A COMMENT ON BRIANÇON-SPEDER POLYNOMIAL

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Abstract. Briançon and Speder gave an example of a μ-constant family of weighted homogeneous polynomials for which μ^* is not constant. In this note we analyze this example. We study similar weighted homogeneous polynomials and determine the number of possible different topology of curves which are obtained as generic plane sections.

Contents

1. Introduction

The notion of μ^* invariants of an analytic function germ was introduced by B. Teissier [13]. He showed that a μ^*-constant family $f_t(\mathbf{z})$ is equivalent to the Whitney regularity of the canonical stratification associated with the family, and thus under this condition, the local links are topologically isomorphic. On the other hand, for a family of weighted homogeneous polynomials with isolated singularities, the diffeomorphism type of the link is constant without assuming μ^* constancy as it has a uniform stable radius (Theorem 3.2, Chapter 1 [10], Lemma 2 [12]).

Let $f(x,y,z)=\sum_\nu a_\nu x^{\nu_1}y^{\nu_2}z^{\nu_3}$ be a weighted homogeneous polynomial of degree e under the weight vector $P={}^t(p_1,p_2,p_3)$ with isolated singularity at the origin. We denote the space of such weighted homogeneous polynomials as $\mathcal{W}(P;e)$. Briançon and Speder observed that for $P={}^t(1,2,3)$ and

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$e = 15$, there are polynomials $f_0, f_1 \in \mathcal{W}(P, e)$ which have different μ^* invariants ([1]). Recall that μ^* invariants consist of three integers $(\mu^{(3)}, \mu^{(2)}, \mu^{(1)})$ where $\mu^{(3)}$ is the Milnor number of f and $\mu^{(2)}$ is the Milnor number of a generic plane section and $\mu^{(1)}$ is the multiplicity minus 1 (see [7] for Milnor number). First we show that there are two more classes of polynomials in $\mathcal{W}(P, 15)$ with different local embedded topology of the generic plane section. Then generalizing this observation, we consider weighted homogeneous polynomials in $\mathcal{W}(P, e)$ with weight vector $P = {}^t(1, a, a+1)$, $a \geq 2$ and degree $e = (a+1)(ak+1)$ in §3. We show that the generic plane sections of $f \in \mathcal{W}(P, e)$ have $\sum_{i=0}^{k} |\mathcal{P}(i)|$ different topological types where $\mathcal{P}(i)$ is the set of partitions of the integer i and $|\mathcal{P}(i)|$ is the cardinality of $\mathcal{P}(i)$ (Theorem 2). In §4, we generalize Theorem 2 for more general weight vectors.

2. Briançon-Speder polynomials

Let us recall the Briançon-Speder family given in [1]:

$$f_t(x, y, z) = x^{15} + xy^7 + tzy^6 + z^5,\ t \in \mathbb{C}.$$

This is a weighted homogeneous polynomial of degree $e = 15$ under the weight vector $P = {}^t(1, 2, 3)$ whose Milnor number is given by $\mu = 364$ by Orlik-Milnor formula ([8]). Briançon and Speder have observed that $\mu^{(2)}(f_1) = 26$ and $\mu^{(2)}(f_0) = 28$. Actually the space $\mathcal{W}(P, 15)$ is quite high dimensional. In fact, the monomials of degree 15 with respect to P are listed as follows[1]:

$$y^6z, y^3z^3, z^5; xy^7, xy^4z^2, xyz^4; x^2y^5z, x^2y^2z^3; x^3y^6, x^3y^3z^2, x^3z^4;$$
$$x^4y^4z, x^4yz^3; x^5y^5, x^5y^2z^2; x^6y^3z, x^6z^3; x^7y^4, x^7yz^2; x^8y^2z; x^9y^3, x^9z^2;$$
$$x^{10}yz; x^{11}y^2; x^{12}z; x^{13}y; x^{15}.$$

Let us consider the family

$$F(x, y, z, s_1, s_2) = x^{15} + xy^7 + z(s_2y^3 + z^2)^2 + s_1y^6z.$$

It is easy to see that $F_{|s_1=1,s_2=0}$ is f_1 and $F_{|s_1=0,s_2=0} = f_0$. We are interested in polynomials with $s_1 = 0$, $s_2 \neq 0$. Thus we consider the polynomial

$$f_2 := F(x, y, z, 0, 1) = x^{15} + xy^7 + z(z^2 + y^3)^2.$$

Though f_2 is not Newton non-degenerate, it has an isolated singularity at the origin and its Milnor number is given by $\mu(f_2) = 364$. Take a generic hyperplane section $H : x = z + y$ of f_2 and denote it by $\hat{f}_2(y, z)$. Then

$$\hat{f}_2(y, z) = z(y^3 + z^2)^2 + y^8 + (\text{higher terms}),$$

where "higher terms" is in the sense of Newton boundary. Now it is easy to see, by a direct computation or using [9], that $\mu^{(2)}(f_2) = \mu(f_{2|H}) = 27$.

[1] weighted degree of a monomial $x^ay^bz^c$ with respect to weight vector $P = {}^t(p_1, p_2, p_3)$ is defined by $ap_1 + bp_2 + cp_3$

We easily see, after one toric modification $\pi : X \to \mathbb{C}^2$, using the toric coordinates (u, v) ($y = u^2 v$, $z = u^3 v^2$), that the pull-back of f_2 is given by

$$\pi^* \hat{f}_2(u, v) \equiv u^{15}((v+1)^2 + uv^8) \mod (u^{17})$$

and we see that $\hat{f}_2 = 0$ has a locally irreducible component with two Puiseux pairs and a smooth component which corresponds to the face function $y^6(z + y^2)$.

There is one more class. Consider

$$f_3(x, y, z) = x^{15} + z^3(z^2 + y^3) + xy^7.$$

Then $\mu(f_3) = 364$ and a generic plane section is given by

$$\hat{f}_3(y, z) = z^3(z^2 + y^3) + y^8 + (\text{higher terms})$$

with $\mu^{(2)} = 27$ which is the same with that of f_2. However the local topologies of the generic plane sections of $f_2 = 0$ and $f_3 = 0$ are different. Note that $\hat{f}_3$ is Newton non-degenerate and has two irreducible components C_1, C_2 where C_1 is defined by $z^2 + y^3 + (\text{higher terms}) = 0$ and C_2 is defined by $z^3 + y^5 + (\text{higher terms}) = 0$. Though the two polynomials f_2 and f_3 have the same μ^* invariant $(364, 27, 4)$, their generic plane sections have different local topologies. In §3, we will show that a similar property holds true for polynomials $\mathcal{W}({}^t(1, a, a+1), e)$ with $e = (a+1)(ak+1)$ (Theorem 2). In §4, we generalize Theorem 2 for more general weights.

3. Generalization of Briançon-Speder example

3.1. **Preliminary.** Consider an analytic function $f(z_1, \dots, z_n) = \sum_\nu a_\nu \mathbf{z}^\nu$ which defined by a convergent series. Here $\nu = (\nu_1, \dots, \nu_n)$ and $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$. Consider the convex hull $\Gamma^+(f)$ defined by the union of $\{\nu + \mathbb{R}^n_+ \,|\, a_\nu \neq 0\}$. The Newton boundary of f is defined by the union of compact boundary of $\Gamma^+(f)$ and we denote it by $\Gamma(f)$. Let $P = {}^t(p_1, \dots, p_n) \in \mathbb{Z}^n_+$ be a weight vector. Denote $\min\{P(\nu) = \sum_{i=1}^n p_i \nu_i \,|\, \nu \in \Gamma(f)\}$ by $d(P, f)$ and define $f_P := \sum_{\nu, P(\nu) = d(P,f)} a_\nu \mathbf{z}^\nu$. We say that f is Newton non-degenerate if for any strictly positive P (i.e., $p_i > 0, \forall i$), the polynomial mapping $f_P : \mathbb{C}^{*n} \to \mathbb{C}$ has no critical points.

We prepare a key lemma. Let a, b be positive coprime integers with $1 < a < b$ and let $P = {}^t(a, b)$ be a primitive integral weight vector. Consider a weighted homogeneous polynomial of degree e with respect to P:

$$h(y, z) = z^{1+am}(z^a - \alpha_1 y^b)^{\nu_1} \cdots (z^a - \alpha_\ell y^b)^{\nu_\ell},\ m \geq 0$$

where $\nu_1, \dots, \nu_\ell$ are positive integers and $\alpha_1, \dots, \alpha_\ell$ are mutually distinct non-zero complex numbers. Putting $m_1 := \sum_{j=1}^\ell \nu_j$, we have $e = m_1 ab + (1 + am)b$. Put $k = m + m_1$. Consider another weighted homogeneous polynomial $R(y, z)$ with respect to P whose degree is greater than e. Consider the polynomial

$$f(y, z) = h(y, z) + R(y, z)$$

and put $a_3 = d(P,R) - e$. By the assumption, a_3 is a positive integer. Here $d(P,R)$ is the weighted degree of R with respect to P. We assume that $\beta := d(P,R)/b$ is an integer and the following conditions are satisfied.

$$\{(y,z) \mid z^a - \alpha_j y^b = R(y,z) = 0\} = \{(0,0)\},\ \nu_j \geq 2 \tag{1}$$

$$R(y,0) = cy^\beta,\ c \neq 0, \text{ if } m > 0. \tag{2}$$

Then $f(y,z)$ has an isolated singularity at the origin and is almost Newton non-degenerate in the sense of [9]. Here as an exceptional case, $m = 0$, $\nu_1 = \cdots = \nu_\ell = 1$ and $R = 0$ is also considered. Let Γ be the Newton boundary of $f(y,z)$. Put μ be the Newton number of $\mathrm{Cone}(\Gamma, 0)$. See [5] for the definition. It is given as $\mu = (1+ka)(m_1 b - 1) + \beta a m + 1 = abk^2 + (b-a)k + ma_3$. Put $\mu^{(tot)} = \sum_{j=1}^{\ell}(\nu_j - 1)$.

Lemma 1. *(a) The Milnor number of f is given as $\mu(f) = \mu + a_3\mu^{(tot)}$.*
(b) If $R(y,0) \neq 0$, any polynomial $S(y,z) = \sum_\gamma a_\gamma y^{\gamma_1} z^{\gamma_2}$ for which (γ_1,γ_2) is above the Newton boundary of f can be added to f(y,z): $f_t(y,z) = f(y,z) + tS(y,z)$, $0 \leq t \leq 1$ without changing the local topology.
(c) If $m = 0$ and $R(y,0) = 0$, $S(y,z)$ as in (b) can be added without changing the local topology. Moreover the monomial cy^β can be added to R with any small coefficient c so that the condition (1) holds true.

Proof. Consider an admissible toric modification[2] $\pi : X \to \mathbb{C}^2$. Take a simplex $\sigma = \mathrm{Cone}(P,Q)$ with $Q = {}^t(c,d)$ with $ad - bc = 1$. Let (u,v) be the toric coordinates. Put $f_t(y,z) = f(y,z) + tS(y,z)$. Then $y = u^a v^c$, $z = u^b v^d$ and

$$\pi^* f(u,v) = u^e v^{e'} \left\{ \prod_{i=1}^{\ell} (v - \alpha_i)^{\nu_i} + c_1 u^{a_3} R(v^c, v^d) v^{-e'} \right\}$$

and $\pi^* f_t(u,v) \equiv \pi^* f(u,v)$ modulo (u^{e+a_3+1}) as $d(P,S) > d(P,R)$, $e' = d(Q,h)$, $a_3 := d(P,R) - d(P,h)$ are positive integer. Thus taking (u,v_i) with $v_i = v - \alpha_i$ as coordinates, the Newton principal part of $\pi^* f$ at $\rho_i : (u,v) = (0,\alpha_i)$ is given as $c' u^e (v_i^{\nu_i} + c_1' u^{a_3})$ where c', c_1' are non-zero constants by (1). Thus $\pi^* f$ is Newton non-degenerate at ρ_i. Other face corresponds to $c_2 z^{1+am} y^{m_1 b} + c_2' y^\beta$ with $c_2 = (-1)^\ell \prod_{j=1}^{\ell} \alpha_j, c_2' \neq 0$ which is Newton non-degenerate. Here we are assuming $R(y,0) \neq 0$. Let $\zeta_i(t)$ be the zeta function of $\pi^* f$ at ρ_i. Then $\deg \zeta_i(t) = (e + a_3)(\nu_i - 1)$ by Varchenko formula [14]. By Theorem 3.7 of [9], we get

$$\mu(f) = \mu - e\mu^{(tot)} + \sum_{j=1}^{\ell} \deg\, \zeta_j(t) = \mu + a_3 \mu^{(tot)}.$$

Now we consider the case $R(y,0) = 0$ and $m = 0$. (This includes the case $\forall \nu_j = 1$ and $m = R = 0$.) $\Gamma(f)$ has one face corresponding to $h(y,z)$ and it

[2] An admissible toric modification is a toric modification with respect to a regular simplicial cone subdivision Σ^* of the dual Newton digram $\Gamma^*(f)$ [10].

has one smooth component $z = 0$. The Newton number does not change. By adding small τy^β to R and putting $f_\tau(y, z) = f(y, z) + \tau y^\beta$, $\Gamma(f_\tau)$ gets a new face Ξ corresponding to $czy^{m_1 b} + \tau y^\beta$. However f_τ is Newton non-degenerate and as $\mathrm{Cone}(\Xi, 0)$ has Newton number 1, it does not change the Newton number $\nu(\mathrm{Cone}(\Gamma, 0))$. The change is that the component $z = 0$ changes to the smooth component corresponding to (topologically) $cz + \tau y^{\beta - m_1 b} = 0$. The assertion for the local topology follows from the constancy of $\mu(f_t)$ or $\mu(f_\tau)$ ([6]). □

3.2. **Main result.** Let $a \geq 2$ and k be positive integers. Consider the weight vector $P = {}^t(1, a, a + 1)$, put $e = (a + 1)(ak + 1)$ and consider the space $\mathcal{W}(P, e)$ of weighted homogeneous polynomials of degree e with respect to P having an isolated singularity at the origin[3]. There is a canonical subdivision of $\mathcal{W}(P, e)$ by μ^* or equivalently by $\mu^{(2)}$. Consider the equivalence relation $\overset{c}{\sim}$ in $\mathcal{W}(P, e)$ defined by $f \overset{c}{\sim} f'$ if and only if f and f' are in the same connected component of the μ^*-constant strata in $\mathcal{W}(P, e)$. See [2] for the description of μ^* constant strata. Let $\overline{\mathcal{W}}_c(P, e)$ be the quotient space, i.e., the set of connected components in the usual topology and for $f \in \mathcal{W}(P, e)$, let $[f]$ be the connected component which contains f. For any polynomial $f \in \mathcal{W}(P, e)$, the Milnor number $\mu(f)$ is given by $(e-1)(e/a-1)(e/(a+1)-1)$ by Orlik-Milnor [8]. Thus in our case, $\mu = ak(ak + k + 1)(a^2k + ak + 1)$. We consider the following monomials of degree e: x^e and

$$S_0 = \{z^{ak+1}, z^{a(k-1)+1}y^{a+1}, \dots, zy^{(a+1)k}\}, \text{ and}$$
$$S_1 := \{xy^{(a+1)k+1}, xy^{(a+1)(k-1)+1}z^a, \dots, xyz^{ak}\}.$$

Put $s = z^a$, $t = y^{a+1}$. Note that monomials in S_0 are expressed as zs^it^{k-i}, $0 \leq i \leq k$. Similarly the monomials in S_1 are xys^jt^{k-j}, $0 \leq j \leq k$. There are two typical Newton non-degenerate polynomials in $\mathcal{W}(P, e)$:

$$f_1 = x^e + zy^{(a+1)k} + z^{ak+1} \text{ and } f_0 = x^e + xy^{(a+1)k+1} + z^{ak+1}$$

whose $\mu^{(2)}$ are given as

$$\mu^{(2)}(f_1) = (e/a-1)(e/(a+1)-1) = a^2k^2 + ak^2 + k,\ \mu^{(2)}(f_0) = a^2k^2 + ak^2 + ak.$$

For an integer n, let $\mathcal{P}(n)$ be the set of partitions of n. A partition $A \in \mathcal{P}(m_1)$ corresponds to a collection of positive integers $A := \{\nu_1, \dots, \nu_\ell\}$ with $\nu_1 + \cdots + \nu_\ell = m_1$ and $\nu_j \geq 1, \forall j$. For a given a partition $A = \{\nu_1, \dots, \nu_\ell\} \in \mathcal{P}(m_1)$, $0 \leq m_1 \leq k$ and mutually distinct non-zero complex numbers $\{\alpha_1, \dots, \alpha_\ell\}$, we associate a weighted homogeneous polynomial of degree e with respect to P:

$$h_A(y, z) = z^{1+ma} \prod_{j=1}^{\ell} (z^a - \alpha_j y^{a+1})^{\nu_j}, \quad m = k - m_1. \tag{3}$$

[3]The space of weighted homogeneous polynomials of degree e is isomorphic to $\mathbb{C}^n$ where n is the number of monomials with weighted degree e and $\mathcal{W}(P, e)$ is a Zariski open subspace.

If $m_1 = 0$, $\mathcal{P}(0) = \{\emptyset\}$ and we define $h_\emptyset = z^{1+ka}$. We consider also polynomials

$$h'_A(x,y,z) = h_A(y,z) + xR(y,z)$$
$$f_A(x,y,z) = \lambda x^e + h'_A(x,y,z).$$

where $R(y,z)$ is a weighted homogeneous polynomial of degree $e-1$ with respect to the weight vector $P' := {}^t(a, a+1)$ such that

$$(\star) \quad \begin{cases} \{(y,z) \,|\, z^a - \alpha_j y^{a+1} = R(y,z) = 0\} = \{(0,0)\}, & \text{if } \nu_j \geq 2 \\ R(y,0) \neq 0, & \text{if } m \neq 0. \end{cases}$$

Note that f_A is a weighted homogeneous polynomial of degree e and if $m > 0$, $(\star)$ says that f_A has monomial $xy^{(a+1)k+1}$. This implies, with a generic λ, that f_A has an isolated singularity at the origin by Bertini theorem (see for example [4]). Thus $f_A \in \mathcal{W}(P,e)$. The topology of the generic plane section of f_A does not depend on the choice of $\alpha_1, \dots, \alpha_\ell$ and R. This is easily shown using an admissible toric modification and Lemma 1. In fact, generic plane sections are given by substituting $x = ay + bz$ in f_A, which is nothing but $h_A(y,z) + (ay+bz)R(y,z)$ for a fixed non-zero a, b. They are family of almost Newton non-degenerate family in the se sense of [9] and thus their Milnor numbers are constant and do not depend on $\{\alpha_1, \dots, \alpha_\ell\}$ or on the choice of $R(y,z)$. This implies f_A is μ^*-constant family and their topology is constant by [13].

Thus the class $[f_A] \in \overline{\mathcal{W}}_c(P,e)$ does not depend on the choice of $\alpha_1, \dots, \nu_\ell$ or $R(y,z)$. Let $\mathcal{P}^{(tot)}(k) = \cup_{i=0}^k \mathcal{P}(i)$. In this way, we get a correspondence $\psi : \mathcal{P}^{(tot)}(k) \to \overline{\mathcal{W}}_c(P,e)$. For $\emptyset \in \mathcal{P}(0)$, $\psi(\emptyset) = [\lambda x^e + z^{1+ak} + xy^{(a+1)k+1}]$ by definition. Now we are ready to state our main result.

Theorem 2. *(a) The correspondence $\psi : \mathcal{P}^{(tot)}(k) \to \overline{\mathcal{W}}_c(P,e)$ is bijective. Namely for any $f \in \mathcal{W}(P,e)$, there are unique m_1 and A with $0 \leq m_1 \leq k$, $A \in \mathcal{P}(m_1)$, so that $[f] = [f_A]$. In particular, the number of connected components of μ^*-constant strata which intersect with $\mathcal{W}(P,e)$ is $\sum_{m_1=0}^k |\mathcal{P}(m_1)|$.*
(b) For f_A described by (6), $\mu^{(2)}(f_A)$ is given as

$$\mu(\hat{f}_A) = a^2k^2 + a^2k + k + m(a-1) + (a-1)\mu^{(tot)}$$
$$= a^2k^2 + a^2k + ak - (a-1)\ell.$$

Thus $\mu^{(2)}(f_1) \leq \mu^{(2)}(f_A) \leq \mu^{(2)}(f_0)$ and the set of values of $\mu^{(2)}$ on $\mathcal{W}(P,e)$ is given by $\{a^2k^2 + a^2k + k + (a-1)\xi \,|\, \xi = 0, 1, \dots, k\}$.

Proof. To see the surjectivity of ψ, take any $f \in \mathcal{W}(P,e)$.
First we observe that z^{ak+1} has a non-zero coefficient in $f(0,y,z)$. Otherwise, f is singular on $x = y = 0$. This follows also by the result of J. Fernandez de Bobadilla and T. Pełka [3] (which says that μ-constant implies equi-multiplicity) as $ak+1$ is the multiplicity of f.

Consider the factorization

$$f(0,y,z) \quad = \quad cz^{1+ma}\prod_{j=1}^{\ell}(z^a-\alpha_j y^{a+1})^{\nu_j}. \tag{4}$$

Putting $m_1 = \sum_{j=1}^{\ell}\nu_j$, we have $k = m + m_1$. Put $h_A(y,z) = f(0,y,z)$. The decomposition $A \in \mathcal{P}(m_1)$ is uniquely determined by the factorization of $f(0,y,z)$. Put $J_f(y,z) := \frac{\partial f}{\partial x}|_{x=0}$ be the partial sum of f over monomials which are divisible by x but not by x^2. We take $R(y,z) = J_f(y,z)$. Note that $R = J_f$ is a weighted homogeneous polynomial of variables y,z having degree $e-1$ with respect to P'. Then f is expressed as as

$$f(x,y,z) = xJ_f(y,z) + h_A(y,z) + \lambda_0 x^e + x^2 S(x,y,z)$$

where S is a weighted homogeneous polynomial of degree $e-2$ with $S(x,0,0) = 0$. Let λ_0 be the coefficient of x^e in f. Using Bertini theorem and the openness of $\mathcal{W}(P,e)$, we may choose a λ sufficiently near λ_0 so that $f'(x,y,z) := f(x,y,z)+(\lambda-\lambda_0)x^e$ is in $\mathcal{W}(P,e)$ and $[f] = [f']$. We assume also $f_A(x,y,z) := \lambda x^e + h_A(y,z) + xR(y,z)$ has an isolated singularity at the origin choosing $|\lambda-\lambda_0|$ sufficiently small. We assert $[f_A] = [f']$.

To see this assertion, we take a generic plane $x = \delta z + \gamma y$. Then the plane section $\hat{f}_A$ of f_A is described as

$$\begin{aligned}\hat{f}_A(y,z) &= h_A(y,z) + \gamma y^{(a+1)k+2} + yJ_f(y,z) + (\text{higher terms})\\ &= z\prod_{j=1}^{\ell}(z^a-\alpha_j y^{a+1})^{\nu_j} + y^{(a+1)k+2} + yJ_f(y,z) + (\text{higher terms})\end{aligned}$$

Note that $x^2S(x,y,z)$ does not give any effect on the generic plane section and $\hat{f}'(y,z) \equiv \hat{f}_A(y,z)$ modulo (higher terms) which are above the Newton boundary of $\hat{f}_A$. Let $M_1,\dots,M_q$ be the monomials in $x^2S(x,y,z)$. Let $\mathcal{M}$ be the space of polynomials $g(x,y,z) = \lambda x^e + h_A(y,z) + xR(y,z) + \sum_{i=1}^{q} t_i M_i$ which have an isolated singularity at the origin. We identify $\mathcal{M}$ with an open subset of $\mathbb{C}^q$ by $g \mapsto (t_1,\dots,t_q)$. Let $(\tau_1,\dots,\tau_q)$ be the coefficient of $f'(x,y,z)$ i.e., $x^2S(x,y,z) = \sum_{i=1}^{q}\tau_i M_i$. Note that $(0,\dots,0),(\tau_1,\dots,\tau_q) \in \mathcal{M}$.

Assertion 3. *There is a piecewise analytic path $\rho(s)$, $0 \le s \le 1$ from $(0,\dots,0)$ to $(\tau_1,\dots,\tau_q)$ in $\mathcal{M}$. (This corresponds to a family of polynomials in $\mathcal{W}(P,e)$ from f_A to f'.)*

The existence assertion follows from the fact that $\mathcal{M}$ is a Zariski open subset of $\mathbb{C}^q$. See for example Theorem 6.1, [5] for Zariski openness. As $\sum_{i=1}^{q} t_i M_i$ does not effect to the generic plane section, $\mu^{(2)}$ is constant along generic plane sections of this path and the assertion $\psi(f_A) = \psi(f')$ follows by Lemma 1. Figure 1 shows the Newton boundary of $\hat{f}_A$. We now show

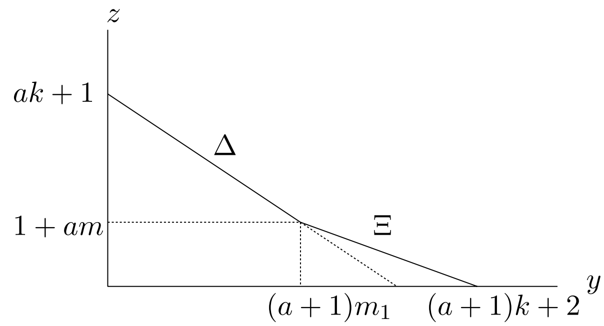

FIGURE 1. Newton boundary of $\hat{f}_A$

the assertion (b). By Lemma 1 and Theorem 3.7 of [9], we get

$$\mu(\hat{f}_A) = (\nu(\mathrm{Cone}(\Gamma(\hat{f}_A), 0)) - e\mu^{(tot)}) + \sum_{j=1}^{\ell} \deg \zeta_j(t).$$

Here $\nu(\Xi)$ is the Newton number of a cone Ξ ([5]). Note that

$$\begin{aligned}
&\nu(\mathrm{Cone}(\Gamma(\hat{f}_A), 0) = (ak+1)((a+1)m_1 - 1) + am((a+1)k+2) + 1 \\
&= \mu^{(2)}(f_1) + (a-1)m, \text{ and} \\
&\sum_{j=1}^{\ell} \deg \zeta_j(t) = \sum_{j=1}^{\ell} (e+a-1)(\nu_j - 1) = (e+a-1)\mu^{(tot)}
\end{aligned}$$

Here recall $f_1 = x^e + zy^{(a+1)k} + z^{ak+1}$. Thus $\mu^{(2)}(f)$, $f \in \mathcal{W}(P,e)$ takes minimal value $a^2k^2 + a^2k + k$ for f_1 and $\mu(\hat{f}_A) = \mu^{(2)}(f_1) + (a-1)m + (a-1)\mu^{(tot)}$. Note that we can also write this as $\mu(\hat{f}_A) = \mu^{(2)}(f_0) - (a-1)m_1 + (a-1)\mu^{(tot)} = \mu^{(2)}(f_0) - (a-1)\ell$ where $\mu^{(2)}(f_0) = a^2k^2 + ak^2 + ak$. This expression says $\mu^{(2)}(f)$ takes its maximal value for f_0. The assertion of the possible values of $\mu^{(2)}$ is also obvious from the above expression. For example, we can take $m = 0, \dots, k$ with $\mu^{(tot)} = 0$. This completes the proof of assertion (b).

Now we consider the injectivity of ψ. Recall that two plane curve germs C, C' are topologically equivalent if and only if they have same number of irreducible components $C = C_1 + \cdots + C_r$ and $C' = C_1' + \cdots + C_r'$ and under the obvious correspondence $C_j \mapsto C_j', 1 \le j \le r$, C_j and C_j' have the same Puiseux pairs and the intersection numbers coincide, that is, $C_i \cdot C_j = C_i' \cdot C_j'$ for any $i \neq j$. See [6] and Theorem 5.5.8 of [15]. Consider f_A which is described by (4). We consider an admissible toric modification for $\hat{f}_A$. The Newton boundary $\Gamma(f_A)$ has two faces Δ and Ξ as in Figure 1 and there are two weight vectors P, Q associated with Δ and Ξ respectively. Namely $P = {}^t(1, a, a+1)$ and $Q = {}^t(\frac{(a+1)m+2}{s}, \frac{1+am}{s})$ where $s = \gcd(1+am, (a+1)m+2)$. After an admissible toric modification $\pi : X \to \mathbb{C}^2$, $\hat{f}_A(y,z) = 0$ splits into several components which divide into two groups. The components in the first group are intersecting with the exceptional divisor $\hat{E}(P)$ at one of ρ_j, $j = 1, \dots, \ell$: they are topologically equivalent to the curve $C_j : v_j^{\nu_j} - c_j u^{a-1} = 0$. See the argument in the proof of Lemma 1. If $r_j := \gcd(\nu_j, a-1) > 1$, C_j has r_j irreducible components of type $v_j^{\nu_j/r_j} - u^{(a-1)/r_j} = 0$ at ρ_j. They have two Puiseux pairs determined by the weights if $\nu_j/r_j > 1$. Two Puiseux pairs are uniquely described by two weight vectors P and $P_j := {}^t((a-1)/r_j, \nu_j/r_j)$. See Remark 7.3, [11]. First Puiseux pair is always $(a, a+1)$. If $\nu_j = 1$, the corresponding component is smooth at X and they have only one Puiseux pair $(a, a+1)$. If $\nu_j > 1$ and $\nu_j = r_j$, r_j components at ρ_j are smooth and transversal to the exceptional divisor $\hat{E}(P)$ but these components are tangent each other. All of them have also one Puiseux pair $(a, a+1)$. If $\nu_j/r_j > 1$, r_j components have two Puiseux pairs where the first pair is always $(a, a+1)$. The second group of components of $\hat{f}_A(y,z) = 0$ corresponds to the curve $(\hat{f}_A)_\Xi(u,v) = 0$. It has s components with one Puiseux pair $(\frac{1+am}{s}, \frac{(a+1)m+2}{s})$. Obviously this weight vector is different from $(a, a+1)$. Recall that the weight vectors are normal primitive vectors orthogonal to the faces. If $m = 0$, there is only one component which is already smooth.

Now we are ready to show the injectivity. Suppose that $\hat{f}_A$ and $\hat{f}_B$ are topologically equivalent. We denote the corresponding factorization of f_A and f_B as

$$\begin{aligned}
f_A(0,y,z) &= c z^{1+m_A} \prod_{j=1}^{\ell_A} (z^a - \alpha_j y^{a+1})^{\nu_{A,j}}, \\
m_{A,1} &= \sum_{j=1}^{\ell_A} \nu_{A,j},\ m_A = k - m_{A,1}, \\
f_B(0,y,z) &= c z^{1+m_B} \prod_{j=1}^{\ell_B} (z^a - \alpha'_j y^{a+1})^{\nu_{B,j}}, \\
m_{B,1} &= \sum_{j=1}^{\ell_B} \nu_{B,j},\ m_B = k - m_{B,1}.
\end{aligned}$$

We need $m_A = m_B$ for the second group to be isomorphic by the above consideration. Then also by the above discussion, we can easily conclude $\ell_A = \ell_B$, $m_{A,1} = m_{B,1}$ and $A, B \in \mathcal{P}(m_1)$ and their partitions coincide (i.e., $A = B$, $\nu_{A,j} = \nu_{B,j}$, $1 \le j \le \ell_A$). This completes the proof of assertion (a). □

Remark 4. Let $\mathcal{M}(\mu^*)$, $\mu^* = (\mu^{(3)}, \mu^{(2)}, \mu^{(1)})$ be the set of strata i.e., space of polynomials of a given μ^* invariants with degree $\le N$ with N large enough. ($\mu^{(3)}$ is given by Orlik-Milnor formula.) If f_A and f_B are connected by a μ^*-constant path g_t with $g_0 = f_A, g_1 = f_B$ in the space of (not necessarily weighted homogeneous) polynomials, then the family $\hat{g}_t$ given by generic plane sections of the original family g_t is also a μ^*-constant family of plane curves from $\hat{f}_A$ to $\hat{f}_B$. Then the local topologies of $\hat{f}_A$ and $\hat{f}_B$ do not change ([6, 13]). Thus $A = B$ by Theorem 2. Thus the number of connected components of $\mathcal{M}(\mu^*)$ which intersect with $\mathcal{W}(P, e)$ is equal to the number of connected components of $\mathcal{M}(\mu^*) \cap \mathcal{W}(P, e)$. We propose here one conjecture.

Conjecture. Every $f \in \mathcal{M}(\mu^*)$ can be written as $f_A +$ (higher terms) for some $A \in \mathcal{P}(m_1)$, $0 \le m_1 \le k$, after a change of coordinates.

3.3. **Examples.** 1. $k = 1$. In this case, $\mathcal{W}(P, e)$ has only two components f_1 and f_0. For example, in the case $a = 2$, $f_1 = x^9 + z^3 + zy^3$, $f_0 = x^9 + z^3 + xy^4$. $\mu = 56$ and $\mu^{(2)}$ are $10, 11$ respectively.

2. $k = 2$. As $|\mathcal{P}^{(tot)}(2)| = |\mathcal{P}(0)| + |\mathcal{P}(1)| + |\mathcal{P}(2)| = 1 + 1 + 2 = 4$. This is exactly the same situation as in Briançon-Speder's example.

3. $k = 3$. As $|\mathcal{P}^{(tot)}(3)| = \sum_{i=0}^{3} |\mathcal{P}(i)| = 7$, we have 7 cases. For $a = 2$, we can take

$$m = 0: \begin{cases} A = \{1+1+1\}, f_1(x, y, z) = x^{21} + z^7 + zy^9 \\ A = \{2+1\},\ f_2(x, y, z) = x^{21} + z(z^2 + y^3)^2(z^2 + 2y^3) + xy^{10} \\ A = \{3\},\ f_3(x, y, z) = x^{21} + z(z^2 + y^3)^3 + xy^{10} \end{cases}$$

$$m = 1: \begin{cases} A = \{1+1\},\ f_4(x, y, z) = x^{21} + z^7 + z^5y^3 + z^3y^6 + xy^{10} \\ A = \{2\},\ f_5(x, y, z) = x^{21} + z^3(z^2 + y^3)^2 + xy^{10} \end{cases}$$

$$m = 2: \ A = \{1\},\ f_6(x, y, z) = x^{21} + z^7 + z^5y^3 + xy^{10}$$

$$m = 3: A = \{\emptyset\},\ f_7(x, y, z) = x^{21} + z^7 + xy^{10}$$

$\mu = 1140$ and $\mu^{(2)}$ are $57, 58, 59, 58, 59, 59, 60$ respectively.

4. Some more generalization

In the previous section, we have considered a certain restricted weight vector $P = {}^t(1, a, a+1)$. This is not so essential and we generalize this part. Consider weighted homogeneous polynomial of degree e with respect to a weight vector $P = {}^t(p_1, p_2, p_3)$. We denote the space of such polynomials by $\widetilde{\mathcal{W}}(P, e)$. Then $\mathcal{W}(P, e)$ is the subspace of $\widetilde{\mathcal{W}}(P, e)$ whose polynomials have an isolated singularity at the origin. We are interested in the case $\mathcal{W}(P, e) \ne \emptyset$. We assume the following condition throughout this section.

$$p_1 < p_2 < p_3,\ p_1 | e, p_3 | e,\ p_2 \not| e,\ \gcd(p_2, p_3) = 1. \tag{5}$$

Let $\mathcal{N}$ be the set of monomials $\mathbf{w}^{\mathbf{a}} = x^{a_1}y^{a_2}z^{a_3}$, $\mathbf{w} = (x, y, z)$, with weighted degree e, that is $d(P, \mathbf{w}^{\mathbf{a}}) = P(\mathbf{a}) = p_1a_1 + p_2a_2 + p_3a_3 = e$. By (5), monomials $x^{e/p_1}, z^{e/p_3} \in \mathcal{N}$. In fact, they are maximal and minimal degree monomials in $\mathcal{N}$ in the ordinary degree. A general polynomial f in $\widetilde{\mathcal{W}}(P, e)$ is written as $f = \sum_{\mathcal{N}} c_{\mathbf{a}}\mathbf{w}^{\mathbf{a}}$ where the sum is taken for $\mathbf{w}^{\mathbf{a}} \in \mathcal{N}$. Let $\mathcal{N}_0 := \{\mathbf{w}^{\mathbf{a}} \in \mathcal{N} | a_1 = 0\}$ and $\mathcal{N}_1 = \{\mathbf{w}^{\mathbf{a}} \in \mathcal{N} | a_1 = 1\}$. Consider two polynomials:

$$h_0(y,z) = \sum_{\mathcal{N}_0} c_{\mathbf{a}} y^{a_2} z^{a_3}$$
$$h_1(x,y,z) = \sum_{\mathcal{N}_1} c_{\mathbf{a}} x y^{a_2} z^{a_3}.$$

In the previous section, h_1 is denoted as $xJ_f(y,z)$. Note that h_0, h_1 are weighted homogeneous polynomials of degree e with respect to the weight vector P. Put $t = y^{p_3}$, $s = z^{p_2}$. Then h_0 and h_1 can be written as

$$h_0(y,z) = z^{m_0} \sum_{i+j=k} c_{0,i,j} t^i s^j,$$
$$(kp_2 + m_0)p_3 = e,\ m_0 < p_2$$
$$h_1(y,z) = x y^{n_2} z^{n_3} \sum_{\ell+m=k'} c_{1,\ell,m} t^\ell s^m$$
$$n_2 < p_3, n_3 < p_2,\ p_1 + p_2 n_2 + p_3 n_3 + k' p_2 p_3 = e.$$

We divide the situation into two cases.

(i) $n_3 = 0$
(ii) $n_3 > 0$ and $m_0 = 1$

Remark 5. *If $m_0 > 1$ and $n_3 > 0$, $z = x = 0$ is a singular locus of the any polynomial $f \in \widetilde{\mathcal{W}}(P,e)$. This follows from the fact that partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ do not contain any monomials of type y^ν. In the case $P = {}^t(1, a, a+1)$ and $e = (1+ka)(a+1)$, $m_0 = 1$ and z^{1+ka}, $xy^{3k+1} \in \mathcal{N}$, that is $m_0 = 1$ and $n_3 = 0$.*

4.1. **Case (i).** If $n_3 = 0$, a generic polynomial in $\widetilde{\mathcal{W}}(P,e)$ contains the monomial $xy^{n_2+p_3k'} \in \mathcal{N}$. For $f \in \mathcal{W}(P,e)$, $h_0(y,z) = f(0,y,z)$ is factorized as

$$h_0(y,z) = z^{m_0+mp_2} \prod_{j=1}^{\ell} (z^{p_2} - \alpha_j y^{p_3})^{\nu_j},\ \exists m,\ k \ge m \ge 0 \tag{6}$$

$$m_1 = \sum_{j=1}^{\ell} \nu_j,\ m = k - m_1 \tag{7}$$

and it gives a partition $A = \{\nu_1, \dots, \nu_\ell\} \in \mathcal{P}(m_1)$. If $m_0 > 1$ or $m_0 = 1$ and $m > 0$, the coefficient of $xy^{n_2+p_3k'}$ in f is non-zero as f has an isolated singularity at the origin. Here $p_1 + (n_2 + p_3k')p_2 = e$. Put $\beta = n_2 + p_3k'$. Then $(1+\beta)p_2 = e + p_2 - p_1$. In the case $n_3 = 0$, we can consider the correspondence $\psi : \mathcal{P}^{(tot)}(k) \to \overline{\mathcal{W}}_c(P,e)$ in exactly the same way as in the previous section and Theorem 2 is sharpened as follows.

Theorem 6. *Assume $n_3 = 0$.*
(a) The correspondence $\psi : \mathcal{P}^{(tot)}(k) \to \overline{\mathcal{W}}_c(P,e)$ is bijective. Namely for

any $f \in \mathcal{W}(P,e)$, there are unique m_1 and A with $0 \le m_1 \le k$, $A \in \mathcal{P}(m_1)$, so that $[f] = [f_A]$. In particular, the number of connected components of μ^*-constant strata which intersect with $\mathcal{W}(P,e)$ is $\sum_{m_1=0}^{k} |\mathcal{P}(m_1)|$.
(b) For $f_A(x,y,z) = h_A(y,z) + xy^\beta + \lambda x^{e/p_1}$ where h_A is defined by h_0 (6), the Milnor number of any generic plane section $\hat{f}_A$ is given as

$$\mu(\hat{f}_A) = (m_0 + p_2 k)(p_3 m_1 - 1) + (\beta + 1)(m_0 + m p_2 - 1) + 1 + (p_2 - p_1)\mu^{(tot)}$$

where $m = k - m_1$.

Proof. The assertion (a) is proved by exactly the same argument as that of Theorem 2. For the calculation of Milnor number, we use a suitable toric modification $\pi : X \to \mathbb{C}^2$ as in the proof of Theorem 2. In a suitable toric coordinates chart $\mathrm{Cone}\,(P,Q)$, the pull back of the generic plane section $\hat{f}_A$ is described as

$$\hat{f}_A(y,z) = (\delta y + \gamma z)y^\beta + f_A(y,z) + (\delta y + \gamma z)^{e/p_1}$$

$$\pi^* \hat{f}_A(u,v) \equiv u^e \left(\prod_{j=1}^{\ell} (v - \alpha_j)^{\nu_j} + c u^{p_2 - p_1} \right) \quad \mathrm{mod}\ (u^{e + p_2 - p_1 + 1}),\ c \ne 0.$$

On the exceptional divisor $\hat{E}(P)$, there are ℓ points $\{\rho_j := (0, \alpha_j) \,|\, j = 1, \dots, \ell\}$ in the toric coordinates which are the intersection of strict transform of $f = 0$ and $\hat{E}(P)$ and the local defining equation at ρ_j takes the form: $u^e(v_j^{\nu_j} + c u^{p_2 - p_1}) + (\text{higher terms}) = 0$ with $v_j = v - \alpha_j$. Thus putting $\zeta_j(t)$ be the local zeta function of $\pi^* \hat{f}_A$ at ρ_j,

$$\nu(\mathrm{Cone}(\Gamma(\hat{f}_A))) = m_0 + p_2 k)(p_3 m_1 - 1) + (\beta + 1)(m_0 + m p_2 - 1) + 1$$
$$\deg\, \zeta_j(t) = (e + p_2 - p_1)(\nu_j - 1)$$

and the calculation goes in the same way as that of Theorem 2. □

4.2. **Case (ii).** If $n_3 \ne 0$, $f \in \widetilde{\mathcal{W}}(P,e)$ has an isolated singularity only if $z = 0$ is a simple root of $f(0,y,z) = 0$ and it is factored as

$$f(0,y,z) = z \prod_{j=1}^{\ell} (z^{p_2} - \alpha_j y^{p_3})^{\nu_j}.$$

Otherwise, $x = z = 0$ is non-isolated critical locus of f. Thus we can only consider $\psi : \mathcal{P}(k) \to \overline{\mathcal{W}}_c(P,e)$. By the same argument, we have

Theorem 7. *Assume that $n_3 > 0$ and $m_0 = 1$. Then $\psi : \mathcal{P}(k) \to \overline{\mathcal{W}}_c(P,e)$ is a bijective correspondence.*

4.3. **Examples.** We give some examples for each case of §4.1 Case (i) and §4.2 Case (ii) described above.
Example of (i-1). Assuming first $m_0 = 1$, to have a monomial $xy^\beta \in \mathcal{N}$, we need a monomial $xy^{\beta'}$ with $\beta' < p_3$ with $d(P, xy^{\beta'}) = p_3$. That is $p_3 = p_1 + \beta' p_2$. We have two typical polynomials as in the subsection 3.2:

$f_1(x,y,z) = x^{e/p_1} + z(y^{kp_3} + z^{kp_2})$ and $f_0(x,y,z) = x^{e/p_1} + z^{p_2 k+1} + xy^{\beta'+kp_3}$. Here k is an arbitrary positive integer. Assume $p_1 = 1$ for simplicity. Then we have $P = (1, p_2, jp_2+1), j = 1, 2, \dots$. The case $p_2 = a, j = 1$ corresponds to $P = {}^t(1, a, a+1)$. If $p_1 > 1$, we need $p_1 | e = p_3(1+kp_2)$ and $p_3 = p_1 + \beta' p_2$. For example, $p_1 = 2$ and take p_2, k to be odd integers. As a simple example, we take, $P = {}^t(2,3,5), k = 3, e = 50$. As typical polynomials, we have $f_1(x,y,z) = x^{25} + z(y^{15} + z^9)$ and $f_0(x,y,z) = x^{25} + z^{10} + xy^{16}$ and $\mu(f_1) = \mu(f_0) = 3384$.

Example of (i-2). Assume that $m_0 > 1$ and there exists $\beta' < p_3$ such that $p_1 + \beta' p_2 = m_0 p_3$. For example, $P = {}^t(1,3,5), k = 2, e = 40, m_0 = 2$. A typical one is $f(x,y,z) = x^{40} + z^2(y^{10} + z^6) + xy^{13}$, $\mu = 3367$.

Example of (ii). In this case, we have one typical polynomial $f_1(x,y,z) = x^{e/p_1} + z(y^{kp_3} + z^{kp_2}) + xh_1(y,z)$ with $h_1(y,0) = 0$. For example, $P = {}^t(1,3,5)$, $k = 2$ and $e = 35$. f_1 can be $x^{35} + z(y^{10} + z^6) + xy^3 z^5$ and $\mu(f_1) = 2176$.

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