

A COMMENT ON BRIANÇON-SPEDER POLYNOMIAL II

MUTSUO OKA

ABSTRACT. In our previous paper on Briançon-Speder Polynomials, we gave a classification of the connected components of μ^* -constant strata on the space of weighted homogeneous polynomials of three variables which generalize Briançon and Speder type polynomials. In this paper, we give further remarks on the μ^* -constant strata of higher dimensional weighted homogeneous polynomials.

1. INTRODUCTION

The notion of μ^* invariants of an analytic function germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ was introduced by B. Teissier [17, 18]. Recall that μ^* invariants consist of n integers $(\mu^{(n)}, \dots, \mu^{(1)})$ where $\mu^{(j)}$ is the Milnor number of f^H where f^H is the restriction of f to a generic subspace H of dimension j passing through the origin. Note that μ^* -constant strata can be defined as semi-algebraic sets (or equivalently by constructible sets) ([2]). Therefore if f and g are in the same connected component of a μ^* -constant stratum, then they can be joined by a complex analytic (or more precisely by a piecewise complex analytic) path in this component. By a result of Teissier [17, 18], a μ^* -constant complex analytic family f_t , $t \in U$ with $f_0 = f$, $f_1 = g$ and U is a neighborhood of $[0, 1]$ in \mathbb{C} gives a canonical Whitney regular stratification $\mathcal{S} = \{\mathcal{V} \setminus \mathcal{U}, \mathcal{U}\}$, which implies that the local topologies of f and g are isomorphic. Here $\mathcal{V} = \{(\mathbf{z}, t) \in \mathbb{C}^n \times U \mid f_t(\mathbf{z}) = 0\}$ and $\mathcal{U} = \{0\} \times U$. On the other hand, if f and g belong to different connected components, then their topologies (of various j -plane sections) may be different in general. Thus it is very important to know the structure of connected components of μ^* constant strata. For $n \neq 3$, a μ -constant family is topologically constant by Lê-Ramanujam theorem [9] but if $f_t(\mathbf{z})$ is not a μ^* -constant family, then the families given by the generic j -plane sections are not necessarily μ -constant families and therefore the topology of the j -plane section may change.

In our previous paper [16], we gave the structure of the connected components of μ^* -constant strata on the space of certain weighted homogeneous polynomials of three variables. In the three variables case, the hyperplane sections are plane curves and the topology of a plane curve is described by the Puiseux pairs of the irreducible components and linking numbers among

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them ([8, 20]). Note that μ -constant families of plane curves have always a constant multiplicity ($=\mu^{(1)} + 1$) by Fernández de Bobadilla and Peřka [4].

In this note, we consider μ^* -constant strata for weighted homogeneous polynomials of n variables for arbitrary $n \geq 4$. The situation is much more complicated in higher dimensions.

Our main results are as follows. First we divide the situation into two cases $p_1 < p_2$ (Case 1) or $p_1 = p_2$ (Case 2) under the assumption $p_1 \leq p_2 \leq \dots \leq p_n$ where $P = {}^t(p_1, \dots, p_n)$ is the weight vector of f . In Case 1, our main result is Theorem 9 and in Case (W2), our main result is described in Theorem 17 under the assumption (b).

2. PRELIMINARIES

Let $f(\mathbf{z})$ be a weighted homogeneous polynomial of degree e under the weight vector $P = {}^t(p_1, \dots, p_n)$ with an isolated singularity at the origin. We denote the space of such weighted homogeneous polynomials as $\mathcal{W}(P, e)$. We first prepare a key lemma. Recall that the degree of a monomial $M = z_1^{\nu_1} \dots z_n^{\nu_n}$ with respect to the weight vector P is defined by $\deg_P M = \sum_{i=1}^n p_i \nu_i$ and for a given polynomial $R(\mathbf{z})$, $d(P, R)$ is the minimal degree of the monomials in R with respect to P . Let $R(\mathbf{z}) = \sum_{\nu} a_{\nu} \mathbf{z}^{\nu}$ where $\nu := (\nu_1, \dots, \nu_n)$ and $\mathbf{z}^{\nu} := z_1^{\nu_1} \dots z_n^{\nu_n}$. Then the face function $R_P(\mathbf{z})$ of R with respect to P is defined by the partial sum of monomials which have P -degree $d(P, R)$ i.e., $R_P(\mathbf{z}) = \sum_{\deg_P \mathbf{z}^{\nu} = d(P, R)} a_{\nu} \mathbf{z}^{\nu}$. The following assertion results also from fact (2), p.445, [5] but for the consistency of the paper, we give a brief direct proof.

Lemma 1. *Suppose that f is a weighted homogeneous polynomial of degree e with respect to a weight $P = {}^t(p_1, \dots, p_n)$ and f has an isolated singularity at the origin, i.e., $f \in \mathcal{W}(P, e)$. Suppose that $R(\mathbf{z})$ is a polynomial with $d(P, R) > e$. Consider the family $f_t = f(\mathbf{z}) + tR(\mathbf{z})$ for $0 \leq t \leq 1$. Then $\mu(f_t) = \mu(f)$ for any $t, 0 \leq t \leq 1$.*

Proof. Consider the \mathbb{C}^* -action on \mathbb{C}^n defined by $s \circ \mathbf{z} = (s^{p_1} z_1, \dots, s^{p_n} z_n)$, $s \in \mathbb{C}^*$ which is associated with the weight vector P . Consider also the unit sphere S^{2n-1} centered at the origin. Consider the links $V_j' := V_j \cap S^{2n-1}$, $j = 1, \dots, n$ in S^{2n-1} where $V_j = \{\mathbf{z} \in \mathbb{C}^n \mid \frac{\partial f}{\partial z_j}(\mathbf{z}) = 0\}$. As f has an isolated singularity at the origin, $\cap_{i=1}^n V_i' = \emptyset$. Take a sufficiently small S^1 -invariant open tubular neighborhood U_j of V_j' . This is possible as $\frac{\partial f}{\partial z_j}(s \circ \mathbf{z}) = s^{e-p_j} \frac{\partial f}{\partial z_j}(\mathbf{z})$. Here we consider $S^1 \subset \mathbb{C}^*$. Let $W_1 := S^{2n-1} \setminus U_1$ and we put $W_j := \bigcap_{i=1}^{j-1} \overline{U}_i \setminus U_j$ for $j = 2, \dots, n$. Note that W_j is S^1 -invariant for $j = 1, \dots, n$. We assume that U_j ($1 \leq j \leq n$) are small enough so that $\cap_{i=1}^n \overline{U}_i = \emptyset$ and $\cup_{j=1}^n W_j = S^{2n-1}$. Note that W_j is a compact subset. Consider the punctured cone $W_j(\varepsilon) := \{s \circ \mathbf{z} \mid 0 < |s| \leq \varepsilon, \mathbf{z} \in W_j\}$. Put $e' := d(P, R) > e$. Put $M_j := \min\{|\frac{\partial f}{\partial z_j}(\mathbf{z})| \mid \mathbf{z} \in W_j\}$ and $M_j' := \max\{|\frac{\partial R_P}{\partial z_j}(\mathbf{z})| \mid \mathbf{z} \in W_j\}$. Then by the assumption $d(P, R) > e$, there exists a positive number ε_0 so that

$|\frac{\partial f}{\partial z_j}(s \circ \mathbf{z})| \geq M_j s^{\epsilon-p_j}$ and $|\frac{\partial R}{\partial z_j}(s \circ \mathbf{z})| \leq 2M'_j s^{\epsilon'-p_j}$ for $\mathbf{z} \in W_j(\epsilon_0)$. Take a sufficiently small $r_0 \ll \epsilon_0$ so that $S_r^{2n-1} \subset \cup_{j=1}^n W_j(\epsilon_0)$ for any $0 < r \leq r_0$ and

$$|\frac{\partial f}{\partial z_j}(\mathbf{z})| > |\frac{\partial R}{\partial z_j}(\mathbf{z})|, \quad \forall \mathbf{z} \in W_j(\epsilon_0).$$

This implies

$$\frac{\partial f_t}{\partial z_j}(\mathbf{z}) \neq 0 \quad (\mathbf{z} \in W_j(\epsilon_0), j = 1, \dots, n, 0 \leq t \leq 1).$$

Consider the jacobian $J(f_t) = (\frac{\partial f_t}{\partial z_1}(\mathbf{z}), \dots, \frac{\partial f_t}{\partial z_n}(\mathbf{z}))$ of f_t . Then the normalized jacobian mapping $\varphi_t : S_{r_0}^{2n-1} \rightarrow S^{2n-1}$ is defined for any $0 \leq t \leq 1$ by $\varphi_t(\mathbf{z}) = J(f_t)/\|J(f_t)\|(\mathbf{z})$. As the Milnor number of f_t is given as the mapping degree of φ_t with a sufficiently small r_0 (see [10]), this proves the assertion. \square

Remark 2. *In Lemma 1, we do not assume that f is Newton non-degenerate but it is enough to assume that $f^{-1}(0)$ has an isolated singularity at the origin. The assertion is well-known for a function which is Newton non-degenerate and convenient ([14]).*

For the stability of the Milnor fibration of f_t , $0 \leq t \leq 1$, we have to show the existence of a uniform stable radius*. The existence of a uniform stable radius of Milnor fibration for the family f_t , $0 \leq t \leq 1$, is more subtle. We show the existence for Newton non-degenerate weighted homogeneous polynomials.

Proposition 3. *Assume that $f(\mathbf{z})$ is Newton non-degenerate and $R(\mathbf{z})$ is strictly above the Newton boundary $\Gamma(f)$. That is, $d(Q, R) > d(Q, f)$ for any strictly positive weight vector Q . Then there exists a uniform stable radius r_0 for f_t , $0 \leq t \leq 1$.*

Proof. This is essentially proved with a little different assumption in [13]. We give a simple direct proof for the reader's convenience. Assume the assertion is not true. Then using Curve Selection Lemma ([10]), there exists an analytic family $(\mathbf{z}(s), t(s))$, $0 \leq s \leq 1$ such that $V(f_{t(s)})$ and the sphere of radius $\|\mathbf{z}(s)\|$ are not transverse at $(\mathbf{z}, t) = (\mathbf{z}(s), t(s))$ and $\mathbf{z}(0) = 0$, $0 \leq t(0) \leq 1$. Then we have a meromorphic function $\lambda(s)$ such that $\text{grad } f_{t(s)}(\mathbf{z}(s)) = \lambda(s)\mathbf{z}(s)$. Namely

$$(1) \quad \overline{\frac{\partial f_{t(s)}}{\partial z_j}}(\mathbf{z}(s)) = \lambda(s)z_j(s), \quad j = 1, \dots, n,$$

where the bar denotes the complex conjugacy. Let $I = \{1 \leq i \leq n \mid z_i(s) \neq 0\}$. For simplicity, we assume $I = \{1, 2, \dots, m\}$. First we note that $\lambda(s) \neq 0$,

* r_0 is a uniform stable radius for f_t , $0 \leq t \leq 1$ if $f_t^{-1}(0)$ intersects transversely with the sphere S_r^{2n-1} for any $0 < r \leq r_0$ and any $0 \leq t \leq 1$.

because $\text{grad} f_{t(s)}(\mathbf{z}(s)) \neq 0$ by Lemma 1. Consider their Taylor and Laurent expansions:

$$\begin{aligned} z_j(s) &= a_j s^{q_j} + (\text{higher terms}), \quad a_j \in \mathbb{C}^*, \quad q_j \geq 1, \quad j \in I \\ \lambda(s) &= bs^\nu + (\text{higher terms}), \quad b \in \mathbb{C}^*, \quad \nu \in \mathbb{Z}. \end{aligned}$$

Put $Q = {}^t(q_1, \dots, q_m)$, $\mathbf{a} = (a_1, \dots, a_m)$. Put also $\tilde{Q} := (q_1, \dots, q_m, N, \dots, N) \in \mathbb{Z}^n$ with a sufficiently large N so that $d(\tilde{Q}, f) = d(Q, f^I)$, $d(\tilde{Q}, R) = d(Q, R^I)$ and $(f_t)_{\tilde{Q}} = f_{\tilde{Q}}^I$. Here $f^I = f|_{\mathbb{C}^I}$ where $\mathbb{C}^I := \{\mathbf{z} \mid z_j = 0, j \notin I\}$. By the assumption, we have $d(\tilde{Q}, R) > d(\tilde{Q}, f)$. As $f_{t(s)}(\mathbf{z}(s)) \equiv 0$ and $d(\tilde{Q}, f) < d(\tilde{Q}, R)$, we have $f_{\tilde{Q}}^I(\mathbf{a}) = 0$. Note that

$$(2) \quad \overline{\frac{\partial f_{t(s)}}{\partial z_j}}(\mathbf{z}(s)) = \overline{\frac{\partial f_{\tilde{Q}}^I}{\partial z_j}}(\mathbf{a}) s^{d(Q, f) - q_j} + (\text{higher terms}).$$

By the Newton non-degeneracy of f , there is some $j_0 \in I$ such that $\overline{\frac{\partial f_{\tilde{Q}}^I}{\partial z_{j_0}}}(\mathbf{a}) \neq 0$. Then by (1) we must have the equality:

$$(3) \quad d(Q, f^I) - q_{j_0} = \nu + q_{j_0}, \quad \overline{\frac{\partial f_{\tilde{Q}}^I}{\partial z_{j_0}}}(\mathbf{a}) = ba_{j_0}.$$

If $q_j < q_{j_0}$ for some j , $d(Q, f^I) - q_j > \nu + q_j$ which gives the equality $ba_j = 0$ by (1) and we get a contradiction to $b \neq 0$. Thus we have $q_{j_0} \leq q_j$ for $\forall j \in I$ and comparing the leading terms of (1), we get

$$\overline{\frac{\partial f_{\tilde{Q}}^I}{\partial z_j}}(\mathbf{a}) = \begin{cases} 0, & \text{if } q_j > q_{j_0} \\ ba_j, & \text{if } q_j = q_{j_0}. \end{cases}$$

Put $I_0 = \{i \in I \mid q_i = q_{j_0}\}$. Multiplying $q_j \bar{a}_j$ to both sides of this equality and summing up for $j \in I_0$, we get

$$(4) \quad \sum_{j \in I_0} q_j \bar{a}_j \overline{\frac{\partial f_{\tilde{Q}}^I}{\partial z_j}}(\mathbf{a}) = b \sum_{j \in I_0} q_j |a_j|^2.$$

The left hand side coincides with $\sum_{j \in I} q_j \bar{a}_j \overline{\frac{\partial f_{\tilde{Q}}^I}{\partial z_j}}(\mathbf{a})$ which is equal to $d(Q, f^I) \overline{f_{\tilde{Q}}^I}(\mathbf{a})$ by Euler equality and it vanishes as we have seen in the above discussion. However the right hand side of (4) is clearly not zero which is a contradiction. \square

Lemma 4. *Let again $f \in \mathcal{W}(P, e)$ and let $V = f^{-1}(0)$. For any $K \subset \{1, \dots, n\}$ with $|K| = n - 1$, $V^K = (f^K)^{-1}(0) \subset \mathbb{C}^K$ has at most 1-dimensional singularity.*

Proof. We may assume for simplicity that $K = \{2, \dots, n\}$. Let $h_1(\mathbf{z}^K)$ be the polynomial of z_2, \dots, z_n defined by $h_1(\mathbf{z}^K) := \frac{\partial f}{\partial z_1}(0, \mathbf{z}^K)$. The locus of singular points of f^K is given by

$$\Sigma(f^K) := \{\mathbf{z}^K \in \mathbb{C}^K \mid \frac{\partial f^K}{\partial z_2}(\mathbf{z}^K) = \dots = \frac{\partial f^K}{\partial z_n}(\mathbf{z}^K) = 0\}$$

and the set of singular points of f in \mathbb{C}^K is the intersection of $\Sigma(f^K)$ and $\Sigma_1 := \{\mathbf{z}^K \mid h_1(\mathbf{z}^K) = 0\}$. As $f \in \mathcal{W}(P, e)$, $\Sigma(f^K) \cap \Sigma_1 = \{0\}$. This implies $\dim \Sigma(f^K) \leq 1$ and the assertion is proved. \square

3. RESULT IN THE PREVIOUS PAPER

For an integer n , let $\mathcal{P}(n)$ be the set of partitions of n by positive integers. Namely a partition $A \in \mathcal{P}(n)$ corresponds to a collection of positive integers $A := \{\nu_1, \dots, \nu_\ell\}$ with $\nu_1 + \dots + \nu_\ell = n$ and $\nu_j \geq 1$ for any j . We recall the main result in the previous paper [16]. In [1], Briançon and Speder considered polynomials of three variables in $\mathcal{W}(P, e)$ with $P = {}^t(1, 2, 3)$ and $e = 15$. In [16], we consider similar weighted homogeneous polynomials of three variables. Let $a \geq 2$ and k be positive integers. Consider a weight vector $P = {}^t(1, a, a + 1)$, and $e = (a + 1)(ak + 1)$ and consider the space $\mathcal{W}(P, e)$ of weighted homogeneous polynomials of degree e with respect to P having an isolated singularity at the origin. For a given partition $A = \{\nu_1, \dots, \nu_\ell\} \in \mathcal{P}(m_1)$, $0 \leq m_1 \leq k$, choose mutually distinct non-zero complex numbers $\{a_1, \dots, a_\ell\}$ and we associate a weighted homogeneous polynomial of degree e with respect to P :

$$h_A(y, z) = z^{1+ma} \prod_{j=1}^{\ell} (z^a - a_j y^{a+1})^{\nu_j}, \quad m_1 = \sum_{i=1}^{\ell} \nu_i, \quad m = k - m_1.$$

If $m_1 = 0$, $\mathcal{P}(0) = \{\emptyset\}$ and we define $h_\emptyset = z^{1+ka}$. Then we consider polynomials

$$\begin{aligned} h'_A(x, y, z) &= h_A(y, z) + xR(y, z) \\ f_A(x, y, z) &= \lambda x^e + h'_A(x, y, z), \quad \lambda \in \mathbb{C}^* \end{aligned}$$

where $R(y, z)$ is a weighted homogeneous polynomial with $d(P, R) = e - 1$ such that

$$(\star) \quad \begin{cases} \{(y, z) \mid z^a - a_j y^{a+1} = R(y, z) = 0\} = \{(0, 0)\}, & \text{if } \nu_j \geq 2 \\ R(y, 0) \neq 0, & \text{if } m \neq 0. \end{cases}$$

Note that f_A is a weighted homogeneous polynomial of degree e , and if $m > 0$, (\star) says that f_A has monomial $xy^{(a+1)k+1}$. Easiest and simplest choice of $R(y, z)$ is $R(y, z) = y^{(a+1)k+1}$. With a generic λ , the polynomial f_A has an isolated singularity at the origin by Bertini theorem (see for example page 137, [6]). Thus $f_A \in \mathcal{W}(P, e)$. The topology of the generic plane section of f_A does not depend on the choice of a_1, \dots, a_ℓ and R satisfying (\star) . Considering a_1, \dots, a_ℓ and λ as parameters, $f_A(x, y, z)$ is a family of almost Newton non-degenerate polynomials in the sense of [15] and thus their Milnor numbers are constant and do not depend on $\{a_1, \dots, a_\ell\}$ or on the choice of $R(y, z)$. This implies that f_A is a μ^* -constant family with respect to the mutually distinct parameters a_1, \dots, a_ℓ and generic coefficients of $R(y, z)$, and therefore the topology of $f_A = 0$ is constant by [17]. Let $\overline{\mathcal{W}}_c(P, e)$ be the set of connected components of μ^* -constant strata in

$\mathcal{W}(P, e)$. In [16], we defined a correspondence $\psi : \mathcal{P}^{(tot)}(k) \rightarrow \overline{\mathcal{W}}_c(P, e)$ by $\psi(A) = [f_A]$ for $A \in \mathcal{P}^{(tot)}(k)$. Here $\mathcal{P}^{(tot)}(k) = \bigcup_{i=0}^k \mathcal{P}(i)$ and $[f_A]$ is the connected component of the μ^* -constant stratum which contains f_A .

Theorem 5 ([16]). *The correspondence $\psi : \mathcal{P}^{(tot)}(k) \rightarrow \overline{\mathcal{W}}_c(P, e)$ is bijective. Namely for any $f \in \mathcal{W}(P, e)$, there are unique $m_1, 0 \leq m_1 \leq k$ and $A \in \mathcal{P}(m_1)$ so that $[f] = \psi(A) = [f_A]$. In particular, the number of connected components of μ^* -constant strata in $\mathcal{W}(P, e)$ is $\sum_{m_1=0}^k |\mathcal{P}(m_1)|$.*

4. GENERALIZATION TO HIGHER DIMENSIONS

In this section, we consider polynomials of n variables with $n \geq 4$. Let $P = {}^t(p_1, \dots, p_n)$. Recall that $\mathcal{W}(P, e)$ is the space of weighted homogeneous polynomials $f(z_1, \dots, z_n)$ of weighted degree e with respect to the weight P which have isolated singularities at the origin. It seems very difficult to classify μ^* -constant strata completely for the space of $\mathcal{W}(P, e)$ with arbitrary $n \geq 4$. We divide the situation into two classes and give examples which shows the complexity of this problem in higher dimensions.

4.1. Two classes. First we consider the space of polynomials $\widetilde{\mathcal{W}}(P, e)$ defined by

$$\widetilde{\mathcal{W}}(P, e) := \{g \in \mathbb{C}[z_1, \dots, z_n] \mid d(P, g) = e, g_P \in \mathcal{W}(P, e)\}$$

where g_P is the face function of g by the weight P . Thus we have the inclusion $\mathcal{W}(P, e) \subset \widetilde{\mathcal{W}}(P, e)$. However a polynomial $g \in \widetilde{\mathcal{W}}(P, e)$ can have monomials with P -degree greater than e . Note that for $g \in \widetilde{\mathcal{W}}(P, e)$, $\mu(g) = \mu(g_P)$ by Lemma 1. In general, a generic hyperplane section of $g \in \mathcal{W}(P, e)$ get monomials which have larger degree than e and it does not stay in $\mathcal{W}(P_{n-1}, e)$. Here $P_{n-1} := {}^t(p_2, \dots, p_n)$. We consider also $\mathcal{W}(P, e)^{(1)}$ which is the set of polynomials g such that its P -face function g_P has singularities of dimension one and $g = 0$ has an isolated singularity at the origin. Thus in general, for $g \in \mathcal{W}(P, e)$, its generic hyperplane section is contained in either $\widetilde{\mathcal{W}}(P_{n-1}, e)$ or $\mathcal{W}(P_{n-1}, e)^{(1)}$ by Lemma 4. The latter case happens if $p_1 < p_j, j = 2, \dots, n$ and $g|_{z_1=0}$ has one-dimensional singularities. There are two cases which give different geometrical behaviors for hyperplane sections. Let us consider weighted homogeneous polynomials with respect to $P = {}^t(p_1, \dots, p_n)$. For simplicity, we assume hereafter $p_1 \leq p_2 \leq \dots \leq p_n$. We divide the situation into two cases by the first two integers p_1, p_2 :

- (W1) $p_1 < p_2$.
- (W2) $p_1 = p_2$.

Remark 6. *Consider the case $P = {}^t(1, \dots, 1)$. This is a special case of (W2). Then $\mathcal{W}(P, e)$ is nothing but the space of smooth projective hypersurface of degree e up to a scalar multiplication in \mathbb{P}^{n-1} . Any generic hyperplane section gives a smooth projective hypersurface of dimension $n - 2$ in $\mathcal{W}(P_{n-1}, e)$. Thus for any $f \in \mathcal{W}(P, e)$, $\mu^{(j)}(f) = (e - 1)^j$ for any $1 \leq j \leq n$ by [11] and $\mathcal{W}(P, e)$ has a unique μ^* -constant stratum and it is connected.*

4.2. **Case (W1).** We will give an example such that several different $\mu^{(n-1)}$ can be taken by generic hyperplane sections in $\mathcal{W}(P, e)$. Suppose $P = {}^t(p_1, \dots, p_n)$ is a weight vector with $p_1 < p_2$ and $p_2 \leq \dots \leq p_n$. Note first that P_{n-1} may not be a primitive integral vector. Let d be the greatest common divisor of p_2, \dots, p_n and put $P'_{n-1} := (p'_2, \dots, p'_n)$ with $p'_i = p_i/d$. Then P'_{n-1} is a primitive integral vector. In the definition of $\mathcal{W}(P_{n-1}, e)$, we do not assume P_{n-1} to be primitive and it is identical with $\mathcal{W}(P'_{n-1}, e/d)$. Note that the Milnor numbers given by Milnor-Orlik's formula for the two weight vectors coincide. Put $e' = e/d$. Then $\mathcal{W}(P_{n-1}, e) = \mathcal{W}(P'_{n-1}, e')$ and for $f \in \mathcal{W}(P_{n-1}, e)$,

$$\mu(f) = \prod_{i=2}^n (e/p_i - 1) = \prod_{i=2}^n ((e/d)/(p_i/d) - 1) = \prod_{i=2}^n (e'/p'_i - 1).$$

Thus we do not need to assume the weight vector to be primitive for the calculation of the Milnor number.

Suppose that $\mathcal{W}(P, e) \neq \emptyset$ and $\mathcal{W}(P_{n-1}, e) \neq \emptyset$ so that there is a weighted homogeneous polynomial of $n-1$ variables z_2, \dots, z_n of degree e with respect to P_{n-1} which has an isolated singularity at the origin.

Consider a weighted homogeneous polynomial $f(\mathbf{z}) = f(z_2, \dots, z_n)$ of degree e with respect to P_{n-1} and assume that $f = 0$ has either an isolated singular point at the origin or has some non-isolated singular points outside of $\{\mathbf{z} \mid z_2 \cdots z_n = 0\}$. They correspond to isolated singular points of $f = 0$ in the weighted projective space $\mathbb{P}^{n-2}(P'_{n-1})^\dagger$. Consider a polynomial $g \in \mathcal{W}(P, e)$ which is written as $g(z_1, \mathbf{z}) = R(z_1, \mathbf{z}) + f(\mathbf{z})$ where $\mathbf{z} = (z_2, \dots, z_n)$ and $z_1 \mid R(z_1, \mathbf{z})$. In this case, we note first

Proposition 7. *Assume that $g \in \mathcal{W}(P, e)$ and $f \notin \mathcal{W}(P_{n-1}, e)$. Then $d = 1$ and $P'_{n-1} = P_{n-1}$.*

Proof. By the assumption, $e = d(P_{n-1}, f)$ and $e \equiv 0$ modulo d . On the other hand, g has an isolated singularity at the origin. Thus for any singular point $q = (0, \rho) \in \{z_2 \cdots z_n \neq 0\}$ of f , q is a regular point of g . This implies that there exists some monomial $M := z_1 z_2^{a_2} \cdots z_n^{a_n}$ in $R(z_1, \mathbf{z})$ so that $\frac{\partial g}{\partial z_1}(0, \rho) \neq 0$. As $e = d(P, M) \equiv p_1$ modulo d and P is primitive, this is possible only if $d = 1$. \square

Consider an admissible toric modification $\pi : X \rightarrow \mathbb{C}^{n-1}$ for a generic hyperplane section g^H . (Recall that g^H is the restriction of g to H .) Assume $f \in \mathcal{W}(P_{n-1}, e)^{(1)}$. Let \tilde{f} be the defining function of the strict transform of $g^H = 0$ to X . Take a toric coordinate chart \mathbb{C}_σ^{n-1} associated with a regular cone

$$\sigma = \text{Cone}(Q_1, Q_2, \dots, Q_{n-1}), \quad Q_1 = P_{n-1},$$

${}^\dagger \mathbb{P}^{n-2}(P'_{n-1})$ is the quotient space of $\mathbb{C}^{n-1} \setminus \{0\}$ by the standard \mathbb{C}^* -action associated with P'_{n-1} .

with coordinates (u_1, \dots, u_{n-1}) . Let $\hat{E}(P_{n-1})$ be the exceptional divisor which is associated with P_{n-1} . Let $E(P_{n-1}) = \hat{E}(P_{n-1}) \cap \tilde{V}$ where \tilde{V} is the strict transform of the hypersurface $f = 0$ and assume their singular points ρ_1, \dots, ρ_s are in the torus $\hat{E}(P_{n-1})^*$ where

$$\hat{E}(P_{n-1})^* = \{(0, u_2, \dots, u_{n-1}) \mid u_2 \dots u_{n-1} \neq 0\}.$$

We will show that these singular points correspond to the singular points of the weighted projective hypersurface $V(f)$ in $\mathbb{P}^{n-2}(P_{n-1})^*$. Here

$$\mathbb{P}^{n-2}(P_{n-1})^* := \{[w_1 : \dots : w_{n-1}] \mid w_1 \dots w_{n-1} \neq 0\}.$$

Consider $(n-2)$ -dimensional torus in the parallel hyperplanes to $\hat{E}(P_{n-1})$:

$$\hat{E}_\tau^* := \{\mathbf{u}_\sigma = (u_1, \dots, u_{n-1}) \mid u_1 = \tau, u_2 \dots u_{n-1} \neq 0\}, \tau \neq 0.$$

It is clear that $\hat{E}(P_{n-1})^*$ and \hat{E}_τ^* are isomorphic. The restriction of π to the toric coordinate space \mathbb{C}_σ^{n-1} is denoted by π_σ . Recall that π_σ is defined by $\pi_\sigma(\mathbf{u}_\sigma) = (z_1, \dots, z_{n-1})$ with

$$z_j = u_1^{p_{j,1}} u_2^{p_{j,2}} \dots u_{n-1}^{p_{j,n-1}}, \quad 1 \leq j \leq n-1, \quad \text{where } \sigma = (p_{i,j})_{1 \leq i, j \leq n-1}.$$

Here we identify σ with the unimodular matrix (Q_1, \dots, Q_{n-1}) , where $Q_j = {}^t(p_{1,j}, \dots, p_{n,j})$, $1 \leq j \leq n-1$. As $P_{n-1} = Q_1$ and $P_{n-1} = {}^t(p_2, \dots, p_n)$, $p_j = p_{j-1,1}$, $j = 2, \dots, n$. We consider the \mathbb{C}^* -action on \mathbb{C}_σ^{n-1} defined by

$$\tau \circ (u_1, \dots, u_{n-1}) = (\tau u_1, u_2, \dots, u_{n-1}).$$

Consider the restriction of π_σ on the torus $\pi'_\sigma : \mathbb{C}_\sigma^{*(n-1)} \rightarrow \mathbb{C}^{*(n-1)}$. Then π'_σ is a group isomorphism. On the base space \mathbb{C}^{n-1} , we consider the \mathbb{C}^* -action associated with P_{n-1} , which is defined by $\tau \circ \mathbf{z} = (\tau^{p_2} z_1, \dots, \tau^{p_n} z_{n-1})$.

We can write the pull back of f as

$$(5) \quad \pi_\sigma^* f(\mathbf{u}_\sigma) = u_1^e \tilde{f}(\mathbf{u}_\sigma), \quad \tilde{f}(\mathbf{u}_\sigma) = h(u_2, \dots, u_{n-1}) \prod_{j=2}^{n-1} u_j^{d(Q_j)}$$

and $E(P_{n-1})^*$ is defined by $\tilde{f} = 0$ (or equivalently $h = 0$) in $\hat{E}(P_{n-1})^*$. Note that π_σ is an \mathbb{C}^* -equivariant mapping i.e., $\pi_\sigma(\tau \circ \mathbf{u}_\sigma) = \tau \circ \pi_\sigma(\mathbf{u}_\sigma)$. We consider the associated map:

$$\psi := \omega \circ \pi_{\sigma|_{\hat{E}_1^*}} : \hat{E}_1^* \rightarrow \mathbb{P}^{n-2}(P_{n-1})^*.$$

Here ω is the quotient map $\omega : \mathbb{C}^{n-1} \setminus \{0\} \rightarrow \mathbb{P}^{n-2}(P_{n-1})^*$.

Proposition 8 (Proposition 3.2, [3]). *Two hypersurface germs $(\tilde{f} = 0, \rho_i) \subset \hat{E}_1^*$ and $(f = 0, \psi(\rho_i)) \subset \mathbb{P}^{n-2}(P_{n-1})^*$ are isomorphic by ψ for $i = 1, \dots, s$.*

Proof. Note that $(\hat{E}(P_{n-1})^*, V(\tilde{f}))$ is canonically isomorphic to $(\hat{E}_1^*, V'(\tilde{f}))$ by the translation $(0, u_2, \dots, u_{n-1}) \mapsto (1, u_2, \dots, u_{n-1})$ where

$$V'(\tilde{f}) := \{(1, u_2, \dots, u_{n-1}) \in \mathbb{C}_\sigma^{*(n-1)} \mid \tilde{f}(u_2, \dots, u_{n-1}) = 0\}.$$

Note also that both \mathbb{C}^* -actions on $\mathbb{C}_\sigma^{*(n-1)}$ and $\mathbb{C}^{*(n-1)}$ are free, \hat{E}_1^* is canonically isomorphic to the quotient space of $\mathbb{C}_\sigma^{*(n-1)}$ and ψ is also isomorphic

to the quotient space of \mathbb{C}_σ^{*n} which is $\mathbb{P}^{n-2}(P_{n-1})^*$. As $\psi(V'(\tilde{f})) = (f = 0) \subset \mathbb{P}^{n-2}(P_{n-1})^*$ by (5), the assertion follows. \square

Theorem 9. *Let f and g be as above and assume $g^H \in \mathcal{W}(P_{n-1}, e)^{(1)}$. We assume also that the singular points ρ_1, \dots, ρ_s of $\tilde{f} = 0$ are in the torus $\hat{E}(P_{n-1})^*$ and they are Newton non-degenerate. Then g^H is an almost Newton non-degenerate function in the sense of [15] and it has Milnor number*

$$\mu(g^H) = \prod_{i=2}^n (e/p_i - 1) + (p_2 - p_1)\mu^{tot}$$

where μ^{tot} is the sum of Milnor numbers $\mu(\tilde{f}, \rho_i)$ of \tilde{f} at ρ_i ($1 \leq i \leq s$).

Proof. Note that a generic hyperplane section g^H of g is given by the substitution z_1 by $a_2 z_2 + \dots + a_n z_n$ with generic coefficients a_2, \dots, a_n . Let H be the hyperplane $H = \{z_1 = \sum_{i=2}^n a_i z_i\}$. Take $\mathbf{z} := (z_2, \dots, z_n)$ as the coordinates of H . Note that the P_{n-1} -face function of g^H , $(g^H)_{P_{n-1}}$, is given by $f(\mathbf{z})$ as $d(P_{n-1}, R') > e$ where $R'(\mathbf{z}) := R(\sum_{i=2}^n a_i z_i, \mathbf{z})$. Consider an admissible toric modification $\pi : X \rightarrow \mathbb{C}^{n-1}$ of $f = 0$ and take a toric coordinate chart \mathbb{C}_σ^{n-1} associated with a regular cone $\sigma = \text{Cone}(Q_1, Q_2, \dots, Q_{n-1})$ with coordinates (u_1, \dots, u_{n-1}) and $Q_1 = P_{n-1}$. Let us write $R(z_1, \mathbf{z}) = \sum_{i=1}^m z_1^i R_i(\mathbf{z})$. Here m is an integer such that $mp_1 \leq e$ and R_i is a weighted homogeneous polynomial with $d(P_{n-1}, R_i) = e - ip_1$. As $(0, \rho_i)$ is not a singular point of g , we need to have $R_1(\rho_i) \neq 0$ for $i = 1, \dots, s$. Assume that $\nu_0 = \max\{i \mid 2 \leq i, p_i = p_2\}$. Then after the substitution $z_1 = \sum_{i=2}^n a_i z_i$, the lowest term from $R(z_1, \mathbf{z})$ is given by $(\sum_{i=2}^{\nu_0} a_i z_i) R_1(\mathbf{z})$. As $d(P, R_1(\mathbf{z})) = e - p_1$,

$$d(P_{n-1}, g^H) = e, \quad \text{and} \quad d(P_{n-1}, (\sum_{i=2}^{\nu_0} a_i z_i) R_1(\mathbf{z})) = (e - p_1) + p_2.$$

Thus the pull-back of g^H by π is written as

$$(\sharp) \pi^* g^H = u_1^e \prod_{i=2}^{n-1} u_i^{d(Q_i, g^H)} \left(\tilde{f}(u_2, \dots, u_{n-1}) + cu_1^{p_2 - p_1} + (\text{higher terms}) \right).$$

Here $\tilde{f}(u_2, \dots, u_{n-1})$ is the defining polynomial of the strict transform of f and c is a non-zero constant which depends on a_2, \dots, a_{ν_0} . By the assumption, at each singular point ρ_i we can find an admissible analytic coordinates change $\mathbf{v}_i = (v_{i,2}, \dots, v_{i,(n-1)})$ of $\mathbf{u} = (u_2, \dots, u_{n-1})$ so that (u_1, \mathbf{v}_i) are local analytic coordinates centered at ρ_i and $\tilde{f}(\mathbf{v}_i)$ is Newton non-degenerate at ρ_i . This implies that the Milnor number of \tilde{f} at ρ_i is equal to that of f at $\psi(\rho_i)$ in $\mathbb{P}^{n-2}(P_{n-1})$. Using coordinates (u_1, \mathbf{v}_i) ($\mathbf{v}_i := (v_{i,2}, \dots, v_{i,n-1})$), (\sharp) is written in the neighborhood of ρ_i as

$$(\sharp)_i : \pi^* g^H = u_1^e \prod_{i=2}^{n-1} u_i^{d(Q_i, g^H)} \left(\tilde{f}(\mathbf{v}_i) + cu_1^{p_2 - p_1} + (\text{higher terms}) \right).$$

Thus by $(\sharp)_i$, π^*g^H is Newton non-degenerate and it behaves exactly as Lê-Yomdin singularity and the argument of the proof of Theorem 3.3 of [3] works. We will give an outline of the proof briefly so that the readers may skip the reference [4]. The key points are the following equalities.

$$(6) \quad -1 + (-1)^{n-1}\mu(g^H) = \deg \zeta_{(g^H,0)}(t),$$

$$(7) \quad \deg \zeta_{(g^H,0)}(t) = -1 + (-1)^{n-1} \left(\prod_{i=2}^n (e/p_i - 1) - e \sum_{j=1}^s \mu(f, \rho_j) \right) \\ + \sum_{j=1}^s \deg \zeta_{(\pi^*g^H, \rho_j)}(t).$$

Here $\zeta_{(\pi^*g^H, \rho_j)}(t)$ is the zeta-function of π^*g^H at ρ_j . As g^H has an isolated singularity at the origin, the equality (5) follows from the definition of the zeta function and by the connectedness of the Milnor fiber of g^H by Milnor [10]. The equality (6) follows from Theorem 8 of [15]. Now the assertion of Theorem 9 is reduced to the following Lemma.

Lemma 10.

$$\deg \zeta_{(\pi^*g^H, \rho_j)}(t) = (-1)^{n-1}(e + (p_2 - p_1))\mu(\tilde{f}, \rho_j).$$

Proof. For the proof, we use Varchenko formula for the zeta functions ([19]). See §1 of [15] or Theorem(5.3), p.147 ([14]) for further detail. First we recall Varchenko formula for a Newton non-degenerate function germ $h(z_1, \dots, z_n)$ at the origin.

$$(V) \quad \zeta(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{Q \in \mathcal{P}_I} (1 - t^{d(Q, h^I)})^{-\beta(Q)/d(Q, h^I)}$$

where h^I is the restriction of h to the coordinate subspace \mathbb{C}^I and $d(Q, h^I)$ is the minimal value of the monomials in h^I with respect to the weight vector Q . Here $\mathbb{C}^I := \{z \mid z_j = 0, j \notin I\}$. For $I \subset \{1, \dots, n\}$ such that h^I is not zero, \mathcal{P}_I is the set of primitive integral weight vectors of the coordinate subspace \mathbb{C}^I which correspond to the maximal faces of $\Gamma(h^I)$. The integer $\beta(Q)$ in (V) is defined as follows:

$$\beta(Q) = (-1)^{|I|-1} |I|! \text{Vol}_{|I|} C(\Delta(Q; h^I), 0).$$

Here $\Delta(Q; h^I)$ is the face of $\Gamma(h^I)$ where Q takes the minimal value and $C(\Delta(Q; h^I), 0)$ is the cone over $\Delta(Q; h^I)$ with vertex at the origin. Using $\Delta := \Delta(Q)$, we use also the notations $\beta(\Delta) := \beta(Q)$ and $h_\Delta^I := h_Q^I$.

Now we consider $\tilde{f}(\mathbf{v}_i)$ and $\pi^*g^H(u_1, \mathbf{v}_i)$. They are Newton non-degenerate at ρ_i and their Newton principal parts are given respectively by the polynomials

$$\tilde{f}_{New}(\mathbf{v}_i), \quad u_1^c (\tilde{f}_{New}(\mathbf{v}_i) + cu_1^{p_2-p_1}), \quad c \neq 0$$

where $\tilde{f}_{New}(\mathbf{v}_i)$ is the Newton principal part of $\tilde{f}(\mathbf{v}_i)$. Let $\mathcal{I} = \{I \subset \{2, \dots, n-1\} \mid \tilde{f}^I \neq 0\}$ and

$$\mathcal{P}_I = \{\Delta \mid \Delta \subset \Gamma(\tilde{f}^I, \rho_i), \dim \Delta = |I| - 1\}.$$

These data are used to describe the zeta function $\zeta_{(\tilde{f}, \rho_i)}(t)$. Let $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{P}}_I$ be the corresponding data for g^H . Then we observe easily that

$$\begin{aligned} \tilde{\mathcal{I}} &= \{\tilde{I} := I \cup \{1\} \mid I \in \mathcal{I}\} \cup \{J\}, \quad J = \{1\}, \\ \tilde{\mathcal{P}}_I &= \{\tilde{\Delta} := (e + p_2 - p_1)e_1 + (ee_1 + \Delta) \mid \Delta \in \mathcal{P}_I\} \cup \{(e + p_2 - p_1)e_1\} \end{aligned}$$

where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$ and ‘+’ is the Minkowski sum in \mathbb{R}^{n-1} . The face function $(\pi^*g^H)_{\tilde{\Delta}}$ is $u_1^e(\tilde{f}_{\Delta}(\mathbf{v}_i) + cu_1^{p_2-p_1})$. Then by a simple determinant calculation, we have

$$(8) \quad \beta(\tilde{\Delta}) = -\beta(\Delta) \times (e + p_2 - p_1).$$

Thus we have

$$\deg \zeta_{(\pi^*g^H, \rho_i)}(t) = (e + p_2 - p_1) - \sum_{I \in \mathcal{I}} \sum_{\tilde{\Delta} \in \tilde{\mathcal{P}}_I} \beta(\tilde{\Delta})$$

where the first term corresponds to $J = \{1\}$. Thus using (7), we get

$$\begin{aligned} \deg \zeta_{(\pi^*g^H, \rho_i)}(t) &= (e + p_2 - p_1) + \sum_{I \in \mathcal{I}} \sum_{\Delta \in \mathcal{P}_I} (e + p_2 - p_1)\beta(\Delta) \\ &= (e + p_2 - p_1) - (e + p_2 - p_1) \deg \zeta_{(\tilde{f}, \rho_i)}(t) \\ &= (e + p_2 - p_1) + (e + p_2 - p_1)((-1)^{n-1}\mu(\tilde{f}, \rho_i) - 1) \\ &= (-1)^{n-1}(e + p_2 - p_1)\mu(\tilde{f}, \rho_i). \end{aligned}$$

□

Thus combining (5) and (6), the assertion for Milnor number of g^H follows. □

Example 11. For simplicity, we consider the case $n = 4$. We consider the weight $P_3 = {}^t(2, 3, 4)$. Take a weighted homogenous polynomial $f(x, y, z)$ of degree 16 with 1-dimensional singularity on \mathbb{C}^3 . We will show that there exist at least four strata. For example,

$$\begin{aligned} f_1(x, y, z) &= x^8 - 3x^2z^3 - 2xy^2z^2 + y^4z + 3z^4 \\ f_2(x, y, z) &= x^8 - \frac{6}{7}x^2y^4 - \frac{2}{7}xy^2z^2 + y^4z - 3x^2z^3 + \frac{15}{7}z^4 \\ f_3(x, y, z) &= x^8 - \frac{18}{5}x^2y^4 - \frac{54}{5}xy^2z^2 + 9y^4z + 5x^2z^3 - \frac{3}{5}z^4 \\ f_4(x, y, z) &= x^8 - \frac{5}{3}x^3y^2z - \frac{13}{9}x^2y^4 + \frac{2}{3}x^2z^3 - \frac{13}{9}xy^2z^2 + 3y^4z - \frac{1}{9}z^4. \end{aligned}$$

f_i has 1-dimensional singularity along the \mathbb{C}^* -orbit of $\rho = (1, 1, 1)$ which is A_i singularity in $\mathbb{P}^2(P_3)$ for $i = 1, 2, 3$. f_4 has $2A_1$ singularities on the orbit

of $\rho_1 = (1, 1, 1), \rho_2 = (1, -1, 2)$ respectively. Consider polynomials

$$g_i(w, x, y, z) = w^{16} + cw y^5 + f_i(x, y, z), \quad i = 1, 2, 3, 4$$

with a general coefficient c . Then $g_i, i = 1, \dots, 4$, are weighted homogenous polynomials of degree 16 under the weight vector $P = {}^t(1, 2, 3, 4)$. Note that g_i has an isolated singularity at the origin and thus $g_i \in \mathcal{W}(P, 16)$ with $\mu(g_i) = 1365$ for $i = 1, \dots, 4$. Here the variables are ordered as (w, x, y, z) so that $\deg_P w = 1$. A generic hyperplane section g_i^H is given by substituting $w = ax + by + cz$ in g_i and its P_3 principal part is given by $f_i(x, y, z)$ which has non isolated singularities by our construction. However g_i^H has an isolated singularity by the help of monomial axy^5 . This implies that $g_i^H \in \mathcal{W}(P_3, 16)^{(1)}$. On the other hand, for a generic member $h \in \mathcal{W}(P, 16)$, we can assume that h^H has a Newton non-degenerate principal part and $h^H \in \widetilde{\mathcal{W}}(P_3, 16)$. Thus $\mu(h^H) = 91$ by Milnor-Orlik's formula. On the other hand, $\mu^{(3)}(g_i)$ is given by $91 + i = 92, 93, 94$ for $i = 1, 2, 3$ respectively and $\mu^{(3)}(g_4) = 93$ by Theorem 9. It is interesting to observe that generic plane sections of f_1, \dots, f_4 (or of $g_i^H, i = 1, 2, 3, 4$) and of h^H are in $\widetilde{\mathcal{W}}(P_2, 16)$, and their $P_2 = {}^t(3, 4)$ principal parts are isomorphic to the curve $ay^4z + bz^4 = 0, a, b \neq 0$. Thus $\mu^{(2)}(g_i) = 13$ for any i .

Example 12. Consider the weight vector $P = {}^t(1, k, \dots, k)$ with $k \geq 2$. Take $g \in \mathcal{W}(P, e)$. We assume that $\mathcal{W}(P_{n-1}, e) \neq \emptyset$. In particular, $e \equiv 0$ modulo k . Note also that there are no monomial of degree e of the form $z_1 R(z_2, \dots, z_n)$ as $\deg_P R \equiv 0$ modulo k . This implies that the generic hyperplane section g^H of any $g \in \mathcal{W}(P, e)$ is contained in $\widetilde{\mathcal{W}}(P_{n-1}, e) = \widetilde{\mathcal{W}}(P_{n-1}/k, e/k)$ where $P_{n-1}/k = {}^t(1, \dots, 1)$. Thus its principal part corresponds to a smooth projective hypersurface in \mathbb{P}^{n-2} . As in Remark 6, any following generic hyperplane sections correspond to smooth projective hypersurfaces of degree e/k . Thus $\mathcal{W}(P, e)$ has a unique μ^* -constant stratum and it is connected.

4.3. Case (W2). Consider $P = {}^t(p_1, \dots, p_n)$ with $p_1 \leq \dots \leq p_n$ as before and assume that $p_1 = \dots = p_s < p_{s+1}$ with $s \geq 2$. Let $\widetilde{\mathcal{W}}(P_j, e)'$ be the image of $\mathcal{W}(P_n, e)$ by the sequences of generic hyperplane sections in the space of polynomials of z_{n-j+1}, \dots, z_n .

(b) (**Conjecture**) For any $g \in \mathcal{W}(P_n, e)$ and for any $(n-j)$ -dimensional generic hyperplane H with $j < s$, the face function $(g^H)_{P_{n-j}}$ is contained in $\mathcal{W}(P_{n-j}, e)$. This means $(g^H)_{P_{n-j}}$ has an isolated singularity.

Note that $(n-j)$ -dimensional generic hyperplane g^H is the same as the j -times repeated generic hyperplane sections of g .

Remark 13. For $g \in \mathcal{W}(P_{n-j+1}, e)$, the generic hyperplane section g^H has an isolated singularity at the origin but the face function $(g^H)_{P_{n-j}}$ may not have an isolated singularity if (b) does not hold. We do not know if for $g \in \widetilde{\mathcal{W}}(P_{n-j+1}, e)$ and if there exists some generic hyperplane H such

that $(g^H)_{P_{n-j}}$ has isolated singularity, the assertion holds for any generic hyperplane in the sense of Teissier [17]. This is related with conjecture (b2) at the end of this paper.

Theorem 14. *Assume $p_1 = \dots = p_s < p_{s+1} \leq \dots \leq p_n$ with $s \geq 2$. Under the above conjecture (b), we have the sequence of generic hyperplane sections*

$$(GHS) : \mathcal{W}(P, e) \longrightarrow \widetilde{\mathcal{W}}(P_{n-1}, e) \dots \longrightarrow \widetilde{\mathcal{W}}(P_{n-s+1}, e)$$

with $P_{n-s+1} = {}^t(p_s, p_{s+1}, \dots, p_n)$. The image of one connected component of μ^* -constant strata of $\widetilde{\mathcal{W}}(P_j, e)$ is contained in a connected component of μ^* -constant strata of $\widetilde{\mathcal{W}}(P_{j-1}, e)$ for $j \geq n - s + 2$.

This means cardinality of $\overline{\mathcal{W}}_c(P, e)$ is less than or equal to the number connected components of μ^* -constant strata of $\widetilde{\mathcal{W}}(P_{n-s+1}, e)$.

Proof. The assertion follows from (b) and Lemma 1. \square

Example 15. *Let $a \geq 2$ and k be positive integers. Consider the weight vector $P = {}^t(1, \dots, 1, a, a+1)$ and $e = (a+1)(ak+1)$ and consider the space $\mathcal{W}(P, e)$ of weighted homogeneous polynomials of degree e with respect to P having an isolated singularity at the origin. A generic hyperplane section is defined by substituting $z_1 \mapsto \sum_{j=2}^n c_j z_j$ for generic coefficients c_2, \dots, c_n and the restriction of f to the hyperplane $H = \{z_1 - \sum_{j=2}^n c_j z_j = 0\}$ is denoted by f^H . We say that a hyperplane H is Milnor number generic if $\mu(f^H)$ is minimal among all hyperplane sections. Milnor number generic hyperplanes are easier to be handled for our purpose. A generic hyperplane in the sense of Teissier [17] is Milnor number generic.*

For a given partition $A = \{\nu_1, \dots, \nu_\ell\} \in \mathcal{P}(m_1)$, $0 \leq m_1 \leq k$ and mutually distinct non-zero complex numbers $\{a_1, \dots, a_\ell\}$ and $b \neq 0$, we associate a weighted homogeneous polynomial of degree e of variables z_{n-1}, z_n with respect to $P_2 = {}^t(a, a+1)$ with fixed generic coefficients b, a_1, \dots, a_ℓ :

$$(9) \quad h_A(z_{n-1}, z_n) = b z_n^{1+ma} \prod_{j=1}^{\ell} (z_n^a - a_j z_{n-1}^{a+1})^{\nu_j}, \text{ where } m = k - m_1.$$

If $m_1 = 0$, $\mathcal{P}(0) = \{\emptyset\}$ and we define $h_\emptyset(z_{n-1}, z_n) = z_n^{1+ka}$. We define for any $j \leq n-2$,

$$(10) \quad \begin{aligned} g_{j,A}(\mathbf{z}_j) &= g'_{j,A}(\mathbf{z}_j) + h_A(z_{n-1}, z_n), \quad g'_{j,A}(0, \dots, 0, z_{n-1}, z_n) = 0, \\ f_{j,A}(\mathbf{z}_j) &= \sum_{i=j}^{n-2} (t_i z_i^e + s_i z_i z_{n-1}^{(a+1)k+1}) + h_A(z_{n-1}, z_n) \end{aligned}$$

where $\mathbf{z}_j = (z_j, \dots, z_n)$, t_i, s_i are generic and $g_{j,A}, f_{j,A}$ are weighted homogeneous polynomials of degree e with respect to P_{n-j+1} . Furthermore $f_{j,A}$ have an isolated singularity at the origin. Put

$$S_k(z_{n-1}, z_n) := \frac{\partial g'_{j,A}}{\partial z_k}(0, \dots, 0, z_{n-1}, z_n), \quad k = j, \dots, n-2.$$

This implies $g'_{j,A}(\mathbf{z}_j) = \sum_{k=j}^{n-2} z_k S_k(z_{n-1}, z_n) + R_j(\mathbf{z}_j)$ where $R_j \equiv 0$ modulo \mathcal{M}_j^2 and \mathcal{M}_j is the ideal (z_j, \dots, z_{n-2}) . Consider the two-dimensional coordinate subspace $L := \{(0, \dots, 0, z_{n-1}, z_n)\}$. We do not assume $g_{j,A}$ has an isolated singularity at the origin but we assume that $g_{j,A}$ satisfies the following condition.

Assume $g(\mathbf{z}_j)$ is weighted homogeneous polynomial of degree e with respect to P_{n-j+1} such that $g^L = h_A$. We say g satisfies (FC) condition if the following condition is satisfied.

(FC): $g(\mathbf{z}_j)$ has no singularity on $L \setminus \{0\}$ as a function of variables $\{z_j, \dots, z_n\}$.

Note that this condition is equivalent to the non existence of the common solution of $T_j(z_{n-1}, z_n) = \dots = T_{n-2}(z_{n-1}, z_n) = 0$ with $\Sigma(h_A) \setminus \{(0, 0)\}$ where $\Sigma(h_A)$ is the set of the singular points of $h_A = 0$ and $T_k(z_{n-1}, z_n) := \frac{\partial g}{\partial z_k}(0, \dots, 0, z_{n-1}, z_n)$, for $k = j, \dots, n-2$. (FC) condition for $j = n-2$ corresponds to (\star) on page 5 in the case $n = 3$. $f_{j,A}$ is a typical such example.

Observation 16. For any $g_{j,A}$, $j < n-2$ and let $g_{j,A}^H$ be a generic hyperplane section of $g_{j,A}$. Then the face function $(g_{j,A}^H)_{P_{n-j}}$ also satisfies the face condition (FC) and $(g_{j,A}^H)_{P_{n-j}}(0, \dots, 0, z_{n-1}, z_n) = h_A(z_{n-1}, z_n)$ (stays unchanged). Let g_A be a polynomial of z_{n-2}, z_{n-1}, z_n obtained by a chain of generic hyperplane sections of $g_{1,A}$. Then g_A satisfies (FC) condition and the generic plane section of g_A is a plane curve with an isolated singularity at the origin and its topological equivalence class is determined only by h_A or equivalently A by the argument in the proof of Theorem 5. See the proof of Theorem 2, [16].

Proof. It is enough to observe that

$$(g_{j,A}^H)_{P_{n-j}}(\mathbf{z}_{j+1}) \equiv h_A(z_{n-1}, z_n) + \sum_{k=j+1}^{n-2} z_k S'_k(z_{n-1}, z_n) \text{ modulo } \mathcal{M}_{j+1}^2$$

and S'_k is a generic linear combination of S_j, \dots, S_{n-2} . \square

In the above observation, it is not necessary for g_A to have an isolated singularity at the origin.

Theorem 17 (Generalization of Theorem 5). *Under the conjecture (b), for $P = {}^t(1, \dots, 1, a, a+1)$ and $e = (a+1)(ak+1)$ as above, there is a correspondence $\psi : \mathcal{P}^{(tot)}(k) \rightarrow \overline{\mathcal{W}}_c(P, e)$ which is bijective.*

Proof. For any $A \in \mathcal{P}^{(tot)}(k)$, we choose $f_{1,A}$ as (10). We define $\psi(A)$ by $[f_{1,A}]$. Here $[f_{1,A}]$ is the connected component of the μ^* -constant strata which contains $f_{1,A}$. For a fixed $A = \{\nu_1, \dots, \nu_\ell\} \in \mathcal{P}(m_1) \subset \mathcal{P}^{(tot)}(k)$ with

$m_1 = \sum_{j=1}^{\ell} \nu_j$, define

$$\mathcal{W}(P_j, e, A) = \{f \in \mathcal{W}(P_j, e) \mid f(0, \dots, 0, z_{n-1}, z_n) = h_A(z_{n-1}, z_n, \mathbf{x}), \exists \mathbf{x}\}$$

where $\mathbf{x} = (x_0, \dots, x_{\ell})$, $\mathbf{x} \in \mathbb{C}^{*(\ell+1)}$, $x_j \neq x_i$, $i \neq j$, $i, j \geq 1$, $P_j = {}^t(p_{n-j+1}, \dots, P_n)$, and

$$h_A(z_{n-1}, z_n, \mathbf{x}) := x_0 z_n^{1+m_1} \prod_{j=1}^{\ell} (z_n^a - x_j z_{n-1}^{a+1})^{\nu_j}.$$

Note that μ^* of $f \in \mathcal{W}(P, e, A)$ is uniquely determined by A . By Observation 16, $\mathcal{W}(P, e, A)$ is mapped into $\widetilde{\mathcal{W}}(P_3, e, A)$ by a sequence of generic hyperplane sections and by Theorem 6, [16], the composition

$$\Psi : \mathcal{P}^{(tot)}(k) \xrightarrow{\psi} \overline{\mathcal{W}}_c(P, e) = \overline{\mathcal{W}}_c(P_n, e) \rightarrow \overline{\mathcal{W}}_c(P_{n-1}, e) \rightarrow \dots \rightarrow \overline{\mathcal{W}}_c(P_3, e)$$

is bijective. Thus it is enough to show that the composition map $\psi' : \overline{\mathcal{W}}_c(P, e) \rightarrow \overline{\mathcal{W}}(P_3, e)$ is bijective. It is clearly surjective. To prove the injectivity of ψ' , it is enough to show that $\mathcal{W}(P, e, A)$ is connected. We fix $h_A, f_{1,A}$ as in (9) and (10). We will show that $\forall f \in \mathcal{W}(P, e, A)$ can be connected to some $f_{1,A}$ in $\mathcal{W}(P, e, A)$. Take an arbitrary $f \in \mathcal{W}(P, e, A)$. Assume

$$f(0, \dots, 0, z_{n-1}, z_n) = h_A(z_{n-1}, z_n, \mathbf{x}_0), \exists \mathbf{x}_0 \in \mathbb{C}^{\ell+1}.$$

Let Δ_A be the supporting edge of h_A and let $\Gamma_A = \Gamma(f_{1,A})$. We assume that the integral points on $\Gamma_A \setminus \Delta_A$ are $\{\mathbf{n}_1, \dots, \mathbf{n}_N\}$. These points are independent of A . A generic polynomial $g \in \mathcal{W}(P, e, A)$ is written as

$$g(\mathbf{z}, \mathbf{c}, \mathbf{x}) = \sum_{i=1}^N c_i \mathbf{z}^{\mathbf{n}_i} + h_A(z_{n-1}, z_n, \mathbf{x}).$$

Here $\mathbf{c} = (c_1, \dots, c_N) \in \mathbb{C}^N$ and $\mathbf{x} = (x_0, \dots, x_{\ell}) \in \mathbb{C}^{\ell+1}$. Let $\Xi := \cup_{i \neq j} L_{ij}$ where

$$L_{ij} := \{\mathbf{x} \in \mathbb{C}^{\ell+1} \mid x_i = x_j\}, \quad 1 \leq i, j \leq \ell, \quad i \neq j,$$

Consider the set

$$\Omega(\mathcal{P}, e, A) := \{(\mathbf{c}, \mathbf{x}) \in \mathbb{C}^N \times (\mathbb{C}^{*(\ell+1)} \setminus \Xi) \mid g(\mathbf{z}, \mathbf{c}, \mathbf{x}) \text{ is Newton non-degenerate on any face } \Delta' \neq \Delta_A \text{ of } \Gamma_A.\}$$

Here Δ' can be Γ_A itself. Then $\Omega(P, e, A)$ is a Zariski open set. See the appendix, [12] and Theorem 6.1, [7]. We may assume that $f \in \Omega(P, e, A)$ by a small perturbation if necessary. For the definition of Newton non-degeneracy, see [7, 12, 14]. It is well-known that a Zariski open set is path-connected and we can take a piecewise linear path $(\mathbf{c}(s), \mathbf{x}(s))$, $0 \leq s \leq 1$ so that $f(\mathbf{z}, \mathbf{c}(0), \mathbf{x}(0)) = f_{1,A}(\mathbf{z})$ and $f(\mathbf{z}, \mathbf{c}(1), \mathbf{x}(1)) = f(\mathbf{z})$. Now we give a simple proof for this assertion for reader's convenience. Consider the pencil

$$f_{\tau}(\mathbf{z}) = \tau f(\mathbf{z}, \mathbf{c}(1), \mathbf{x}(1)) + (1 - \tau) f_{1,A}(\mathbf{z}), \quad \tau \in \mathbb{C}.$$

Note that $f_0 = f_{1,A}$ and $f_1 = f$. By the Zariski openness, there exists a finite set $B \subset \mathbb{C}$ such that $f_\tau \in \Omega(P, e, A)$ for any $\tau \in \mathbb{C} \setminus B$. Thus we can take a piecewise linear path $\phi : [0, 1] \rightarrow \mathbb{C} \setminus B$ with $\phi(0) = 0, \phi(1) = 1$ in the complex plane \mathbb{C} and define $f_s := f_{\phi(s)}, 0 \leq s \leq 1$. This gives a desired path from f to $f_{1,A}$ in $\mathcal{W}(P, e, A)$ and the assertion is proved. \square

Remark 18. Note that the number of the μ^* -constant connected components of $\widetilde{\mathcal{W}}(P_2, e)''$ and $\widetilde{\mathcal{W}}(P_2, e)'$ (=the image of $\widetilde{\mathcal{W}}(P_n, e)$ by generic hyperplane sections) are different in general. Here $\widetilde{\mathcal{W}}(P_2, e)''$ is the set of polynomials f which have an isolated singularity at the origin and f_{P_2} is P_2 -principal part of a generic section of a polynomial in $\mathcal{W}(P_3, e)$. Consider, for example, $P_3 = {}^t(1, 4, 5)$ and $k = 2$ and $e = 45$. By Theorem 5, the number of the connected component of $\widetilde{\mathcal{W}}(P_2, 45)'$ is $4 = |\mathcal{P}^{(tot)}(k)|$. Corresponding $\mu^{(2)}$ can be 82, 85, 85, 88. Consider a monomial $z_{n-2}^a z_{n-1}^b z_n^c$ of degree $e' \geq e$ in $f \in \widetilde{\mathcal{W}}(P_3, e)'$ with $a \geq 1$. This gives monomial $z_{n-1}^{a+b} z_n^c$ of degree $e' + 3a \geq 48$ under a hyperplane section. (By the same argument, if f has already a monomial $z_{n-1}^{b'} z_n^{c'}$, as f comes from some $g \in \mathcal{W}(P_n, e)$ after $(n-3)$ times generic hyperplane sections, we can easily see that $4b' + 5c' \geq 48$.) Thus in $\widetilde{\mathcal{W}}(P_2, 45)'$, any lowest degree monomial whose degree is greater than 45 has P_2 -degree 48. As $\widetilde{\mathcal{W}}(P_2, 45)'' \setminus \widetilde{\mathcal{W}}(P_2, 45)'$ has two monomials $z^6 y^4, z^7 y^3$ which have P_2 -degrees 46 and 47 respectively with $y = z_{n-1}$ and $z = z_n$, $\mu^{(2)}$ can take also 83 and 84. We can take for example, polynomials $g_1, g_2 \in \widetilde{\mathcal{W}}(P_2, 45)'' \setminus \widetilde{\mathcal{W}}(P_2, 45)'$ as

$$g_1(y, z) = z(z^4 - y^5)^2 + z^6 y^4, \quad g_2(y, z) = z(z^4 - y^5)^2 + z^7 y^3.$$

Without the conjecture (b), we have

Proposition 19. Assume that $P = {}^t(1, \dots, 1, a, a+1)$ and $e = (a+1)(ak+1)$. Then the cardinality of $\overline{\mathcal{W}}_c(P, e)$ is greater than or equal to $|\mathcal{P}^{(tot)}(k)|$.

Proof. For any $A \in \mathcal{P}^{(tot)}(k)$, we consider the mapping $A \mapsto [f_{1,A}]$. As the generic hyperplane sections of $f_{1,A}$ satisfy (FC) condition and Observation 16 says that their generic two-dimensional sections are isomorphic to the plane section of $f_{n-2,A}$. Thus assertion follows. \square

Remark 20. In the situation of Theorem 17, the following sections are Milnor number generic.

$$f_{1,A} \xrightarrow{z_1=0} f_{2,A} \xrightarrow{z_2=0} f_{3,A} \xrightarrow{z_3=0} \dots \xrightarrow{z_{n-3}=0} f_{n-2,A}.$$

As a problem which is related to Lemma 1 and Conjecture (b), we give another conjecture:

(b2) (**Conjecture**) Suppose we have a μ -constant family $g_t = f_t + R_t, 0 \leq t \leq 1$ and f_t are weighted homogeneous polynomials of variables z_1, \dots, z_n of degree e with respect to a weight vector $P = {}^t(p_1, \dots, p_n)$ and R_t is a family of polynomials with $\deg_P R_t > e, \forall t$. Assume that $f_t, t \neq 0$ has isolated

singularity at the origin. Then f_0 has also an isolated singularity at the origin.

Correction. In our previous paper [16], §3.3, there was an error. The correct $\mu^{(2)}$ of $x^9 + z^3 + zy^3$ and $x^9 + z^3 + xy^4$ are 7 and 8 respectively.

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EMERITUS PROFESSOR, TOKYO INSTITUTE OF TECHNOLOGY, 152-8551 OHOKAYAMA,
MEGURO-KU, TOKYO

Email address: okamutsuo@gmail.com