $\mathcal{A}' = \{1, \dots, \hat{j}, \dots, m\}$ and $|\mathbf{u}_{q\bullet}| + |\mathbf{v}_{q\bullet}| = N_{r+q}$, then*

$$T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon) = \prod_{\substack{q=1 \\ q \neq j}}^m ((-1)^{N_{r+q}} N_{r+q}!) \quad (69)$$

$$\times \prod_{p=1}^{\alpha} \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m c_{q, a'_p}^{v_{q,p}} \right) \prod_{p=1}^{\beta} \left(\binom{-N_{b_p} - 1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} c_{j, b_p}^{-N_{b_p} - 1 - |\mathbf{u}_{\bullet p}|} \prod_{\substack{q=1 \\ q \neq j}}^m c_{q, b_p}^{u_{q,p}} \right).$$

ii) *Otherwise, we have $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon) = 0$.*

PROOF. i) Taking the limit of (66) when $\mathbf{s} \rightarrow (-N_1, \dots, -N_{r+m})$, we get an expression of $T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon)$. Note that this expression contains an integral over a domain

given by an empty product, so by convention, the corresponding integral is 1. The rest of the terms are straightforward to compute using Lemma 22.

ii) The second case follows from Lemma 28. \square

This last proposition shows that there are only finitely many terms contributing to the series on the right-hand side of (67). By plugging (69) into (67), we get (64).

5.2. Computation of $C_{j, \mathcal{A}, k}^1(c, N_\theta)$

From (68), it is enough to study the derivative value $T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-N_\theta, \varepsilon)$ to obtain (65). The computations are done in a similar fashion as in §5.1.

PROPOSITION 30. *Let $\mathcal{A}' \subseteq \llbracket 1, m \rrbracket \setminus \{j\}$.*

i) *If $\mathcal{A}' = \{1, \dots, \widehat{j}, \dots, m\}$ and $|\mathbf{u}_{q\bullet}| + |\mathbf{v}_{q\bullet}| = N_{r+q}$ ($1 \leq q \neq j \leq m$), then there exists $T^*_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\varepsilon) = \lambda \ln \varepsilon$ with $\lambda \in \mathbb{C}$ such that*

$$\begin{aligned} T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-N_\theta, \varepsilon) &= T^*_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\varepsilon) + \prod_{\substack{q=1 \\ q \neq j}}^m ((-1)^{N_{r+q}} N_{r+q}!) \prod_{p=1}^\alpha \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m C_{q, a_p}^{v_{q,p}} \right) \\ &\quad \times \left(\sum_{p=1}^\beta \theta_{b_p} (h_{N_{b_p}} - h_{N_{b_p} + |\mathbf{u}_{\bullet p}|} + \ln c_{j, b_p}) + \sum_{\substack{q=1 \\ q \neq j}}^m \theta_{r+q} (\gamma - h_{N_{r+q}}) \right) \\ &\quad \times \prod_{p=1}^\beta \left(\binom{-N_{b_p} - 1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} c_{j, b_p}^{-N_{b_p} - 1 - |\mathbf{u}_{\bullet p}|} \prod_{\substack{q=1 \\ q \neq j}}^m C_{q, b_p}^{u_{q,p}} \right). \end{aligned}$$

ii) *If $\mathcal{A}' = \{1, \dots, \widehat{j}, \dots, m\}$, and if there exists an integer $\ell \in \mathcal{A}'$ such that $|\mathbf{u}_{\ell\bullet}| + |\mathbf{v}_{\ell\bullet}| \neq N_{r+\ell}$ and $|\mathbf{u}_{q\bullet}| + |\mathbf{v}_{q\bullet}| = N_{r+q}$ ($1 \leq q \neq j, \ell \leq m$), then $T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-N_\theta, \varepsilon) \in \mathbb{C} \varepsilon^{|\mathbf{u}_{\ell\bullet}| + |\mathbf{v}_{\ell\bullet}| - N_{r+\ell}}$ (as a polynomial in ε).*

iii) *If $\mathcal{A}' = \llbracket 1, m \rrbracket \setminus \{j, \ell\}$ and $|\mathbf{u}_{q\bullet}| + |\mathbf{v}_{q\bullet}| = N_{r+q}$ ($1 \leq q \neq j, \ell \leq m$), then there exists a function with no constant term $T^*_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\varepsilon) = \mu \ln \varepsilon + \sum_{n=i}^{+\infty} \lambda_n \varepsilon^n$ defined for all $0 < \varepsilon \ll 1$ where $i \in \mathbb{Z}_{\leq 0}$, and such that*

$$\begin{aligned} T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-N_\theta, \varepsilon) &= T^*_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(\varepsilon) + \theta_{r+\ell} \prod_{\substack{q=1 \\ q \neq j}}^m ((-1)^{N_{r+q}} N_{r+q}!) \prod_{p=1}^\alpha \left(\binom{k_p}{\mathbf{v}_{\bullet p}} \prod_{q=1}^m C_{q, a_p}^{v_{q,p}} \right) \\ &\quad \times W_{\mathcal{B}, j, \ell, \mathbf{n}(N, \mathbf{u}), N_{\ell+r} - |\mathbf{v}_{\ell\bullet}|}(c) \prod_{p=1}^\beta \left(\binom{-N_{b_p} - 1}{|\mathbf{u}_{\bullet p}|} \binom{|\mathbf{u}_{\bullet p}|}{\mathbf{u}_{\bullet p}} \prod_{\substack{q=1 \\ q \neq j, \ell}}^m C_{q, b_p}^{u_{q,p}} \right) \end{aligned}$$

where $W_\bullet(c)$ corresponds to the constant defined in (76), and where $\mathbf{n}(N, \mathbf{u})$ is the tuple defined in Proposition 27.

iv) Otherwise, we have $T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon) = 0$.

PROOF. By (66), we obtain a formula for the derivative $\partial_z T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N} + z\boldsymbol{\theta}, \varepsilon)$. Taking the limit of the resulting formula at $z = 0$, and using Lemmata 22, 23, and 40, we prove the four claims of the proposition. \square

We now compute $C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{N})$. By (68) and Proposition 30, we have

$$\begin{aligned}
C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{N}) &= C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{N}) \left(\theta_{r+j}(h_{N_{r+j}} - \gamma) + \sum_{p=1}^{\beta} \theta_{b_p}(h_{N_{b_p}} - \gamma) \right) + G^*(\varepsilon) \\
&+ \sum_{\substack{v_{1,1} + \dots + v_{m,1} = k_1 \\ \dots \\ v_{1,\alpha} + \dots + v_{m,\alpha} = k_\alpha}} (-1)^{N_{r+1} + \dots + \widehat{N_{r+j}} + \dots + N_{r+m}} N_{r+j}! \\
&\times \left(\sum_{\substack{\widehat{j} \\ u_{1,1} + \dots + u_{1,\beta} = N_{r+1} - |\mathbf{v}_{1,\bullet}| \\ \dots \\ u_{m,1} + \dots + u_{m,\beta} = N_{r+m} - |\mathbf{v}_{m,\bullet}|}} + \sum_{\substack{\ell=1 \\ \ell \neq j}}^m \sum_{\substack{\widehat{j, \ell} \\ u_{1,1} + \dots + u_{1,\beta} = N_{r+1} - |\mathbf{v}_{1,\bullet}| \\ \dots \\ u_{m,1} + \dots + u_{m,\beta} = N_{r+m} - |\mathbf{v}_{m,\bullet}|}} \right) \\
&\left(T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon) - T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}^*(-\mathbf{N}, \varepsilon) \right)
\end{aligned} \tag{70}$$

where $G^*(\varepsilon) = \mu \ln \varepsilon + \sum_{n \geq i} \lambda_n \varepsilon^n$ is a function with no constant term, defined for $0 < \varepsilon \ll 1$, with $i \in \mathbb{Z}_{\leq 0}$. In the last line of (70), we understand that \mathcal{A}' is either $\llbracket 1, m \rrbracket \setminus \{j\}$ or $\llbracket 1, m \rrbracket \setminus \{j, \ell\}$. Since $C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{N})$ is independent of ε , it follows from Lemma 26 that $G^*(\varepsilon) = 0$. Therefore the constant term (with respect to ε) of the right-hand side of (70) is equal to the coefficient $C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{N})$. By replacing $T'_{\mathcal{A}', \mathbf{u}, \mathbf{v}}(-\mathbf{N}, \varepsilon) - T_{\mathcal{A}', \mathbf{u}, \mathbf{v}}^*(-\mathbf{N}, \varepsilon)$ with their corresponding values obtained in Proposition 30, we get (65).

6. Some values and derivative values of Hurwitz and generalized Barnes zeta functions

In this section, we give explicit formulae for several of the terms appearing in (43), in specific cases. We start by recalling some well-known formulae for the Hurwitz zeta function in §6.1. In §6.2, we prove a formula for the generalized Barnes zeta function when the coefficients are rational numbers. Finally, we use Theorems 20 and 21 to establish Theorem B.

6.1. Particular Hurwitz zeta values and derivative values

Let us first recall that the values at nonpositive integers of the Hurwitz zeta function are given by Bernoulli polynomials by the relation $\zeta(-N, x) = -\frac{B_{N+1}(x)}{N+1}$ ($\Re(x) > 0$, $N \in \mathbb{Z}_{\geq 0}$), where the Bernoulli polynomials are given by the generating series $\frac{ze^{xz}}{e^z - 1}$. Recall that the Hurwitz zeta function also satisfies the *multiplication formula* (see [MA98])

$$\sum_{n=0}^{k-1} \zeta\left(s, x + \frac{n}{k}\right) = k^s \zeta(s, kx) \quad (\Re(x) > 0, s \neq 1, k \in \mathbb{N}). \quad (71)$$

Little is known about derivative values of $\zeta(s, x)$ in general. We have $\zeta'(-1, 1) = \frac{1}{12} - \ln A$ and $\zeta'(0, x) = \ln \Gamma(x) - \frac{\ln(2\pi)}{2}$ ($\Re(x) > 0$) where $A = 1.2824271\dots$ is the Glaisher–Kinkelin constant. Also, by writing $\zeta(s, x) = \zeta(s, x+1) + \frac{1}{x^s}$, we have

$$\zeta'(-N, x+1) = \zeta'(-N, x) + x^N \ln x \quad (\Re(x) > 0, N \in \mathbb{Z}_{\geq 0}). \quad (72)$$

6.2. Explicit formula for generalized Barnes zeta functions with rational coefficients

In this subsection, we fix $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{Q}_{>0}^r$, $\mathbf{x} = (x_1, \dots, x_r) \in H_0^r$, and $\mathbf{M} = (M_1, \dots, M_r) \in \mathbb{Z}_{\geq 0}^r$. We establish an explicit relation between the generalized Barnes zeta function $\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{c})$ and the Hurwitz zeta function $\zeta(s, x)$. To that end, we adapt a strategy used by Aoki and Sakane in [SA22], which originally led to formulae for higher derivative values of classical Barnes zeta functions with rational coefficients.

NOTATION. Let $c_i = \frac{p_i}{q_i}$ where $p_i, q_i \in \mathbb{N}$ are coprime integers ($1 \leq i \leq r$). We set $\tilde{c} := \frac{\text{lcm}(p_1, \dots, p_r)}{\text{gcd}(q_1, \dots, q_r)}$, and $\tilde{c}_i := \frac{\tilde{c}}{c_i} \in \mathbb{N}$ ($1 \leq i \leq r$).

PROPOSITION 31. *We have*

$$\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{c}) = \frac{\prod_{i=1}^r \tilde{c}_i^{M_i}}{\tilde{c}^s} \sum_{\substack{0 \leq u_1 \leq \tilde{c}_1 - 1 \\ \dots \\ 0 \leq u_r \leq \tilde{c}_r - 1}} \zeta_B\left(s, \mathbf{M}, \left(\frac{x_i + u_i}{\tilde{c}_i}\right)_{i \in [1, r]} \mid (1, \dots, 1)\right). \quad (73)$$

PROOF. By the analytic continuation principle, it is enough to check that (73) holds for all $s \in \mathbb{C}$ such that $\sigma \gg 1$. Let $q = \text{lcm}(q_1, \dots, q_r)$ and $p = \text{lcm}(qc_1, \dots, qc_r)$. By [SA22, Proposition 2], we know that $\frac{p}{q} = \tilde{c}$. Using the decomposition $\mathbb{Z}_{\geq 0} = \bigsqcup_{u_i=0}^{\tilde{c}_i-1} (\tilde{c}_i \mathbb{Z}_{\geq 0} + u_i)$ for each $1 \leq i \leq r$, we get

$$\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{c}) = \tilde{c}^{-s} \sum_{\substack{0 \leq u_1 \leq \tilde{c}_1 - 1 \\ \dots \\ 0 \leq u_r \leq \tilde{c}_r - 1}} \sum_{\substack{n_1 = u_1 \pmod{\tilde{c}_1} \\ \dots \\ n_r = u_r \pmod{\tilde{c}_r}}} \frac{\prod_{i=1}^r (n_i + x_i)^{M_i}}{\left(\frac{q}{p} \sum_{i=1}^r (c_i x_i + c_i u_i + c_i (n_i - u_i))\right)^s}.$$

By a change of variables $n_i \rightarrow \tilde{c}_i n_i + u_i$ ($1 \leq i \leq r$), we get

$$\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{c}) = \tilde{c}^{-s} \sum_{\substack{0 \leq u_1 \leq \tilde{c}_1 - 1 \\ \dots \\ 0 \leq u_r \leq \tilde{c}_r - 1}} \sum_{\substack{n_1 \geq 0 \\ \dots \\ n_r \geq 0}} \frac{\prod_{i=1}^r (\tilde{c}_i n_i + u_i + x_i)^{M_i}}{\left(\frac{q}{p} \sum_{i=1}^r (c_i x_i + c_i u_i + c_i \tilde{c}_i n_i)\right)^s}.$$

By factoring $\tilde{c}_i^{M_i}$ in the numerator, and by noticing that $\frac{qc_i \tilde{c}_i}{p} = 1$ ($1 \leq i \leq r$), we get

$$\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{c}) = \frac{\prod_{i=1}^r \tilde{c}_i^{M_i}}{\tilde{c}^s} \sum_{\substack{0 \leq u_1 \leq \tilde{c}_1 - 1 \\ \dots \\ 0 \leq u_r \leq \tilde{c}_r - 1}} \sum_{\substack{n_1 \geq 0 \\ \dots \\ n_r \geq 0}} \frac{\prod_{i=1}^r \left(n_i + \frac{u_i + x_i}{\tilde{c}_i} \right)^{M_i}}{\left(\sum_{i=1}^r \left(n_i + \frac{q}{p} (c_i x_i + c_i u_i) \right) \right)^s}.$$

Since $\frac{u_i + x_i}{\tilde{c}_i} = \frac{q}{p} (c_i x_i + c_i u_i)$ ($1 \leq i \leq r$), we find that (73) holds when $\sigma \gg 1$. \square

We now establish a formula for $\zeta_B(s, \mathbf{M}, \mathbf{x}|(1, \dots, 1))$, by slightly modifying Onodera's original proof of a formula for certain Barnes zeta functions with numerator terms (see [Ono21, Proposition 4.1]).

PROPOSITION 32. *Let $s \in \mathbb{C} \setminus \mathbb{N}$. We have*

$$\begin{aligned} \zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{1}) &= \sum_{\emptyset \neq \mathcal{A} \subsetneq \llbracket 1, r \rrbracket} (-1)^{\beta + |\mathbf{M}|_{|\mathcal{B}|} - 1} \prod_{i=1}^{\beta} M_{b_i}! \\ &\quad \times \sum_{k' + k_1 + \dots + k_{\alpha} = |\mathbf{M}|_{|\mathcal{B}|} + \beta - 1} (-1)^{k'} \frac{\zeta(s - k', |\mathbf{x}|)}{k'!} \prod_{i=1}^{\alpha} \frac{\zeta(-M_{a_i} - k_i, x_{a_i})}{k_i!}. \end{aligned} \quad (74)$$

Before proving this proposition, we first state a lemma provided by Onodera.

LEMMA 33 ([Ono21, Lemma 4.2]). *Let $\ell \in \mathbb{Z}$. We have*

$$\begin{aligned} &\sum_{\substack{n_1, \dots, n_r \geq 0 \\ n_1 + \dots + n_r = \ell}} \prod_{i=1}^r (n_i + x_i)^{M_i} + (-1)^{r-1} \sum_{\substack{n_1, \dots, n_r \leq -1 \\ n_1 + \dots + n_r = \ell}} \prod_{i=1}^r (n_i + x_i)^{M_i} \\ &= \sum_{\emptyset \neq \mathcal{A} \subsetneq \llbracket 1, r \rrbracket} \prod_{i=1}^{\beta} M_{b_i}! \sum_{k' + k_1 + \dots + k_{\alpha} = |\mathbf{M}|_{|\mathcal{B}|} + \beta - 1} \frac{(|\mathbf{x}| + \ell)^{k'}}{k'!} \prod_{i=1}^{\alpha} \frac{(-1)^{k_i} \zeta(-M_{a_i} - k_i, x_{a_i})}{k_i!}. \end{aligned}$$

PROOF OF PROPOSITION 32. On the convergence domain of $\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{1})$ we have

$$\zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{1}) = \sum_{n'=0}^{+\infty} \frac{1}{(n' + |\mathbf{x}|)^s} \sum_{\substack{n_1, \dots, n_r \geq 0 \\ n_1 + \dots + n_r = n'}} \prod_{i=1}^r (n_i + x_i)^{M_i}.$$

By Lemma 33, we get

$$\begin{aligned} \zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{1}) &= \sum_{\emptyset \neq \mathcal{A} \subsetneq \llbracket 1, r \rrbracket} \prod_{i=1}^{\beta} M_{b_i}! \sum_{k_1 + \dots + k_{\alpha} + k' = |\mathbf{M}|_{|\mathcal{B}|} + \beta - 1} \prod_{i=1}^{\alpha} \frac{(-1)^{k_i} \zeta(-M_{a_i} - k_i, x_i)}{k_i!} \\ &\quad \times \sum_{n'=0}^{+\infty} \frac{(n' + |\mathbf{x}|)^{k'}}{k'! (n' + |\mathbf{x}|)^s}. \end{aligned}$$

Since $\zeta(s - k', |\mathbf{x}|) = \sum_{n'=0}^{+\infty} \frac{(n' + |\mathbf{x}|)^{k'}}{(n' + |\mathbf{x}|)^s}$, we obtain (74). \square

By plugging (74) into (73), we finally get

$$\begin{aligned} \zeta_B(s, \mathbf{M}, \mathbf{x}|\mathbf{c}) &= \frac{\prod_{i=1}^r \tilde{c}_i^{M_i}}{\tilde{c}^s} \sum_{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket} (-1)^{\beta + |\mathbf{M}|_{|\mathcal{B}|} - 1} \prod_{i=1}^{\beta} M_{b_i}! \\ &\times \sum_{\substack{0 \leq u_1 \leq \tilde{c}_1 - 1 \\ \dots \\ 0 \leq u_r \leq \tilde{c}_r - 1}} \sum_{\substack{k' + k_1 + \dots + k_\alpha \\ = |\mathbf{M}|_{|\mathcal{B}|} + \beta - 1}} \frac{(-1)^{k'} \zeta\left(s - k', \frac{\langle \mathbf{c}, \mathbf{x} + \mathbf{u} \rangle}{\tilde{c}}\right)}{k'!} \prod_{i=1}^{\alpha} \frac{\zeta\left(-M_{a_i} - k_i, \frac{x_{a_i} + u_{a_i}}{\tilde{c}_{a_i}}\right)}{k_i!}, \end{aligned}$$

where we set $\langle \mathbf{c}, \mathbf{x} + \mathbf{u} \rangle := c_1(x_1 + u_1) + \dots + c_r(x_r + u_r)$. By evaluating the derivative with respect to s at nonpositive integers, we get:

COROLLARY 34. *Let $N \in \mathbb{Z}_{\geq 0}$,*

$$\begin{aligned} \zeta'_B(-N, \mathbf{M}, \mathbf{x}|\mathbf{c}) &= \tilde{c}^N \prod_{i=1}^r \tilde{c}_i^{M_i} \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 0 \leq u_1 \leq \tilde{c}_1 - 1 \\ \dots \\ 0 \leq u_r \leq \tilde{c}_r - 1}} (-1)^{\beta + |\mathbf{M}|_{|\mathcal{B}|} - 1} \prod_{i=1}^{\beta} M_{b_i}! \\ &\times \sum_{k' + k_1 + \dots + k_\alpha = |\mathbf{M}|_{|\mathcal{B}|} + \beta - 1} \frac{(-1)^{k'}}{k'!} \prod_{i=1}^{\alpha} \frac{\zeta\left(-M_{a_i} - k_i, \frac{x_{a_i} + u_{a_i}}{\tilde{c}_{a_i}}\right)}{k_i!} \\ &\times \left[\zeta'\left(-N - k', \frac{\langle \mathbf{c}, \mathbf{x} + \mathbf{u} \rangle}{\tilde{c}}\right) - \zeta\left(-N - k', \frac{\langle \mathbf{c}, \mathbf{x} + \mathbf{u} \rangle}{\tilde{c}}\right) \ln \tilde{c} \right]. \end{aligned}$$

EXAMPLE 35. Let us denote by A the Glaisher–Kinkelin constant. We have

$$\begin{aligned} \zeta'_B(0, \mathbf{0}, \mathbf{1}|(1, 1)) &= \frac{\ln(2\pi)}{2} - \ln A + \frac{1}{12}, & \zeta'_B(0, \mathbf{0}, \mathbf{1}|(1, 2)) &= \frac{\ln 2}{4} + \frac{\ln \pi}{2} - \frac{\ln A}{2} + \frac{1}{24}, \\ \zeta'_B(0, \mathbf{0}, \mathbf{1}|(1, 3)) &= -\frac{5 \ln 3}{9} + \ln(2\pi) - \frac{\ln A}{3} - \frac{2 \ln \Gamma\left(\frac{2}{3}\right)}{3} - \frac{\ln \Gamma\left(\frac{1}{3}\right)}{3} + \frac{1}{36}, \\ \zeta'_B(0, \mathbf{0}, \mathbf{1}|(2, 3)) &= -\frac{11 \ln 3}{18} + \frac{13 \ln 2}{12} + \frac{4 \ln \pi}{3} - \frac{\ln A}{6} - \frac{2 \ln \Gamma\left(\frac{2}{3}\right)}{3} - \ln \Gamma\left(\frac{1}{3}\right) + \frac{1}{72}. \end{aligned}$$

6.3. Application: Formulae for special values of Witten zeta functions

Here we compute the derivative values at $s = 0$ of Witten zeta functions of rank 2. We begin with the following lemma, which simplifies (43) where $\mathbf{N} = \mathbf{0}$.

LEMMA 36. *We have*

$$\begin{aligned} \zeta'(\mathbf{c}, \mathbf{x}, \mathbf{0}) &= \sum_{j=1}^m \theta_{r+j} \zeta'_B(0, \mathbf{0}, \mathbf{x}|\mathbf{c}_{j\bullet}) \\ &+ \sum_{\substack{\emptyset \neq \mathcal{A} \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \frac{(-1)^\beta \theta_{r+j}}{|\theta|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} \sum_{k_1 + \dots + k_\alpha = \beta} \frac{\prod_{p=1}^{\alpha} c_{j, a_p}^{k_p}}{\prod_{p=1}^{\beta} c_{j, b_p}} \sum_{i=1}^{\alpha} \theta_{a_i} \frac{\zeta'(-k_i, x_{a_i})}{k_i!} \prod_{\substack{p=1 \\ p \neq i}}^{\alpha} \frac{\zeta(-k_p, x_{a_p})}{k_p!} \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\emptyset \neq A \subseteq \llbracket 1, r \rrbracket \\ 1 \leq j \leq m}} \frac{(-1)^\beta \theta_{r+j}}{|\theta|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket}} \sum_{k_1 + \dots + k_\alpha = \beta} \prod_{p=1}^{\alpha} \frac{\zeta(-k_p, x_{a_p})}{k_p!} \left[\frac{\prod_{p=1}^{\alpha} c_{j, a_p}^{k_p}}{\prod_{p=1}^{\beta} c_{j, b_p}} \sum_{p=1}^{\beta} \theta_{b_p} \ln c_{j, b_p} \right. \\
& \quad \left. + \sum_{\substack{\ell=1 \\ \ell \neq j}}^m \theta_{r+\ell} \sum_{\substack{v_{j,1} + v_{\ell,1} = k_1 \\ \dots \\ v_{j,\alpha} + v_{\ell,\alpha} = k_\alpha}} W_{\mathcal{B}, j, \ell, \mathbf{0}, -|\mathbf{v}_{\ell \bullet}|}(\mathbf{c}) \prod_{p=1}^{\alpha} \left(\binom{k_p}{v_{j,p}} c_{j, a_p}^{v_{j,p}} c_{\ell, a_p}^{v_{\ell,p}} \right) \right].
\end{aligned}$$

PROOF. It directly follows from (43), and from the two following equalities

$$\begin{aligned}
C_{j, \mathcal{A}, \mathbf{k}}^0(\mathbf{c}, \mathbf{0}) &= \frac{\prod_{p=1}^{\alpha} c_{j, a_p}^{k_p}}{\prod_{p=1}^{\beta} c_{j, b_p}}, \\
C_{j, \mathcal{A}, \mathbf{k}}^1(\mathbf{c}, \mathbf{0}) &= \frac{\prod_{p=1}^{\alpha} c_{j, a_p}^{k_p}}{\prod_{p=1}^{\beta} c_{j, b_p}} |\theta|_{|\mathcal{B} \cup \llbracket r+1, r+m \rrbracket} \gamma + \frac{\prod_{p=1}^{\alpha} c_{j, a_p}^{k_p}}{\prod_{p=1}^{\beta} c_{j, b_p}} \sum_{p=1}^{\beta} \theta_{b_p} \ln c_{j, b_p} \\
& \quad + \sum_{\substack{\ell=1 \\ \ell \neq j}}^m \theta_{r+\ell} \sum_{\substack{v_{j,1} + v_{\ell,1} = k_1 \\ \dots \\ v_{j,\alpha} + v_{\ell,\alpha} = k_\alpha}} W_{\mathcal{B}, j, \ell, \mathbf{0}, -|\mathbf{v}_{\ell \bullet}|}(\mathbf{c}) \prod_{p=1}^{\alpha} \left(\binom{k_p}{v_{j,p}} c_{j, a_p}^{v_{j,p}} c_{\ell, a_p}^{v_{\ell,p}} \right).
\end{aligned}$$

□

Using the expressions of Witten zeta functions of rank 2 in (2), (3), (4), and (5), we have

$$\begin{aligned}
\zeta_{\text{PU}_3}(s) &= \sum_{j=1}^3 2^s \zeta \left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right), \left(\frac{j}{3}, \frac{j}{3} \right), (s, s, s) \right), \quad \zeta_{\text{Spin}_5}(s) = 6^s \zeta \left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix} \right), (1, 1), (s, \dots, s) \right), \\
\zeta_{\text{SO}_5}(s) &= 3^s \zeta \left(\left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix} \right), \left(\frac{1}{2}, 1 \right), (s, \dots, s) \right), \quad \zeta_{\text{G}_2}(s) = 120^s \zeta \left(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{smallmatrix} \right), (1, 1), (s, \dots, s) \right).
\end{aligned}$$

We apply Theorem 20 and Lemma 36 with the above data to obtain explicit formulae for the special values of each Witten zeta function. To clear out some constants in the formulae for the derivative values, we use Examples 35 and 39, (72), the multiplication formula (71), and Euler's reflection formula. We then obtain the last row of Table 1.

7. Application: Asymptotic formula for $r_{\text{G}_2}(n)$

Bridges, Brindle, Bringmann, and Franke [BBBF24] established a more general variant of Meinardus' theorem. We apply their results to obtain an asymptotic formula for the number of n -dimensional representations of the Lie group G_2 .

7.1. Statement of the Meinardus-type theorem

Let $f : \mathbb{N} \rightarrow \mathbb{Z}_{\geq 0}$. We set for all $q = e^{-z}$ ($z \in H_0$), the functions

$$G_f(z) = \sum_{n \geq 0} p_f(n) q^n := \prod_{n \geq 1} \frac{1}{(1 - q^n)^{f(n)}}, \quad L_f(s) := \sum_{n \geq 1} \frac{f(n)}{n^s}.$$

Let $\Lambda := \mathbb{Z}_{\geq 0} \setminus f^{-1}(\{0\})$. We assume that:

- (P1) Let $\alpha > 0$ be the largest pole of L_f . There exists an integer $L \in \mathbb{Z}_{\geq 0}$ such that, for every prime number p , $|\Lambda \setminus (p\mathbb{Z}_{\geq 0} \cap \Lambda)| \geq L > \frac{\alpha}{2}$.
- (P2) There exists a real number $R \in \mathbb{R}_{\geq 0}$ such that L_f is meromorphic on $\overline{H}_{-R} = \{z \in \mathbb{C} \mid \Re(z) \geq -R\}$, and is holomorphic on the line $(\Re(z) = -R)$. We also assume that the meromorphic function $L_f^*(s) := \Gamma(s)\zeta(s+1)L_f(s)$ has only real poles $\alpha := \gamma_1 > \dots$, and that these poles are simple except at $s = 0$, where the pole may be a double pole.
- (P3) There exists a real number $a < \frac{\pi}{2}$ such that, on each vertical strip $\sigma_1 \leq \sigma \leq \sigma_2$ contained in the domain of holomorphy of L_f , we have

$$L_f(s) \Big|_{|\tau| \rightarrow +\infty} = O_{\sigma_1, \sigma_2} \left(e^{a|\tau|} \right) \quad (s = \sigma + i\tau).$$

THEOREM 37 ([**BBBF24**, Theorem 4.4]). *We assume conditions (P1), (P2), and (P3). Let L be the real number from (P1) and R be the real number from (P2). Moreover, we assume that L_f has only two poles $\alpha > \beta > 0$ in H_0 , and that there exists $\ell \in \mathbb{N}$ satisfying the inequality $\frac{\ell+1}{\ell}\beta < \alpha < \frac{\ell}{\ell-1}\beta$. Then we have*

$$p_f(n) \Big|_{n \rightarrow +\infty} \frac{C}{n^b} \exp \left(A_1 n^{\frac{\alpha}{\alpha+1}} + A_2 n^{\frac{\beta}{\alpha+1}} + \sum_{j=3}^{\ell+1} A_j n^{\frac{(j-1)\beta}{\alpha+1} + \frac{j-2}{\alpha+1} + 2-j} \right) \\ \times \left(1 + \sum_{j=2}^N \frac{\tilde{B}_j}{n^{\nu_j}} + O_{L,R} \left(n^{-\min\{\frac{2L-\alpha}{2(\alpha+1)}, \frac{R}{\alpha+1}\}} \right) \right),$$

with $(A_j)_{j \geq 3}$ explicit constants, $0 < \nu_2 < \dots$ running over the positive elements of $\mathcal{N} + \mathcal{M}$ described in formulae [**BBBF24**, (1.9), (1.10)].

7.2. Proof of Theorem C

As seen in (6), we know that the generating series of $r_{G_2}(n)$ corresponds to an infinite product. The associated L -function is $L_{f_{G_2}}(s) = \zeta_{G_2}(s) = \sum_{n_1, n_2 \geq 1} P(n_1, n_2)^{-s}$, where $P(n_1, n_2) := \frac{n_1 n_2 (n_1 + n_2) (n_1 + 2n_2) (n_1 + 3n_2) (2n_1 + 3n_2)}{5!}$. Following notations of [**BBBF24**], we denote $L_{f_{G_2}}^*(s) = \zeta_{G_2}(s)\Gamma(s)\zeta(s+1)$. From [**KMT11**, Theorem 3.1], we have that $\zeta_{G_2}(s)$ has only two poles in H_0 , at $\alpha := \frac{1}{3}$ and $\beta := \frac{1}{5}$, and the integer $\ell = 2$ satisfies the hypothesis of Theorem 37. Moreover, $L_{f_{G_2}}^*(s)$ has a pole of order 2 at $s = 0$. We set $\omega_\alpha := \text{Res}_{s=\frac{1}{3}} \zeta_{G_2}(s)$ and $\omega_\beta := \text{Res}_{s=\frac{1}{5}} \zeta_{G_2}(s)$. Let $\Lambda := \mathbb{Z}_{\geq 0} \setminus f_{G_2}^{-1}(\{0\})$. We now check the conditions (P1), (P2), and (P3) of Theorem 37:

- (P1) For any prime number p , we have $|\Lambda \setminus (p\mathbb{Z}_{\geq 0} \cap \Lambda)| = +\infty$. Indeed, each sequence defined in the following

$$\begin{aligned} u(2) &:= (P(8k+1, 1))_{k \in \mathbb{Z}_{\geq 0}}, & u(3) &:= (P(9k+1, 1))_{k \in \mathbb{Z}_{\geq 0}}, \\ u(5) &:= (P(25k+1, 1))_{k \in \mathbb{Z}_{\geq 0}}, & u(p) &:= (P(kp+1, 1))_{k \in \mathbb{Z}_{\geq 0}} \quad \text{if } p \geq 7. \end{aligned}$$

is strictly increasing, and for every prime p , we have $u(p) \subseteq \Lambda \setminus (p\mathbb{Z}_{\geq 0} \cap \Lambda)$. Therefore, any real number $L \geq \frac{1}{6}$ satisfies condition (P1).

(P2) Thanks to [KMT11, Theorem 3.1], we know that the poles of $\zeta_{G_2}(s)$ belong to the set $\mathcal{S}_{G_2} := \{1/5\} \cup \{k/3 \mid k \in \mathbb{Z}_{\leq 1}, k \neq 0 \pmod{3}\}$. Also, Theorem 6 implies that all the poles are simple poles.

(P3) The polynomial P satisfies the H₀S condition of the article [Ess97]. Therefore, from [Ess97, Theorem 3], $\zeta_{G_2}(s)$ admits a polynomial bound on each vertical strip.

In the conditions (P1) and (P2), L, R are arbitrarily large. From the definition of the sets (1.8), (1.9), and (1.10) in [BBBF24], we find that the sequence ν_j in [BBBF24, Theorem 4.4] corresponds to $\nu_2 = \frac{1}{20}, \nu_3 = \frac{2}{20}, \nu_4 = \frac{3}{20}, \dots$. Applying Theorem 37 to the integer sequence $f_{G_2}(n)$, we get Theorem C with the coefficients:

$$\begin{aligned} C &= 2^{\frac{7}{4}} 15^{\frac{5}{12}} \pi^2 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{-\frac{3}{16}}, \quad b = \frac{9}{16}, \quad K_2 = \frac{3\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{20}}}, \\ A_1 &= 4 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}}, \quad A_2 = \frac{\omega_\beta \Gamma\left(\frac{1}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{1}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{20}}}, \\ A_3 &= \frac{2K_2^2}{3 \left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{3}{4}}} - \frac{\omega_\beta \Gamma\left(\frac{6}{5}\right) \zeta\left(\frac{6}{5}\right)}{\left(\omega_\alpha \Gamma\left(\frac{4}{3}\right) \zeta\left(\frac{4}{3}\right) \right)^{\frac{9}{10}}} K_2. \end{aligned}$$

These coefficients were recently explicitly computed by Au in [Au25].

8. Annex

We introduce a few elementary but technical tools used to derive our main results.

PROPOSITION 38. *Let $\mathbb{K} = \mathbb{Q}(c)$ be the field generated by the coefficients $c_{q,p}$. Let $\mathcal{B} = \{b_1, \dots, b_\beta\} \subsetneq \llbracket 1, r \rrbracket$, $j, \ell \in \llbracket 1, m \rrbracket$, $n' \in \mathbb{Z}$, $\mathbf{n} = (n_1, \dots, n_\beta) \in \mathbb{Z}_{\geq 0}^\beta$. We have the following partial fraction decomposition, in the y variable,*

$$\begin{aligned} & y^{n'-1} \times \prod_{p=1}^{\beta} (c_{j,b_p} + c_{\ell,b_p} y)^{-n_p-1} \\ &= \tilde{Q}_{\mathcal{B},j,\ell,\mathbf{n},n'}(y) + \sum_{k=1}^{n'} \frac{U_{\mathcal{B},j,\ell,\mathbf{n},n',k}}{y^k} + \sum_{p=1}^{\beta} \sum_{k=1}^{|\mathbf{n}|+\beta} \frac{V_{\mathcal{B},j,\ell,\mathbf{n},n',p,k}}{(c_{j,b_p} + c_{\ell,b_p} y)^k} \end{aligned} \quad (75)$$

where $U_{\mathcal{B},j,\ell,\mathbf{n},n',k}, V_{\mathcal{B},j,\ell,\mathbf{n},n',p,k} \in \mathbb{K}$, and where $\tilde{Q}_{\mathcal{B},j,\ell,\mathbf{n},n'}(y) \in \mathbb{K}[y]$ is a polynomial.

NOTATION. Let $Q_{\mathcal{B},j,\ell,\mathbf{n},n'}(y)$ denote the antiderivative of the polynomial $\tilde{Q}_{\mathcal{B},j,\ell,\mathbf{n},n'}(y)$, normalized so that it vanishes at $y = 1$. Let

$$W_{\mathcal{B},j,\ell,\mathbf{n},n'}(c) := -Q_{\mathcal{B},j,\ell,\mathbf{n},n'}(0) - \sum_{k=2}^{n'} \frac{U_{\mathcal{B},j,\ell,\mathbf{n},n',k}}{k-1} \quad (76)$$

$$\begin{aligned}
 & - \sum_{p=1}^{\beta} \frac{1}{c_{\ell, b_p}} \sum_{k=2}^{|\mathbf{n}|+\beta} \frac{V_{\mathcal{B}, j, \ell, \mathbf{n}, n', p, k}}{k-1} \left((c_{j, b_p} + c_{\ell, b_p})^{1-k} - c_{j, b_p}^{1-k} \right) \\
 & + \sum_{p=1}^{\beta} \frac{1}{c_{\ell, b_p}} V_{\mathcal{B}, j, \ell, \mathbf{n}, n', p, 1} \ln \left(1 + \frac{c_{\ell, b_p}}{c_{j, b_p}} \right).
 \end{aligned}$$

EXAMPLE 39. Let $j \neq \ell$, then $W_{\emptyset, j, \ell, \emptyset, 0}(\mathbf{c}) = 0$, and

$$W_{\{p\}, j, \ell, 0, 0}(\mathbf{c}) = -\frac{1}{c_{j, b_p}} \ln \left(1 + \frac{c_{\ell, b_p}}{c_{j, b_p}} \right), \quad W_{\{p\}, j, \ell, 0, 1}(\mathbf{c}) = \frac{1}{c_{\ell, b_p}} \ln \left(1 + \frac{c_{\ell, b_p}}{c_{j, b_p}} \right).$$

LEMMA 40. There exists a function with no constant term $G_{\mathcal{B}, j, \ell, \mathbf{n}, n'}^*(\varepsilon) = \mu \ln \varepsilon + \sum_{n \geq i} \lambda_n \varepsilon^n$ defined for all $0 < \varepsilon \ll 1$ with $i \in \mathbb{Z}$, and such that

$$\int_{\varepsilon}^1 y^{-1-n'} \times \prod_{p=1}^{\beta} (c_{j, b_p} + c_{\ell, b_p} y)^{-1-n_p} dy = G_{\mathcal{B}, j, \ell, \mathbf{n}, n'}^*(\varepsilon) + W_{\mathcal{B}, j, \ell, \mathbf{n}, n'}(\mathbf{c}),$$

where $W_{\mathcal{B}, j, \ell, \mathbf{n}, n'}(\mathbf{c})$ is the constant defined in (76).

PROOF. Plugging (75) into the integrand, we get

$$\begin{aligned}
 & \int_{\varepsilon}^1 y^{-1-n'} \times \prod_{p=1}^{\beta} (c_{j, b_p} + c_{\ell, b_p} y)^{-1-n_p} dy = -U_{\mathcal{B}, j, \ell, \mathbf{n}, n', 1} \ln \varepsilon \\
 & - Q_{\mathcal{B}, j, \ell, \mathbf{n}, n'}(\varepsilon) - \sum_{k=2}^{n'} \frac{U_{\mathcal{B}, j, \ell, \mathbf{n}, n', k} (1 - \varepsilon^{1-k})}{k-1} \\
 & + \frac{1}{c_{\ell, b_p}} \sum_{p=1}^{\beta} V_{\mathcal{B}, j, \ell, \mathbf{n}, n', p, 1} \left(\ln \left(1 + \frac{c_{\ell, b_p}}{c_{j, b_p}} \right) - \ln \left(1 + \frac{c_{\ell, b_p}}{c_{j, b_p}} \varepsilon \right) \right) \\
 & - \frac{1}{c_{\ell, b_p}} \sum_{p=1}^{\beta} \sum_{k=2}^{|\mathbf{n}|+\beta} \frac{V_{\mathcal{B}, j, \ell, \mathbf{n}, n', p, k}}{k-1} \left((c_{j, b_p} + c_{\ell, b_p})^{1-k} - (c_{j, b_p} + c_{\ell, b_p} \varepsilon)^{1-k} \right).
 \end{aligned}$$

We conclude via a Taylor expansion of $(c_{j, b_p} + c_{\ell, b_p} \varepsilon)^{1-k}$ and $\ln \left(1 + \frac{c_{\ell, b_p}}{c_{j, b_p}} \varepsilon \right)$ near $\varepsilon = 0$. \square

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