Asymptotic expansion for a class of second-order evolution equations in the energy space and its applications

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Abstract. In this paper, we mainly discuss asymptotic profiles of solutions to a class of abstract second-order evolution equations of the form $u'' + Au + u' = 0$ in real Hilbert spaces, where A is a nonnegative selfadjoint operator. The main result is the asymptotic expansion for all initial data belonging to the energy space, which is naturally expected. This is an improvement for the author's previous work [Math. Ann. Vol. 380 (2021), 1–19] (required sufficient regularity of the initial data). As an application, we focus our attention to damped wave equations in an exterior domain Ω in $\mathbb{R}^N$ ($N \geq 2$) with the Dirichlet boundary condition on $\partial\Omega$. By using the asymptotic expansion in the present paper, we could derive the optimal decay rates of the energy functional of solutions to damped wave equations. Moreover, some decay estimates of the local energy (energy functional restricted in a compact subset of Ω) can be observed via the asymptotic expansion.

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1 Introduction

In this paper, we consider the initial-value problem of the following second-order evolution equation in a real Hilbert space H:

$$\begin{cases} u''(t) + Au(t) + u'(t) = 0 \quad \text{in } \mathbb{R}_+, \\ (u, u')(0) = (u_0, u_1), \end{cases} \tag{1.1}$$

where $\mathbb{R}_+ = (0, \infty)$ and A is a nonnegative selfadjoint operator in H (with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$) endowed with domain $D(A)$. The pair (u_0, u_1) belongs to the energy space $\mathcal{H} = D(A^{1/2}) \times H$ and in this case a weak solution $u \in C([0, \infty); D(A^{1/2})) \cap C^1([0, \infty); H)$ of the problem (1.1) uniquely exists (see, e.g., Cazenave–Haraux [1] or Pazy [26]). The aim of the present paper is to provide the asymptotic expansion of weak solutions to (1.1) for arbitrary initial data in the energy space $\mathcal{H}$.

First, the energy functional

$$E(u; t) = \|u'(t)\|^2 + \|A^{1/2}u(t)\|^2$$

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for the solution u of (1.1) plays a central role to analyse properties of u. In particular, the energy identity

$$E(u;t) + 2\int_0^t \|u'(s)\|^2\,ds = E(u;0), \quad t > 0$$

is well-known. In Ikehata–Nishihara [14], they tried to analyse the problem (1.1) via the energy method with a technique inspired by Morawetz [21] and found that the weak solution u of (1.1) has the asymptotic behavior $v(t) = e^{-tA}(u_0 + u_1)$ with the estimate

$$\|u(t) - v(t)\| \le C_\delta \|(u_0, u_1)\|_{\mathcal{H}} (1+t)^{-1} \big(1 + \log(1+t)\big)^{1/2+\delta}, \quad t \ge 0$$

for (u_0, u_1) belonging to the energy space $\mathcal{H}$ (with the norm $\|(u_0, u_1)\|_{\mathcal{H}}^2 = \|u_0\|_{D(A^{1/2})}^2 + \|u_1\|^2$), where $\{e^{-tA}\}_{t\ge 0}$ is the C_0-semigroup on H generated by $-A$. Actually, the discussion in [14] was motivated by the observation of the behavior of weak solutions to the initial-boundary value problem of the damped wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ (u, \partial_t u)(x,0) = (u_0(x), u_1(x)) & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $\Omega = \mathbb{R}^N$ ($N \in \mathbb{N}$) or Ω is a connected exterior domain in $\mathbb{R}^N$ ($N \ge 2$) with a smooth boundary $\partial\Omega$; note that $\mathbb{N}$ denotes the set of all positive integers. The logarithmic factor on the right-hand side of the above inequality is removed by Chill–Haraux [2]. In Radu–Todorova–Yordanov [27], applying spectral analysis, the authors proved the following estimate

$$\|u(t) - v(t)\| \le Ct^{-1}\big(\|e^{-\frac{t}{2}A}u_0\| + \|e^{-\frac{t}{2}A}u_1\|\big) + Ce^{-\frac{t}{16}}\big(\|u_0\| + \|(1+A^{1/2})^{-1}u_1\|\big), \quad t > 0.$$

As a consequence of these previous works, it has been clarified that the C_0-semigroup $\{e^{-tA}\}_{t\ge 0}$ indicates the asymptotic behavior of weak solutions to (1.1).

In the recent research Sobajima [30], it is shown that if the pair satisfies $(u_0, u_1) \in D(A^{n-\frac{1}{2}}) \times D(A^{n-\frac{1}{2}})$ for some $n \in \mathbb{N}$, then the solution u of (1.1) satisfies

$$\left\| u(t) - \sum_{\ell=0}^{n-1} v_\ell(t) \right\| \le C_n t^{-(n-\frac{1}{2})} \|(u_0, u_1)\|_{D(A^{n-\frac{1}{2}}) \times D(A^{n-\frac{1}{2}})}, \quad t > 0, \tag{1.3}$$

where $v_0(t) = e^{-tA}(u_0 + u_1)$ and for $\ell \in \mathbb{N}$,

$$v_\ell(t) = A^\ell \left(\sum_{j=0}^{\ell} \binom{2\ell}{\ell+j} \frac{(-tA)^j}{j!} e^{-tA}(u_0 + u_1) - \sum_{k=0}^{\ell-1} \binom{2\ell-1}{\ell+k} \frac{(-tA)^k}{k!} e^{-tA} u_0 \right). \tag{1.4}$$

The above work can be regarded as a generalization of the result in Takeda [34] about the asymptotic expansion (with a different representation) for the Cauchy problem of the damped wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ (u, \partial_t u)(x,0) = (u_0(x), u_1(x)) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.5}$$

It is remarkable that the asymptotic expansion in [30] is also applicable to the exterior problem of the damped wave equation (1.2).

Of course, there are many previous works dealing with the problem (1.5) via the solution formula with the Fourier transform. In the pioneering work [20] by Matsumura, it is shown that the solution u of (1.5) satisfies

$$\|\partial_t^k \partial_x^\alpha u(t)\|_{L^2} \le C(1+t)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{2})-k-\frac{|\alpha|}{2}} \|(u_0,u_1)\|_{(H^m\cap L^q)\times(H^{m-1}\cap L^q)}, \quad t \ge 0$$

(and also L^∞-estimates) with $q \in [1,2]$ and $m = k + |\alpha|$, where $k \in \mathbb{Z}_{\ge 0}$, $\alpha \in \mathbb{Z}_{\ge 0}^N$ ($\mathbb{Z}_{\ge 0} = \mathbb{N} \cup \{0\}$ and $|\alpha| = \sum_{k=1}^N \alpha_k$). Here we have used the notations of the Lebesgue spaces $L^q = L^q(\mathbb{R}^N)$ and the Sobolev spaces $H^m = H^m(\mathbb{R}^N)$. Although there are many subsequent papers for asymptotic behavior of weak solutions to (1.5) and also (1.2), we will not enter details (for subsequent papers related to the Matsumura estimates and the asymptotic behavior in various situations and their applications to nonlinear problems, see e.g., Hsiao–Liu [8], Kawashima–Nakao–Ono [19], Karch [18], Yang–Milani [35], Ikehata–Ohta [15], Nishihara [23], Narazaki [22], Hosono–Ogawa [7], Ikeda–Inui–Okamoto–Wakasugi [9], Ikeda–Taniguchi–Wakasugi [13] and the references therein).

It should be pointed out that the asymptotic behavior can be found in [14],[2] and also [27] with initial data in the energy space $\mathcal{H}$, but the estimate (1.4) with $n = 1$ requires the additional regularity $u_1 \in D(A^{1/2})$. The purpose of the present paper is to fill the gap explained above, that is, to justify the asymptotic expansion with initial data in energy space $\mathcal{H}$ *without any additional regularity*.

To state the result of the present paper, we give the precise definition of weak solutions to (1.1) and the corresponding asymptotic profiles.

Definition 1.1 (Definition of weak solutions)**.** Let A be nonnegative and selfadjoint in the Hilbert space H with domain $D(A)$. For $(u_0,u_1) \in \mathcal{H} = D(A^{1/2}) \times H$, the function $u : [0,\infty) \to H$ is called a *weak solution* of (1.1) if u belongs to the class $C^1([0,\infty);H) \cap C([0,\infty);D(A^{1/2}))$ and $\mathcal{U}(\cdot) = (u(\cdot), u'(\cdot)) \in C([0,\infty);\mathcal{H})$ satisfies $\mathcal{U}(t) = e^{t\mathcal{L}}(u_0,u_1)$ for all $t \ge 0$, where $\{e^{t\mathcal{L}}\}_{t\ge 0}$ is the C_0-semigroup on $\mathcal{H}$ generated by $\mathcal{L}(f,g) = (g, -Af - g)$ endowed with domain $D(\mathcal{L}) = D(A) \times D(A^{1/2})$.

Definition 1.2 ([30])**.** For $(u_0,u_1) \in \mathcal{H}$, define a family of functions $\{v_\ell(\cdot)\}_{\ell\in\mathbb{Z}_{\ge 0}}$ as

$$v_0(t) = e^{-tA}(u_0+u_1),$$
$$v_\ell(t) = A^\ell \left(\sum_{j=0}^{\ell} \binom{2\ell}{\ell+j} \frac{(-tA)^j}{j!} e^{-tA}(u_0+u_1) - \sum_{k=0}^{\ell-1} \binom{2\ell-1}{\ell+k} \frac{(-tA)^k}{k!} e^{-tA} u_0 \right), \quad \ell \in \mathbb{N}.$$

Also we define another family $\{V_n(\cdot)\}_{n\in\mathbb{Z}_{\ge 0}}$ as $V_0(t) = 0$ and $V_n(t) = \sum_{\ell=0}^{n-1} v_\ell(t)$ for $n \in \mathbb{N}$.

The following is the main result of the present paper describing asymptotic expansion for general initial data.

Theorem 1.1. *For every $n \in \mathbb{Z}_{\geq 0}$, there exists a positive constant $C_{1,n}$ (depending only on n) such that the following assertion holds: For every $(u_0, u_1) \in \mathcal{H}$, the corresponding weak solution u of (1.1) satisfies*

$$\|u(t) - V_n(t)\|^2 + tE\left(u - V_n; t\right) \leq C_{1,n} t^{-2n} \|(u_0, u_1)\|_{\mathcal{H}}^2, \quad t \geq 1$$

where $\{V_n(\cdot)\}_{n \in \mathbb{Z}_{\geq 0}}$ is given in Definition 1.2.

Remark 1.1. For the case $n = 1$, the estimate in Theorem 1.1 matches the result of [2] and [27]. We can also see the decay estimate for the energy functional $E(u; \cdot)$ for solution u to (1.1).

Remark 1.2. The short-time estimate ($t \ll 1$) depends on the regularity of the initial data. In contrast, for the large-time behavior, we can observe all asymptotic profiles without any additional regularity.

Incidentally, the above asymptotic expansion stated in Theorem 1.1 has several applications to the exterior problem of the damped wave equation (1.2). In this case, the corresponding choice is $H = L^2(\Omega)$ and $A = -\Delta_\Omega$, where Δ_Ω is so-called Dirichlet Laplacian in $L^2(\Omega)$ endowed with domain $D(\Delta_\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$. To ensure only such a domain characterization, we have assumed that $\partial\Omega$ is smooth ($C^{1,1}$ is sufficient). Hereafter, we will use the corresponding energy space

$$\mathbf{E}(\Omega) = D((-\Delta_\Omega)^{1/2}) \times H = H_0^1(\Omega) \times L^2(\Omega)$$

and

$$\mathbf{L}^q(\Omega) = L^q(\Omega) \times L^q(\Omega)$$

for additional integrability conditions. As is expected, the Dirichlet heat semigroup $\{e^{t\Delta_\Omega}\}_{t \geq 0}$ plays a crucial role for the analysis of (1.2). To describe the detail, we introduce the energy functional $E_\Omega(u; \cdot)$ for the problem (1.2) as

$$E_\Omega(u; t) = \int_\Omega \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx, \quad t \geq 0$$

for $u \in C^1([0,\infty); L^2(\Omega)) \cap C([0,\infty); H_0^1(\Omega))$. Applying Theorem 1.1 with the usual L^p-L^q estimate ($1 \leq p \leq q \leq \infty$)

$$\|e^{t\Delta_\Omega} f\|_{L^q(\Omega)} \leq C t^{-\frac{N}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\Omega)}, \quad t > 0,$$

we can find the following assertion.

Corollary 1.2. *For every $q \in [1, 2]$ and $m \in \mathbb{Z}_{\geq 0}$, there exists a positive constant $C_{2,q,m}$ such that the following assertion holds: If $(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^q(\Omega)$, then the corresponding weak solution of (1.2) satisfies*

$$\|u(t) - V_m(t)\|_{L^2(\Omega)}^2 + tE_\Omega\left(u - V_m; t\right) \leq C_{2,q,m} t^{-N\left(\frac{1}{q} - \frac{1}{2}\right) - 2m} \|(u_0, u_1)\|_{\mathbf{E}(\Omega) \cap \mathbf{L}^q(\Omega)}^2, \quad t \geq 1,$$

where $\{V_m\}_{m \in \mathbb{Z}_{\geq 0}}$ is given in Definition 1.2 with $A = -\Delta_\Omega$ in $L^2(\Omega)$.

Remark 1.3. Ono [25] gives the estimates

$$\|u(t)\|_{L^2(\Omega)}^2 + (1+t)E_\Omega(u;t) \le \begin{cases} C(1+t)^{-\frac{N}{2}} & \text{if } N \ge 3, \\ C_\delta(1+t)^{-1+\delta} & \text{if } N = 2 \end{cases}$$

(note that $C_\delta \to \infty$ as $\delta \to 0$) under the assumption $(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^1(\Omega)$, which can be understood as a generalization of the Matsumura estimates. For the two-dimensional exterior problem, Saeki–Ikehata [28] found the decay estimate $E_\Omega(u;t) \le C(1+t)^{-2}$ also for non-compactly supported initial data in $\mathbf{E}(\Omega)$ with the additional condition $d_\Omega(u_0 + u_1) \in L^2(\Omega)$ with $d_\Omega(x) = |x|\log(\frac{2|x|}{r_\Omega})$ and $r_\Omega = \inf\{|x| \ ; \ x \in \partial\Omega\}$.

The case of two-dimensional exterior domains is something exceptional. Actually, in Grigor'yan and Saloff-Coste [6], it is observed that Dirichlet heat semigroups $\{e^{t\Delta_\Omega}\}_{t\ge0}$ in two-dimensional exterior domains satisfy a peculiar behavior which originates from the recurrence property of the Brownian motion in $\mathbb{R}^2$. This can be translated into the following inequality for $\{e^{t\Delta_\Omega}\}_{\ge0}$:

$$\|e^{t\Delta_\Omega} f\|_{L^2(\Omega)} \le Ct^{-\frac{1}{2}}\big(1+\log(1+t)\big)^{-1} \left\| \left(1 + \log\frac{|x|}{r_\Omega}\right) f \right\|_{L^1(\Omega)}, \quad t > 0 \tag{1.6}$$

(for its alternative proof via the comparison principle, see also Appendix in Ikeda–Sobajima–Taniguchi–Wakasugi [12]). If we use (1.6) instead of the usual L^p-L^q estimates, then the following assertion can be found out.

Corollary 1.3. *Let Ω be a two-dimensional exterior domain. Then for every $m \in \mathbb{Z}_{\ge0}$, there exists a positive constant $C_{3,m}$ such that the following assertion holds: If $(u_0, u_1) \in \mathbf{E}(\Omega)$ with*

$$M = \left\| \left(1 + \log\frac{|x|}{r_\Omega}\right) u_0 \right\|_{L^1(\Omega)} + \left\| \left(1 + \log\frac{|x|}{r_\Omega}\right) u_1 \right\|_{L^1(\Omega)} < \infty,$$

then the corresponding weak solution u of (1.2) satisfies

$$\|u(t) - V_m(t)\|_{L^2(\Omega)}^2 + tE_\Omega\left(u - V_m; t\right) \le C_{3,m}t^{-1-2m}(\log t)^{-2}\big(\|(u_0,u_1)\|_{\mathbf{E}(\Omega)}^2 + M^2\big), \quad t \ge 2$$

where $\{V_m(\cdot)\}_{m\in\mathbb{Z}_{\ge0}}$ is given in Definition 1.2 with $A = -\Delta_\Omega$ in $L^2(\Omega)$.

Remark 1.4. The case $m = 0$ can be found in Ikeda–Sobajima–Taniguchi–Wakasugi [12]. Corollary 1.3 also provides a slight improvement in the estimates for $m \ge 1$.

We also provide the optimality of these estimates in the following sense. The essential tool for the proof is the test function method with positive harmonic functions satisfying Dirichlet boundary condition (similar treatment can be found in Sobajima [32] for one-dimensional Schrödinger semigroups).

Proposition 1.4. *Let u be the weak solution of (1.2) with $(u_0, u_1) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$ satisfying $u_0 \ge 0, u_1 \ge 0$ and $u_0 + u_1 \not\equiv 0$. Then the following assertions hold:*

(i) *If $\Omega = \mathbb{R}^N$ $(N \in \mathbb{N})$ or Ω is an exterior domain in $\mathbb{R}^N$ $(N \geq 3)$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty} \left(t^{\frac{N}{2}} \|u(t)\|_{L^2(\Omega)}^2 \right), \quad \limsup_{t\to\infty} \left(t^{\frac{N}{2}+1} E_\Omega(u;t) \right).$$

(ii) *If Ω is a two-dimensional exterior domain, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty} \left(t(\log t)^2 \|u(t)\|_{L^2(\Omega)}^2 \right), \quad \limsup_{t\to\infty} \left(t^2 (\log t)^2 E_\Omega(u;t) \right).$$

Remark 1.5. We will prove the same conclusion with slightly generalized initial data in Section 3.

Another application of Theorem 1.1 is the estimation for a localized version of the energy functional

$$E_{\Omega,R}(u;t) = \int_{\Omega\cap B(0,R)} \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx, \quad t \geq 0$$

with $R > R_\Omega = \sup\{|x| \ ; \ x \in \partial\Omega\}$ (or $R_{\mathbb{R}^N} = 0$), where $B(y,r)$ denotes the N-dimensional ball centred at $y \in \mathbb{R}^N$ with radius r. This is the so-called "local energy". The consequence is as follows:

Proposition 1.5. *Let Ω be an exterior domain in $\mathbb{R}^N$ $(N \geq 2)$. Let u be the weak solution of (1.2) with $(u_0, u_1) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. Then there exists a positive constant C_4 such that u satisfies for every $R > R_\Omega$,*

$$E_{\Omega,R}(u;t) \leq \begin{cases} C_4 R^{N-2} t^{-N} & \text{if } N \geq 3, \\ C_4 \left(1 + \log \frac{R}{r_\Omega}\right)^2 t^{-2} (\log t)^{-4} & \text{if } N = 2 \end{cases}$$

for every $t \geq 2$, where $R_\Omega = \sup\{|x| \ ; \ x \in \partial\Omega\}$ and $r_\Omega = \inf\{|x| \ ; \ x \in \partial\Omega\}$.

Remark 1.6. In Shibata [29] (for $N \geq 3$) and Dan–Shibata [3] (for $N \geq 2$), the same estimate has been derived without logarithmic factor. More precisely, the following estimates were shown: If $(u_0, u_1) \in \mathbf{E}(\Omega)$ satisfies $\operatorname{supp} u_0 \cup \operatorname{supp} u_1 \subset B(0,R)$, then for every $t \geq 1$,

$$E_{\Omega,R}(u;t) \leq C_R t^{-N} \|(u_0,u_1)\|_{\mathbf{E}(\Omega)}^2.$$

The method in [3] (and also [29]) is the asymptotic analysis of resolvent operators related to the damped wave equation. Proposition 1.5 for the case $N \geq 3$ may be regarded as an alternative proof of their result. A novelty here is the appearance of the logarithmic factor $(\log t)^{-4}$ in the estimate for the two-dimensional case, which is slightly faster than the decay rate of the energy functional $E_\Omega(u;\cdot)$ in Corollary 1.3. But we do not know whether its decay rate is optimal or not.

Here we briefly explain the crucial idea in the present paper. The derivation of asymptotic profiles has been provided in [30]. However, if we directly apply the decomposition in [30], then the initial data are required to be regular enough. Therefore we further introduce a procedure for the regularization. Here we use a pair of reasonable operators $I_n, J_n : H \to D(A^n)$ satisfying $1_H = I_n + A^n J_n$, where 1_H denotes the identity map from H to itself (see Lemma 2.7). This gives the decomposition $u = u_* + A^n u_{**}$ with the respective initial data

$$(u_*, u_*')(0) = (I_n u_0, I_n u_1), \quad (u_{**}, u_{**}')(0) = (J_n u_0, J_n u_1).$$

This expression is reasonable because u_* and u_{**} are regular enough ($u_*, u_{**} \in D(A^{n+\frac{1}{2}})$) and $A^n u_{**}$ can be expected to have an extra decay property coming from the factor A^n. At this moment, we can apply the procedure in [30] to u_*. The rest is a straightforward computation based on the energy method.

For the optimality of decay rates for energy functionals, we use the test function method with positive harmonic functions satisfying the Dirichlet boundary condition for *linear* equations together with the Gagliardo–Nirenberg and Nash inequalities. In the case of two-dimensional exterior domain, we further need a Nash-type inequality involving a logarithmic weight function found in [12].

The present paper is organized as follows. At the beginning of Section 2, we start with the basic facts in the analysis of the second-order evolution equation (1.1) from the view point of the auxiliary quantity

$$\|u(t)\|_{\mathcal{E}^\sharp}^2 = \|A^{1/2}u(t)\|^2 + \left\|u'(t) + \frac{1}{2}u(t)\right\|^2 + \frac{1}{4}\|u(t)\|^2 \tag{1.7}$$

(a similar explanation can be found in Sobajima [33]). In the middle of Section 2, we would give an alternative proof of asymptotic expansion for regular initial data stated in [30] via the use of the auxiliary quantity $\|u(t)\|_{\mathcal{E}^\sharp}^2$. Then at the end of Section 2, we shall give a proof of Theorem 1.1 via the regularization with I_n and J_n explained above.

As applications of Theorem 1.1, we discuss the profile of weak solutions to the exterior problem of the damped wave equation (1.2) in Section 3. First we prove the decay estimates for L^2-norm and the energy $E_\Omega(u;\cdot)$. We also show the optimality of their decay rates via the test function method with positive harmonic functions satisfying the Dirichlet boundary condition. Finally, we discuss decay estimates of the local energy $E_{\Omega,R}(u;t)$ for weak solutions to (1.2).

2 Analysis for the abstract evolution equation

Here we consider the initial-value problem of the second-order evolution equation in a real (or even complex) Hilbert space H (with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$), which is of the form

$$\begin{cases} u''(t) + Au(t) + u'(t) = 0, \quad t \in \mathbb{R}_+, \\ (u, u')(0) = (u_0, u_1) \in D(A^{1/2}) \times H. \end{cases} \tag{2.1}$$

Here A is a nonnegative selfadjoint operator in H endowed with domain $D(A)$. We discuss the asymptotic expansion of weak solutions to (2.1).

2.1 Basic facts with a derivation

As usual, existence and uniqueness of weak solutions to (2.1) is obtained by the Hille–Yosida theorem. Here we use the following space and operator; note that the choice of the norm for the product space is slightly different from the usual one but equivalent to it.

Definition 2.1. **(i)** Define $\mathcal{H} = D(A^{1/2}) \times H$ as a Hilbert space with the inner product

$$\big\langle (f_1, g_1), (f_2, g_2) \big\rangle_{\mathcal{H}} = \langle A^{1/2} f_1, A^{1/2} f_2 \rangle + \frac{1}{4} \langle f_1, f_2 \rangle + \Big\langle g_1 + \frac{1}{2} f_1, g_2 + \frac{1}{2} f_2 \Big\rangle$$

for $(f_1, g_1), (f_2, g_2) \in \mathcal{H}$; note that the norm $\| \cdot \|_{\mathcal{H}}$ is equivalent to the usual norm:

$$\|A^{1/2} f\|^2 + \frac{1}{6} \|f\|^2 + \frac{1}{4} \|g\|^2 \le \|(f, g)\|_{\mathcal{H}}^2 \le \|A^{1/2} f\|^2 + \|f\|^2 + \frac{3}{2} \|g\|^2.$$

(ii) Define the densely defined operator $\mathcal{L} : D(\mathcal{L}) \, (\subset \mathcal{H}) \to \mathcal{H}$ as

$$\begin{cases} D(\mathcal{L}) = D(A) \times D(A^{1/2}) \\ \mathcal{L}(f, g) = (g, -Af - g); \end{cases}$$

note that by using $\mathcal{L}$ we can reduce the second-order evolution equation (2.1) to the first-order equation $\frac{d}{dt}(u, \widetilde{u}) = \mathcal{L}(u, \widetilde{u})$ equipped with the initial condition $(u, \widetilde{u})|_{t=0} = (u_0, u_1)$.

(iii) We also define for $\xi \in C^1([0, \infty); H) \cap C([0, \infty); D(A^{1/2}))$,

$$\|\xi(t)\|_{\mathcal{E}^\sharp} = \|(\xi(t), \xi'(t))\|_{\mathcal{H}}.$$

Remark 2.1. The following identity suggests that the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ might represents a reasonable dissipative structure for the problem (2.1): for every $(f, g) \in D(\mathcal{L})$,

$$\begin{aligned} \big\langle \mathcal{L}(f, g), (f, g) \big\rangle_{\mathcal{H}} &= \big\langle (g, -Af - g), (f, g) \big\rangle_{\mathcal{H}} \\ &= \langle A^{1/2} g, A^{1/2} f \rangle + \Big\langle -Af - \frac{1}{2} g, g + \frac{1}{2} f \Big\rangle + \frac{1}{4} \langle g, f \rangle \\ &= -\frac{1}{2} \Big(\|A^{1/2} f\|^2 + \|g\|^2 \Big). \end{aligned}$$

The solvability of the abstract problem (2.1) is verified by the Hille–Yosida theorem.

Proposition 2.1. *The operator $-\mathcal{L}$ is m-accretive in $\mathcal{H}$, and therefore, $\mathcal{L}$ generates a C_0-semigroup $\{e^{t\mathcal{L}}\}_{t \ge 0}$ of contractions on $\mathcal{H}$. In particular, for every $n \in \mathbb{Z}_{\ge 0}$, one has*

$$e^{t\mathcal{L}}(u_0, u_1) \in \bigcap_{k=0}^{n} C^{n-k}([0, \infty); D(\mathcal{L}^k))$$

provided $(u_0, u_1) \in D(\mathcal{L}^n)$.

The following lemma is the merit of the use of the quantity $\|u(t)\|^2_{\mathcal{E}^\sharp}$.

Lemma 2.2. *Let u be the weak solution of (2.1) with $(u_0, u_1) \in D(\mathcal{L})$. Then one has*

$$\frac{d}{dt}E(u;t) + 2\|u'(t)\|^2 = 0, \quad t \geq 0, \tag{2.2}$$

$$\frac{d}{dt}\|u(t)\|^2_{\mathcal{E}^\sharp} + E(u;t) = 0, \quad t \geq 0. \tag{2.3}$$

Proof. For the convenience, we also give a proof of the well-known identity (2.2). By the standard computation with the equation in (2.1), we have

$$\begin{aligned}\frac{d}{dt}E(u;t) &= 2\langle u'(t), u''(t)\rangle + 2\langle A^{1/2}u'(t), A^{1/2}u(t)\rangle \\ &= 2\langle u'(t), u''(t) + Au(t)\rangle \\ &= -2\|u'(t)\|^2.\end{aligned}$$

For the second identity (2.3), we argue in the following way:

$$\frac{d}{dt}\|u(t)\|^2_{\mathcal{E}^\sharp} = \frac{d}{dt}\|e^{t\mathcal{L}}(u_0, u_1)\|^2_{\mathcal{H}} = 2\left\langle \mathcal{L}e^{t\mathcal{L}}(u_0, u_1), e^{t\mathcal{L}}(u_0, u_1)\right\rangle_{\mathcal{H}} = -E(u;t),$$

where we have used the computation in Remark 2.1. □

Therefore we can observe the well-known estimate of $\|u(t)\|^2 + (1+t)E(u;t)$ for the solution u to (2.1) via the identities in Lemma 2.2.

Lemma 2.3. *Let u be the weak solution of (2.1) with $(u_0, u_1) \in \mathcal{H}$. Then*

$$2\|u(t)\|^2_{\mathcal{E}^\sharp} + tE(u;t) + 2\int_0^t \left(\|A^{1/2}u(s)\|^2 + (1+s)\|u'(s)\|^2\right) ds \leq 2\|(u_0, u_1)\|^2_{\mathcal{H}}, \quad t \geq 0.$$

Decay estimates for higher derivatives of the weak solution u to (2.1) are proved via an iterative procedure. The second lemma is helpful to derive a desired estimate for each step in such an iteration.

Lemma 2.4. *Let w be the weak solution of $w'' + Aw + w' = 0$ with $(w, w')(0) = (w_0, w_1) \in \mathcal{H}$. Then for every $k > 0$, there exists a positive constant $C_{5,k}$ depending only on k such that*

$$\begin{aligned}&(1+t)^k\left(\|w(t)\|^2_{\mathcal{E}^\sharp} + tE(w;t)\right) + \int_0^t (1+s)^k\left(\|A^{1/2}w(s)\|^2 + (1+s)\|w'(s)\|^2\right) ds \\ &\leq C_{5,k}\left(\|(w_0, w_1)\|^2_{\mathcal{H}} + \int_0^t (1+s)^{k-1}\|w(s)\|^2\, ds\right), \quad t \geq 0.\end{aligned}$$

Proof. This is a direct consequence of the following computation:

$$\begin{aligned}
&\frac{d}{dt}\left[(4k+t)^k\|w(t)\|_{\mathcal{E}^\sharp}^2+\frac{(4k+t)^{k+1}}{2(k+1)}E(w;t)\right]\\
&=(4k+t)^k\left[\frac{k}{4k+t}\|w(t)\|_{\mathcal{E}^\sharp}^2-E(w;t)+\frac{1}{2}E(w;t)-\frac{4k+t}{k+1}\|w'(t)\|^2\right]\\
&\le(4k+t)^k\left[\frac{k}{4k+t}\|w\|^2-\frac{1}{8}E(w;t)-\frac{4k+t}{k+1}\|w'(t)\|^2\right].
\end{aligned}$$

Integrating it over $[0,t]$, we deduce the desired estimate. □

Decay estimates for $u^{(n)}=\frac{d^n}{dt^n}u$ and $A^{n/2}u$ of the weak solution u to (2.1) are stated as follows.

Lemma 2.5. *For every $n\in\mathbb{N}$, there exists a positive constant $C_{6,n}$ such that the following holds: for every $(u_0,u_1)\in D(\mathcal{L}^n)$, the corresponding (strong) solution u of (2.1) satisfies*

$$\left(\|u^{(n)}(t)\|^2+(1+t)E(u^{(n)};t)\right)\le C_{6,n}(1+t)^{-2n}\|(u_0,u_1)\|_{D(\mathcal{L}^n)}^2,\quad t\ge0 \tag{2.4}$$

and

$$\left(\|A^{n/2}u(t)\|^2+(1+t)E(A^{n/2}u;t)\right)\le C_{6,n}(1+t)^{-n}\|(u_0,u_1)\|_{D(\mathcal{L}^n)}^2,\quad t\ge0. \tag{2.5}$$

Proof. We first show (2.4) by proving

$$\begin{aligned}
&(1+t)^{2n}\left(\|u^{(n)}(t)\|_{\mathcal{E}^\sharp}^2+tE(u^{(n)};t)\right)\\
&+\int_0^t(1+s)^{2n}\left(\|A^{1/2}u^{(n)}(s)\|^2+(1+s)\|u^{(n+1)}(s)\|^2\right)ds\le\widetilde{C}_n\|(u_0,u_1)\|_{D(\mathcal{L}^n)}^2
\end{aligned} \tag{2.6}$$

for some series of positive constants $\{\widetilde{C}_n\}_{n\in\mathbb{Z}_{\ge0}}$. We argue by induction as explained above. In view of Lemma 2.3, we already have (2.6) with $n=0$. For the estimate (2.6) with $n\in\mathbb{N}$, we assume the validity of (2.6) with $n=m-1$ holding with $m\in\mathbb{N}$. Then Lemma 2.4 with $(w,k)=(u^{(m)},2m)$ gives for every $t\ge0$,

$$\begin{aligned}
&(1+t)^{2m}\left(\|u^{(m)}(t)\|_{\mathcal{E}^\sharp}^2+tE(u^{(m)};t)\right)\\
&+\int_0^t(1+s)^{2m}\left(\|A^{1/2}u^{(m)}(s)\|^2+(1+s)\|u^{(m+1)}(s)\|^2\right)ds\\
&\le C_{5,2m}\left(\|(u^{(m)}(0),u^{(m+1)}(0))\|_{\mathcal{H}}^2+\int_0^t(1+s)^{2m-1}\|u^{(m)}(s)\|^2\,ds\right)\\
&\le C_{5,2m}\left(\|\mathcal{L}^m(u_0,u_1)\|_{\mathcal{H}}^2+\widetilde{C}_{m-1}\|(u_0,u_1)\|_{D(\mathcal{L}^{m-1})}^2\right).
\end{aligned}$$

For (2.5), a similar induction argument via Lemma 2.4 with $(w,k)=(A^{m/2}u,m)$ provides the desired inequality. □

2.2 Asymptotic expansion

Here we consider the asymptotic expansion for weak solutions to (2.1) as $t \to \infty$. Namely, we would explain how to construct a family $\{v_\ell(t)\}_{\ell \in \mathbb{Z}_{\geq 0}}$ satisfying for every $n \in \mathbb{Z}_{\geq 0}$,

$$\|v_n(t)\| = O(t^{-n}), \quad \|u(t) - V_n(t)\| = O(t^{-n}) \quad \text{as } t \to \infty,$$

where $V_0(t) = 0$ and $V_n(t) = \sum_{\ell=0}^{n-1} v_\ell(t)$ for $n \in \mathbb{N}$.

2.2.1 Asymptotic expansion for regular initial data (revisited)

If the initial data are regular enough, then the asymptotic profiles of (regular) solutions to (2.1) have been found in [30] as in Definition 1.2. The profiles $\{v_\ell(\cdot)\}_{\ell \in \mathbb{Z}_{\geq 0}}$ are successively derived by the following procedure. Put $v_{0,0}(t) = e^{-tA}(u_0 + u_1)$. Then using the equation (2.1), we have

$$u''(t) + A\big(u(t) - v_{0,0}(t)\big) + \big(u(t) - v_{0,0}(t)\big)' = 0, \quad t \in \mathbb{R}_+.$$

Putting $U_1(t) = \int_0^t \big(u(s) - v_{0,0}(s)\big)\, ds$ formally and integrating both sides of the above equality, we arrive at for every $t > 0$,

$$\begin{aligned} u'(t) + AU_1(t) + u(t) - v_{0,0}(t) &= u'(0) + u(0) - v_{0,0}(0) \\ &= u_1 + u_0 - (u_0 + u_1) \\ &= 0. \end{aligned}$$

Then rewriting the equation in term of the unknown function U_1, we deduce

$$\begin{cases} U_1''(t) + AU_1(t) + U_1'(t) = -v_{0,0}'(t) & \text{in } \mathbb{R}_+, \\ (U_1, U_1')(0) = (0, -u_1). \end{cases}$$

This consideration actually can be justified that U_1' decays faster than $v_{0,0}$. Similarly, define $\{v_{\ell,0}\}_{\ell \in \mathbb{N}}$ successively via the following relation

$$\begin{cases} v_{\ell,0}'(t) + Av_{\ell,0}(t) = -v_{\ell-1,0}'(t) & \text{in } \mathbb{R}_+, \\ v_{\ell,0}(0) = (-1)^\ell u_1. \end{cases}$$

Finally, we obtain the decomposition

$$\begin{aligned} u(t) &= v_{0,0}(t) + U_1'(t) \\ &= v_{0,0}(t) + v_{1,0}'(t) + U_2''(t) \\ &\vdots \\ &= \sum_{\ell=0}^{n-1} \frac{d^\ell}{dt^\ell} v_{\ell,0}(t) + \frac{d^n}{dt^n} U_n(t), \end{aligned} \tag{2.7}$$

where $\{U_n\}_{\ell\in\mathbb{N}}$ are determined by the solution of the following problems:

$$\begin{cases} U_n''(t) + AU_n(t) + U_n'(t) = -v_{n-1,0}'(t) & \text{in } \mathbb{R}_+, \\ (U_n, U_n')(0) = (0, (-1)^n u_1). \end{cases} \tag{2.8}$$

The expression

$$v_{\ell,0}(t) = (-1)^\ell \left[\sum_{j=0}^{\ell} \binom{\ell}{j} \frac{(-tA)^j}{j!} e^{-tA}(u_0+u_1) - \sum_{j=0}^{\ell-1} \binom{\ell-1}{j} \frac{(-tA)^j}{j!} e^{-tA}u_0 \right] \quad (\ell \in \mathbb{N})$$

can be found via a direct computation with the Duhamel formula

$$v_{\ell,0}(t) = (-1)^\ell e^{-tA}u_1 - \int_0^t e^{-(t-s)A} v_{\ell-1,0}'(s)\, ds, \quad t \in \mathbb{R}_+$$

and the induction. Therefore the ℓ-th order asymptotic profile of the solution u to (2.1) is expected to be the following:

$$v_\ell(t) = \frac{d^\ell}{dt^\ell} v_{\ell,0}, \quad \ell \in \mathbb{Z}_{\geq 0}.$$

(for the detail, see [30]). In [30], it is proved that if the initial data are regular ($(u_0,u_1) \in D(A^{n-\frac{1}{2}}) \times D(A^{n-\frac{1}{2}})$), then this family actually gives the asymptotic profiles of the solution u to (2.1). The precise assertion is the following.

Theorem 2.6 ([30, Theorem 1.1])**.** *For every $n \in \mathbb{N}$, there exists a positive constant $C_{7,n}$ such that the following assertion holds: If $(u_0,u_1) \in D(A^{n-\frac{1}{2}}) \times D(A^{n-\frac{1}{2}})$, then the corresponding solution u of (2.1) satisfies*

$$\|u(t) - V_n(t)\| \leq C_{7,n}(1+t)^{-(n-\frac{1}{2})} \|(u_0,u_1)\|_{D(A^{n-1/2})\times D(A^{n-1/2})}, \quad t \geq 0,$$

where the family $\{V_n(\cdot)\}_{n\in\mathbb{N}}$ is given in Definition 1.2.

Remark 2.2. One can see that the asymptotic profiles $\{v_\ell(\cdot)\}_{\ell\in\mathbb{Z}_{\geq 0}}$ satisfy

$$\|v_\ell(t)\| \leq C_\ell t^{-\ell} \|(u_0,u_1)\|_{H\times H}, \quad t \geq 1$$

with some constants C_ℓ depending only on ℓ. Therefore Theorem 2.6 actually provides a justification of asymptotic expansion.

Here we provide a brief explanation of the proof of Theorem 2.6 by using the quantity $\|\cdot\|_{\mathcal{E}^\sharp}$. This is not a novelty, but some arguments based on energy methods are slightly simplified.

Sketch of the proof of Theorem 2.6. Let $n \in \mathbb{N}$. To derive the estimate for $\frac{d^n}{dt^n}U_n(t)$, we start with the usual energy estimate for U_n and then proceed a similar argument as the proof of Lemma 2.5.

Put for $\ell, k \in \mathbb{Z}_{\geq 0}$, $v_{0,k}(t) = (-1)^k e^{-tA}(u_0 + u_1)$ and

$$v_{\ell,k}(t) = (-1)^{\ell+k}\left[\sum_{j=0}^{\ell}\binom{\ell+k}{j+k}\frac{(-tA)^j}{j!}e^{-tA}(u_0+u_1) - \sum_{j=0}^{\ell-1}\binom{\ell+k-1}{j+k}\frac{(-tA)^j}{j!}e^{-tA}u_0\right],$$

which satisfies $\frac{d^k}{dt^k}v_{\ell,0} = A^k v_{\ell,k}$. Using (2.8), we can find a couple of equalities

$$\begin{aligned}
&\frac{d}{dt}E(U_n;t) + 2\|U_n'(t)\|^2 = -2\left\langle Av_{n-1,1}(t), U_n'(t)\right\rangle,\\
&\frac{d}{dt}\|U_n(t)\|_{\mathcal{E}^\sharp}^2 + E(U_n;t) = -2\left\langle Av_{n-1,1}(t), U_n'(t)\right\rangle - \left\langle A^{1/2}v_{n-1,1}(t), A^{1/2}U_n(t)\right\rangle.
\end{aligned}$$

Therefore by a straightforward calculation, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\|U_n(t)\|_{\mathcal{E}^\sharp}^2 + \frac{4+t}{2}E(U_n;t)\right] + E(U_n;t) + (4+t)\|U_n'(t)\|^2\\
&\leq 2\|Av_{n-1,1}(t)\|\|U_n'(t)\| + \|A^{1/2}v_{n-1,1}(t)\|\|A^{1/2}U_n(t)\|\\
&\quad + \frac{1}{2}E(U_n;t) + (4+t)\|Av_{n-1,1}(t)\|\|U_n'(t)\|\\
&\leq \|A^{1/2}v_{n-1,1}(t)\|^2 + \frac{12+t}{2}\|Av_{n-1,1}(t)\|^2 + \frac{3}{4}E(U_n;t) + \frac{4+t}{2}\|U_n'(t)\|^2
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{d}{dt}\left[\|U_n(t)\|_{\mathcal{E}^\sharp}^2 + \frac{4+t}{2}E(U_n;t)\right] + \frac{1}{4}E(U_n;t) + \frac{4+t}{2}\|U_n'(t)\|^2\\
&\leq \|A^{1/2}v_{n-1,1}(t)\|^2 + \frac{12+t}{2}\|Av_{n-1,1}(t)\|^2.
\end{aligned}$$

Since the right-hand side of the above inequality is integrable on $(0,\infty)$ via the estimate

$$\int_0^\infty (1+t)^m \|A^{\frac{m+1}{2}}e^{-tA}f\|^2\,dt \leq C_m\|f\|_{D(A^{m/2})}^2 \quad (m \in \mathbb{Z}_{\geq 0},\ f \in D(A^{m/2}))$$

for some positive constant C_m (see e.g., [30, Lemma 2.1]), we arrive at

$$\begin{aligned}
&\|U_n(t)\|_{\mathcal{E}^\sharp}^2 + (1+t)E(U_n;t)\\
&+ \int_0^t \left(\|A^{1/2}U_n(t)\|^2 + (1+s)\|U_n'(s)\|^2\right)ds \leq C(\|u_0+u_1\|_{D(A^{1/2})}^2 + \|u_1\|_{D(A^{1/2})}^2).
\end{aligned}$$

This is the desired estimate for $U_n(t)$. To find the estimate for $U_n'(t)$, we proceed a similar argument as Lemma 2.4 (but with inhomogeneous terms). Putting $w_1(t) = U_n'(t)$ for simplicity, we can see that w_1 satisfies

$$\begin{cases}
w_1''(t) + Aw_1(t) + w_1'(t) = -v_{n-1,0}''(t) = -A^2 v_{n-1,2}(t) \quad \text{in } \mathbb{R}_+,\\
(w_1, w_1')(0) = ((-1)^n u_1, -(-1)^n u_1 - v_{n-1,0}'(0)).
\end{cases}$$

Noting that $(1+t)\|w_1(t)\|^2$ is integrable on $(0,\infty)$ and $\|w_1(t)\|^2_{\mathcal{E}^\sharp} \le E(w_1;t) + 2\|w_1(t)\|^2$, we further find

$$
\begin{aligned}
&\frac{d}{dt}\left[(8+t)^2\|w(t)\|^2_{\mathcal{E}^\sharp} + \frac{(8+t)^3}{6}E(w;t)\right] \\
&= 2(8+t)\|w(t)\|^2_{\mathcal{E}^\sharp} - (8+t)^2\Big[E(w;t) + 2\left\langle A^2 v_{n-1,2}(t), w_1'(t)\right\rangle + \left\langle A^{3/2}v_{n-1,2}(t), A^{1/2}w_1(t)\right\rangle\Big] \\
&\quad + \frac{1}{2}(8+t)^2E(w;t) - \frac{1}{3}(8+t)^3\Big[\|w'(t)\|^2 + \left\langle A^2 v_{n-1,2}(t), w_1'(t)\right\rangle\Big] \\
&\le 4(8+t)\|w(t)\|^2 - \frac{1}{3}(2+t)(8+t)E(w;t) - \frac{1}{6}(8+t)^3\|w'(t)\|^2 \\
&\quad + (8+t)^2\Big[6\|A^2 v_{n-1,2}(t)\|^2 + 3\|A^{3/2}v_{n-1,2}(t)\|^2\Big] + \frac{1}{6}(8+t)^3\|A^2 v_{n-1,2}(t)\|^2.
\end{aligned}
$$

This shows that $(1+t)^3E(U_n';t)$ is bounded and $(1+t)^3\|U_n''(t)\|^2$ is integrable on $(0,\infty)$. Repeating this argument for the m-th derivative of U_n $(m=2,\dots,n-1)$, we can find that $(1+t)^{2n-1}E(\frac{d^{n-1}}{dt^{n-1}}U_n;t)$ is bounded. Therefore, we can conclude that

$$
\begin{aligned}
(1+t)^{2n-1}\|u(t)-V_n(t)\|^2 &= (1+t)^{2n-1}\left\|\tfrac{d^n}{dt^n}U_n(t)\right\|^2 \\
&\le (1+t)^{2n-1}E(\tfrac{d^{n-1}}{dt^{n-1}}U_n;t) \\
&\le C
\end{aligned}
$$

for some positive constant C. It should be pointed out that to complete all procedure stated above, we have to assume the regularity $(u_0,u_1)\in D(A^{n-\frac{1}{2}})\times D(A^{n-\frac{1}{2}})$. □

Remark 2.3. The above procedure also provides the following inequality

$$
(1+t)^{2n}\Big(\|u(t)-V_n(t)\|^2_{\mathcal{E}^\sharp} + (1+t)E(u-V_n;t)\Big) \le \widetilde{C}_{7,n}\big(\|u_0\|^2_{D(A^n)} + \|u_1\|^2_{D(A^n)}\big), \quad t\ge 0.
$$

2.2.2 Asymptotic expansion for general initial data

In contrast, Theorem 1.1 asserts the same consequence as that of Theorem 2.6 without any additional regularity for the initial data. To prove Theorem 1.1, the following decomposition of identity 1_H via the resolvent operator $J=(1_H+A)^{-1}$ is crucial.

Lemma 2.7. *For every $n\in\mathbb{N}$, there exists a pair of linear bounded operators (I_n,J_n) from H to $D(A^n)$ such that $1_H = I_n + A^nJ_n$.*

Proof. Observing that

$$
\begin{aligned}
1_H &= (1_H+A)^{2n-1}J^{2n-1} \\
&= \sum_{k=0}^{n-1}\binom{2n-1}{k}A^kJ^{2n-1} + \sum_{k=n}^{2n-1}\binom{2n-1}{k}A^kJ^{2n-1} \\
&= J^n\left(\sum_{k=0}^{n-1}\binom{2n-1}{k}A^kJ^{n-1}\right) + A^nJ^n\left(\sum_{k=0}^{n-1}\binom{2n-1}{n+k}A^kJ^{n-1}\right),
\end{aligned}
$$

we can choose I_n and J_n respectively as

$$I_n = J^n \left(\sum_{k=0}^{n-1} \binom{2n-1}{k} A^k J^{n-1} \right), \quad J_n = J^n \left(\sum_{k=0}^{n-1} \binom{2n-1}{n+k} A^k J^{n-1} \right).$$

□

By using I_n and J_n, we decompose the solution u of (2.1) as

$$u(t) = I_n u(t) + A^n J_n u(t). \tag{2.9}$$

Then we can see that the decomposition in [30] (explained in the beginning of Section 2.2.1) is applicable to the term $I_n u$.

Lemma 2.8. *Let u be the solution of (2.1) with $(u_0, u_1) \in \mathcal{H}$ and let $n \in \mathbb{N}$. Put U_* as the solution of the following problem:*

$$\begin{cases} U_*''(t) + AU_*(t) + U_*'(t) = -I_n v_{n-1,0}'(t) & \text{in } \mathbb{R}_+, \\ (U_*, U_*')(0) = (0, (-1)^n I_n u_1), \end{cases}$$

where $v_{n-1,0}$ is given in the beginning of Section 2.2.1. Then one has

$$I_n u(t) = \sum_{\ell=0}^{n-1} I_n v_\ell(t) + \frac{d^n}{dt^n} U_*(t) = I_n V_n(t) + \frac{d^n}{dt^n} U_*(t).$$

Proof. Since $u_* = I_n u$ satisfies the following problem (especially, the regularity of the initial data):

$$\begin{cases} u_*''(t) + Au_*(t) + u_*'(t) = 0 \quad \text{in } \mathbb{R}_+, \\ (u_*, u_*')(0) = (I_n u_0, I_n u_1) \in D(A^{n+1/2}) \times D(A^n), \end{cases}$$

the decomposition in [30] is applicable, which is the desired assertion. □

Proof of Theorem 1.1. We employ Lemmas 2.7 and 2.8 to the weak solution u of (2.1). Then

$$\begin{aligned} u(t) - V_n(t) &= I_n u(t) + A^n J_n u(t) - I_n V_n(t) - A^n J_n V_n(t) \\ &= \frac{d^n}{dt^n} U_*(t) + A^n J_n u(t) - A^n J_n V_n(t). \end{aligned}$$

Hereafter we always consider estimates for $t \geq 1$. Applying the procedure of Theorem 2.6 and also Remark 2.3, we can deduce

$$\left\| \frac{d^n}{dt^n} U_*(t) \right\|^2 + tE\left(\frac{d^n}{dt^n} U_*; t \right) \leq \widetilde{C}_{7,n} t^{-2n} \|(I_n u_0, I_n u_1)\|^2_{D(A^n) \times D(A^n)}.$$

Moreover, since $J_n u$ satisfies $(J_n u, J_n u')(0) = (J_n u_0, J_n u_1) \in D(\mathcal{L}^{2n})$, Lemma 2.5 gives for $t \geq 1$,

$$\|A^n J_n u(t)\|^2 + tE(A^n J_n u; t) \leq C_{6,2n} t^{-2n} \|(J_n u_0, J_n u_1)\|^2_{D(\mathcal{L}^{2n})}.$$

Furthermore, the boundedness of $(-tA)^k e^{-tA}$ in t (for all k) gives for $\ell = 0, \ldots, n-1$,

$$\|A^n J_n v_\ell(t)\|^2 + tE(A^n J_n v_\ell; t) \leq C t^{-2n} \|(u_0, u_1)\|^2_{H \times H}$$

for some positive constant C (depending only on n and ℓ). These yield a reasonable estimate for $A^n J_n V_n(t)$. Combining these inequalities, we arrive at the desired inequality. The proof is complete. □

3 Application to the damped wave equation

In this section, we consider the following initial-boundary value problem of the N-dimensional damped wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ (u, \partial_t u)(x,0) = (u_0(x), u_1(x)) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where Ω is either the whole space $\mathbb{R}^N$ or an exterior (connected) domain with a smooth boundary $\partial\Omega$. We shall discuss several properties of weak solutions to (3.1) as an application of the asymptotic expansion in Theorem 1.1 under the assumption

$$(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^q(\Omega) = (H_0^1(\Omega) \cap L^q(\Omega)) \times (L^2(\Omega) \cap L^q(\Omega)) \tag{3.2}$$

with $q \in [1, 2)$. Here we fix

$$v_{00}(x) = u_0(x) + u_1(x), \quad x \in \Omega.$$

For the Cauchy problem ($\Omega = \mathbb{R}^N$ and ignoring the boundary condition in (3.1)), we use $\Delta = \Delta_{\mathbb{R}^N}$ (endowed with domain $D(\Delta) = H^2(\mathbb{R}^N)$) and the usual heat semigroup

$$e^{t\Delta} f(x) = e^{t\Delta_{\mathbb{R}^N}} f(x) = (4\pi t)^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy, \quad x \in \mathbb{R}^N.$$

For the exterior problem, we use the Dirichlet heat semigroup $\{e^{t\Delta_\Omega}\}_{t \geq 0}$ which is generated by the Dirichlet Laplacian Δ_Ω endowed with domain $D(\Delta_\Omega) = H^2(\Omega) \cap H_0^1(\Omega)$, and we assume $0 \notin \overline{\Omega}$ and set positive constants r_Ω and R_Ω as $r_\Omega = \min_{\partial\Omega} |x|$ and $R_\Omega = \max_{\partial\Omega} |x|$, that is, $B(0, r_\Omega) \subset \Omega^c \subset B(0, R_\Omega)$ (for the convenience put $r_{\mathbb{R}^N} = 0$, $R_{\mathbb{R}^N} = \infty$). We discuss the large-time behavior of weak solution u and also the one for the energy functional

$$E_\Omega(u; t) = \int_\Omega \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \right) dx.$$

Here we again use the same notations as in Definition 1.2 with $A = -\Delta_\Omega$, that is,

$$v_0(t) = e^{t\Delta_\Omega} v_{00},$$
$$v_\ell(t) = (-\Delta_\Omega)^\ell \left(\sum_{j=0}^{\ell} \binom{2\ell}{\ell+j} \frac{(t\Delta_\Omega)^j}{j!} e^{t\Delta_\Omega} v_{00} - \sum_{k=0}^{\ell-1} \binom{2\ell-1}{\ell+k} \frac{(t\Delta_\Omega)^k}{k!} e^{t\Delta_\Omega} u_0 \right), \quad \ell \in \mathbb{N}$$

and also $V_0(t) = 0$ and $V_m(t) = \sum_{\ell=0}^{m-1} v_\ell(t)$ for $\ell \in \mathbb{N}$.

3.1 Decay estimates for L^2-norm and the energy functional

The comparison principle shows that $|e^{t\Delta_\Omega} f| \le e^{t\Delta_{\mathbb{R}^N}} |f_*|$ for any $f \in L^2(\Omega)$, where $f_* : \mathbb{R}^N \to \mathbb{R}$ is the zero-extension of f. This immediately gives the usual L^2-L^q estimate

$$\|e^{t\Delta} f\|_{L^2(\Omega)} \le C t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{2})} \|f\|_{L^q(\Omega)}, \quad t > 0$$

for $f \in L^q(\Omega)$ with $q \in [1, 2]$. Therefore under the condition (3.2), we additionally have the decay property

$$\|e^{t\Delta_\Omega} v_{00}\|_{L^2(\Omega)} + \|e^{t\Delta_\Omega} u_0\|_{L^2(\Omega)} \le C t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{2})} \|(u_0, u_1)\|_{\mathbf{L}^q(\Omega)}.$$

In this case, we can also find the respective decay properties for $u - V_m$ ($m \in \mathbb{N}$). These can be formulated as the following.

Proposition 3.1. *For every $q \in [1, 2)$ and $m \in \mathbb{Z}_{\ge 0}$, there exists a positive constant $C_{8,q,m}$ such that the following assertion holds: If $(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^q(\Omega)$, then the corresponding weak solution u of (3.1) satisfies*

$$\|u(t) - V_m(t)\|^2_{L^2(\Omega)} + t E_\Omega(u - V_m; t) \le C_{8,q,m} t^{-N(\frac{1}{q}-\frac{1}{2})-2m} \|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)\cap\mathbf{L}^q(\Omega)}, \quad t \ge 1, \tag{3.3}$$

where the family $\{V_m(\cdot)\}_{m\in\mathbb{Z}_{\ge 0}}$ is given in Definition 1.2 with $A = -\Delta_\Omega$.

Proof. Fix $m_0 \in \mathbb{N}$ satisfying

$$\frac{N}{2}\left(\frac{1}{q} - \frac{1}{2}\right) < m_0 \le \frac{N}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + 1.$$

Applying Theorem 1.1 with $n = m + m_0$, we have

$$\begin{aligned} \|u(t) - V_{m+m_0}(t)\|^2_{L^2(\Omega)} + t E_\Omega\left(u - V_{m+m_0}; t\right) &\le C_{1,m+m_0} t^{-2m_0-2m} \|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)} \\ &\le C_{1,m+m_0} t^{-N(\frac{1}{q}-\frac{1}{2})-2m} \|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)}. \end{aligned}$$

On the one hand, note that $V_{m+m_0} - V_m = \sum_{\ell=m}^{m+m_0-1} v_\ell$ and for any $\ell = m, m+1, \cdots, m+m_0-1$, we have

$$\begin{aligned} \|v_\ell(t)\|^2_{L^2(\Omega)} + t E_\Omega\left(v_\ell; t\right) &\le C t^{-2\ell} \left(\|e^{\frac{t}{2}\Delta_\Omega} v_{00}\|^2_{L^2(\Omega)} + \|e^{\frac{t}{2}\Delta_\Omega} u_0\|^2_{L^2(\Omega)} \right) \\ &\le C t^{-N(\frac{1}{q}-\frac{1}{2})-2\ell} \left(\|(u_0, u_1)\|^2_{\mathbf{L}^q(\Omega)} \right). \end{aligned}$$

Combining these estimates, we can reach the desired inequality. □

3.1.1 Optimality of decay rates for $\Omega = \mathbb{R}^N$ $(N \in \mathbb{N})$

Proposition 3.1 with $m = 0$ actually provides the optimal decay rate for the L^2-norm and the energy of weak solutions with general initial data except the case of two-dimensional exterior domain, which will be discussed later.

The following proposition gives the optimality of the decay rate for the case $\Omega = \mathbb{R}^N$ $(N \in \mathbb{N})$.

Proposition 3.2. *Let $\Omega = \mathbb{R}^N (N \in \mathbb{N})$ and let u be the weak solution of (3.1) with $(u_0, u_1) \in \mathbf{E}(\mathbb{R}^N) \cap \mathbf{L}^1(\mathbb{R}^N)$. Then the following assertions hold:*

(i) *If $\int_{\mathbb{R}^N} (u_0 + u_1)\, dx \neq 0$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty} \left(t^{\frac{N}{2}} \|u(t)\|^2_{L^2(\mathbb{R}^N)} \right), \quad \limsup_{t\to\infty} \left(t^{\frac{N}{2}+1} E_{\mathbb{R}^N}(u;t) \right).$$

(ii) *If $u_1 = -u_0$ and $\int_{\mathbb{R}^N} u_0\, dx \neq 0$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty} \left(t^{\frac{N}{2}+2} \|u(t)\|^2_{L^2(\mathbb{R}^N)} \right), \quad \limsup_{t\to\infty} \left(t^{\frac{N}{2}+3} E_{\mathbb{R}^N}(u;t) \right).$$

To prove this, we use the following Sobolev-type inequality.

Lemma 3.3. *There exists a positive constant C_9 such that for every $f \in H^1_0(\Omega) \cap L^1(\Omega)$,*

$$\|f\|^{1+\frac{2}{N}}_{L^{1+\frac{2}{N}}(\Omega)} \leq C_9 \|f\|^{\frac{N^2+4}{N(N+2)}}_{L^1(\Omega)} \|\nabla f\|^{\frac{4}{N+2}}_{L^2(\Omega)}.$$

Proof. If $N = 2$, then this is nothing but the Nash inequality

$$\|f\|^{2+\frac{4}{N}}_{L^2(\Omega)} \leq C_{\mathrm{Nash}} \|f\|^{\frac{4}{N}}_{L^1(\Omega)} \|\nabla f\|^2_{L^2(\Omega)}. \tag{3.4}$$

If $N \geq 3$, then we just adapt (3.4) with the Hölder inequality $\|f\|_{L^{1+\frac{2}{N}}(\Omega)} \leq \|f\|^{\frac{N-2}{N+2}}_{L^1(\Omega)} \|f\|^{\frac{4}{N+2}}_{L^2(\Omega)}$. If $N = 1$, then we combine (3.4) with the Gagliardo–Nirenberg inequality of the form $\|f\|_{L^3(\mathbb{R})} \leq C_{\mathrm{GN}} \|f\|^{\frac{5}{6}}_{L^2(\mathbb{R})} \|f'\|^{\frac{1}{6}}_{L^2(\mathbb{R})}$. The proof is complete. □

Proof of Proposition 3.2. **(i)** First note that Proposition 3.1 with $(q, m) = (1, 0), (1, 1)$ gives that

$$t^{\frac{N}{2}} \|u(t)\|^2_{L^2(\mathbb{R}^N)} + t^{\frac{N}{2}+1} E_{\mathbb{R}^N}(u;t) \leq C_{8,1,0} \|(u_0, u_1)\|^2_{\mathbf{E}(\mathbb{R}^N)\cap\mathbf{L}^1(\mathbb{R}^N)}, \quad t \geq 1$$

and

$$t^{\frac{N}{2}} \|u(t) - v_0(t)\|^2_{L^2(\Omega)} + t^{\frac{N}{2}+1} E_{\mathbb{R}^N}(u - v_0;t) \leq C_{8,1,1} t^{-2} \|(u_0, u_1)\|^2_{\mathbf{E}(\mathbb{R}^N)\cap\mathbf{L}^1(\mathbb{R}^N)}, \quad t \geq 1,$$

where $v_0(t) = e^{t\Delta_{\mathbb{R}^N}}(u_0 + u_1)$. The first inequality assures that both quantities in the assertion are finite. Moreover, we see that

$$t^{\frac{N}{2}+1}\|\nabla v_0(t)\|^2_{L^2(\mathbb{R}^N)} \le 2t^{\frac{N}{2}}\|v_0(t/2)\|^2_{L^2(\mathbb{R}^N)}.$$

Therefore it suffices to show only the lower bound for $\|\nabla v_0(t)\|_{L^2(\mathbb{R}^N)}$. Here we employ the strategy of the test function method in Ikeda–Sobajima [10]. Fix a non-increasing function $\eta \in C^\infty(\mathbb{R})$ satisfying $\mathbb{1}_{(-\infty,\frac{1}{2}]} \le \eta \le \mathbb{1}_{(-\infty,1]}$ and set for the parameter $\rho > 0$,

$$\zeta_\rho(x,t) = \eta\big(\xi_\rho(x,t)\big), \quad \xi_\rho(x,t) = \frac{|x|^2 + t}{\rho},$$

where $\mathbb{1}_D$ denotes the indicator function on D. Since the dominated convergence theorem gives

$$\lim_{\rho\to\infty}\int_{\mathbb{R}^N}(u_0+u_1)\zeta_\rho\,dx = \int_{\mathbb{R}^N}(u_0+u_1)\,dx \neq 0,$$

we can choose a positive constant ρ_* such that for every $\rho \ge \rho_*$,

$$\left|\int_{\mathbb{R}^N}(u_0+u_1)\zeta_\rho\,dx\right| \ge c_0 = \frac{1}{2}\left|\int_{\mathbb{R}^N}(u_0+u_1)\,dx\right| > 0.$$

Multiplying the equation $\partial_t v_0 - \Delta v_0 = 0$ to ζ_ρ and integrating it over $\mathbb{R}^N$, we see from integration by parts that

$$0 = \int_{\mathbb{R}^N}(\partial_t v_0 - \Delta v_0)\zeta_\rho\,dx = \frac{d}{dt}\int_{\mathbb{R}^N} v_0\zeta_\rho\,dx - \int_{\mathbb{R}^N} v_0(\partial_t\zeta_\rho + \Delta\zeta_\rho)\,dx.$$

Integrating it again over $\mathbb{R}_+$ and using the Hölder inequality (and putting $\mathcal{Q} = \mathbb{R}^N \times \mathbb{R}_+$), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(u_0+u_1)\zeta_\rho\,dx\right|^{1+\frac{2}{N}} &= \left|\iint_{\mathcal{Q}} v_0(\partial_t\zeta_\rho+\Delta\zeta_\rho)\,dxdt\right|^{1+\frac{2}{N}} \\ &\le \left(\iint_{\mathcal{Q}}|\partial_t\zeta_\rho+\Delta\zeta_\rho|^{\frac{N}{2}+1}\,dxdt\right)^{\frac{2}{N}}\rho Y'(\rho),\end{aligned}$$

where we have set

$$Y(T) = \int_0^T\left(\iint_{\mathcal{Q}}|v_0|^{1+\frac{2}{N}}\mathbb{1}_{[\frac{1}{2},1]}(\xi_\rho)\,dxdt\right)\frac{d\rho}{\rho} \quad \left(\le \log 2\int_0^T \|v_0(t)\|^{1+\frac{2}{N}}_{L^{1+\frac{2}{N}}(\mathbb{R}^N)}\,dt\right).$$

Noting that

$$\iint_{\mathcal{Q}}|\partial_t\zeta_\rho+\Delta\zeta_\rho|^{\frac{N}{2}+1}\,dxdt \le C$$

for some positive constant C independent of ρ, we arrive at the following lower bound for Y with the logarithmic function:

$$Y(\rho_*) + \frac{c_0^{1+\frac{2}{N}}}{C^{\frac{2}{N}}}\log\frac{T}{\rho_*} \le Y(T) \le \log 2\int_0^T \|v_0(t)\|^{1+\frac{2}{N}}_{L^{1+\frac{2}{N}}(\mathbb{R}^N)}\,dt, \quad T > \rho_*.$$

Applying Lemma 3.3, we deduce

$$\int_0^T \|\nabla v_0(t)\|_{L^2(\mathbb{R}^N)}^{\frac{4}{N+2}}\,dt \geq \delta \log T, \quad T > \rho_*$$

for some positive constant $\delta > 0$. This gives $\limsup_{t\to\infty} \left(t^{\frac{N}{2}+1}\|\nabla v_0(t)\|_{L^2(\mathbb{R}^N)}^2\right) > 0$.

(ii) In this case, the first asymptotic profile vanishes, that is, $v_0(t) = e^{t\Delta}(u_0+u_1) = 0$. Proposition 3.1 with $(q,m) = (1,1), (1,2)$ gives that

$$t^{\frac{N}{2}+2}\|u(t)\|_{L^2(\mathbb{R}^N)}^2 + t^{\frac{N}{2}+3}E_{\mathbb{R}^N}(u;t) \leq C_{8,1,1}\|u_0\|_{H_0^1(\Omega)\cap L^1(\Omega)}^2, \quad t \geq 1$$

and

$$t^{\frac{N}{2}+2}\|u(t)-v_1(t)\|_{L^2(\mathbb{R}^N)}^2 + t^{\frac{N}{2}+3}E_{\mathbb{R}^N}(u-v_1;t) \leq C_{8,1,2}t^{-2}\|u_0\|_{H_0^1(\Omega)\cap L^1(\Omega)}^2, \quad t \geq 1$$

and therefore the quantities in the assertion are both finite. Observe that the second asymptotic profile of u is given by $v_1(t) = \Delta e^{t\Delta}u_0$. As in the proof of **(i)**, we can find

$$\limsup_{t\to\infty}\left(t^{\frac{N}{2}+1}\|\nabla e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}^2\right) > 0.$$

Using the interpolation estimate $\|\nabla f\|_{L^2(\mathbb{R}^N)}^3 \leq \|f\|_{L^2(\mathbb{R}^N)}^2\|\nabla\Delta f\|_{L^2(\mathbb{R}^N)}$ and the L^2-L^1 estimate, we also have

$$\begin{aligned}\left(t^{\frac{N}{2}+1}\|\nabla e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}^2\right)^3 &\leq \left(t^{\frac{N}{4}}\|e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}\right)^4\left(t^{\frac{N}{2}+3}\|\nabla\Delta e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}^2\right)\\ &\leq C\|u_0\|_{L^1(\mathbb{R}^N)}^4\left(t^{\frac{N}{2}+3}\|\nabla v_1(t)\|_{L^2(\mathbb{R}^N)}^2\right).\end{aligned}$$

Therefore we find that $t^{\frac{N}{2}+3}\|\nabla v_1(t)\|_{L^2(\mathbb{R}^N)}^2$ and also $t^{\frac{N}{2}+2}\|v_1(t)\|_{L^2(\mathbb{R}^N)}^2$ are bounded below by positive constants. The proof is complete. □

Remark 3.1. The assumption $u_1 = -u_0$ in Proposition 3.2 **(ii)** can be weakened in the following sense. Roughly speaking, a similar phenomenon occurs when $e^{t\Delta}(u_0+u_1)$ decays faster than $\Delta e^{t\Delta}u_0$. For example, if $u_0 \in H^1(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} u_0\,dx \neq 0$ and $u_1 = -u_0 + \Delta^2 w_0$ for some $w_0 \in H^4(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, then the profiles $v_0(t) = e^{t\Delta}(u_0+u_1)$ and $v_1(t) = \Delta e^{t\Delta}u_0 - (2+t\Delta)\Delta e^{t\Delta}(u_0+u_1)$ satisfy

$$\begin{aligned}\|v_0(t)\|_{L^2(\mathbb{R}^N)}^2 = O(t^{-\frac{N}{2}-4}), \quad E_{\mathbb{R}^N}(v_0;t) = O(t^{-\frac{N}{2}-5}),\\ E_{\mathbb{R}^N}(v_1;t) = \|\nabla\Delta e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}^2 + O(t^{-\frac{N}{2}-5}).\end{aligned}$$

However, the same argument as in the proof of Proposition 3.2 shows that the following quantities for V_1 are both finite and strictly positive:

$$\limsup_{t\to\infty}\left(t^{\frac{N}{2}+2}\|\Delta e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}^2\right), \quad \limsup_{t\to\infty}\left(t^{\frac{N}{2}+3}\|\nabla\Delta e^{t\Delta}u_0\|_{L^2(\mathbb{R}^N)}^2\right).$$

In this case, actually we could check that the asymptotic behavior of u is clarified as $\Delta e^{t\Delta}u_0$ which differs from the usual one $v_0(t) = e^{t\Delta}(u_0+u_1)$. This phenomenon occurs also for the exterior problems.

3.1.2 Optimality of decay rates for the exterior problem ($N \geq 3$)

In this case we employ the test function method with positive harmonic functions satisfying the Dirichlet boundary condition. The following lemma describes the asymptotic behavior of the positive harmonic function at spatial infinity similar to $1 - |x|^{2-N}$. The construction is well-known (see e.g., Jost [17, Section 22]), but for the reader's convenience, we give a sketch of the proof.

Lemma 3.4. *Let Ω be an exterior domain in $\mathbb{R}^N$ ($N \geq 3$). Then there exists a unique solution $h_\Omega \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ of the problem*

$$\begin{cases} \Delta h_\Omega(x) = 0, & x \in \Omega, \\ h_\Omega(x) = 0, & x \in \partial\Omega, \\ h_\Omega(x) > 0, & x \in \Omega \end{cases} \tag{3.5}$$

satisfying $\lim_{|x|\to\infty} h_\Omega(x) = 1$. Moreover, h_Ω satisfies

$$|\nabla h_\Omega(x)| \leq \frac{2^{N-1} N R_\Omega^{N-2}}{|x|^{N-1}}, \quad x \in B(0, 2R_\Omega)^c.$$

Proof. For $n \in \mathbb{N}$ satisfying $n > R_\Omega$, take the unique minimizer $\widetilde{h}_n$ for the functional

$$\int_{B(0,n)} |\nabla h|^2 \, dx$$

restricted to the closed convex set $\{h \in H_0^1(B(0,n)) \; ; \; h \geq \mathbb{1}_{\Omega^c}\}$. Letting $n \to \infty$, we have the increasing limit $\widetilde{h}(x) = \lim_{n\to\infty} h_n(x)$ which satisfies

$$\frac{r_\Omega^{N-2}}{|x|^{N-2}} \leq \widetilde{h}(x) \leq \frac{R_\Omega^{N-2}}{|x|^{N-2}}, \quad x \in \Omega.$$

This is verified via the Stampacchia's theorem and the maximum principle. Then we can see that the function $1 - \widetilde{h}(x)$ satisfies (3.5). If $|x_0| \geq 2R_\Omega$, then the mean-value theorem for harmonic functions (see e.g., Evans [4, Section 2.2]) gives

$$|\nabla h_\Omega(x_0)| \leq \frac{2N}{|x_0|} \|h_\Omega\|_{L^\infty(\partial B(x_0, \frac{|x_0|}{2}))} \leq \frac{2^{N-1} N R_\Omega^{N-2}}{|x_0|^{N-1}}.$$

The proof is complete. □

Remark 3.2. If $N = 2$, then the above procedure gives $\widetilde{h} \equiv 1$ ($h_\Omega \equiv 0$) which is not meaningful. We will discuss the case of two-dimensional exterior domain later.

The optimality of decay estimates in Proposition 3.1 with $m = 0, 1$ can be found via the use of h_Ω.

Proposition 3.5. *Let Ω be an exterior domain in $\mathbb{R}^N$ $(N \geq 3)$ and let u be the weak solution of (3.1) with $(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^1(\Omega)$. Let h_Ω be given in Lemma 3.4. Then the following assertions hold:*

(i) *If $\int_\Omega (u_0 + u_1) h_\Omega \, dx \neq 0$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty} \Big(t^{\frac{N}{2}} \|u(t)\|^2_{L^2(\Omega)} \Big), \quad \limsup_{t\to\infty} \Big(t^{\frac{N}{2}+1} E_\Omega(u;t) \Big).$$

(ii) *If $u_1 = -u_0$ and $\int_\Omega u_0 h_\Omega \, dx \neq 0$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty} \Big(t^{\frac{N}{2}+2} \|u(t)\|^2_{L^2(\Omega)} \Big), \quad \limsup_{t\to\infty} \Big(t^{\frac{N}{2}+3} E_\Omega(u;t) \Big).$$

Proof. We first note that by Proposition 3.1, all quantities are verified to be finite. We would only explain the difference compared to the proof of Proposition 3.2. For **(i)**, fix η as in the proof of Proposition 3.2 and

$$\widetilde{\zeta}_\rho(x,t) = \eta\big(\widetilde{\xi}_\rho(x,t)\big), \quad \widetilde{\xi}_\rho(x,t) = \frac{[(|x| - 2R_\Omega)_+]^2 + t}{\rho}.$$

Then we can check that

$$|\partial_t(\widetilde{\zeta}_\rho h_\Omega) + \Delta(\widetilde{\zeta}_\rho h_\Omega)| \leq \big(|\partial_t \widetilde{\zeta}_\rho| + |\Delta \widetilde{\zeta}_\rho|\big) h_\Omega + 2|\nabla \widetilde{\zeta}_\rho \cdot \nabla h_\Omega| \leq \frac{K}{\rho} \mathbb{1}_{[\frac{1}{2},1]}(\widetilde{\xi}_\rho) h_\Omega$$

for some positive constant K. Using the above estimate, we can proceed the test function method with the functional

$$Y_\Omega(T) = \int_0^T \left(\iint_{\Omega\times(0,\infty)} |v_0|^{1+\frac{2}{N}} \mathbb{1}_{[\frac{1}{2},1]}(\widetilde{\xi}_\rho) h_\Omega \, dxdt \right) \frac{d\rho}{\rho}.$$

Using Lemma 3.3 with $h_\Omega \leq 1$, we could find the lower bound of $t^{\frac{N}{2}+1} \|\nabla v_0(t)\|^2_{L^2(\Omega)}$ and also $t^{\frac{N}{2}} \|v_0(t)\|^2_{L^2(\Omega)}$. The proof of **(ii)** is completely the same as the one of Proposition 3.2. □

3.1.3 Decay estimates for the two-dimensional exterior problem

In the two-dimensional case, the series of estimates in Proposition 3.1 can be improved by restricting the choice of initial data. To describe this, we use *unbounded* positive harmonic functions satisfying the Dirichlet boundary condition, which is the reason why the two-dimensional case is exceptional.

Lemma 3.6. *If Ω is a two-dimensional exterior domain, then there exists a unique solution $h_\Omega \in C^\infty(\Omega)$ of the problem*

$$\begin{cases} \Delta h_\Omega(x) = 0, & x \in \Omega, \\ h_\Omega(x) = 0, & x \in \partial\Omega, \\ h_\Omega(x) > 0, & x \in \Omega \end{cases} \tag{3.6}$$

satisfying $\log\left(\frac{|x|}{R_\Omega}\right) \le h_\Omega(x) \le \log\left(\frac{|x|}{r_\Omega}\right)$ *and* $|x|^2\nabla h_\Omega(x) - x \in L^\infty(\Omega;\mathbb{R}^2)$. *In particular, there exists a positive constant* $R_\Omega^\sharp (> R_\Omega)$ *such that for every* $x \in B(0,R_\Omega^\sharp)^c$,

$$\left|\frac{h_\Omega(x)}{\log|x|} - 1\right| + |x\cdot\nabla h_\Omega(x) - 1| \le \frac{1}{2}.$$

Proof. The function h_Ω can be constructed via the Kelvin transform $K : y \mapsto \frac{y}{|y|^2}$ and the Green function on the set $\{K(x)\ ;\ x \in \Omega\} \cup \{0\}$. □

Hereafter we shall use the non-degenerate positive harmonic function

$$H_\Omega(x) = 1 + h_\Omega(x), \quad x \in \Omega,$$

where h_Ω is given in Lemma 3.6. Then we introduce the weighted L^1-space

$$L^1_{d\mu} = \{f \in L^1(\Omega)\ ;\ H_\Omega f \in L^1(\Omega)\}, \quad \|f\|_{L^1_{d\mu}} = \|H_\Omega f\|_{L^1(\Omega)};$$

note that $f \in L^1_{d\mu}$ is equivalent to the condition $(1+\log\frac{|x|}{r_\Omega})f \in L^1(\Omega)$, which appears in Introduction. We list several properties of the Dirichlet heat semigroup $\{e^{t\Delta_\Omega}\}_{t\ge0}$ on $L^2(\Omega)$ (see e.g., [12, Appendix]).

Lemma 3.7. *The following inequalities hold:*

(i) *For every $f \in L^2(\Omega)\cap L^1_{d\mu}$,*

$$\|e^{t\Delta_\Omega}f\|_{L^1_{d\mu}} \le \|f\|_{L^1_{d\mu}}, \quad t \ge 0.$$

(ii) *There exists a positive constant C_{10} such that for every $f \in L^2(\Omega)$,*

$$\left\|H_\Omega^{-1}e^{t\Delta_\Omega}f\right\|_{L^\infty(\Omega)} \le C_{10}t^{-\frac{1}{2}}\big(1+\log(1+t)\big)^{-1}\|f\|_{L^2(\Omega)}, \quad t > 0.$$

(iii) *For every $f \in L^2(\Omega)\cap L^1_{d\mu}$,*

$$\|e^{t\Delta_\Omega}f\|_{L^2(\Omega)} \le C'_{10}t^{-\frac{1}{2}}\big(1+\log(1+t)\big)^{-1}\|f\|_{L^1_{d\mu}}, \quad t > 0,$$

where the constant C'_{10} can be chosen as the same constant C_{10} in **(ii)**.

Remark 3.3. The estimates **(ii)** and **(iii)** are due to Grigor'yan and Saloff-Coste [6].

We also need a modification of the Nash inequality with the measure $d\mu = H_\Omega\,dx$.

Lemma 3.8 ([12]). *There exists a positive constant C_{11} such that for every $f \in H^1_0(\Omega) \cap L^1_{d\mu}$,*

$$\|H_\Omega^{\frac{1}{2}} f\|^2_{L^2(\Omega)} \le C_{11}\|f\|_{L^1_{d\mu}}\|\nabla f\|_{L^2(\Omega)}.$$

Decay estimates of the L^2-norm and the energy for this case and their optimality are the following. Here we use the notation

$$\mathbf{L}^1_{d\mu} = L^1_{d\mu} \times L^1_{d\mu}.$$

Proposition 3.9. *Let Ω be a two-dimensional exterior domain and let u be the weak solution of (3.1) with $(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^1_{d\mu}$. Then the following assertions hold:*

(i) *For every $m \in \mathbb{Z}_{\ge 0}$, there exists a positive constant $C_{12,m}$ depending only on Ω and m such that*

$$\|u(t) - V_m(t)\|^2_{L^2(\Omega)} + tE_\Omega(u - V_m; t) \le C_{12,m}t^{-1-2m}(\log t)^{-2}\|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)\cap\mathbf{L}^1_{d\mu}}, \quad t \ge 2.$$

where the family $\{V_m(\cdot)\}_{m\in\mathbb{Z}_{\ge 0}}$ is given in Definition 1.2 with $A = -\Delta_\Omega$.

(ii) *If $\int_\Omega (u_0 + u_1)h_\Omega\,dx \ne 0$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty}\Big(t(\log t)^2\|u(t)\|^2_{L^2(\Omega)}\Big), \quad \limsup_{t\to\infty}\Big(t^2(\log t)^2 E_\Omega(u; t)\Big).$$

(iii) *If $u_1 = -u_0$ and $\int_\Omega u_0 h_\Omega\,dx \ne 0$, then the following quantities are both finite and strictly positive:*

$$\limsup_{t\to\infty}\Big(t^3(\log t)^2\|u(t)\|^2_{L^2(\Omega)}\Big), \quad \limsup_{t\to\infty}\Big(t^4(\log t)^2 E_\Omega(u; t)\Big).$$

Proof. **(i)** Applying Theorem 1.1 with $n = m + 1$, we find the following inequality:

$$\|u(t) - V_{m+1}(t)\|^2_{L^2(\Omega)} + tE_\Omega(u - V_{m+1}; t) \le C_{1,m+1}t^{-2m-2}\|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)}, \quad t \ge 1.$$

Noting that $u(t) - V_{m+1}(t) = u(t) - V_m(t) - v_m(t)$, we check the estimate for $v_m(t)$. By Lemma 3.7, we have for $t \ge 2$,

$$\begin{aligned}\|v_m(t)\|^2_{L^2(\Omega)} + tE_\Omega(v_m; t) &\le Ct^{-2m}\Big(\|e^{\frac{t}{2}\Delta_\Omega}v_{00}\|^2_{L^2(\Omega)} + \|e^{\frac{t}{2}\Delta_\Omega}u_0\|^2_{L^2(\Omega)}\Big)\\ &\le C_m t^{-1-2m}(\log t)^{-2}\|(u_0, u_1)\|^2_{\mathbf{L}^1_{d\mu}}\end{aligned}$$

for some constant C_m.

(ii) In this case, we adopt the strategy of the test function method in Ikeda–Sobajima [11] to find a lower bound for $v_0(t)$. Let η, $\widetilde{\zeta}_\rho$ and $\widetilde{\xi}_\rho$ be as in the proof of Proposition 3.5 and let h_Ω be given in Lemma 3.6. Then we see that

$$|\partial_t(\widetilde{\zeta}_\rho h_\Omega) + \Delta(\widetilde{\zeta}_\rho h_\Omega)| \le \big(|\partial_t\widetilde{\zeta}_\rho| + |\Delta\widetilde{\zeta}_\rho|\big)h_\Omega + 2|\nabla\widetilde{\zeta}_\rho \cdot \nabla h_\Omega| \le \frac{K}{\rho}\mathbb{1}_{[\frac{1}{2},1]}(\widetilde{\xi}_\rho)(1+h_\Omega)$$

for some positive constant K; note that the difference compared to the case of Proposition 3.2 ($N \ge 3$) is just $h_\Omega \notin L^\infty(\Omega)$. Choose $\rho_* \ge 2$ such that for every $\rho > \rho_*$,

$$\left|\int_\Omega v_0\widetilde{\zeta}_\rho h_\Omega\,dx\right| \ge c_0 = \frac{1}{2}\left|\int_\Omega v_0 h_\Omega\,dx\right|.$$

Then a computation similar to the proof of Proposition 3.2 gives for every $\rho > \rho_*$

$$c_0^2 \le \left(\iint_{\Omega\times\mathbb{R}_+} |\partial_t\widetilde{\zeta}_\rho + \Delta\widetilde{\zeta}_\rho|^2(1+h_\Omega)\,dxdt\right)\rho\widetilde{Y}'(\rho) \le K\rho\log\rho\widetilde{Y}'(\rho),$$

where we have put

$$\widetilde{Y}(T) = \int_0^T \left(\iint_{\Omega\times\mathbb{R}_+} |v_0|^2\mathbb{1}_{[1/2,1]}(\widetilde{\xi}_\rho)(1+h_\Omega)\,dxdt\right)\frac{d\rho}{\rho}.$$

This implies that

$$\widetilde{Y}(\rho_*) + \frac{c_0^2}{K}\log\left(\frac{\log T}{\log\rho_*}\right) \le \widetilde{Y}(T) \le \log 2\int_0^T \|H_\Omega^{\frac{1}{2}}v_0(t)\|_{L^2(\Omega)}^2\,dt.$$

Therefore Lemmas 3.8 and 3.7 give

$$\frac{c_0^2}{KC_{11}\log 2\|v_0\|_{L^1_{d\mu}}}\log\left(\frac{\log T}{\log\rho_*}\right) \le \int_0^T \|\nabla v_0(t)\|_{L^2(\Omega)}\,dt.$$

This implies $\limsup_{t\to\infty}\big(t^2(\log t)^2E_\Omega(v;t)\big) > 0$. The estimate for L^2-norm can be found via $\|\nabla v_0(t)\|_{L^2(\Omega)}^2 \le 2t^{-1}\|v_0(t/2)\|_{L^2(\Omega)}^2$.

(iii) In this case we note again that $v_0(t) \equiv 0$ and therefore $V_1(t) = v_1(t) = \Delta e^{t\Delta_\Omega}u_0$ actually gives the asymptotic profile of u. As in **(ii)**, in this case we have

$$\limsup_{t\to\infty}\left(t^2(\log t)^2\|\nabla e^{t\Delta_\Omega}u_0\|_{L^2(\Omega)}^2\right) > 0.$$

Then it is enough to observe that

$$\begin{aligned}t^2(\log t)^2\|\nabla e^{t\Delta_\Omega}u_0\|_{L^2(\Omega)}^2 &\le \big(t(\log t)^2\|e^{t\Delta_\Omega}u_0\|_{L^2(\Omega)}^2\big)^{\frac{1}{2}}\big(t^3(\log t)^2\|\Delta e^{t\Delta_\Omega}u_0\|_{L^2(\Omega)}^2\big)^{\frac{1}{2}}\\ &\le C_7\|u_0\|_{L^1_{d\mu}}\big(t^3(\log t)^2\|v_1(t)\|_{L^2(\Omega)}^2\big)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}t^3(\log t)^2\|v_1(t)\|_{L^2(\Omega)}^2 &\le \big(t^{\frac{1}{2}}(\log t)\|\nabla e^{t\Delta_\Omega}u_0\|_{L^2(\Omega)}\big)^{\frac{2}{3}}\big(t^4(\log t)^2\|\nabla\Delta e^{t\Delta_\Omega}u_0\|_{L^2(\Omega)}^2\big)^{\frac{2}{3}}\\ &\le (C_7\|u_0\|_{L^1_{d\mu}})^{\frac{2}{3}}\big(t^4(\log t)^2E_\Omega(v_1;t)\big)^{\frac{2}{3}}.\end{aligned}$$

The proof is complete. □

3.2 Estimates for the local energy

For $R > R_\Omega$, we consider the large-time behavior of a localized version of the energy functional given by

$$E_{\Omega,R}(u;t) = \int_{\Omega\cap B(0,R)} \Big(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \Big)\, dx$$

for $R > R_\Omega$ as an application of asymptotic expansion (Theorem 1.1). Here we again use the space $\mathbf{L}^q(\Omega) = L^q(\Omega) \times L^q(\Omega)$ and do not restrict ourselves to the case of compactly supported functions. The first assertion is for the large-time behavior of the local energy in $\Omega = \mathbb{R}^N$.

Proposition 3.10. *Let $\Omega = \mathbb{R}^N$ $(N \in \mathbb{N})$. For every $q \in [1,2)$ and $m \in \mathbb{Z}_{\geq 0}$, there exists a positive constant $C_{10,q,m}$ such that the following assertion holds: If $(u_0,u_1) \in \mathbf{E}(\mathbb{R}^N) \cap \mathbf{L}^q(\mathbb{R}^N)$, then the corresponding weak solution of (3.1) satisfies for every $R > 0$,*

$$E_{\mathbb{R}^N,R}(u - V_m;t) \leq C_{10,q,m} R^N t^{-\frac{N}{q}-2m-1} \|(u_0,u_1)\|^2_{\mathbf{E}(\mathbb{R}^N)\cap\mathbf{L}^q(\mathbb{R}^N)}, \quad t \geq 1,$$

where the family $\{V_m(\cdot)\}_{m\in\mathbb{Z}_{\geq 0}}$ in Definition 1.2 with $A = -\Delta = -\Delta_{\mathbb{R}^N}$.

Proof. Fix $R > 0$ arbitrary. We apply Theorem 1.1 with $n = m + N$, that is, $u - V_{m+N}$ satisfies

$$E_{\mathbb{R}^N,R}(u - V_{m+N};t) \leq E_{\mathbb{R}^N}(u - V_{m+N};t) \leq C_{4,m+N} t^{-2N-2m-1} \|(u_0,u_1)\|^2_{\mathbf{E}(\mathbb{R}^N)}, \quad t \geq 1.$$

Therefore it suffices to consider the estimates for v_ℓ $(\ell = m, \dots, m+N-1)$. By the basic property of heat semigroup, we see for $t \geq 1$,

$$\begin{aligned} E_{\mathbb{R}^N,R}(v_\ell;t) &\leq \Big(\|\partial_t v_\ell(t)\|^2_{L^\infty(\mathbb{R}^N)} + \|\nabla v_\ell(t)\|^2_{L^\infty(\mathbb{R}^N)} \Big) \|\mathbb{1}_{B(0,R)}\|_{L^1(\mathbb{R}^N)} \\ &\leq C R^N t^{-\frac{N}{q}-2\ell-1} \|(u_0,u_1)\|^2_{\mathbf{L}^q(\mathbb{R}^N)}. \end{aligned}$$

Consequently, the estimate for $E_{\mathbb{R}^N,R}(u - V_m;t)$ only reflects the worst one $E_{\mathbb{R}^N,R}(v_m;t)$. The proof is complete. □

For the exterior problem, the gradient of the heat kernel does not have a good decay property in general (see e.g., Jimbo–Sakaguchi [16] and also Georgiev–Taniguchi [5]). The following assertion can be regarded as a modification of the result in [29, 3]. A precise dependence of the parameter R for the constant is explicitly given.

Proposition 3.11. *Let Ω be an exterior domain $(N \geq 2)$. For every $q \in [1,2)$ and $m \in \mathbb{Z}_{\geq 0}$, there exists a positive constant $C_{13,q,m}$ such that the following assertion holds: If $(u_0,u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^q(\Omega)$, then the corresponding weak solution u of (3.1) satisfies for every $R > R_\Omega$,*

$$E_{\Omega,R}(u - V_m;t) \leq C_{13,q,m} R^{N-2} t^{-\frac{N}{q}-2m} \|(u_0,u_1)\|^2_{\mathbf{E}(\Omega)\cap\mathbf{L}^q(\Omega)}, \quad t \geq 1.$$

where the family $\{V_m(\cdot)\}_{m\in\mathbb{Z}_{\geq 0}}$ is given in Definition 1.2 with $A = -\Delta_\Omega$.

Proof. Fix $R > R_\Omega$ arbitrary. We apply Theorem 1.1 with $n = m + N - 2$, that is, $u - V_{m+N-2}$ satisfies

$$E_{\Omega,R}(u - V_{m+N-2}; t) \le E_\Omega(u - V_{m+N-2}; t) \le C_{1,m+N-2} t^{-2N-2m+3} \|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)}, \quad t \ge 1.$$

Therefore it again suffices to consider the estimates for v_ℓ ($\ell = m, \dots, m+N-3$). Let η be as in the proof of Proposition 3.2 and put

$$\zeta_R(x) = \eta\left(\frac{|x|^2}{2R^2}\right)$$

which satisfies $\mathbb{1}_{B(0,R)} \le \zeta_R \le \mathbb{1}_{B(0,2R)}$. Then by integration by parts, we can compute

$$\begin{aligned}
\int_{\Omega\cap B(0,R)} |\nabla v_\ell|^2\,dx &\le \int_\Omega |\nabla v_\ell|^2 \zeta_R\,dx \\
&= \frac{1}{2}\int_\Omega v_\ell^2 \Delta\zeta_R\,dx - \int_\Omega v_\ell \Delta v_\ell \zeta_R\,dx \\
&\le \frac{1}{2}\|v_\ell\|^2_{L^\infty(\Omega)}\|\Delta\zeta_R\|_{L^1(\Omega)} + \|v_\ell\|_{L^N(\Omega)}\|\Delta v_\ell\|_{L^N(\Omega)}\|\zeta_R\|_{L^{1^{**}}(\Omega)} \\
&\le CR^{N-2}t^{-\frac{N}{q}-2\ell}\|(u_0,u_1)\|^2_{\mathbf{L}^q(\Omega)}
\end{aligned}$$

where we have put $1^{**} = \frac{N}{N-2}$ ($N \ge 3$) and $1^{**} = \infty$ ($N = 2$). And also

$$\begin{aligned}
\int_{\Omega\cap B(0,R)} |\partial_t v_\ell|^2\,dx &\le \|\partial_t v_\ell\|^2_{L^N(\Omega)}\|\mathbb{1}_{B(0,R)}\|_{L^{1^{**}}(\Omega)} \\
&\le CR^{N-2}t^{-N(\frac{1}{q}-\frac{1}{N})-2\ell-2}\|(u_0,u_1)\|^2_{\mathbf{L}^q(\Omega)} \\
&= CR^{N-2}t^{-\frac{N}{q}-2\ell-1}\|(u_0,u_1)\|^2_{\mathbf{L}^q(\Omega)}.
\end{aligned}$$

As a consequence, it is clarified that the factor $E_{\Omega,R}(v_m; t)$ has the worst decay rate $t^{-\frac{N}{q}-2m}$. The proof is complete. □

To close the paper, we finally discuss some refinement of the decay rate of $E_{\Omega,R}(u,t)$ when Ω is a two-dimensional exterior domain. Thanks to the decay estimate for the Dirichlet heat semigroup involving the logarithmic function with initial values in $L^1_{d\mu}$ (Lemma 3.7), we also obtain a decay estimate involving the logarithmically decaying factor $(\log t)^{-4}$.

Proposition 3.12. *Let Ω be a two-dimensional exterior domain and let u be the solution of (3.1) with $(u_0, u_1) \in \mathbf{E}(\Omega) \cap \mathbf{L}^1_{d\mu}$. Then for every $m \in \mathbb{Z}_{\ge 0}$, there exists a positive constant $C_{14,m}$ (depending only on Ω and m) such that for every $R > R_\Omega$,*

$$E_{\Omega,R}(u - V_m; t) \le C_{14,m}\left(1 + \log\frac{R}{r_\Omega}\right)^2 t^{-2-2m}(\log t)^{-4}\|(u_0,u_1)\|^2_{\mathbf{E}(\Omega)\cap\mathbf{L}^1_{d\mu}}, \quad t \ge 2,$$

where the family $\{V_m\}_{m\in\mathbb{Z}_{\ge 0}}$ is given in Definition 1.2 with $A = -\Delta_\Omega$ and $r_\Omega = \inf_{x\in\partial\Omega}|x|$.

Proof. Fix $R > R_\Omega$ arbitrary. We apply Proposition 3.11 with m replaced with $m+1$. Then we already have the following estimate for $u - V_{m+1} = u - V_m - v_m$:

$$E_{\Omega,R}(u - V_{m+1}; t) \le C_{13,1,m+1} t^{-2m-4} \|(u_0, u_1)\|^2_{\mathbf{E}(\Omega)\cap \mathbf{L}^1(\Omega)}, \quad t \ge 1.$$

Then we consider the estimate for $E_{\Omega,R}(v_m; t)$ via Lemma 3.7. Using the same function ζ_R as in Proposition 3.11, we see that

$$\begin{aligned}
&\int_{\Omega\cap B(0,R)} |\nabla v_m(t)|^2 \, dx \\
&\le \frac{1}{2}\int_\Omega v_m(t)^2 \Delta\zeta_R \, dx - \int_\Omega v_m(t)\Delta v_m(t)\zeta_R \, dx \\
&\le \frac{1}{2}\|H_\Omega^{-1} v_m(t)\|^2_{L^\infty(\Omega)} \|H_\Omega^2 \Delta\zeta_R\|_{L^1(\Omega)} + \|H_\Omega^{-1} v_m(t)\|_{L^2(\Omega)} \|H_\Omega^{-1}\Delta v_m(t)\|_{L^2(\Omega)} \|H_\Omega^2 \zeta_R\|_{L^\infty(\Omega)} \\
&\le C\left(1 + \log\frac{R}{r_\Omega}\right)^2 \left(\|H_\Omega^{-1} v_m(t)\|^2_{L^\infty(\Omega)} + \|H_\Omega^{-1} v_m(t)\|_{L^2(\Omega)} \|H_\Omega^{-1}\Delta v_m(t)\|_{L^2(\Omega)}\right).
\end{aligned}$$

Here we put $v_{m*}(t)$ as

$$v_{m*}(t) = \sum_{j=0}^{m} \binom{2m}{m+j} \frac{(t\Delta)^j}{j!} e^{\frac{2t}{3}\Delta_\Omega}(u_0 + u_1) - \sum_{k=0}^{m-1} \binom{2m-1}{m+k} \frac{(t\Delta)^k}{k!} e^{\frac{2t}{3}\Delta_\Omega} u_0$$

which satisfies $v_m(t) = e^{\frac{t}{3}\Delta_\Omega}(-\Delta)^m v_{m*}(t)$. Putting $s = t/3$, we can control the first term on the right hand side of the above inequality as

$$\begin{aligned}
\|H_\Omega^{-1} v_m(t)\|^2_{L^\infty(\Omega)} &\le C_{10} s^{-1} (\log s)^{-2} \|(-\Delta)^m v_{m*}(t)\|^2_{L^2(\Omega)} \\
&\le C C_{10} s^{-1} (\log s)^{-2-2m} \left(\|e^{s\Delta_\Omega}(u_0 + u_1)\|^2_{L^2(\Omega)} + \|e^{s\Delta_\Omega} u_0\|^2_{L^2(\Omega)}\right) \\
&\le C' C_{10}^2 s^{-2-2m} (\log s)^{-4} \|(u_0, u_1)\|^2_{\mathbf{L}^1_{d\mu}}.
\end{aligned}$$

On the one hand, we see that for $t > 2R_\Omega^2$ and $f \in L^2(\Omega) \cap L^1_{d\mu}$,

$$\begin{aligned}
\|H_\Omega^{-1} e^{s\Delta_\Omega} f\|_{L^2(\Omega)} &\le \|H_\Omega^{-1} e^{s\Delta_\Omega} f\|_{L^2(\Omega\cap B(0,\sqrt{s}))} + \|H_\Omega^{-1} e^{s\Delta_\Omega} f\|_{L^2(B(0,\sqrt{s})^c)} \\
&\le \|H_\Omega^{-1} e^{s\Delta_\Omega} f\|_{L^\infty(\Omega)} \|\mathbb{1}\|_{L^2(\Omega\cap B(0,\sqrt{s}))} + \|H_\Omega^{-1}\|_{L^\infty(B(0,\sqrt{s})^c)} \|e^{s\Delta_\Omega} f\|_{L^2(\Omega)} \\
&\le C C_{10} (\log s)^{-1} \|f\|_{L^2(\Omega)}.
\end{aligned}$$

We also see that the remaining term can be estimated as follows: for $\tau = 0, 1$,

$$\begin{aligned}
\|H_\Omega^{-1}(-\Delta)^\tau v_m(t)\|_{L^2(\Omega)} &\le C C_{10} (\log s)^{-1} \|(-\Delta)^{m+\tau} v_{m*}(t)\|_{L^2(\Omega)} \\
&\le C' C_{10} s^{-m-\tau} (\log s)^{-1} \left(\|e^{\frac{t}{3}\Delta_\Omega}(u_0 + u_1)\|_{L^2(\Omega)} + \|e^{\frac{t}{3}\Delta_\Omega} u_0\|_{L^2(\Omega)}\right) \\
&\le C'' C_{10}^2 s^{-m-\tau-\frac{1}{2}} (\log s)^{-2} \|(u_0, u_1)\|_{\mathbf{L}^1_{d\mu}}.
\end{aligned}$$

The proof is complete. □

Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest The author declares that he has no conflict of interest.

References

[1] T. Cazenave, A. Haraux, "An introduction to semilinear evolution equations". Oxford Lecture Series in Mathematics and its Applications **13**. The Clarendon Press, Oxford University Press, New York, 1998. MR1691574

[2] R. Chill, A. Haraux, *An optimal estimate for the difference of solutions of two abstract evolution equations*, J. Differential Equations **193** (2003), 385–395. MR1998640

[3] W. Dan, Y. Shibata, *On a local energy decay of solutions of a dissipative wave equation*, Funkcial. Ekvac. **38** (1995), 545–568. MR1374437

[4] L.C. Evans, "Partial differential equations, Second edition," Grad. Stud. Math. **19**, American Mathematical Society, Providence, RI, 2010. MR2597943

[5] V. Georgiev, K. Taniguchi, *On fractional Leibniz rule for Dirichlet Laplacian in exterior domain*, Discrete Contin. Dyn. Syst. **39** (2019), 1101–1115. MR3918208

[6] A. Grigor'yan, L. Saloff-Coste, *Dirichlet heat kernel in the exterior of a compact set*, Comm. Pure Appl. Math. **55** (2002), 93–133. MR1857881

[7] T. Hosono, T. Ogawa, *Large time behavior and L^p-L^q estimate of solutions of 2-dimensional nonlinear damped wave equations*, J. Differential Equations **203** (2004), 82–118. MR2070387

[8] L. Hsiao, T.-P. Liu, *Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping*, Comm. Math. Phys. **43** (1992), 599–605. MR1145602

[9] M. Ikeda, T. Inui, M. Okamoto, Y. Wakasugi, *L^p-L^q estimates for the damped wave equation and the critical exponent for the nonlinear problem with slowly decaying data*, Commun. Pure Appl. Anal. **18** (2019), 1967–2008. MR3927427

[10] M. Ikeda, M. Sobajima, *Sharp upper bound for lifespan of solutions to some critical semilinear parabolic, dispersive and hyperbolic equations via a test function method*, Nonlinear Anal. **182** (2019), 57–74. MR3894246

[11] M. Ikeda, M. Sobajima, *Remark on upper bound for lifespan of solutions to semilinear evolution equations in a two-dimensional exterior domain*, J. Math. Anal. Appl. **470** (2019), 318–326. MR3865139

[12] M. Ikeda, M. Sobajima, K. Taniguchi, Y. Wakasugi, *Lifespan estimates for semilinear damped wave equation in a two-dimensional exterior domain*, Calc. Var. Partial Differential Equations **63** (2024), no. 9, Paper No. 240, 23 pp. MR4821889

[13] M. Ikeda, K. Taniguchi, Y. Wakasugi, *Global existence and asymptotic behavior for semilinear damped wave equations on measure spaces*, Evol. Equ. Control Theory **13** (2024), 1101–1125. MR4757314

[14] R. Ikehata, K. Nishihara, *Diffusion phenomenon for second order linear evolution equations*, Studia Math. **158** (2003), 153–161. MR2013737

[15] R. Ikehata, M. Ohta, *Critical exponents for semilinear dissipative wave equations in* $\mathbf{R}^N$, J. Math. Anal. Appl. **269** (2002), 87–97. MR1907875

[16] S. Jimbo, S. Sakaguchi, *Movement of hot spots over unbounded domains in* $\mathbb{R}^N$, J. Math. Anal. Appl. **182** (1994), 810–835. MR1272155

[17] J. Jost, "Postmodern analysis", Third edition Universitext, Springer-Verlag, Berlin, 2005. MR2166001

[18] G. Karch, *Selfsimilar profiles in large time asymptotics of solutions to damped wave equations*, Studia Math. **143** (2000), 175–197. MR1813366

[19] S. Kawashima, M. Nakao, K. Ono, *On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term*, J. Math. Soc. Japan **47** (1995), 617–653. MR1348752

[20] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci. **12** (1976/77), 169–189. MR0420031

[21] C.S. Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Comm. Pure Appl. Math. **14** (1961), 561–568. MR0132908

[22] T. Narazaki, L^p-L^q *estimates for damped wave equations and their applications to semilinear problem*, J. Math. Soc. Japan **56** (2004), 585–626. MR2048476

[23] K. Nishihara, L^p-L^q *estimates of solutions to the damped wave equation in* 3*-dimensional space and their application*, Math. Z. **244** (2003), 631–649. MR1992029

[24] T. Ogawa, H. Takeda, *Non-existence of weak solutions to nonlinear damped wave equations in exterior domains*, Nonlinear Anal. **70** (2009), 3696–3701. MR2504456

[25] K. Ono, *Decay estimates for dissipative wave equations in exterior domains*, J. Math. Anal. Appl. **286** (2003), 540–562. MR2008848

[26] A. Pazy, "Semigroups of linear operators and applications to partial differential equations," Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983. MR0710486

[27] P. Radu, G. Todorova, B. Yordanov, *Diffusion phenomenon in Hilbert spaces and applications*, J. Differential Equations **250** (2011), 4200–4218. MR2776911

[28] A. Saeki, R. Ikehata, *Remarks on the decay rate for the energy of the dissipative linear wave equations in exterior domains*, SUT J. Math. **36** (2000), 267–277. MR1810467

[29] Y. Shibata, *On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain*, Tsukuba J. Math. **7** (1983), 1–68. MR0703667

[30] M. Sobajima, *Higher order asymptotic expansion of solutions to abstract linear hyperbolic equations*, Math. Ann. **380** (2021), 1–19. MR4263676

[31] M. Sobajima, *Global existence of solutions to a weakly coupled critical parabolic system in two-dimensional exterior domains*, J. Math. Anal. Appl. **501** (2021), 125214, 20 pp. MR4242051

[32] M. Sobajima, *Optimal decay for one-dimensional damped wave equations with potentials via a variant of Nash inequality*, J. Differential Equations **408** (2024), 117–139. MR4768279

[33] M. Sobajima, *The effect of inhomogeneous terms for asymptotic profiles of solutions to second-order abstract evolution equations with time-dependent dissipation*, J. Math. Anal. Appl. **542** (2025), 128794. MR4793767

[34] H. Takeda, *Higher-order expansion of solutions for a damped wave equation*, Asymptot. Anal. **94** (2015), 1–31. MR3385242

[35] H. Yang, A. Milani, *On the diffusion phenomenon of quasilinear hyperbolic waves*, Bull. Sci. Math. **124** (2000), 415–433. MR1781556