INTERLACING OF ZEROS OF PERIOD POLYNOMIALS

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Abstract. By a lot of previous work, it is known that the zeros of the period polynomial for a newform $f \in S_k(\Gamma_0(N))$ all lie on the circle $|z| = 1/\sqrt{N}$. In this paper we show that these zeros satisfy various interlacing properties for fixed N and varying k when either k or N is large. We also present a complete result when $N = 1$. Lastly, we establish the interlacing properties when k is fixed and N varies.

1. Introduction

Let $f \in S_k(\Gamma_0(N))$ be a normalized newform of even weight k and level N. Suppose the Fourier expansion of f is given by $f(z) = \sum_{n=1}^{\infty} a_n(f) q^n$. The associated L-function of f is defined as $L(f, s) := \sum_{n=1}^{\infty} a_n(f)/n^s$. The completed L-function,

$$\Lambda(f, s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f, s),$$

satisfies the functional equation $\Lambda(f, s) = \varepsilon(f)\Lambda(f, k - s)$. Here, $\varepsilon(f) = \pm 1$ is called the sign of f [8, Section 1]. The period polynomial associated to f is the degree $k - 2$ polynomial

$$r_f(z) = \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau.$$

Using L-functions we may rewrite the period polynomial as

$$r_f(z) = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=0}^{k-2} \frac{(2\pi i z)^n}{n!} L(f, k - n - 1),$$

or equivalently as

$$r_f(z) = i^{k-1} N^{-(k-1)/2} \sum_{n=0}^{k-2} \binom{k-2}{n} \left(i\sqrt{N} z\right)^n \Lambda(f, k - 1 - n). \tag{1.1}$$

The functional equation for $\Lambda(f, s)$ gives the following functional equation for $r_f(z)$:

$$r_f(z) = -i^k \varepsilon(f) \left(\sqrt{N} z\right)^{(k-2)/2} r_f\left(-\frac{1}{Nz}\right). \tag{1.2}$$

This implies that if ρ is a zero of $r_f(z)$ then so is $-1/(N\rho)$. An analogue of the Riemann hypothesis indicates that zeros of $r_f(z)$ should all lie on the center of symmetry: the circle $|z| = 1/\sqrt{N}$. When $N = 1$, this was proved by El-Guindy and Raj [4]. For a general level N, Jin et. al. [8] showed that all the zeros of $r_f(z)$ are on the circle $|z| = 1/\sqrt{N}$. This result was therein called as the "Riemann hypothesis" for period polynomials of modular forms. Actually, more information was obtained in

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[8] on the zeros of $r_f(z)$: for example, they showed that the zeros become equidistributed when either k or N is large.

This paper is devoted to further studying properties of the zeros of period polynomials, especially the relationship between zeros of period polynomials associated to different newforms. This work is motivated by works of [11] and [6], in which two types of interlacing properties were established for the zeros of Eisenstein series, respectively.

We now recall the definitions of these two interlacing properties. First, the standard or regular interlacing is given as follows.

Definition 1.1 ([7], Definition 1.3). *Let $I \subset \mathbb{R}$ be an interval. Let $m, n \in \mathbb{N}$ be such that $|m-n| \leq 1$. Suppose that $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are two strictly increasing ordered sets of points in I. Then, X and Y interlace if and only if the following conditions hold:*

(1) if $m > n$, then $x_i < y_i < x_{i+1}$ for all $1 \leq i \leq n$;
(2) if $m < n$, then $y_i < x_i < y_{i+1}$ for all $1 \leq i \leq m$;
(3) if $m = n$, then either
  (a) $x_i < y_i$ for all $1 \leq i \leq n$ and $y_i < x_{i+1}$ for all $1 \leq i \leq n-1$, or
  (b) $y_i < x_i$ for all $1 \leq i \leq n$ and $x_i < y_{i+1}$ for all $1 \leq i \leq n-1$.

Notice that Definition 1.1 is symmetric on the sets X and Y. Besides the standard interlacing property, we also consider the Stieltjes interlacing, which we now define.

Definition 1.2. *Let $I \subset \mathbb{R}$ be an interval. Let $m, n \in \mathbb{N}$ such that $m \geq n-1$. Suppose that $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are two strictly increasing ordered sets of points in I. Then*

(1) we say X Stieltjes interlaces with Y if there lies at least one element of X strictly between any two elements of Y,
(2) we say X strongly Stieltjes interlaces with Y if (1) is satisfied and furthermore, $x_1 < y_1$ and $x_m > y_n$.

We want to point out that the notions of standard interlacing and Stieltjes interlacing have their origins in the theory of orthogonal polynomials; see Szegő [13, Theorems 3.3.2–3.3.3]. We mention a few results that have established various interlacing properties for zeros of certain modular forms: Nozaki [11] showed the standard interlacing between zeros of Eisenstein series E_{k+12} and E_k; in [6] the Stieltjes interlacing was established between zeros of E_k and E_ℓ with $k > \ell$; the strong Stieltjes interlacing was shown in [5] for the zeros of j_m and j_n with $m > n$, where $j_m = mT_m(j - 744)$ is essentially the modular function obtained by applying the mth Hecke operator on the Klein j-invariant. To state our results we need to fix some notation. We first introduce the following notation of sample angles, following [8, Theorem 1.2].

Definition 1.3. *Let $k = 2m+2$ with $m \geq 1$. When $\varepsilon(f) = 1$, $\theta_{k,\ell}$ denotes the (unique) solution in the interval $[0, 2\pi)$ to the equation*

$$m\theta_{k,\ell} = \frac{\pi}{2} + \ell\pi + \frac{2\pi}{\sqrt{N}} \sin \theta_{k,\ell} \tag{1.3}$$

for $0 \leq \ell \leq 2m-1$.

Definition 1.4. *Let $k = 2m + 2$ with $m \geq 1$. When $\varepsilon(f) = -1$, $\phi_{k,\ell}$ denotes the (unique) solution in the interval $[0, 2\pi)$ to the equation*

$$m\phi_{k,\ell} = \ell\pi + \frac{2\pi}{\sqrt{N}} \sin \phi_{k,\ell}$$

for $0 \leq \ell \leq 2m - 1$.

We will show later that these sample angles are very close to the corresponding actual angles in A_f (Definition 1.5) below as long as N or k is sufficiently large. We next introduce the notation of angles for actual zeros of $r_f(z)$.

Definition 1.5. *By the result of [8] we will write the zeros of $r_f(z)$ as*

$$\frac{1}{i\sqrt{N}} e^{i\theta^*_{k,\ell}}, \quad \text{or} \quad \frac{1}{i\sqrt{N}} e^{i\phi^*_{k,\ell}},$$

depending on whether $\varepsilon(f) = 1$ or $\varepsilon(f) = -1$. Here, $k = 2m+2$, $0 \leq \ell \leq 2m-1$, $0 \leq \theta^*_{k,\ell}, \phi^*_{k,\ell} < 2\pi$, and each $\theta^*_{k,\ell}$ (resp. $\phi^*_{k,\ell}$) denotes the angle closest to the sample angle $\theta_{k,\ell}$ (resp. $\phi_{k,\ell}$) in Definition 1.3 (resp. Definition 1.4). Note that the angles $\theta^*_{k,\ell}$ or $\phi^*_{k,\ell}$ are the angles of the actual zeros of $r_f(z)$ plus $\pi/2$. For each newform $f \in S_k(\Gamma_0(N))$ we define:

(1) if $\varepsilon(f) = 1$, then

$$A_f := \{\theta^*_{k,\ell}\}_{\ell=0}^{m-1};$$

(2) if $\varepsilon(f) = -1$, then

$$A_f := \{\phi^*_{k,\ell}\}_{\ell=1}^{m-1}. \tag{1.4}$$

Remark. *(1) We only consider half of the actual zeros or actual angles because (1.2) implies that if $e^{i\theta^*}/i\sqrt{N}$ is a zero of $r_f(z)$ then so is $e^{i(2\pi-\theta^*)}/i\sqrt{N}$. This means that*

$$\theta^*_{k,2m-1-\ell} = 2\pi - \theta^*_{k,\ell} \text{ for } 0 \leq \ell \leq m-1 \quad \text{and} \quad \phi^*_{k,2m-\ell} = 2\pi - \phi^*_{k,\ell} \text{ for } 1 \leq \ell \leq m-1.$$

(2) When $\varepsilon(f) = -1$, it follows from (1.1) and the functional equation that $\pm 1/i\sqrt{N}$ are always zeros of $r_f(z)$. This is the reason why, in (1.4), we have removed the angles $\phi^*_{k,0} = 0$ because they are always the same as the sample angle $\phi_{k,0} = 0$.
(3) We shall show later (Lemma 3.10) that, as long as k or N is reasonably large, these actual angles $\theta^*_{k,\ell}$ or $\phi^*_{k,\ell}$ are well-defined, i.e. they are uniquely determined and are ordered by their indices ℓ.
(4) Strictly speaking, we should include f in the above definitions of various angles because both sample and actual angles are related to the specific newform f. However, to lighten the burden of notation we simply drop f throughout the paper.

We shall establish the interlacing properties of zeros of period polynomials in various scenarios. First, we consider the case when either k or N is large.

Theorem 1.6. *Suppose either $k' > k \geq 78$, or $N \geq 335464$ and $k' > k \geq 6$. Let $f, h \in S_k(\Gamma_0(N))$, $g \in S_{k+2}(\Gamma_0(N))$ and $f' \in S_{k'}(\Gamma_0(N))$ be newforms. Then*

(1) If $\varepsilon(g) = \varepsilon(f)$, the set A_g interlaces with the set A_f.
(2) If $\varepsilon(f') = \varepsilon(f)$, the set $A_{f'}$ strongly Stieltjes interlaces with the set A_f.
(3) If $\varepsilon(h) \neq \varepsilon(f)$, the set A_h interlaces with the set A_f.
(4) If $\varepsilon(f') = -1$, $\varepsilon(f) = 1$, then the set $A_{f'}$ Stieltjes interlaces with A_f.
(5) If $\varepsilon(f') = 1$, $\varepsilon(f) = -1$, then the set $A_{f'}$ strongly Stieltjes interlaces with A_f.

Remark. *(1) The conditions in parts (1) and (3) are not satisfied for* $N = 1$*, as* $\varepsilon(f) = 1$ *when* $k \equiv 0 \pmod 4$*, and* $\varepsilon(f) = -1$ *when* $k \equiv 2 \pmod 4$*, but could be satisfied for some other values of* N*.*
(2) It is not hard to see from definitions of interlacing that Theorem 1.6 part (2) implies part (1) by taking $k' = k + 2$*.*

Next, we consider the case when $k = 4$.

Theorem 1.7. *Let* $f \in S_4(\Gamma_0(N))$*,* $g \in S_6(\Gamma_0(N))$*, and* $f' \in S_{k'}(\Gamma_0(N))$ *be newforms with* $k' \geq 6$*. When* $N \geq \lceil 335464^{4.41} \rceil$*,*

(1) If $\varepsilon(g) = \varepsilon(f)$*, the set* A_g *interlaces with the set* A_f*.*
(2) If $\varepsilon(f') = \varepsilon(f)$*, the set* $A_{f'}$ *strongly Stieltjes interlaces with the set* A_f*.*
(3) If $\varepsilon(f') \neq \varepsilon(f)$ *and* $k' \geq 10$*, then the set* $A_{f'}$ *strongly Stieltjes interlaces with* A_f*.*

Remark. *(1) In Theorem 1.7 one has* $|A_f| = 1$ *when* $\varepsilon(f) = 1$*, and* $|A_f| = 0$ *when* $\varepsilon(f) = -1$*.*
(2) We have numerically verified Theorems 1.6 and 1.7 in small cases for $k' \leq 104$ *and* $N = 1, 2, 3, 10$ *using Pari/GP* [9, checking_pari.sage]*. Thus, it is natural to conjecture that Theorems 1.6 and 1.7 hold true in general; see the following Theorem 1.8 for a complete result when* $N = 1$*.*

Theorem 1.8. *Suppose* $k' > k \geq 12$*,* $k, k' \neq 14$*, and* $N = 1$*. Let* $f \in S_k(\Gamma_0(1))$*,* $f' \in S_{k'}(\Gamma_0(1))$ *be newforms. Then*

(1) If $\varepsilon(f') = \varepsilon(f)$*, the set* $A_{f'}$ *strongly Stieltjes interlaces with the set* A_f*.*
(2) If $\varepsilon(f') = -1$*,* $\varepsilon(f) = 1$*, then the set* $A_{f'}$ *Stieltjes interlaces with* A_f*.*
(3) If $\varepsilon(f') = 1$*,* $\varepsilon(f) = -1$*, then the set* $A_{f'}$ *strongly Stieltjes interlaces with* A_f*.*

Lastly, we also establish some interlacing results between $A_{f'}$ and A_f when $k' = k$ and $N' \neq N$ (that is in the level aspect); see Propositions 9.3, 9.6 and 9.9 for the precise statements.

An argument similar to that of [5, Corollary 6.1] or [6, Proposition 7.2], by counting the number of elements in A_f, reveals the following corollary on indivisibility between period polynomials. For brevity we only state it for $N = 1$.

Corollary 1.9. *Suppose* $k' > k \geq 12$*,* $k, k' \neq 14$*, and* $N = 1$*. Let* $f \in S_k(\Gamma_0(1))$*,* $f' \in S_{k'}(\Gamma_0(1))$ *be newforms. Then*

(1) if $\varepsilon(f') = \varepsilon(f) = 1$ *and* $2k > k' + 4$ *then* $r_f(z) \nmid r_{f'}(z)$*;*
(2) if $\varepsilon(f') = \varepsilon(f) = -1$ *and* $2k > k' + 6$ *then* $r_f(z) \nmid r_{f'}(z)$*;*
(3) if $\varepsilon(f') = -\varepsilon(f) = -1$ *and* $2k > k' + 8$*, then* $r_f(z) \nmid r_{f'}(z)$*;*
(4) if $\varepsilon(f') = -\varepsilon(f) = 1$*, then* $r_f(z) \nmid r_{f'}(z)$*.*

Proof. Proofs of (1)-(3) are similar, so we only present the proof of (1) here. Suppose on the contrary that $r_f(z) \mid r_{f'}(z)$. Then we have $A_f \subseteq A_{f'}$. On the other hand, by the Stieltjes interlacing between $A_{f'}$ and A_f established in Theorem 1.8, we know that $A_{f'}$ must have at least another $|A_f| - 1$ elements strictly between elements of A_f. Therefore, $|A_{f'}| \geq |A_f| + |A_f| - 1$. Since $\varepsilon(f') = \varepsilon(f) = 1$, we obtain $m' \geq 2m - 1$, or equivalently $(k' - 2)/2 \geq k - 3$, or $k' \geq 2k - 4$, a contradiction.

(4) In this case, since $i/\sqrt{N}$ (corresponding to the actual angle 0) is a zero of $r_f(z)$, by Lemma 3.11 (2) for $k' \geq 78$ and computer check for $k' < 78$ it is never a zero of $r_{f'}(z)$, so $r_f(z) \nmid r_{f'}(z)$. ☐

The main idea of proving various interlacing properties is as follows. The first step is to show these interlacing properties for the sets of sample angles. Then, we need to show that the distances between sample angles and actual angles are small enough for the sets $A_{f'}$ and A_f of actual angles to retain these properties. Lastly, there are two key ingredients in establishing the strong Stieltjes interlacing between $A_{f'}$ and A_f: one is to show that the distance between any two adjacent elements in $A_{f'}$ is always smaller than those of A_f, and the other is to make sure that the two extreme elements of $A_{f'}$ are outside the range of A_f.

We now give an outline of the paper. In Section 2, we will make explicit and strengthen the estimates in [8] on the distances between angles of the actual zeros of $r_f(z)$ in Definition 1.5 and the sample angles in Definitions 1.3 and 1.4. In Section 3 we will establish various bounds for the distance between sample angles and the distance between actual angles to be used throughout the paper. We will prove parts (1) and (2) of Theorem 1.6 when either N or k is large enough in Sections 4 and 5, respectively. Section 6 will treat the remaining parts of Theorem 1.6. Next, we will prove Theorem 1.7 in Section 7, and will show Theorem 1.8 in Section 8. Finally, we shall establish interlacing properties in the level aspect in Section 9.

2. Distances between actual and sample angles

Following [8], we will make explicit the constants involved in [8, Theorem 1.2], thus provide explicit bounds for the distances between the sample angles (Definitions 1.3-1.4) and actual angles (Definition 1.5). Let us first treat the case when $m \geq 2$, or equivalently $k = 2m+2 \geq 6$. By the functional equation, we see that [8, (1.6)]

$$r_f\left(\frac{z}{i\sqrt{N}}\right) = i^{k-1} N^{-(k-1)/2} \varepsilon(f) z^m \left(P_f(z) + \varepsilon(f) P_f\left(\frac{1}{z}\right)\right),$$

where

$$P_f(z) = (2m)! \left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1} L(f, 2m+1) Q_f(z),$$

and [8, (6.1)]

$$Q_f(z) = z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right) + S_1(z) + S_2(z) + S_3(z),$$

with

$$S_1(z) = z^m \sum_{j=1}^{m-1} \frac{1}{j!} \left(\frac{2\pi}{z\sqrt{N}}\right)^j \left(\frac{L(f, 2m+1-j)}{L(f, 2m+1)} - 1\right),$$

$$S_2(z) = -z^m \sum_{j=m}^{\infty} \frac{1}{j!} \left(\frac{2\pi}{z\sqrt{N}}\right)^j,$$

and

$$S_3(z) = \frac{1}{2(m!)^2} \left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1} \frac{\Lambda\left(f, \frac{k}{2}\right)}{L(f, 2m+1)}.$$

When $\varepsilon(f) = 1$, the zeros of the period polynomial $r_f(z)$ are located at $1/i\sqrt{N}$ times the zeros of $\mathrm{Re}(Q_f(z))$; when $\varepsilon(f) = -1$ the zeros of $r_f(z)$ are $1/i\sqrt{N}$ times the zeros of $\mathrm{Im}(Q_f(z))$, see [8, Section 7]. For $z = e^{i\theta}$ on the unit circle,

$$\left| Q_f(z) - \exp\left(im\theta + \frac{2\pi}{\sqrt{N}} e^{-i\theta}\right)\right| \leq |S_1(z)| + |S_2(z)| + |S_3(z)|.$$

By [8, (6.3)], (2.6), and the fact that [8, Section 6]

$$\Lambda\left(\frac{k}{2}\right) \leq \left(\frac{\sqrt{N}}{2\pi}\right)^{m+2} (m+1)!\zeta\left(\frac{3}{2}\right)^2,$$

we obtain

$$\begin{aligned}&|S_1(z)| + |S_2(z)| + |S_3(z)| \\ \leq& \frac{16}{5}\frac{1}{2^m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \frac{17}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1} + \frac{m+1}{2(m!)}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}\left(\frac{\zeta(\frac{3}{2})\zeta(\frac{5}{2})}{\zeta(5)}\right)^2,\end{aligned}$$

for all $m \geq 2$ and $N \geq 1$. For later application we define

$$B_{k,N} := \frac{16}{5}\frac{1}{2^m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right) - 1\right) + \left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}\left(\frac{17}{4}\frac{1}{(m-1)!} + \frac{m+1}{2(m!)}\left(\frac{\zeta(\frac{3}{2})\zeta(\frac{5}{2})}{\zeta(5)}\right)^2\right). \quad (2.1)$$

Thus, for all $z = e^{i\theta}$

$$\left|Q_f(z) - z^m \exp\left(\frac{2\pi}{z\sqrt{N}}\right)\right| = |E(z)| < B_{k,N},$$

where $E(z) = S_1(z) + S_2(z) + S_3(z)$. Recall that, for each $0 \leq \ell \leq m-1$, $\theta_{k,\ell} < \pi$ denotes a sample angle in (1.3) and $\theta^*_{k,\ell}$ denotes the angle of the closest actual zero (Definition 1.5). Note that $e^{i\theta^*_{k,\ell}}$ is also a zero of the real part of $Q_f(z)$ ([8, Section 7]):

$$\begin{aligned}\mathrm{Re}(Q_f(z)) &= \mathrm{Re}\left(\exp\left(im\theta + \frac{2\pi}{\sqrt{N}}(\cos\theta - i\sin\theta)\right)\right) + \mathrm{Re}(E(z)) \\ &= \exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta\right)\cdot\left(\cos\left(m\theta - \frac{2\pi}{\sqrt{N}}\sin\theta\right)\right) + \mathrm{Re}(E(z)).\end{aligned} \quad (2.2)$$

Define

$$\delta_{k,N} := \frac{\exp\left(\frac{2\pi}{\sqrt{N}}\right)\cdot\frac{\pi}{2}\cdot B_{k,N}}{m - \frac{2\pi}{\sqrt{N}}}. \quad (2.3)$$

Then, under the assumptions of Theorem 1.6, that is either $N \geq 335464$ and $k' > k \geq 6$, or $k' > k \geq 78$, we have

$$0 < \delta_{k,N} < \frac{289.596}{2^m\sqrt{N}}, \quad \text{or} \quad 0 < \delta_{k,N} < \frac{2.434 \times 10^7}{2^m\sqrt{N}},$$

respectively, by [9, zgap.sage, 4bigO.sage]. Also, note that under these assumptions

$$0 < \frac{4\pi}{\sqrt{N}}\cos\left(\frac{2\theta_{k,\ell} + \delta_{k,N}}{2}\right)\sin\left(-\frac{\delta_{k,N}}{2}\right) + m\delta_{k,N} < \frac{\pi}{2}. \quad (2.4)$$

Now, we want to show that for each $0 \leq \ell \leq 2m-1$

$$\left|\theta_{k,\ell} - \theta^*_{k,\ell}\right| < \delta_{k,N}. \quad (2.5)$$

It suffices to show that if we plug $\theta_{k,\ell} \pm \delta_{k,N}$ into (2.2) then there is a sign change. We first do this for $\theta_{k,\ell} + \delta_{k,N}$:

$$\begin{aligned}
&\cos\left(m\left(\theta_{k,\ell}+\delta_{k,N}\right)-\frac{2\pi}{\sqrt{N}}\sin(\theta_{k,\ell}+\delta_{k,N})\right)\\
&=\cos\left(\left(m\theta_{k,\ell}-\frac{2\pi}{\sqrt{N}}\sin\theta_{k,\ell}\right)+\frac{2\pi}{\sqrt{N}}(\sin\theta_{k,\ell}-\sin(\theta_{k,\ell}+\delta_{k,N}))+m\delta_{k,N}\right)\\
&=\cos\left(\frac{\pi}{2}+\ell\pi+\frac{2\pi}{\sqrt{N}}(\sin\theta_{k,\ell}-\sin(\theta_{k,\ell}+\delta_{k,N}))+m\delta_{k,N}\right) \quad (\text{by (1.3)})\\
&=\pm\sin\left(\frac{4\pi}{\sqrt{N}}\cos\left(\frac{2\theta_{k,\ell}+\delta_{k,N}}{2}\right)\sin\left(-\frac{\delta_{k,N}}{2}\right)+m\delta_{k,N}\right).
\end{aligned}$$

As $\sin(-\delta_{k,N}/2) \geq -\delta_{k,N}/2$, we get

$$\frac{4\pi}{\sqrt{N}}\cos\left(\frac{2\theta_{k,\ell}+\delta_{k,N}}{2}\right)\sin\left(-\frac{\delta_{k,N}}{2}\right)+m\delta_{k,N}\geq\left(m-\frac{2\pi}{\sqrt{N}}\right)\delta_{k,N}.$$

As for $0 \leq x \leq \frac{\pi}{2}$, $\sin x \geq 2x/\pi$, by (2.3) and (2.4), we have

$$\begin{aligned}
&\exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell}\right)\sin\left(\frac{4\pi}{\sqrt{N}}\cos\left(\frac{2\theta_{k,\ell}+\delta_{k,N}}{2}\right)\sin\left(-\frac{\delta_{k,N}}{2}\right)+m\delta_{k,N}\right)\\
&\geq\exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell}\right)\sin\left(\left(m-\frac{2\pi}{\sqrt{N}}\right)\delta_{k,N}\right)\\
&\geq\exp\left(\frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell}\right)\exp\left(\frac{2\pi}{\sqrt{N}}\right)\cdot B_{k,N}\\
&>|E(z)|\geq|\mathrm{Re}(E(z))|.
\end{aligned}$$

In conjunction with (2.2), this means that $\mathrm{Re}(Q_f(z))$ for $z = e^{i(\theta_{k,\ell}+\delta_{k,N})}/i\sqrt{N}$ has the same sign as $\cos\left(m(\theta_{k,\ell}+\delta_{k,N}) - \frac{2\pi}{\sqrt{N}}\sin(\theta_{k,\ell}+\delta_{k,N})\right)$. Similarly, repeating the above calculations for $\theta_{k,\ell} - \delta_{k,N}$, we get

$$\begin{aligned}
&\cos\left(m(\theta_{k,\ell}-\delta_{k,N})-\frac{2\pi}{\sqrt{N}}\sin(\theta_{k,\ell}-\delta_{k,N})\right)\\
&=\pm\sin\left(\frac{4\pi}{\sqrt{N}}\cos\left(\frac{2\theta_{k,\ell}-\delta_{k,N}}{2}\right)\sin\left(\frac{\delta_{k,N}}{2}\right)-m\delta_{k,N}\right)\\
&=\mp\sin\left(\frac{4\pi}{\sqrt{N}}\cos\left(\frac{2\theta_{k,\ell}-\delta_{k,N}}{2}\right)\sin\left(-\frac{\delta_{k,N}}{2}\right)+m\delta_{k,N}\right),
\end{aligned}$$

which has the opposite sign as $\cos\left(m(\theta_{k,\ell}+\delta_{k,N}) - \frac{2\pi}{\sqrt{N}}\sin(\theta_{k,\ell}+\delta_{k,N})\right)$. Thus, following the same argument as above we find that $\mathrm{Re}(Q_f(z))$ for $z = e^{i(\theta_{k,\ell}-\delta_{k,N})}/i\sqrt{N}$ has a sign opposite to $\cos\left(m(\theta_{k,\ell}+\delta_{k,N}) - \frac{2\pi}{\sqrt{N}}\sin(\theta_{k,\ell}+\delta_{k,N})\right)$. Therefore, we have established a sign change and thus have completed the proof of (2.5).

When $\varepsilon(f) = -1$, since $e^{i\phi^*_{k,\ell}}$ is a zero of the imaginary part of $Q_f(z)$ ([8, Section 7]), we obtain a similar bound for $0 \leq \ell \leq 2m-1$

$$\left|\phi_{k,\ell} - \phi^*_{k,\ell}\right| < \delta_{k,N}.$$

Thus, we have established explicit bounds for $\left|\theta_{k,\ell} - \theta^*_{k,\ell}\right|$ and $\left|\phi_{k,\ell} - \phi^*_{k,\ell}\right|$ when $k \geq 6$. To get a meaningful bound when $k = 4$, or equivalently when $m = 1$, we need to take a different approach, by making explicit the constants in [8, Section 3]. When $\varepsilon(f) = -1$, we already know that the only two zeros of $r_f(z)$ are given by $\pm 1/i\sqrt{N}$. On the other hand, when $\varepsilon(f) = 1$, from [8, Section 3] we know that the two zeros of $r_f(z)$ are located at $\pm e^{i\theta^*_{4,0}}/i\sqrt{N}$, where $0 < \theta^*_{4,0} < \pi$ and $\cos\theta^*_{4,0} = -\frac{\Lambda(f,2)}{\Lambda(f,3)}$. Now, we need to give some explicit estimates on the values of $\Lambda(f, 2)$ and $\Lambda(f, 3)$.

First, we give a lower bound of $\Lambda(f, 3)$. More generally, we have the following lower bound for $\Lambda(f, k/2 + 1)$.

Lemma 2.1. *Suppose $f \in S_k(\Gamma_0(N))$ is a normalized newform for $k \geq 4$. Then*

$$\Lambda\left(f, \frac{k}{2}+1\right) > \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+1} \Gamma\left(\frac{k}{2}+1\right) \frac{\zeta(3)^2}{\zeta\left(\frac{3}{2}\right)^2}, \quad \textit{and} \quad L(f, k-1) > \frac{\zeta(k-1)^2}{\zeta\left(\frac{k-1}{2}\right)^2}. \tag{2.6}$$

Proof. Since $k/2+1 > (k+1)/2$ is within the region of convergence of the Euler product of $L(f, s)$, we can write

$$\begin{aligned} L(f,s) &= \prod_{p\nmid N}\left(1 - a_p(f)p^{-s} + p^{k-1-2s}\right)^{-1} \prod_{p\mid N}\left(1 - a_p(f)p^{-s}\right)^{-1} \\ &= \prod_{p\nmid N}\left(1 - \alpha_p(f)p^{-s}\right)^{-1}\left(1 - \beta_p(f)p^{-s}\right)^{-1} \prod_{p\mid N}\left(1 - a_p(f)p^{-s}\right)^{-1}, \end{aligned}$$

where $|\alpha_p(f)| = |\beta_p(f)| = p^{(k-1)/2}$ by Deligne's Theorem if $p \nmid N$ and Li [10, Theorem 3] $|a_p(f)| \leq p^{(k-1)/2}$ if $p \mid N$. Therefore,

$$\begin{aligned} \Lambda\left(f, \frac{k}{2}+1\right) &= \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+1} \Gamma(k/2+1) L\left(f, \frac{k}{2}+1\right) \\ &\geq \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+1} \Gamma(k/2+1) \prod_{p\nmid N}(1 + p^{(k-1)/2}p^{-k/2-1})^{-2} \prod_{p\mid N}(1 + p^{(k-1)/2}p^{-k/2-1})^{-1} \\ &> \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+1} \Gamma(k/2+1) \prod_{p\nmid N}(1 + p^{-3/2})^{-2} \prod_{p\mid N}(1 + p^{-3/2})^{-2} \\ &= \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+1} \Gamma(k/2+1) \prod_{p}(1 + p^{-3/2})^{-2} \\ &= \left(\frac{\sqrt{N}}{2\pi}\right)^{k/2+1} \Gamma(k/2+1) \frac{\zeta(3)^2}{\zeta\left(\frac{3}{2}\right)^2}. \end{aligned}$$

A similar argument gives

$$L(f, k-1) > \frac{\zeta(k-1)^2}{\zeta\left(\frac{k-1}{2}\right)^2}.$$

This completes the proof. □

Next, we give an upper bound of $\Lambda(f,2)$. The original bound $\Lambda(f,2) \ll N^{5/4+\varepsilon}$ in [8, Section 3] which utilizes the Phrágmen-Lindelöf principle is inexplicit. Following [2, Lemma 2.4] we shall provide an explicit albeit weaker bound.

Lemma 2.2. *Let $f \in S_k(\Gamma_0(N))$ be a normalized newform, such that $\varepsilon(f) = 1$. Then*

$$\Lambda\left(f, \frac{k}{2}\right) \le 2\left(\frac{\sqrt{N}}{2\pi}\right)^{k/2} \Gamma\left(\frac{k}{2}\right) \times \left(2\sqrt{k}\ln(ek) + 9 \cdot 2^{k/2}\Gamma\left(\frac{k}{2}\right)\left(\frac{\sqrt{N}}{\pi}\right)^{3/4}\Gamma\left(\frac{3}{4}\right)\right). \tag{2.7}$$

Proof. Following the proof of [12, Theorem 3.66] or [3, Theorem 5.10.2], we obtain

$$\begin{aligned}
\Lambda\left(f, \frac{k}{2}\right) &= (\varepsilon(f)+1)N^{k/4}\int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy)y^{k/2-1}\,dy \\
&= (\varepsilon(f)+1)N^{k/4}\sum_{n=1}^{\infty} a_n(f)\int_{\frac{1}{\sqrt{N}}}^{\infty} e^{-2\pi ny}y^{k/2-1}\,dy \\
&= (\varepsilon(f)+1)N^{k/4}(2\pi)^{-k/2}\sum_{n=1}^{\infty}\frac{a_n(f)}{n^{k/2}}\int_{\frac{2\pi n}{\sqrt{N}}} e^{-x}x^{k/2-1}\,dx \quad (x = 2\pi ny) \\
&\le 2\left(\frac{\sqrt{N}}{2\pi}\right)^{k/2}\cdot\sum_{n=1}^{\infty}\frac{|a_n(f)|}{n^{k/2}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x}x^{k/2-1}\,dx \\
&\le 2\left(\frac{\sqrt{N}}{2\pi}\right)^{k/2}\cdot\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/2}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x}x^{k/2-1}\,dx \quad (|a_n(f)| \le d(n)n^{(k-1)/2}).
\end{aligned}$$

Now, splitting the sum into $n \le k$ and $n \ge k+1$ we get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/2}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x}x^{k/2-1}\,dx \\
=&\sum_{n=1}^{k}\frac{d(n)}{n^{1/2}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x}x^{k/2-1}\,dx + \sum_{n=k+1}^{\infty}\frac{d(n)}{n^{1/2}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x}x^{k/2-1}\,dx \\
\le&\sum_{n=1}^{k}\frac{d(n)}{n^{1/2}}\int_{0}^{\infty} e^{-x}x^{k/2-1}\,dx + \sum_{n=k+1}^{\infty}\frac{d(n)}{n^{1/2}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x}x^{k/2-1}\,dx \\
\le&\Gamma\left(\frac{k}{2}\right)\sum_{n=1}^{k}\frac{d(n)}{n^{1/2}} + \sum_{n=k+1}^{\infty}\frac{d(n)}{n^{1/2}}e^{-\frac{\pi n}{\sqrt{N}}}\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x/2}x^{k/2-1}\,dx.
\end{aligned}$$

Now,

$$\sum_{n=1}^{k}\frac{d(n)}{n^{1/2}} = \sum_{mn\le k}\frac{1}{(mn)^{1/2}} \le \sum_{m\le k}\frac{1}{m^{1/2}}\int_0^{k/m} u^{-1/2}\,du = 2k^{1/2}\sum_{m\le k}\frac{1}{m} \le 2k^{1/2}\ln(ek),$$

and

$$\int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x/2}x^{k/2-1}\,dx = 2^{k/2}\int_{\frac{\pi n}{\sqrt{N}}}^{\infty} e^{-t}t^{k/2-1}\,dt \quad \leq 2^{k/2}\Gamma\left(\frac{k}{2}\right).$$

As $d(n) \leq 9n^{1/4}$ for $n \geq 1$ ([1, Lemma 4.2]), we get

$$\begin{aligned}\sum_{n=k+1}^{\infty} \frac{d(n)}{n^{1/2}} e^{-\frac{\pi n}{\sqrt{N}}} \int_{\frac{2\pi n}{\sqrt{N}}}^{\infty} e^{-x/2}x^{k/2-1}\,dx <& 9\cdot 2^{k/2}\Gamma(k/2)\sum_{n=k+1}^{\infty} n^{-1/4}e^{-\pi n/\sqrt{N}} \\ <& 9\cdot 2^{k/2}\Gamma(k/2)\int_0^{\infty} x^{-1/4}e^{-\pi x/\sqrt{N}}\,dx \\ =& 9\cdot 2^{k/2}\Gamma(k/2)\left(\frac{\sqrt{N}}{\pi}\right)^{3/4}\Gamma(3/4).\end{aligned}$$

The desired upper bound (2.7) is obtained by combining the above calculations. ◻

We summarize the above calculations and make explicit [8, Theorem 1.2] in the next result.

Proposition 2.3. *Retain the notation and assumptions of Definition 1.5 and Theorem 1.6. Then the following hold.*

(1) Suppose that $k = 4$, or equivalently $m = 1$. If $\varepsilon(f) = -1$, then A_f is empty and the zeros of $r_f(z)$ are given by $\pm i/\sqrt{N}$, or equivalently $\phi^*_{4,0} = 0$ and $\phi^*_{4,1} = \pi$. If $\varepsilon(f) = 1$, then the unique element of A_f satisfies

$$\left|\theta^*_{4,0} - \frac{\pi}{2}\right| < \frac{C(4,N)}{N^{1/8}},$$

for

$$C(4,N) := \left(\frac{\pi\cdot\zeta\left(\frac{3}{2}\right)}{\zeta(3)}\right)^2\left(\frac{4\ln(4e)}{N^{3/8}} + \frac{36\cdot\Gamma\left(\frac{3}{4}\right)}{\pi^{3/4}}\right).$$

(2) If $k, k' \geq 6$, then for all $0 \leq \ell \leq m-1$ and $0 \leq \ell' \leq m'-1$

$$\left|\theta^*_{k,\ell} - \theta_{k,\ell}\right|,\ \left|\phi^*_{k',\ell'} - \phi_{k',\ell'}\right| < \frac{C(k,N)}{2^m\sqrt{N}}, \tag{2.8}$$

where

$$C(k,N) := \frac{2^{m-1}\pi\sqrt{N}}{m - \frac{2\pi}{\sqrt{N}}}\exp\left(\frac{2\pi}{\sqrt{N}}\right)B_{k,N},$$

and $B_{k,N}$ is defined in (2.1).

We also have the following decreasing properties for the constants $C(k,N)$ introduced in Proposition 2.3. Their proofs are by a straightforward induction and are omitted.

Lemma 2.4. *If $N > N_0 \geq 1$, then $C(4,N) < C(4,N_0)$. Thus, for $k = 4$ and $N > N_0 \geq 1$*

$$\left|\theta^*_{4,0} - \frac{\pi}{2}\right| < \frac{C(4,N_0)}{N_0^{1/8}}.$$

Lemma 2.5. *If either $N \geq 79 > 8\pi^2$ and $k \geq 6$, or $k \geq 54 > 16\pi + 2$, then*

(1) $C(k,N) > C(k+2,N)$,

(2) $C(k,N) > C(k,N+1)$.

Therefore, for $k \geq 6$, $N \geq 335464$, *and* $0 \leq \ell \leq m-1$

$$\left|\theta^*_{k,\ell} - \theta_{k,\ell}\right|, \left|\phi^*_{k,\ell} - \phi_{k,\ell}\right| < \frac{C(6,335464)}{2^m\sqrt{N}} < \frac{289.596}{2^m\sqrt{N}}, \tag{2.9}$$

and for $k \geq 78$, $N \geq 1$, *and* $0 \leq \ell \leq m-1$

$$\left|\theta^*_{k,\ell} - \theta_{k,\ell}\right|, \left|\phi^*_{k,\ell} - \phi_{k,\ell}\right| < \frac{C(78,1)}{2^m\sqrt{N}} < \frac{2.434 \cdot 10^{-7}}{2^m\sqrt{N}}.$$

3. Bounds for the Sample and Actual Angles

In this section, we first present a few preparatory lemmas that will be used in this section. Then, we establish interlacing and bounds for the sample angles and actual angles that will be used throughout the paper.

3.1. **Preparatory Lemmas.** The first two lemmas establish some bounds on the difference between sines and cosines.

Lemma 3.1. *For any two angles* θ_1 *and* θ_2, $|\sin\theta_1 - \sin\theta_2| \leq |\theta_1 - \theta_2|$ *and* $|\cos\theta_1 - \cos\theta_2| \leq |\theta_1 - \theta_2|$.

Proof. It follows immediately from the identity of difference of sines or cosines and the well known fact $|\sin(x)| \leq |x|$. □

Lemma 3.2. *Suppose* $0 \leq \theta_1 < \theta_2 \leq \pi$. *Then*

$$(\theta_2 - \theta_1)\cos\theta_2 < \sin\theta_2 - \sin\theta_1 < (\theta_2 - \theta_1)\cos\theta_1.$$

Proof. As

$$\sin\theta_2 - \sin\theta_1 = \int_{\theta_1}^{\theta_2} \cos\theta d\theta,$$

and $\cos\theta$ is a monotonically decreasing function on $[0,\pi]$, we get

$$(\theta_2 - \theta_1)\cos\theta_1 = \int_{\theta_1}^{\theta_2} \cos\theta_1 d\theta > \int_{\theta_1}^{\theta_2} \cos\theta d\theta > \int_{\theta_1}^{\theta_2} \cos\theta_2 d\theta = (\theta_2 - \theta_1)\cos\theta_2,$$

as desired. □

The next simple observation will be used in Section 5.

Lemma 3.3. *Suppose* $M \in \mathbb{N}$. *Let* $f(m,m') : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ *be a function satisfying:*

(1) $f(M, M+1) > 0$ *for* $M \geq 1$,
(2) $f(m+1, m+2) > f(m, m+1)$ *for all* $m \geq M$,
(3) $f(m, m'+1) > f(m, m')$, $m' \geq m+1$ *for all* $m' \geq m+1 \geq M+1$.

Then, $f(m,m') > 0$ *for all* $m' \geq m+1 \geq M+1$.

Proof. Suppose the above conditions hold. Then by induction, for all $m \geq M$, $f(m, m+1) > 0$ by conditions (1) and (2). On the other hand, again by induction and conditions (2) and (3), for a fixed m, we have that for all $n \in \mathbb{N}$, $f(m, m+n) > 0$, that is for all $m \geq M$, $m' \geq m+1$, $f(m,m') > 0$. □

3.2. **Interlacing and Bounds for the Sample Angles.** The next few lemmas will collect some facts about sample angles. Observe that the sample angles split into m angles in the interval $[0,\pi)$ and m angles in $[\pi, 2\pi)$ in either case of $\varepsilon(f)$.

Lemma 3.4. *We have $\theta_{k,m}, \phi_{k,m} \geq \pi$ and $\theta_{k,m-1}, \phi_{k,m-1} < \pi$. Additionally, $\theta_{k,0} > 0$ and $\phi_{k,0} = 0$.*

Lemma 3.5. *Suppose $m > 2\pi/\sqrt{N}$.*

(1) For $0 \leq \ell \leq m-2$,

$$\theta_{k,\ell} < \theta_{k,\ell+1} \text{ and } \phi_{k,\ell} < \phi_{k,\ell+1}.$$

(2) Furthermore, for $0 \leq \ell \leq m-2$, we may bound the difference of consecutive sample angles by

$$\frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}} < \theta_{k,\ell+1} - \theta_{k,\ell} < \frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell}}, \tag{3.1}$$

$$\frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\phi_{k,\ell+1}} < \phi_{k,\ell+1} - \phi_{k,\ell} < \frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\phi_{k,\ell}}.$$

Proof. (1) First suppose $\varepsilon(f) = 1$ and suppose $\theta_{k,\ell} \geq \theta_{k,\ell+1}$ instead. Taking the difference of (1.3) for ℓ and $\ell+1$ gives

$$m(\theta_{k,\ell} - \theta_{k,\ell+1}) = -\pi + \frac{2\pi}{\sqrt{N}}\left(\sin\theta_{k,\ell} - \sin\theta_{k,\ell+1}\right). \tag{3.2}$$

Using Lemma 3.1,

$$m(\theta_{k,\ell} - \theta_{k,\ell+1}) \leq -\pi + \frac{2\pi}{\sqrt{N}}\left(\theta_{k,\ell} - \theta_{k,\ell+1}\right)$$

which contradicts the assumption that $m > 2\pi/\sqrt{N}$. The argument for the case when $\varepsilon(f) = -1$ is identical.

(2) When $\varepsilon(f) = 1$, by Lemma 3.4 and (1), we have $0 \leq \theta_{k,\ell} < \theta_{k,\ell+1} \leq \theta_{k,m-1} < \pi$. We may immediately apply Lemma 3.2 to (3.2) to see that

$$\pi + \frac{2\pi}{\sqrt{N}}(\theta_{k,\ell+1} - \theta_{k,\ell})\cos\theta_{k,\ell+1} < m(\theta_{k,\ell+1} - \theta_{k,\ell}) < \pi + \frac{2\pi}{\sqrt{N}}(\theta_{k,\ell+1} - \theta_{k,\ell})\cos\theta_{k,\ell}$$

which implies the desired statement. The argument for the $\varepsilon(f) = -1$ case is the same. □

Lemma 3.5 leads to the following simpler bounds.

Lemma 3.6. *Suppose $m > 2\pi/\sqrt{N}$. Then for all $0 \leq \ell \leq m-2$,*

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} < \theta_{k,\ell+1} - \theta_{k,\ell} < \frac{\pi}{m - \frac{2\pi}{\sqrt{N}}}$$

and

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} < \phi_{k,\ell+1} - \phi_{k,\ell} < \frac{\pi}{m - \frac{2\pi}{\sqrt{N}}}.$$

Lemma 3.7. *Let $k' = 2m' + 2 > k = 2m + 2$, and suppose $m > 2\pi/\sqrt{N}$. Then*

(1) $\theta_{k',0} < \theta_{k,0}$, and furthermore, $\theta_{k,0} - \theta_{k',0} \geq \frac{(m'-m)\pi}{2(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})}$;

(2) $\phi_{k',1} < \phi_{k,1}$, and furthermore, $\phi_{k,1} - \phi_{k',1} \geq \frac{(m'-m)\pi}{2(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})}$;

(3) $\theta_{k,m-1} < \theta_{k',m'-1}$, and furthermore, $\theta_{k',m'-1} - \theta_{k,m-1} \geq \frac{(m'-m)\pi}{2(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})}$;

(4) $\phi_{k,m-1} < \phi_{k',m'-1}$*, and furthermore,* $\phi_{k',m'-1} - \phi_{k,m-1} \geq \frac{(m'-m)\pi}{(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})}$.

Proof. By Definition 1.3,

$$m\theta_{k,0} = \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}} \sin\theta_{k,0}, \quad \text{and} \quad m'\theta_{k',0} = \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}} \sin\theta_{k',0}, \tag{3.3}$$

from which we obtain

$$m(\theta_{k',0} - \theta_{k,0}) + (m' - m)\theta_{k',0} = \frac{2\pi}{\sqrt{N}}(\sin\theta_{k',0} - \sin\theta_{k,0}). \tag{3.4}$$

For the sake of contradiction, suppose $\theta_{k',0} \geq \theta_{k,0}$. Then, Lemma 3.1 and Lemma 3.4 gives

$$m(\theta_{k',0} - \theta_{k,0}) < m(\theta_{k',0} - \theta_{k,0}) + (m' - m)\theta_{k',0} \leq \frac{2\pi}{\sqrt{N}}(\theta_{k',0} - \theta_{k,0}).$$

Now, the assumption $\theta_{k',0} \geq \theta_{k,0}$ leads to a contradiction when $m > 2\pi/\sqrt{N}$.

Then since $\theta_{k,0} > \theta_{k',0}$, by (3.4) and Lemma 3.1,

$$m(\theta_{k,0} - \theta_{k',0}) - (m' - m)\theta_{k',0} \geq -\frac{2\pi}{\sqrt{N}}(\theta_{k,0} - \theta_{k',0}),$$

which shows

$$\theta_{k,0} - \theta_{k',0} \geq \frac{(m' - m)\theta_{k',0}}{m + \frac{2\pi}{\sqrt{N}}}.$$

Applying (3.3) yields

$$\theta_{k',0} \geq \frac{\pi}{2m'} > \frac{\pi}{2\left(m' + \frac{2\pi}{\sqrt{N}}\right)}. \tag{3.5}$$

Combining these two equations yields

$$\theta_{k,0} - \theta_{k',0} \geq \frac{(m' - m)\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)}.$$

For part (3), recall the following equations:

$$m\theta_{k,m-1} = \frac{\pi}{2} + (m-1)\pi + \frac{2\pi}{\sqrt{N}} \sin\theta_{k,m-1}, \quad \text{and} \quad m'\theta_{k',m'-1} = \frac{\pi}{2} + (m'-1)\pi + \frac{2\pi}{\sqrt{N}} \sin\theta_{k',m'-1}.$$

Taking their difference yields

$$m(\theta_{k,m-1} - \theta_{k',m'-1}) - (m' - m)\theta_{k',m'-1} = -(m' - m)\pi + \frac{2\pi}{\sqrt{N}}(\sin\theta_{k,m-1} - \sin\theta_{k',m'-1}). \tag{3.6}$$

Suppose on the contrary that $\theta_{k,m-1} \geq \theta_{k',m'-1}$. Then by Lemma 3.1,

$$m(\theta_{k,m-1} - \theta_{k',m'-1}) - (m' - m)\theta_{k',m'-1} \leq -(m' - m)\pi + \frac{2\pi}{\sqrt{N}}(\theta_{k,m-1} - \theta_{k',m'-1}),$$

or

$$(m' - m)\left(\pi - \theta_{k',m'-1}\right) \leq \left(\frac{2\pi}{\sqrt{N}} - m\right)\left(\theta_{k,m-1} - \theta_{k',m'-1}\right).$$

By Lemma 3.4, $\theta_{k',m'-1} < \pi$, thus we get a contradiction since $m > 2\pi/\sqrt{N}$.

As we have shown $\theta_{k',m'-1} > \theta_{k,m-1}$, by (3.6) and Lemma 3.1,

$$m(\theta_{k',m'-1} - \theta_{k,m-1}) + (m'-m)\theta_{k',m'-1} \geq (m'-m)\pi - \frac{2\pi}{\sqrt{N}}\left(\theta_{k',m'-1} - \theta_{k,m-1}\right),$$

which implies

$$\theta_{k',m'-1} - \theta_{k,m-1} \geq \frac{(m'-m)(\pi - \theta_{k',m'-1})}{m + \frac{2\pi}{\sqrt{N}}}. \tag{3.7}$$

On the other hand by (1.3),

$$\begin{aligned}
\pi - \theta_{k',m'-1} &= \pi - \frac{1}{m'}\left(\frac{\pi}{2} + (m'-1)\pi + \frac{2\pi}{\sqrt{N}}\sin\theta_{k',m'-1}\right) \\
&= \frac{\pi m' - \frac{\pi}{2} + \pi - m'\pi}{m'} - \frac{2\pi}{m'\sqrt{N}}\sin\theta_{k',m'-1} \\
&= \frac{\pi}{2m'} - \frac{2\pi}{m'\sqrt{N}}\sin\theta_{k',m'-1} \\
&\geq \frac{\pi}{2m'} - \frac{2\pi}{m'\sqrt{N}}(\pi - \theta_{k',m'-1}),
\end{aligned}$$

where the last inequality comes from the fact that $\sin(x) \leq \pi - x$ on $[0, \pi)$. Thus

$$\pi - \theta_{k',m'-1} \geq \frac{\pi}{2m'\left(1 + \frac{2\pi}{m'\sqrt{N}}\right)} = \frac{\pi}{2\left(m' + \frac{2\pi}{\sqrt{N}}\right)}. \tag{3.8}$$

Finally, combining (3.7) and (3.8) gives

$$\theta_{k',m'-1} - \theta_{k,m-1} \geq \frac{(m'-m)\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)},$$

as desired. The case for ϕ is proved similarly. ∎

In this next lemma, we instead compare θ and ϕ.

Lemma 3.8. *Suppose $m > 2\pi/\sqrt{N}$. Then*

(1) $\phi_{k',1} < \theta_{k,1}$ and $\theta_{k,m-2} < \phi_{k',m'-1}$,
(2) $\theta_{k',0} < \phi_{k,1}$ and $\phi_{k,m-1} < \theta_{k',m'-1}$.

Proof. (1) Taking the difference of Definitions 1.3 and 1.4, we have

$$m(\theta_{k,1} - \phi_{k',1}) = (m'-m)\phi_{k',1} + \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}}(\sin\theta_{k,1} - \sin\phi_{k',1}). \tag{3.9}$$

We must have $\phi_{k',1} < \theta_{k,1}$, otherwise by Lemma 3.1, we have the inequality

$$m(\phi_{k',1} - \theta_{k,1}) \leq -(m'-m)\phi_{k',1} - \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}}(\phi_{k',1} - \theta_{k,1}),$$

which cannot hold for $m > 2\pi/\sqrt{N}$. On the other hand, we similarly have

$$m\left(\phi_{k',m'-1} - \theta_{k,m-2}\right) = \left(m'-m\right)\left(\pi - \phi_{k,m'-1}\right) + \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}}\left(\sin\phi_{k',m'-1} - \sin\theta_{k,m-2}\right). \tag{3.10}$$

We must have $\theta_{k,m-2} < \phi_{k',m'-1}$, otherwise by Lemma 3.1 and (3.10), we get

$$m\left(\theta_{k,m-2} - \phi_{k',m'-1}\right) \leq -\left(m' - m\right)\left(\pi - \phi_{k,m'-1}\right) - \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}}\left(\theta_{k,m-2} - \phi_{k',m'-1}\right),$$

which cannot hold for $m > 2\pi/\sqrt{N}$, by Lemma 3.4.

(2) The proof is similar to (1). □

This next lemma will be useful in Section 6.

Lemma 3.9. *If $m > 2\pi/\sqrt{N}$, then*

(1) $\phi_{k,\ell} < \theta_{k,\ell}$ for all $1 \leq \ell \leq m-1$ and $\theta_{k,\ell} - \phi_{k,\ell} > \frac{\pi}{2(m+2\pi/\sqrt{N})}$,
(2) $\theta_{k,\ell} < \phi_{k,\ell+1}$ for all $0 \leq \ell \leq m-2$ and $\phi_{k,\ell+1} - \theta_{k,\ell} > \frac{\pi}{2(m+2\pi/\sqrt{N})}$.

Proof. Recall Definitions 1.3 and 1.4:

$$m\phi_{k,\ell} = \ell\pi + \frac{2\pi}{\sqrt{N}}\sin\phi_{k,\ell}, \tag{3.11}$$

$$m\theta_{k,\ell} = \frac{\pi}{2} + \ell\pi + \frac{2\pi}{\sqrt{N}}\sin\theta_{k,\ell}, \tag{3.12}$$

$$m\phi_{k,\ell+1} = (\ell+1)\pi + \frac{2\pi}{\sqrt{N}}\sin\phi_{k,\ell+1}. \tag{3.13}$$

(1) Assume for the sake of contradiction that $\phi_{k,\ell} \geq \theta_{k,\ell}$. Taking the difference of (3.11) and (3.12) yields

$$m(\phi_{k,\ell} - \theta_{k,\ell}) = -\frac{\pi}{2} + \frac{2\pi}{\sqrt{N}}(\sin\phi_{k,\ell} - \sin\theta_{k,\ell}). \tag{3.14}$$

By Lemma 3.1,

$$m(\phi_{k,\ell} - \theta_{k,\ell}) + \frac{\pi}{2} \leq \frac{2\pi}{\sqrt{N}}(\phi_{k,\ell} - \theta_{k,\ell}),$$

which is a contradiction for $m > 2\pi/\sqrt{N}$.

Now that $\phi_{k,\ell} < \theta_{k,\ell}$, (3.14) and Lemma 3.1 gives us

$$\theta_{k,\ell} - \phi_{k,\ell} \geq \frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)}.$$

(2) Assume for the sake of contradiction $\theta_{k,\ell} \geq \phi_{k,\ell+1}$ and take the difference of (3.12) and (3.13) to obtain

$$m(\theta_{k,\ell} - \phi_{k,\ell+1}) = \frac{\pi}{2} - \pi + \frac{2\pi}{\sqrt{N}}(\sin\theta_{k,\ell} - \sin\phi_{k,\ell+1}).$$

By Lemma 3.1,

$$m(\theta_{k,\ell} - \phi_{k,\ell+1}) + \frac{\pi}{2} \leq \frac{2\pi}{\sqrt{N}}(\theta_{k,\ell} - \phi_{k,\ell+1}), \tag{3.15}$$

a contradiction for $m > 2\pi/\sqrt{N}$.

We have shown $\theta_{k,\ell} < \phi_{k,\ell+1}$. Thus by (3.15) and 3.1,

$$\phi_{k,\ell+1} - \theta_{k,\ell} \geq \frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)},$$

as desired. □

3.3. **Interlacing and Bounds for the Actual Angles.** The remaining lemmas in this section establish similar properties for the actual angles $\theta^*_{k,\ell}$ and $\phi^*_{k,\ell}$ when either k or N is large enough.

First, we provide conditions to show that each $\theta^*_{k,\ell}$ is ordered and unique, that is, each actual angle $\theta^*_{k,\ell}$ is close to only one sample angle, namely $\theta_{k,\ell}$. Recall that for $k \geq 6$, $C(k,N)$ is defined as in Proposition 2.3.

Lemma 3.10. *Suppose* $\frac{\pi}{m+2\pi/\sqrt{N}} > \frac{2C(k,N)}{2^m\sqrt{N}}$, $k' > k \geq 6$, *and* $m > 2\pi/\sqrt{N}$. *Then the following hold:*

(1) for all $0 \leq \ell \leq m-2$, $\theta^*_{k,\ell} < \theta^*_{k,\ell+1}$;
(2) for all $1 \leq \ell \leq m-2$, $\phi^*_{k,\ell} < \phi^*_{k,\ell+1}$;
(3) for all $0 \leq \ell' \leq m'-2$, $\theta^*_{k',\ell'} < \theta^*_{k',\ell'+1}$;
(4) for all $1 \leq \ell' \leq m'-2$, $\phi^*_{k',\ell'} < \phi^*_{k',\ell'+1}$.

Proof. We will only show (1) and (3), as the proofs for (2) and (4) are similar.

(1) By Lemma 3.6 and (2.8),

$$\begin{aligned}\theta^*_{k,\ell+1} - \theta^*_{k,\ell} &= (\theta^*_{k,\ell+1} - \theta_{k,\ell+1}) + (\theta_{k,\ell+1} - \theta_{k,\ell}) + (\theta_{k,\ell} - \theta^*_{k,\ell}) \\ &\geq (\theta_{k,\ell+1} - \theta_{k,\ell}) - \left|\theta_{k,\ell} - \theta^*_{k,\ell}\right| - \left|\theta^*_{k,\ell+1} - \theta_{k,\ell+1}\right| \\ &> \frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} - \frac{2C(k,N)}{2^m\sqrt{N}}.\end{aligned}$$

(3) Although this statement looks identical to (1), notice that we only impose a condition on k, not k'. It is easy to show with induction on m that

$$\frac{\pi}{m' + \frac{2\pi}{\sqrt{N}}} > \frac{2C(k,N)}{2^{m'}\sqrt{N}}.$$

□

The next few results provide conditions for the ordering between the first (resp. last) elements of $A_{f'}$ and A_f when $\varepsilon(f') = \varepsilon(f)$.

Lemma 3.11. *Let* $f' \in S_{k'}(\Gamma_0(N)), f \in S_k(\Gamma_0(N))$ *be newforms, and* $k' > k \geq 6$. *Suppose* $\varepsilon(f') = \varepsilon(f)$ *and* $m > 2\pi/\sqrt{N}$. *Then*

(1) if $\frac{(m'-m)\pi}{(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}$, *then* $\theta^*_{k',0} < \theta^*_{k,0}$ *and* $\theta^*_{k,m-1} < \theta^*_{k',m'-1}$.
(2) if $\frac{\pi}{m+2\pi/\sqrt{N}} > \frac{2C(k,N)}{2^m\sqrt{N}}$, *then* $\theta^*_{k,0} > 0$ *and* $\theta^*_{k,m-1} < \pi$.
(3) if $\frac{(m'-m)\pi}{(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}$, *then* $\phi^*_{k',1} < \phi^*_{k,1}$ *and* $\phi^*_{k,m-1} < \phi^*_{k',m'-1}$.
(4) if $\frac{\pi}{m+2\pi/\sqrt{N}} > \frac{2C(k,N)}{2^m\sqrt{N}}$, *then* $\phi^*_{k,1} > 0$ *and* $\phi^*_{k,m-1} < \pi$.

Proof. (1) By Lemma 3.7 and (2.8),

$$\begin{aligned}\theta^*_{k,0} - \theta^*_{k',0} &= (\theta^*_{k,0} - \theta_{k,0}) + (\theta_{k,0} - \theta_{k',0}) + (\theta_{k',0} - \theta^*_{k',0}) \\ &\geq \theta_{k,0} - \theta_{k',0} - \left|\theta^*_{k,0} - \theta_{k,0}\right| - \left|\theta^*_{k',0} - \theta_{k',0}\right| \\ &> \frac{(m'-m)\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)} - \frac{C(k,N)}{2^m\sqrt{N}} - \frac{C(k',N)}{2^{m'}\sqrt{N}} \\ &> 0.\end{aligned}$$

A similar argument shows

$$\theta^*_{k',m'-1} - \theta^*_{k,m-1} > \frac{(m'-m)\pi}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)} - \frac{2C(k,N)}{2^m\sqrt{N}} - \frac{2C(k',N)}{2^{m'}\sqrt{N}} > 0.$$

(2) By (3.5) and (2.8),

$$\theta^*_{k,0} > \theta_{k,0} - \left|\theta^*_{k,0} - \theta_{k,0}\right| > \frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)} - \frac{C(k,N)}{2^m\sqrt{N}} > 0.$$

We can similarly show

$$\pi - \theta^*_{k,m-1} > \frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)} - \frac{C(k,N)}{2^m\sqrt{N}} > 0.$$

Parts (3) and (4) follow similarly to (1) and (2). □

The next lemma resembles the previous lemma except we are instead comparing θ^* and ϕ^*.

Lemma 3.12. *Suppose $m > 2\pi/\sqrt{N}$.*

(1) If $\frac{(m'-m)\pi}{(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}$, then $\phi^*_{k',1} < \theta^*_{k,1}$ and $\theta^*_{k,m-2} < \phi^*_{k',m'-1}$.
(2) If $\frac{(m'-m)\pi}{(m+2\pi/\sqrt{N})(m'+2\pi/\sqrt{N})} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}$, then $\theta^*_{k',0} < \phi^*_{k,1}$ and $\phi^*_{k,m-1} < \theta^*_{k',m'-1}$.

Proof. (1) Since $\sin\phi_{k',1} \geq 0$, we have $\phi_{k',1} \geq \pi/m' > \pi/(m'+2\pi/\sqrt{N})$, and thus by (3.9), Lemmas 3.1 and 3.8 part (1),

$$\theta_{k,1} - \phi_{k',1} > \frac{(m'-m)\phi_{k',1} + \frac{\pi}{2}}{m + \frac{2\pi}{\sqrt{N}}} > \frac{(m'-m)\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)}\left(m' + \frac{2\pi}{\sqrt{N}}\right).$$

Thus, in the same way as Lemma 3.11, we may bound the difference of the actual angles:

$$\theta^*_{k,1} - \phi^*_{k',1} > \frac{(m'-m)\pi}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)} - \frac{2C(k,N)}{2^m\sqrt{N}} - \frac{2C(k',N)}{2^{m'}\sqrt{N}}.$$

As in Lemma 3.10, it is easy to see $\pi - \phi_{k',m'-1} \geq \frac{\pi}{m'+2\pi/\sqrt{N}}$. Thus,

$$\left(m'-m\right)\left(\pi - \phi_{k',m'-1}\right) \geq \frac{\left(m'-m\right)\pi}{m' + \frac{2\pi}{\sqrt{N}}} > \frac{\left(m'-m\right)\pi}{2\left(m' + \frac{2\pi}{\sqrt{N}}\right)}.$$

Now, by (3.10) and Lemmas 3.1 and 3.8 part (1),

$$\phi_{k',m'-1} - \theta_{k,m-2} \geq \frac{(m'-m)\left(\pi - \phi_{k',m'-1}\right) + \frac{\pi}{2}}{m + \frac{2\pi}{\sqrt{N}}}.$$

Therefore,

$$\phi_{k',m'-1} - \theta_{k,m-2} > \frac{(m'-m)\left(\pi - \phi_{k',m'-1}\right) + \frac{\pi}{2}}{\left(m + \frac{2\pi}{\sqrt{N}}\right)} > \frac{(m'-m)\,\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)},$$

so that akin to the proof of Lemma 3.11, we may write

$$\phi^*_{k',m'-1} - \theta^*_{k,m-2} > \frac{(m'-m)\pi}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)} - \frac{2C(k,N)}{2^m\sqrt{N}} - \frac{2C(k',N)}{2^{m'}\sqrt{N}}.$$

(2) The proof is similar to (1), using Lemma 3.8 part (2). □

This next lemma will be useful in Section 6.

Lemma 3.13. *If* $\frac{\pi}{2(m+2\pi/\sqrt{N})} > \frac{2C(k,N)}{2^m\sqrt{N}}$, *then*

(1) $\phi^*_{k,\ell} < \theta^*_{k,\ell}$ *for all* $1 \leq \ell \leq m-1$,
(2) $\theta^*_{k,\ell} < \phi^*_{k,\ell+1}$ *for all* $0 \leq \ell \leq m-2$.

Proof. The argument is identical to Lemma 3.11, using Lemma 3.9 and (2.8). □

4. The case $k' > k \geq 6$, $N \geq 335464$, and $\varepsilon(f') = \varepsilon(f)$

In this section, we will consider parts (1) and (2) of Theorem 1.6 when $N \geq 335464$. The first subsection will treat the case when $\varepsilon(f') = \varepsilon(f) = 1$ and the second will give the result for $\varepsilon(f') = \varepsilon(f) = -1$.

For $N \geq 335464$, by (2.9), we have a tighter bound for the distance between actual angles and sample angles. That is, for all $k \geq 6$ and all levels $N \geq 335464$, we have the bound

$$\left|\theta^*_{k,\ell} - \theta_{k,\ell}\right| < \frac{C(6,335464)}{2^m\sqrt{N}},$$

where $C(6,335464) < 289.596$ by Lemma 2.5 and [9, zgap.sage]. In addition,

$$\sqrt{N} \geq 2C(6,335464). \tag{4.1}$$

Lemma 4.1. *If*

$$\frac{\pi\left(m' - m - \frac{4\pi}{\sqrt{N}}\right)}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}},$$

then for all $0 \leq \ell \leq m-2$ *and* $0 \leq \ell' \leq m'-2$,

$$\left|\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'}\right| < \left|\theta^*_{k,\ell+1} - \theta^*_{k,\ell}\right|,$$

and for all $1 \leq \ell \leq m-2$ *and* $1 \leq \ell' \leq m'-2$,

$$\left|\phi^*_{k',\ell'+1} - \phi^*_{k',\ell'}\right| < \left|\phi^*_{k,\ell+1} - \phi^*_{k,\ell}\right|.$$

Proof. By our assumption on N, we have $m > 2\pi/\sqrt{N}$. Recall from Lemmas 3.5 and 3.6,

$$\frac{\pi}{m+\frac{2\pi}{\sqrt{N}}} < |\theta_{k,\ell+1}-\theta_{k,\ell}| < \frac{\pi}{m-\frac{2\pi}{\sqrt{N}}}. \tag{4.2}$$

In addition, as we let

$$\frac{\pi\left(m'-m-\frac{4\pi}{\sqrt{N}}\right)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'-\frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}},$$

we have the equivalent expression

$$\frac{\pi}{m+\frac{2\pi}{\sqrt{N}}} - \frac{2C(k,N)}{2^m\sqrt{N}} > \frac{\pi}{m'-\frac{2\pi}{\sqrt{N}}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}. \tag{4.3}$$

On the other hand, by (2.8) we get:

$$\left|\theta^*_{k',\ell'+1}-\theta^*_{k',\ell'}\right| < \left|\theta_{k',\ell'+1}-\theta_{k',\ell'}\right| + \frac{2C(k',N)}{2^{m'}\sqrt{N}} \tag{4.4}$$

and

$$\left|\theta^*_{k,\ell+1}-\theta^*_{k,\ell}\right| > |\theta_{k,\ell+1}-\theta_{k,\ell}| - \frac{2C(k,N)}{2^m\sqrt{N}}. \tag{4.5}$$

Thus, by (4.2), (4.3), (4.4), and (4.5),

$$\left|\theta^*_{k',\ell'+1}-\theta^*_{k',\ell'}\right| < \left|\theta^*_{k,\ell+1}-\theta^*_{k,\ell}\right|.$$

The same argument proves $\left|\phi^*_{k',\ell'+1}-\phi^*_{k',\ell'}\right| < \left|\phi^*_{k,\ell+1}-\phi^*_{k,\ell}\right|$. □

We need a few more lemmas before we can show parts (1) and (2) of Theorem 1.6 for $N \geq 335464$ and $k \geq 6$.

Lemma 4.2. *For all $m' > m \geq 2$ and $N \geq 335464$,*

$$\frac{\pi(m'-m)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{\pi\left(m'-m-\frac{4\pi}{\sqrt{N}}\right)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'-\frac{2\pi}{\sqrt{N}}\right)}.$$

Proof. This can be shown easily. □

Lemma 4.3. *When $N \geq 335464$, for all $m' > m \geq 2$,*

$$\frac{\pi\left(m'-m-\frac{4\pi}{\sqrt{N}}\right)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'-\frac{2\pi}{\sqrt{N}}\right)} > \frac{3\pi}{5m^2}.$$

Proof. By our assumption on N, $m' - 2\pi/\sqrt{N} > 0$. Thus, it suffices to show

$$5m^2\left(m'-m-\frac{4\pi}{\sqrt{N}}\right) - 3m'\left(m+\frac{2\pi}{\sqrt{N}}\right) > 0.$$

Let

$$f(m,m') := 5m^2\left(m'-m-\frac{4\pi}{\sqrt{N}}\right) - 3m'\left(m+\frac{2\pi}{\sqrt{N}}\right).$$

It is straightforward to apply Lemma 3.3 by taking $M = 2$ to show f is positive. $\square$

4.1. **The case** $\varepsilon(f') = \varepsilon(f) = 1$. The next lemma establishes sufficient conditions for strong Stieltjes interlacing (1.2) to hold (and therefore regular interlacing) for $A_{f'}$ and A_f. Recall $k' = 2m' + 2$ and $k = 2m + 2$ in the following.

Lemma 4.4. *Suppose $k' > k \geq 6$ and $N \geq 335464$. Then, the following statements hold for $0 \leq \ell \leq m - 2$ and $0 \leq \ell' \leq m' - 2$:*

(1) $\theta^*_{k,\ell} < \theta^*_{k,\ell+1}$ and $\theta^*_{k',\ell'} < \theta^*_{k',\ell'+1}$,
(2) $\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} < \theta^*_{k,\ell+1} - \theta^*_{k,\ell}$,
(3) $0 < \theta^*_{k',0} < \theta^*_{k,0}$ and $\theta^*_{k,m-1} < \theta^*_{k',m'-1} < \pi$.

Proof. (1) Since $\frac{2\pi}{\sqrt{N}} < 1$, it is clear that for $m \geq 2$,

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{\pi}{m+1} > \frac{1}{2^m}.$$

By (4.1) and Lemma 2.5, we have $\sqrt{N} \geq 2C(6, 335464) \geq 2C(k, N)$, so

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{1}{2^m} > \frac{2C(k, N)}{2^m\sqrt{N}}.$$

Thus, the desired statement follows from Lemma 3.10.

(2) Firstly, by Lemma 2.5 and (4.1), we have the estimate

$$\frac{2C(k, N)}{2^m\sqrt{N}} + \frac{2C(k', N)}{2^{m'}\sqrt{N}} \leq \frac{2C(6, 335464)}{2^m\sqrt{N}} + \frac{2C(6, 335464)}{2^{m'}\sqrt{N}} \leq \frac{1}{2^m} + \frac{1}{2^{m+1}} = \frac{3}{2^{m+1}}.$$

On the other hand, by Lemma 4.3 and a simple estimate, for $m \geq 2$ we have

$$\frac{\pi\left(m' - m - \frac{4\pi}{\sqrt{N}}\right)}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\right)} > \frac{3\pi}{5m^2} > \frac{3}{2^{m+1}},$$

so that

$$\frac{\pi\left(m' - m - \frac{4\pi}{\sqrt{N}}\right)}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(k, N)}{2^m\sqrt{N}} + \frac{2C(k', N)}{2^{m'}\sqrt{N}}.$$

Thus by Lemma 4.1 and part (1), $\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} < \theta^*_{k,\ell+1} - \theta^*_{k,\ell}$.

(3) By Lemma 4.2 and part (2),

$$\frac{\pi(m' - m)}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)} > \frac{\pi\left(m' - m - \frac{4\pi}{\sqrt{N}}\right)}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(k, N)}{2^m\sqrt{N}} + \frac{2C(k', N)}{2^{m'}\sqrt{N}}. \quad (4.6)$$

We have already shown

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{2C(k, N)}{2^m\sqrt{N}},$$

so by Lemma 3.11, $0 < \theta^*_{k',0} < \theta^*_{k,0}$ and $\theta^*_{k,m-1} < \theta^*_{k',m'-1} < \pi$. $\square$

The next proposition proves Theorem 1.6 part (1) and (2) when $\varepsilon(f') = \varepsilon(f) = 1$ for $N \geq 335464$ and $k \geq 6$.

Proposition 4.5. *Let $k' > k \geq 6$, and $f' \in S_{k'}(\Gamma_0(N)), f \in S_k(\Gamma_0(N))$ be newforms with $\varepsilon(f') = \varepsilon(f) = 1$. Then, $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Proof. Lemma 4.4 part (1) implies that the elements of $A_{f'}$ and A_f have the same order as their ℓ indices. Suppose that for some ℓ, no element from $A_{f'}$ lies in the interval $(\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$. Thus by Lemma 4.4 part (3), we have that $\theta^*_{k',0} < \theta^*_{k,\ell} < \theta^*_{k',m'-1}$ for all $0 \leq \ell \leq m-1$. Then there exists some $0 \leq \ell' \leq m'-2$ such that $\theta^*_{k',\ell'} < \theta^*_{k,\ell} < \theta^*_{k,\ell+1} < \theta^*_{k',\ell'+1}$. This implies that $\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} > \theta^*_{k,\ell+1} - \theta^*_{k,\ell}$, violating Lemma 4.4 part (2). Thus, we have strong Stieltjes interlacing between $A_{f'}$ and A_f. In particular, this implies that for $k' = k+2$, $A_{f'}$ interlaces with A_f. ☐

4.2. **The case $\varepsilon(f') = \varepsilon(f) = -1$.** Here, Lemma 4.6 and Proposition 4.7 are proved identically to Lemma 4.4 and Proposition 4.5, respectively.

Lemma 4.6. *Suppose $k' > k \geq 6$ and $N \geq 335464$. The following statements hold for $1 \leq \ell \leq m-2$ and $1 \leq \ell' \leq m'-2$:*

(1) $\phi^*_{k,\ell} < \phi^*_{k,\ell+1}$ and $\phi^*_{k',\ell'} < \phi^*_{k',\ell'+1}$,
(2) $\phi^*_{k',\ell'+1} - \phi^*_{k',\ell'} < \phi^*_{k,\ell+1} - \phi^*_{k,\ell}$,
(3) $0 < \phi^*_{k',1} < \phi^*_{k,1}$ and $\phi^*_{k,m-1} < \phi^*_{k',m'-1} < \pi$.

Proposition 4.7. *Suppose $\varepsilon(f') = \varepsilon(f) = -1$, $N \geq 335464$, and $k' > k \geq 6$. Then $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Therefore, we have completed the proof of Theorem 1.6 parts (1) and (2) for $N \geq 335464$.

5. The case $k' > k \geq 78$ and $\varepsilon(f') = \varepsilon(f)$

In this section, we will prove Theorem 1.6 parts (1) and (2) under the assumptions that $k' > k \geq 78$ and $\varepsilon(f') = \varepsilon(f)$. By Lemma 2.5 and [9, zgap.sage], for all levels N and $k' > k \geq 78$, we have the bound

$$\left|\theta^*_{k,\ell} - \theta_{k,\ell}\right| < \frac{C(78,1)}{2^m\sqrt{N}} < \frac{2.434 \cdot 10^7}{2^m\sqrt{N}}.$$

Recall that $k' = 2m'+2$ and $k = 2m+2$. The following lemma will be shown later to be a sufficient condition for strong Stieltjes interlacing.

Lemma 5.1. *For all $k' > k \geq 78$ and $N \geq 1$,*

$$\frac{2\pi^2}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}}\right) - 3C\left(78,1\right) \cdot \left(\frac{2^m + 2^{m'}}{2^{m+m'}}\right) > 0. \tag{5.1}$$

Proof. When N increases, the left hand side of (5.1) increases, so we only need to prove this for $N = 1$. Define $f(m, m')$ to be the left hand side of (5.1) with $N = 1$. Then for $M = 38$, it is straightforward to apply Lemma 3.3 to see that f is always positive for $m' > m \geq 38$, or equivalently $k' > k \geq 78$. ☐

5.1. **The case $\varepsilon(f') = \varepsilon(f) = 1$.** Our next lemma compares the first angles of $A_{f'}$ and A_f.

Lemma 5.2. *For all $k' > k \geq 78$ and $N \geq 1$, we have*

(1) $\theta_{k,0} - \theta_{k',0} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}$,
(2) $\theta^*_{k',0} < \theta^*_{k,0}$,
(3) $\theta^*_{k,m-1} < \theta^*_{k',m'-1}$.

Proof. (1) By (5.1),

$$\frac{2\pi^2}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{\pi}{m'-\frac{2\pi}{\sqrt{N}}}\right) > \frac{3C\,(78,1)}{2^m} + \frac{3C\,(78,1)}{2^{m'}}.$$

Since $\pi/(m' - 2\pi/\sqrt{N}) > 0$, we get

$$\frac{2\pi^2}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}}\right) > \frac{3C(78,1)}{2^m} + \frac{3C(78,1)}{2^{m'}},$$

which implies

$$\frac{\pi(m'-m)}{3\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}. \tag{5.2}$$

By Lemma 3.7 part (1), we have

$$\theta_{k,0} - \theta_{k',0} \geq \frac{(m'-m)\pi}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{\pi(m'-m)}{3\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

(2) By Lemma 2.5, $C(78,1) \geq C(k,N) > C(k',N)$,

$$\frac{(m'-m)\pi}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k',N)}{2^{m'}\sqrt{N}}.$$

Thus by Lemma 3.11 part (1),

$$\theta^*_{k,0} - \theta^*_{k',0} > 0,$$

as desired.

(3) This follows similarly to (2) from Lemma 3.7 part (3) and Lemma 3.11 part (3). □

The following lemma resembles Lemma 4.4 part (1).

Lemma 5.3. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then for all $0 \leq \ell \leq m-2$ and $0 \leq \ell' \leq m'-2$, $\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > 0$, and $\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} > 0$.*

Proof. Since $(m' + 2\pi/\sqrt{N})/(m'-m) > 1$ and by (5.2),

$$\frac{\pi}{m+\frac{2\pi}{\sqrt{N}}} > \frac{3C(78,1)}{2^m\sqrt{N}} + \frac{3C(78,1)}{2^{m'}\sqrt{N}} > \frac{2C(78,1)}{2^m\sqrt{N}}. \tag{5.3}$$

Therefore, by Lemma 3.10 parts (1) and (3), we have

$$\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > 0 \quad \text{and} \quad \theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} > 0,$$

as desired. ∎

For each $0 \leq \ell \leq m-2$, we would like to find some $\theta_{k',\hat{\ell}}$ that is close to $\theta_{k,\ell+1}$ where $0 \leq \hat{\ell} \leq m'-2$. We need the following definition:

Definition 5.4. *For each $0 \leq \ell \leq m-2$, we define the set*

$$U_\ell := \left\{ 0 \leq \ell' \leq m'-2 \,:\, \theta_{k',\ell'} < \theta_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} \right\}.$$

We define $\hat{\ell}$ to be the largest element of U_ℓ.

Remark. *By Lemma 5.2 and Lemma 3.5 part (1),*

$$\theta_{k',0} < \theta_{k,0} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} < \theta_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}},$$

so $0 \in U_\ell$ and $\hat{\ell}$ exists.

For each $0 \leq \ell \leq m-2$, we would like a bound on the distance between $\theta_{k,\ell+1}$ and $\theta_{k',\hat{\ell}}$.

Lemma 5.5. *Suppose $k' > k \geq 78$ and $N \geq 1$. For all $0 \leq \ell \leq m-2$,*

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} > \theta_{k,\ell+1} - \theta_{k',\hat{\ell}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Proof. By the definition of $\hat{\ell}$,

$$\theta_{k,\ell+1} - \theta_{k',\hat{\ell}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Suppose for the sake of contradiction that

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} \leq \theta_{k,\ell+1} - \theta_{k',\hat{\ell}}.$$

By Lemma 3.7 part (3), $\theta_{k,m-1} < \theta_{k',m'-1}$. Then by Lemma 3.5 and our assumption, $\theta_{k',m'-1} > \theta_{k,m-1} \geq \theta_{k,\ell+1} > \theta_{k',\hat{\ell}}$, and $\hat{\ell} \leq m'-2$. By Lemma 3.6,

$$\theta_{k',\hat{\ell}+1} - \theta_{k',\hat{\ell}} < \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} \leq \theta_{k,\ell+1} - \theta_{k',\hat{\ell}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}},$$

and thus

$$\theta_{k',\hat{\ell}+1} < \theta_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Therefore $\hat{\ell}+1 \in U_\ell$, a contradiction, and

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} > \theta_{k,\ell+1} - \theta_{k',\hat{\ell}},$$

as desired. ∎

Lemma 5.6. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then for all $0 \leq \ell \leq m-2$, $\theta^*_{k,\ell+1} > \theta^*_{k',\hat{\ell}}$.*

Proof. By Lemma 5.5 and (2.8), we get

$$\begin{aligned}\theta^*_{k,\ell+1} - \theta^*_{k',\hat{\ell}} &\geq (\theta_{k,\ell+1} - \theta_{k',\hat{\ell}}) - \left|\theta_{k',\hat{\ell}} - \theta^*_{k',\hat{\ell}}\right| - \left|\theta^*_{k,\ell+1} - \theta_{k,\ell+1}\right| \\ &> \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} - \frac{C(78,1)}{2^m\sqrt{N}} = 0,\end{aligned}$$

completing the proof. □

Lemma 5.7. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then for all $0 \leq \ell \leq m-2$,*

$$\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > \theta^*_{k',\hat{\ell}+1} - \theta^*_{k',\hat{\ell}}.$$

Proof. By (5.1),

$$\frac{2\pi^2}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}}\right) - \frac{C(78,1)}{2^m} - \frac{C(78,1)}{2^{m'}} > \frac{2C(78,1)}{2^m} + \frac{2C(78,1)}{2^{m'}}.$$

Since $m' > m \geq 38 > \sqrt{2}\pi$,

$$\frac{2\pi^2}{\sqrt{N}\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)} < 1,$$

so that

$$\begin{aligned}\frac{2\pi^2}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m' + \frac{2\pi}{\sqrt{N}}\right)}&\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}}\right) \\ &> \frac{2C(78,1)}{2^m} + \frac{2C(78,1)}{2^{m'}}.\end{aligned} \tag{5.4}$$

Next, since $|\cos(\theta)| \leq 1$,

$$\begin{aligned}\frac{2\pi^2}{\sqrt{N}\left(m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\cos\theta_{k',\hat{\ell}}\right)}&\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}}\right) \\ &> \frac{2C(78,1)}{2^m\sqrt{N}} + \frac{2C(78,1)}{2^{m'}\sqrt{N}}.\end{aligned} \tag{5.5}$$

By Lemma 5.5 and Lemma 3.1,

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} > \theta_{k,\ell+1} - \theta_{k',\hat{\ell}} > \cos\theta_{k',\hat{\ell}} - \cos\theta_{k,\ell+1},$$

giving us

$$\begin{aligned}&\left(m' - \frac{2\pi}{\sqrt{N}}\cos\theta_{k',\hat{\ell}}\right) - \left(m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}\right) \\ &\qquad > \frac{2\pi}{\sqrt{N}}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}}\right).\end{aligned}$$

Therefore,

$$\frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}\cos\theta_{k',\hat{\ell}}}$$

$$> \frac{2\pi^2}{\sqrt{N}\left(m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\cos\theta_{k',\hat{\ell}}\right)}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}}\right).$$

Combining this with (5.5), we get

$$\frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}\cos\theta_{k',\hat{\ell}}} > \frac{2C(78,1)}{2^m\sqrt{N}} + \frac{2C(78,1)}{2^{m'}\sqrt{N}}.$$

Recall Lemma 2.5 states $C(78,1) \geq C(k,N) > C(k',N)$. By (3.1),

$$(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\theta_{k',\hat{\ell}+1} - \theta_{k',\hat{\ell}}) > \frac{2C(78,1)}{2^m\sqrt{N}} + \frac{2C(78,1)}{2^{m'}\sqrt{N}} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}.$$

Therefore by (2.8),

$$\begin{aligned}(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\theta_{k',\hat{\ell}+1} - \theta_{k',\hat{\ell}}) &> (\theta_{k,\ell+1} - \theta^*_{k,\ell+1}) + (\theta^*_{k,\ell} - \theta_{k,\ell}) \\ &\quad + (\theta^*_{k',\hat{\ell}+1} - \theta_{k',\hat{\ell}+1}) + (\theta_{k',\hat{\ell}} - \theta^*_{k',\hat{\ell}}),\end{aligned}$$

which implies

$$\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > \theta^*_{k',\hat{\ell}+1} - \theta^*_{k',\hat{\ell}},$$

our desired result. ☐

The next proposition proves Theorem 1.6 parts (1) and (2) when $\varepsilon(f') = \varepsilon(f) = 1$ for $k' > k \geq 78$ and $N \geq 1$.

Proposition 5.8. *Let $k' > k \geq 78$, $N \geq 1$, and $f' \in S_{k'}(\Gamma_0(N))$, $f \in S_k(\Gamma_0(N))$ be newforms with $\varepsilon(f') = \varepsilon(f) = 1$. Then, $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Proof. As proved in Lemma 5.3, for all $0 \leq \ell \leq m-2$, $\theta^*_{k,\ell} < \theta^*_{k,\ell+1}$. Suppose for some ℓ, no element from $A_{f'}$ lies in the interval $(\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$. In particular, $\theta^*_{k',\hat{\ell}} \notin (\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$. By Lemma 5.6, we have $\theta^*_{k',\hat{\ell}} < \theta^*_{k,\ell+1}$, so $\theta^*_{k',\hat{\ell}} \leq \theta^*_{k,\ell}$. Thus,

$$\theta^*_{k',\hat{\ell}} \leq \theta^*_{k,\ell} < \theta^*_{k,\ell+1}.$$

By Lemma 5.7, $\theta^*_{k',\hat{\ell}+1} - \theta^*_{k',\hat{\ell}} < \theta^*_{k,\ell+1} - \theta^*_{k,\ell}$, which implies that $\theta^*_{k',\hat{\ell}+1} < \theta^*_{k,\ell+1}$. Since $\theta^*_{k',\hat{\ell}+1} \notin (\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$, we have $\theta^*_{k',\hat{\ell}+1} \leq \theta^*_{k,\ell}$. So,

$$\theta^*_{k',\hat{\ell}} < \theta^*_{k',\hat{\ell}+1} \leq \theta^*_{k,\ell} < \theta^*_{k,\ell+1}. \tag{5.6}$$

By (5.3), we have

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}},$$

which by Lemma 3.6 and (2.8) implies

$$\theta_{k,\ell+1} - \theta_{k,\ell} + (\theta_{k,\ell} - \theta^*_{k,\ell}) + (\theta^*_{k',\hat{\ell}+1} - \theta_{k',\hat{\ell}+1}) > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Equivalently,

$$\theta_{k,\ell+1} - \theta^*_{k,\ell} + \theta^*_{k',\hat{\ell}+1} - \theta_{k',\hat{\ell}+1} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Combining this with (5.6), we get

$$\theta_{k,\ell+1} - \theta_{k',\hat{\ell}+1} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Therefore, $\hat{\ell}+1 \in U_\ell$ and is larger than $\hat{\ell}$, contradicting Definition 5.4. This gives us Stieltjes interlacing of ${A_f}'$ and A_f. To complete the proof of strong Stieltjes interlacing, by Lemma 5.2 part (2) and part (3),

$$\theta^*_{k',0} < \theta^*_{k,0} \quad \text{and} \quad \theta^*_{k,m-1} < \theta^*_{k',m'-1}.$$

Thus, strong Stieltjes interlacing between $A_{f'}$ and A_f is proved for $k' > k \geq 78$, $N \geq 1$, and $\varepsilon(f') = \varepsilon(f) = 1$. ☐

5.2. **The case $\varepsilon(f') = \varepsilon(f) = -1$.** This proceeds similarly to the previous section, so we just state the corresponding lemmas without proof.

This first lemma is similar to Lemma 5.2.

Lemma 5.9. *For all $k' > k \geq 78$ and $N \geq 1$, we have*

(1) $\phi_{k,1} - \phi_{k',1} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}$,
(2) $\phi^*_{k',1} < \phi^*_{k,1}$,
(3) $\phi^*_{k,m-1} < \phi^*_{k',m'-1}$.

Here, we have a lemma that matches Lemma 5.3.

Lemma 5.10. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then for all $1 \leq \ell \leq m-2$ and $1 \leq \ell' \leq m'-2$, $\phi^*_{k,\ell+1} - \phi^*_{k,\ell} > 0$, and $\phi^*_{k',\ell'+1} - \phi^*_{k',\ell'} > 0$.*

The next definition is similar to Definition 5.4.

Definition 5.11. *For each $1 \leq \ell \leq m-2$, consider the set*

$$V_\ell = \left\{ 1 \leq \ell' \leq m'-2 \,:\, \phi_{k',\ell'} < \phi_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} \right\}.$$

We define $\tilde{\ell}$ to be the largest element of V_ℓ.

Remark. *By Lemma 5.9 and Lemma 3.5,*

$$\phi_{k',1} < \phi_{k,1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} < \phi_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}},$$

so $1 \in V_\ell$.

The following lemmas match Lemmas 5.5-5.7, respectively.

Lemma 5.12. *Suppose $k' > k \geq 78$ and $N \geq 1$. For all $1 \leq \ell \leq m-2$,*

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} > \phi_{k,\ell+1} - \phi_{k',\tilde{\ell}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Lemma 5.13. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then, $\phi^*_{k,\ell+1} > \phi^*_{k',\tilde{\ell}}$.*

Lemma 5.14. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then, for all $1 \leq \ell \leq m-2$,*

$$\phi^*_{k,\ell+1} - \phi^*_{k,\ell} > \phi^*_{k',\tilde{\ell}+1} - \phi^*_{k',\tilde{\ell}}.$$

The next proposition proves Theorem 1.6 parts (1) and (2) when $\varepsilon(f')=\varepsilon(f)=-1$ for $k'>k\geq 78$ and $N\geq 1$.

Proposition 5.15. *Let $k'>k\geq 78$, and $f'\in S_{k'}(\Gamma_0(N)), f\in S_k(\Gamma_0(N))$ be newforms with $\varepsilon(f')=\varepsilon(f)=-1$. Then, $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Combining Propositions 4.5, 4.7, 5.8, and 5.15, we have completed the proof of Theorem 1.6 for parts (1) and (2).

6. The case when $\varepsilon(f')\neq\varepsilon(f)$

This section is devoted to the proof of the remaining parts of Theorem 1.6. First, we prove part (3). Suppose $f,h\in S_k(\Gamma_0(N))$ and $\varepsilon(f)\neq\varepsilon(h)$. The following lemma is similar to Lemma 4.4.

Lemma 6.1. *Suppose either $k\geq 6$ and $N\geq 335464$, or $k\geq 78$. Then, we have the following:*

(1) for all $1\leq\ell\leq m-1$, $\phi^*_{k,\ell}<\theta^*_{k,\ell}$;
(2) for all $0\leq\ell\leq m-2$, $\theta^*_{k,\ell}<\phi^*_{k,\ell+1}$;
(3) for all $0\leq\ell\leq m-2$, $\theta^*_{k,\ell}<\theta^*_{k,\ell+1}$, and $\phi^*_{k,\ell}<\phi^*_{k,\ell+1}$;
(4) $\theta^*_{k,0}>0$ and $\phi^*_{k,m-1}<\pi$.

Proof. (1) Recall $k=2m+2$, so $m>2\pi/\sqrt{N}$ in either of the supposed conditions. When $N\geq 335464$ and $k\geq 6$, we may take $C(6,335464)<289.596$ as in Section 4. Then, by Lemma 2.5, since

$$\sqrt{N}\geq 2C(6,335464)\geq 2C(k,335464)\quad\text{and}\quad 2\pi/\sqrt{N}<1,$$

we get the following expression:

$$\frac{\pi}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)}>\frac{\pi}{2(m+1)}>\frac{1}{2^m}>\frac{2C(6,335464)}{2^m\sqrt{N}}>\frac{2C(k,N)}{2^m\sqrt{N}}.$$

By Lemma 3.13, we have proved (1) for $N\geq 335464$ and $k\geq 6$.

On the other hand, when $k\geq 78$ and $N\geq 1$, we know that $C(78,1)<2.43\cdot 10^7$ as in Section 5. Applying Lemma 2.5 for $m=38$ we have

$$\frac{\pi}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)}\geq\frac{\pi}{2(m+2\pi)}\approx 0.035>0.00018\approx\frac{2C(78,1)}{2^m}\geq\frac{2C(k,N)}{2^m\sqrt{N}}.$$

By Lemma 3.13 we have proved (1) for $k\geq 78$.

(2)-(4) are clear from the proof of (1) by applying Lemmas 3.13, 3.10, and 3.11, respectively. □

Lemma 6.1 directly implies the following.

Proposition 6.2. *Suppose either $k\geq 78$, or $N\geq 335464$ and $k\geq 6$. Let $f,h\in S_k(\Gamma_0(N))$ be newforms with $\varepsilon(f)=1$, $\varepsilon(h)=-1$. Then, A_h interlaces with A_f.*

Next, we will prove parts (4) and (5) of Theorem 1.6. Let $f'\in S_{k'}(\Gamma_0(N))$ and $f\in S_k(\Gamma_0(N))$ be newforms with $\varepsilon(f')\neq\varepsilon(f)$. Then, we have the following proposition:

Proposition 6.3. *For $k'>k\geq 78$ and $N\geq 1$, or $k'>k\geq 6$ and $N\geq 335464$:*

(1) If $\varepsilon(f')=-1$ and $\varepsilon(f)=1$, $A_{f'}$ Stieltjes interlaces with A_f;
(2) If $\varepsilon(f')=1$ and $\varepsilon(f)=-1$, $A_{f'}$ strongly Stieltjes interlaces with A_f.

The technique of proving this proposition is exceedingly similar to the techniques that have already been established in Sections 4 and 5. Therefore, we will prove some similar lemmas and then give a proof outline of Proposition 6.3.

When $N \geq 335464$, the following lemma mimics Lemma 4.1.

Lemma 6.4. *If*

$$\frac{\pi\left(m'-m-\frac{4\pi}{\sqrt{N}}\right)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'-\frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}},$$

then for all $1 \leq \ell \leq m-2$ *and* $0 \leq \ell' \leq m'-2$,

$$\left|\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'}\right| < \left|\phi^*_{k,\ell+1} - \phi^*_{k,\ell}\right|,$$

and for all $0 \leq \ell \leq m-2$ *and* $1 \leq \ell' \leq m'-2$,

$$\left|\phi^*_{k',\ell'+1} - \phi^*_{k',\ell'}\right| < \left|\theta^*_{k,\ell+1} - \theta^*_{k,\ell}\right|.$$

Proof. Identical to Lemma 4.1. □

Definition 6.5. *For each* $0 \leq \ell \leq m-2$, *we define the set*

$$W_\ell := \left\{0 \leq \ell' \leq m'-2 \,:\, \phi_{k',\ell'} < \theta_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}}\right\}.$$

Let $\check{\ell}$ *denote the largest element of* W_ℓ.

Similarly, for each $1 \leq \ell \leq m-2$, *we define the set*

$$Z_\ell := \left\{0 \leq \ell' \leq m'-2 \,:\, \theta_{k',\ell'} < \phi_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}}\right\}.$$

Let $\bar{\ell}$ *denote the largest element of* Z_ℓ.

Remark. *By Lemma 3.12 and a similar argument to Lemma 5.2, these sets are well defined.*

The following lemma mimics Lemmas 5.5-5.7.

Lemma 6.6. *Suppose* $k' > k \geq 78$ *and* $N \geq 1$. *Then for all* $0 \leq \ell \leq m-2$, *we have*

(1) $\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m'-2\pi/\sqrt{N}} > \theta_{k,\ell+1} - \phi_{k',\check{\ell}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}$,
(2) $\theta^*_{k,\ell+1} > \phi^*_{k',\check{\ell}}$,
(3) $\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > \theta^*_{k',\check{\ell}+1} - \theta^*_{k',\check{\ell}}$.

Proof. (1) By the definition of $\check{\ell}$,

$$\theta_{k,\ell+1} - \phi_{k',\check{\ell}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Suppose for the sake of contradiction that

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m'-\frac{2\pi}{\sqrt{N}}} \leq \theta_{k,\ell+1} - \phi_{k',\check{\ell}}.$$

By Lemma 3.8 part (1), $\theta_{k,m-2} < \phi_{k',m'-1}$. Then by Lemma 3.5 and our assumption, $\phi_{k',m'-1} > \theta_{k,m-2} \geq \theta_{k,\ell+1} > \phi_{k',\check{\ell}}$, and $\check{\ell} \leq m'-2$. By Lemma 3.6,

$$\phi_{k',\check{\ell}+1} - \phi_{k',\check{\ell}} < \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} \leq \theta_{k,\ell+1} - \phi_{k',\check{\ell}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}},$$

and thus

$$\phi_{k',\check{\ell}+1} < \theta_{k,\ell+1} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Therefore, $\check{\ell}+1 \in W_\ell$, a contradiction and

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} > \theta_{k,\ell+1} - \phi_{k',\check{\ell}},$$

as desired.

(2) Identical to the proof of Lemma 5.6.

(3) We will follow the proof of Lemma 5.7. By (2) and Lemma 3.1,

$$\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} > \theta_{k,\ell+1} - \phi_{k',\check{\ell}} > \cos\phi_{k',\check{\ell}} - \cos\theta_{k,\ell+1},$$

giving us

$$\begin{aligned}&\left(m' - \frac{2\pi}{\sqrt{N}}\cos\phi_{k',\check{\ell}}\right) - \left(m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}\right)\\ >&\frac{2\pi}{\sqrt{N}}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}\cos\phi_{k',\check{\ell}}}\\ >&\frac{2\pi^2}{\sqrt{N}\left(m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}\right)\left(m' - \frac{2\pi}{\sqrt{N}}\cos\phi_{k',\check{\ell}}\right)}\left(\frac{m'-m}{\frac{2\pi}{\sqrt{N}}} - \frac{C(78,1)}{2^m\sqrt{N}} - \frac{C(78,1)}{2^{m'}\sqrt{N}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}}\right),\end{aligned}$$

and by (5.4), we get

$$\frac{\pi}{m - \frac{2\pi}{\sqrt{N}}\cos\theta_{k,\ell+1}} - \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}\cos\phi_{k',\check{\ell}}} > \frac{2C(78,1)}{2^m\sqrt{N}} + \frac{2C(78,1)}{2^{m'}\sqrt{N}}.$$

Recall that Lemma 2.5 states $C(78,1) \geq C(k,N) > C(k',N)$. By (3.1),

$$(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\phi_{k',\check{\ell}+1} - \phi_{k',\check{\ell}}) > \frac{2C(78,1)}{2^m\sqrt{N}} + \frac{2C(78,1)}{2^{m'}\sqrt{N}} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}}.$$

Therefore by (2.8),

$$\begin{aligned}&(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\phi_{k',\check{\ell}+1} - \phi_{k',\check{\ell}})\\ >&(\theta_{k,\ell+1} - \theta^*_{k,\ell+1}) + (\theta^*_{k,\ell} - \theta_{k,\ell}) + (\phi^*_{k',\check{\ell}+1} - \phi_{k',\check{\ell}+1}) + (\phi_{k',\check{\ell}} - \phi^*_{k',\check{\ell}}),\end{aligned}$$

which implies

$$\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > \phi^*_{k',\check{\ell}+1} - \phi^*_{k',\check{\ell}},$$

our desired result. $\square$

The next lemma follows from Lemmas 5.12-5.14, by applying the same proof as Lemma 6.6.

Lemma 6.7. *Suppose $k' > k \geq 78$ and $N \geq 1$. Then, for all $1 \leq \ell \leq m-2$,*

(1) $\frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}} + \frac{\pi}{m'-2\pi/\sqrt{N}} > \phi_{k,\ell+1} - \theta_{k',\bar{\ell}} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}$,
(2) $\phi^*_{k,\ell+1} > \theta^*_{k',\bar{\ell}}$,
(3) $\phi^*_{k,\ell+1} - \phi^*_{k,\ell} > \theta^*_{k',\bar{\ell}+1} - \theta^*_{k',\bar{\ell}}$.

Similarly to Lemmas 5.2 and 5.9, we show the smallest element of $A_{f'}$ is less than the smallest element of A_f, and the largest element of $A_{f'}$ is greater than the largest element of A_f.

Lemma 6.8. *Suppose either $k' > k \geq 6$ and $N \geq 335464$, or $k' > k \geq 38$. Then*

(1) $0 < \phi^*_{k',1} < \theta^*_{k,1}$ *and* $\theta^*_{k,m-2} < \phi^*_{k',m'-1} < \pi$,
(2) $0 < \theta^*_{k',0} < \phi^*_{k,1}$ *and* $\phi^*_{k,m-1} < \theta^*_{k',m'-1} < \pi$.

Proof. (1) By (4.6), when $k' > k \geq 6$ and $N \geq 335464$, we have

$$\frac{\pi(m'-m)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(k,N)}{2^m\sqrt{N}} + \frac{2C(k',N)}{2^{m'}\sqrt{N}},$$

and when $k' > k \geq 78$, by (5.2),

$$\frac{\pi(m'-m)}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m'+\frac{2\pi}{\sqrt{N}}\right)} > \frac{C(78,1)}{2^m\sqrt{N}} + \frac{C(78,1)}{2^{m'}\sqrt{N}}.$$

Thus, by Lemma 3.12, the desired statement holds.

(2) The proof is identical. $\square$

Proof of Proposition 6.3. (1) When $k' > k \geq 78$, similarly as the proof of Proposition 5.8 using Lemmas 5.3, 5.10, 6.6, and 6.8 part (1), we can achieve the desired statement. On the other hand, when $N \geq 335464$ and $k' > k \geq 6$, similarly to Proposition 4.5, we can use Lemmas 4.4, 4.6, 6.4, and 6.8 part (1), to show $A_{f'}$ strongly Stieltjes interlaces with $A_f \setminus \{\theta_{k,0}, \theta_{k,m-1}\}$. If $\theta_{k,0} < \phi_{k',1} < \theta_{k,1}$, then $A_{f'}$ Stieltjes interlaces with $A_f \setminus \theta_{k,m-1}$. Else, if $\phi_{k',1} \leq \theta_{k,0}$, by Lemma 6.4, there exists some ℓ' such that $\theta_{k,0} < \phi_{k',\ell'} < \theta_{k,1}$, and again $A_{f'}$ Stieltjes interlaces with $A_f \setminus \theta_{k,m-1}$. We can use a similar argument to add $\theta_{k,m-1}$ back in to obtain that $A_{f'}$ Stieltjes interlaces with A_f.

(2) When $k' > k \geq 78$, following the proof of Proposition 5.8 using Lemmas 5.3, 5.10, 6.7, and 6.8 part (2), we can achieve the desired statement. On the other hand, when $N \geq 335464$ and $k' > k \geq 6$, identically to Proposition 4.5, we can use Lemmas 4.4, 4.6, 6.4, and 6.8 part (2), to show that $A_{f'}$ strongly Stieltjes interlaces with A_f. $\square$

7. The case when $k = 4$

In this section, we will prove Theorem 1.7. For brevity, we will define $N_0 := \lceil 335464^{4.41} \rceil$. We only need to consider the case when $\varepsilon(f) = 1$, since A_f is empty when $\varepsilon(f) = -1$; see Definition 1.5. Recall that $\theta_{4,0} = \pi/2$. Thus, by Proposition 2.3 and Lemma 2.4 we know that for $N \geq N_0$,

$$\left|\frac{\pi}{2} - \theta^*_{4,0}\right| < \frac{C(4,N_0)}{N^{1/8}},$$

where $C(4, N_0) \approx 871.455$ [9, 4bigO.sage].

7.1. **When $\varepsilon(f') = 1$.** To prove strong Stieltjes interlacing between $A_{f'}$ and A_f, it suffices to show $0 < \theta^*_{k',0} < \theta^*_{4,0} < \theta^*_{k',m'-1} < \pi$. The following proposition proves Theorem 1.7 parts (1) and (2).

Proposition 7.1. *For $k = 4$, $k' \geq 6$, and $N \geq N_0 = \lceil 335464^{4.41} \rceil$, the following are true:*

(1) $\theta^*_{k',0} < \theta^*_{4,0} < \theta^*_{k',m'-1}$,
(2) $0 < \theta^*_{k',0}$ and $\theta^*_{k',m'-1} < \pi$.

Proof. We have already shown in Lemma 4.4 that part (2) holds when $N \geq 335464$ and $k' \geq 6$, so the statement certainly holds for $N \geq N_0$. Thus, we need only show $\theta^*_{k',0} < \theta^*_{4,0} < \theta^*_{k',m'-1}$.

Firstly, using Definition 1.3 and Lemma 3.1, we have

$$\theta_{k',0} \leq \frac{\pi}{2\left(m' - \frac{2\pi}{\sqrt{N}}\right)} \quad \text{and} \quad \theta_{k',m'-1} \geq \frac{2\pi m' - \pi}{2\left(m' + \frac{2\pi}{\sqrt{N}}\right)}. \tag{7.1}$$

By Proposition 2.3 and Lemmas 2.4 and 2.5,

$$\begin{aligned}
\theta^*_{4,0} - \theta^*_{k',0} &\geq \theta_{4,0} - \theta_{k',0} - \left|\theta^*_{4,0} - \theta_{4,0}\right| - \left|\theta^*_{k',0} - \theta_{k',0}\right| \\
&\geq \frac{\pi}{2} - \frac{\pi}{2\left(m' - \frac{2\pi}{\sqrt{N}}\right)} - \frac{C(4, N_0)}{N^{1/8}} - \frac{C(k', N_0)}{2^{m'}\sqrt{N}} \\
&\geq \frac{\pi}{2} - \frac{\pi}{2\left(m' - \frac{2\pi}{\sqrt{N}}\right)} - \frac{C(4, N_0)}{N^{1/8}} - \frac{C(6, 335464)}{2^{m'}\sqrt{N}} \\
&\geq \frac{\pi}{2} - \frac{\pi}{2\left(2 - \frac{2\pi}{\sqrt{N_0}}\right)} - \frac{871.455}{N_0^{1/8}} - \frac{221.628}{2^2\sqrt{N_0}} \\
&\approx 0.0015 > 0.
\end{aligned}$$

Next, by Proposition 2.3 and Lemma 2.5 we may bound

$$\begin{aligned}
\theta^*_{k',m'-1} - \theta^*_{4,0} &\geq \theta_{k',m'-1} - \theta_{4,0} - \left|\theta^*_{4,0} - \theta_{4,0}\right| - \left|\theta^*_{k',m'-1} - \theta_{k',m'-1}\right| \\
&\geq \frac{2\pi m' - \pi}{2\left(m' + \frac{2\pi}{\sqrt{N}}\right)} - \frac{\pi}{2} - \frac{C(4, N_0)}{N^{1/8}} - \frac{C(k', N_0)}{2^{m'}\sqrt{N}} \\
&\geq \frac{2\pi m' - \pi}{2\left(m' + \frac{2\pi}{\sqrt{N}}\right)} - \frac{\pi}{2} - \frac{C(4, N_0)}{N^{1/8}} - \frac{C(6, 335464)}{2^{m'}\sqrt{N}} \\
&\geq \frac{4\pi - \pi}{2\left(2 + \frac{2\pi}{\sqrt{N_0}}\right)} - \frac{\pi}{2} - \frac{871.455}{N_0^{1/8}} - \frac{221.628}{2^2\sqrt{N_0}} \\
&\approx 0.0015 > 0.
\end{aligned}$$

So for all $N \geq N_0$ and $k' \geq 6$, $\theta^*_{k',0} < \theta^*_{4,0} < \theta^*_{k',m'-1}$. □

7.2. **When $\varepsilon(f') = -1$.** This section will proceed similarly to above. The following proposition proves Theorem 1.7 part (3).

Proposition 7.2. *For $k' \geq 10$ and $N \geq N_0$, the following hold:*

(1) $\phi^*_{k',1} < \theta^*_{4,0} < \phi^*_{k',m'-1}$.
(2) $0 < \phi^*_{k',1}$ *and* $\phi^*_{k',m'-1} < \pi$.

Proof. We have already shown in Lemma 4.4 that (2) holds when $N \geq 335464$, so the statements certainly hold for $N \geq N_0$. Thus, we need only show (1). Firstly, by Definition 1.4 and Lemma 3.1,

$$\phi_{k',1} \leq \frac{\pi}{m' - \frac{2\pi}{\sqrt{N}}} \quad \text{and} \quad \phi_{k',m'-1} \geq \frac{(m'-1)\pi}{m' + \frac{2\pi}{\sqrt{N}}}. \tag{7.2}$$

Similarly to Proposition 7.1, by Proposition 2.3, Lemmas 2.4 and 2.5,

$$\begin{aligned}\theta^*_{4,0} - \phi^*_{k',1} &\geq \theta_{4,0} - \phi_{k',1} - \left|\theta^*_{4,0} - \theta_{4,0}\right| - \left|\phi^*_{k',1} - \phi_{k',1}\right| \\ &\geq \frac{\pi}{2} - \frac{\pi}{4 - \frac{2\pi}{\sqrt{N_0}}} - \frac{871.455}{N_0^{1/8}} - \frac{221.628}{2^4\sqrt{N_0}} \\ &\approx 0.0015 > 0.\end{aligned}$$

Then identically to Proposition 7.1, by Proposition 2.3 and Lemmas 2.4 and 2.5, we have

$$\begin{aligned}\phi^*_{k',m'-1} - \theta^*_{4,0} &\geq \phi_{k',m'-1} - \theta_{4,0} - \left|\theta^*_{4,0} - \theta_{4,0}\right| - \left|\phi^*_{k',m'-1} - \phi_{k',m'-1}\right| \\ &\geq \frac{3\pi}{4 + \frac{2\pi}{\sqrt{N_0}}} - \frac{\pi}{2} - \frac{871.455}{N_0^{1/8}} - \frac{221.628}{2^4\sqrt{N_0}} \\ &\approx 0.0015 > 0.\end{aligned}$$

Thus $\phi^*_{k',1} < \theta^*_{4,0} < \phi^*_{k',m'-1}$, for $k' \geq 10$ and $N \geq \lceil 335464^{4.41} \rceil$. □

8. The Case $N = 1$

When $k < 78$, the upper bound of the difference between actual and sample angles given by (2.8), is too large. Nevertheless, we can directly compute the actual angles and prove strong Stieltjes interlacing between $A_{f'}$ and A_f for k' large enough. This would allow us to verify finite cases when both k and k' are small. In this way, we prove Theorem 1.8.

Definition 8.1.

$$D(f) := \min\{|x - y| : x \neq y \in A_f \cup 0 \cup \pi\} \text{ and } D := \min\{D(f) : f \in S_k(\Gamma_0(1)), k \leq 76\}.$$

In [9, distance.sage], *we compute* $D > 0.03629$.

Lemma 8.2. *Recall* $k' = 2m' + 2$. *When* $k' \geq 188$, *we have*

$$\frac{\pi}{m' - 2\pi} + \frac{2C(k',1)}{2^{m'}} < D. \tag{8.1}$$

Proof. Recall $C(k, N)$ is given in Proposition 2.3. By [9, zgap.sage], we have $C(188, 1) < 8.9 \cdot 10^6$. Thus, when $m' = 93$, $\frac{\pi}{93 - 2\pi} + \frac{2C(188,1)}{2^{93}} < 0.03623 < D$. By Lemma 2.5, the left side of (8.1) decreases as k' increases, so (8.1) holds for every $k' \geq 188$. □

In the next lemma, we gather various results scattered in (7.1), (7.2), and Lemma 3.7.

Lemma 8.3. *Suppose $k' \geq 188$. Then*

(1) $\theta_{k',0} < \frac{\pi}{2(m'-2\pi)} < \frac{\pi}{m'-2\pi}$,
(2) $\phi_{k',1} < \frac{\pi}{m'-2\pi}$,
(3) $\pi - \theta_{k',m'-1} < \frac{\pi}{2m'} < \frac{\pi}{m'-2\pi}$,
(4) $\pi - \phi_{k',m'-1} < \frac{\pi}{m'} < \frac{\pi}{m'-2\pi}$,
(5) $\theta_{k',\ell'+1} - \theta_{k',\ell'} < \frac{\pi}{m'-2\pi}$, *for all* $0 \leq \ell' \leq m'-2$,
(6) $\phi_{k',\ell'+1} - \phi_{k',\ell'} < \frac{\pi}{m'-2\pi}$, *for all* $1 \leq \ell' \leq m'-2$.

Proposition 8.4. *Suppose $k' \geq 188$ and $k \leq 76$. Let $f' \in S_{k'}(\Gamma_0(1)), f \in S_k(\Gamma_0(1))$ be newforms. Then $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Proof. By Lemmas 5.3 and 5.10, when $k' \geq 188$, $\theta^*_{k',\ell'} < \theta^*_{k',\ell'+1}$ for all $0 \leq \ell' \leq m'-2$, and $\phi^*_{k',\ell'} < \phi^*_{k',\ell'+1}$ for all $1 \leq \ell' \leq m'-2$. Combining (8.1), Lemma 8.3 part (5), and applying (2.8), we get

$$\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} < \theta_{k'\ell'+1} - \theta_{k',\ell'} + \frac{2C(k',1)}{2^{m'}} < \frac{\pi}{m'-2\pi} + \frac{2C(k',1)}{2^{m'}} < D.$$

Similarly, using Lemma 8.3 part (6) results in $\phi^*_{k',\ell'+1} - \phi^*_{k',\ell'} < D$. Thus, regardless of the sign of f' or f, the distances between consecutive angles of $A_{f'}$ are always less than those of A_f. In order to show strong Stieltjes interlacing between $A_{f'}$ and A_f, it suffices to show that the first element of $A_{f'}$ is smaller than the first element of A_f, and the last element of $A_{f'}$ is larger than the last element of A_f. When $\varepsilon(f') = 1$, by Lemma 8.3 part (1), (2.8), and (8.1),

$$\theta^*_{k',0} < \theta_{k',0} + \frac{C(k',1)}{2^{m'}} < \frac{\pi}{m'-2\pi} + \frac{2C(k',1)}{2^{m'}} < D.$$

Similarly, when $\varepsilon(f') = -1$, by Lemma 8.3 part (3), $\phi^*_{k',1} < D$. Thus, the first element of $A_{f'}$ is smaller than the first element of A_f. Next, when $\varepsilon(f') = 1$, by Lemma 8.3 part (2), (2.8), and (8.1),

$$\pi - \theta^*_{k',m'-1} < \pi - \theta_{k',m'-1} + \frac{C(k',1)}{2^{m'}} < \frac{\pi}{m'-2\pi} + \frac{2C(k',1)}{2^{m'}} < D.$$

Similarly, when $\varepsilon(f') = -1$, by Lemma 8.3 part (4), $\pi - \phi^*_{k',m'-1} < D$. Therefore, the last element of $A_{f'}$ is greater than the last element of A_f. This completes the proof that $A_{f'}$ strongly Stieltjes interlaces with A_f. ☐

In [9, checking.sage], we have verified that Theorem 1.8 holds for all $k \leq 76$ and $k' \leq 186$. Thus, with Theorem 1.6 and Proposition 8.4, we have proven Theorem 1.8.

9. Interlacing in the level aspect

In this section, we will consider the interlacing in the level aspect. Let $f' \in S_k(\Gamma_0(N'))$ and $f \in S_k(\Gamma_0(N))$ be newforms such that $N' > N$. Depending on the signs of f' and f, we divide the discussion into three cases.

9.1. **The case $\varepsilon(f') = \varepsilon(f) = 1$.** In this case, we write $N' = N(1+\varepsilon)$ for $2^{-n} < \varepsilon \leq 2^{-n+1}$ and $n \geq 0$. To lighten the burden, we will denote the sample zeros by $\theta_{N,\ell}$ and $\theta_{N',\ell}$, respectively. We first compare sample angles and obtain a result similar to Lemma 3.9.

Lemma 9.1. *Suppose $m > 2\pi/\sqrt{N}$ and $N' > N$. Then*

(1) for all $0 \le \ell \le m-1$, $\theta_{N',\ell} < \theta_{N,\ell}$ *and*

$$\theta_{N,\ell} - \theta_{N',\ell} > \frac{2\pi(N'-N)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m+\frac{2\pi}{\sqrt{N'}}\right)\sqrt{NN'}\left(\sqrt{N}+\sqrt{N'}\right)}.$$

(2) for all $0 \le \ell \le m-2$, $\theta_{N,\ell} < \theta_{N',\ell+1}$ *and*

$$\theta_{N',\ell+1} - \theta_{N,\ell} > \frac{\pi}{m+\frac{2\pi}{\sqrt{N}}}.$$

Proof. (1) By Lemma 3.1 and Definition 1.3, it is not hard to see that $\theta_{N,\ell} > \theta_{N',\ell}$. At the same time,

$$\begin{aligned} m(\theta_{N,\ell} - \theta_{N',\ell}) &= \frac{2\pi}{\sqrt{N}}\left(\sin\theta_{N,\ell} - \sin\theta_{N',\ell}\right) + \left(\frac{2\pi}{\sqrt{N}} - \frac{2\pi}{\sqrt{N'}}\right)\sin\theta_{N',\ell} \\ &> \frac{2\pi}{\sqrt{N}}(\theta_{N',\ell} - \theta_{N,\ell}) + \frac{2\pi(N'-N)}{\sqrt{NN'}\left(\sqrt{N}+\sqrt{N'}\right)}\sin\theta_{N',\ell}. \end{aligned} \tag{9.1}$$

Recall from (3.5) and (3.8) the following bounds for $\theta_{N',\ell}$:

$$\theta_{N',\ell} > \frac{\pi}{2m} > \frac{\pi}{2\left(m+\frac{2\pi}{\sqrt{N'}}\right)}, \quad \pi - \theta_{N',\ell} \ge \frac{\pi}{2\left(m+\frac{2\pi}{\sqrt{N'}}\right)}.$$

When $0 \le x \le \frac{\pi}{2}$, we have $\sin(x) \ge \frac{2x}{\pi}$; if $\frac{\pi}{2} \le x < \pi$, then $\sin(x) \ge \frac{2(\pi-x)}{\pi}$. Thus, by (9.1):

$$\theta_{N,\ell} - \theta_{N',\ell} > \frac{\frac{2\pi(N'-N)}{\sqrt{NN'}(\sqrt{N}+\sqrt{N'})}\sin\theta_{N',\ell}}{m+\frac{2\pi}{\sqrt{N}}} \ge \frac{\frac{2\pi(N'-N)}{\sqrt{NN'}(\sqrt{N}+\sqrt{N'})}\frac{1}{m+\frac{2\pi}{\sqrt{N'}}}}{m+\frac{2\pi}{\sqrt{N}}}.$$

(2) On the other hand, it is not hard to show $\theta_{N',\ell+1} > \theta_{N,\ell}$, and

$$\begin{aligned} m(\theta_{N',\ell+1} - \theta_{N,\ell}) &= \pi + \frac{2\pi}{\sqrt{N'}}\sin\theta_{N',\ell+1} - \frac{2\pi}{\sqrt{N}}\sin\theta_{N,\ell} \\ &> \pi + \frac{2\pi}{\sqrt{N}}\left(\sin\theta_{N',\ell+1} - \sin\theta_{N,\ell}\right). \end{aligned}$$

By Lemma 3.1 again, we obtain

$$\theta_{N',\ell+1} - \theta_{N,\ell} > \frac{\pi}{m+\frac{2\pi}{\sqrt{N}}}.$$

This completes the proof. □

We also have the following result that is similar to Lemma 4.1.

Lemma 9.2. *Suppose* $m > 2\pi/\sqrt{N}$ *and* $N' = N(1+\varepsilon)$ *with* $2^{-n} < \varepsilon \le 2^{-n+1}$ *and* $n \ge 0$.

(1) If $\frac{\pi}{(m+2\pi/\sqrt{N})^2} > \frac{2C(k,N)}{2^{m-n}}(1+2^{-n+1})$, *then for all* $0 \le \ell \le m-1$, $\theta^*_{N,\ell} > \theta^*_{N',\ell}$.

(2) If $\frac{\pi}{m+2\pi/\sqrt{N}} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$, *then for all* $0 \le \ell \le m-2$, $\theta^*_{N',\ell+1} > \theta^*_{N,\ell}$.

Proof. (1) By Lemma 9.1, we obtain

$$\begin{aligned}\theta_{N,\ell} - \theta_{N',\ell} &> \frac{2\pi\varepsilon}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m + \frac{2\pi}{\sqrt{N'}}\right)\sqrt{N}\sqrt{1+\varepsilon}\left(1 + \sqrt{1+\varepsilon}\right)} \\ &> \frac{\pi\varepsilon}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m + \frac{2\pi}{\sqrt{N'}}\right)\sqrt{N}\left(1+\varepsilon\right)}.\end{aligned}$$

Therefore by (2.8),

$$\theta^*_{N,\ell} - \theta^*_{N',\ell} > \frac{\pi\varepsilon}{\left(m + \frac{2\pi}{\sqrt{N}}\right)\left(m + \frac{2\pi}{\sqrt{N'}}\right)\sqrt{N}(1+\varepsilon)} - \frac{C(k,N)}{2^m\sqrt{N}} - \frac{C(k,N')}{2^m\sqrt{N'}},$$

which, combined with Lemma 2.5, implies

$$\theta^*_{N,\ell} - \theta^*_{N',\ell} > \frac{\pi}{\left(m + \frac{2\pi}{\sqrt{N}}\right)^2} - \frac{2C(k,N)}{2^{m-n}}(1 + 2^{-n+1}).$$

(2) This follows immediately from (2.8) and Lemma 9.1. ☐

We are now ready to establish the following interlacing property when $\varepsilon(f') = \varepsilon(f) = 1$.

Proposition 9.3. *Suppose $N' = N(1+\varepsilon)$ where $2^{-n} < \varepsilon \le 2^{-n+1}$ and $n \ge 0$. If $k \ge 78 + 2n$, then*

(1) for all $0 \le \ell \le m-1$, $\theta^*_{N,\ell} > \theta^*_{N',\ell}$.
(2) for all $0 \le \ell \le m-2$, $\theta^*_{N',\ell+1} > \theta^*_{N,\ell}$.
(3) for all $0 \le \ell \le m-2$, $\theta^*_{N,\ell} < \theta^*_{N,\ell+1}$ and $\theta^*_{N',\ell} < \theta^*_{N',\ell+1}$.
(4) $0 < \theta^*_{N,0}$ and $\theta^*_{N',m-1} < \pi$.

Proof. (1) Since $k := 2m+2$, if $k \ge 78 + 2n$, then $m \ge 38 + n$. Combining this with Lemma 2.5,

$$\frac{C(78,1)}{2^{37}}(1+2) > \frac{2C(k,N)}{2^{m-n}}\left(1 + 2^{-n+1}\right).$$

By [9, zgap.sage], $C(78,1) < 2.434 \cdot 10^7$, so that

$$\frac{\pi}{\left(m + \frac{2\pi}{\sqrt{N}}\right)^2} > \frac{2C(78,1)}{2^m}(1+2) > \frac{2C(k,N)}{2^{m-n}}(1 + 2^{-n+1})$$

holds for all $m \ge 38 + n$. Thus by Lemma 9.2 part (1), $\theta^*_{N,\ell} > \theta^*_{N',\ell}$ for all $0 \le \ell \le m-1$.

(2) When $m = 38$ by Lemma 2.5,

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{\pi}{m + 2\pi} \approx 0.0709 > 0.00008 \approx \frac{2C(78,1)}{2^m} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$$

Therefore it is clear that for $m \ge 38 + n$,

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$$

and by Lemma 9.2 part (2), the desired statement holds.

Parts (3) and (4) follow from Lemma 3.11 part (2) and Lemma 5.3. ☐

9.2. **The case $\varepsilon(f') = \varepsilon(f) = -1$.** This section proceeds similarly to the previous subsection. However, the lower bounds for sample angles are slightly different. First, similar to Lemma 9.1, we have the following.

Lemma 9.4. *Suppose $m > 2\pi/\sqrt{N}$ and $N' > N$. Then*

(1) for all $1 \le \ell \le m-1$, $\phi_{N',\ell} < \phi_{N,\ell}$ and

$$\phi_{N,\ell} - \phi_{N',\ell} > \frac{2\pi(N'-N)}{\left(m+\frac{2\pi}{\sqrt{N}}\right)\left(m+\frac{2\pi}{\sqrt{N'}}\right)\sqrt{NN'}\left(\sqrt{N}+\sqrt{N'}\right)}.$$

(2) for all $1 \le \ell \le m-2$, $\phi_{N,\ell} < \phi_{N',\ell+1}$ and

$$\phi_{N',\ell+1} - \phi_{N,\ell} > \frac{\pi}{m+\frac{2\pi}{\sqrt{N}}}.$$

Proof. The proof is almost identical to that of Lemma 9.1, except that we instead use the lower bounds

$$\phi_{k,N} > \frac{\pi}{m} > \frac{\pi}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)} \quad \text{and} \quad \pi - \phi_{k,N} > \frac{\pi}{m+\frac{2\pi}{\sqrt{N}}} > \frac{\pi}{2\left(m+\frac{2\pi}{\sqrt{N}}\right)},$$

which can be obtained similarly as in (3.5) and (3.8). □

The next lemma resembles Lemma 9.2 and is proved identically.

Lemma 9.5. *Suppose $m > 2\pi/\sqrt{N}$ and $N' = N(1+\varepsilon)$ with $2^{-n} < \varepsilon \le 2^{-n+1}$ and $n \ge 0$.*

(1) If $\frac{\pi}{(m+2\pi/\sqrt{N})^2} > \frac{2C(k,N)}{2^{m-n}}(1+2^{-n+1})$, then for all $1 \le \ell \le m-1$, $\phi^*_{N,\ell} > \phi^*_{N',\ell}$.

(2) If $\frac{\pi}{m+2\pi/\sqrt{N}} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$, then for all $1 \le \ell \le m-2$, $\phi^*_{N',\ell+1} > \phi^*_{N,\ell}$.

The following proposition establishes the interlacing between $A_{f'}$ and A_f when $\varepsilon(f') = \varepsilon(f) = -1$, and is similar to Proposition 9.3.

Proposition 9.6. *Suppose $N' = N(1+\varepsilon)$ where $2^{-n} < \varepsilon \le 2^{-n+1}$ and $n \ge 0$. If $k \ge 78+2n$, then*

(1) For all $1 \le \ell \le m-1$, $\phi^*_{N,\ell} > \phi^*_{N',\ell}$,
(2) For all $1 \le \ell \le m-2$, $\phi^*_{N',\ell+1} > \phi^*_{N,\ell}$,
(3) For all $1 \le \ell \le m-2$, $\phi^*_{N,\ell} < \phi^*_{N,\ell+1}$ and $\phi^*_{N',\ell} < \phi^*_{N',\ell+1}$,
(4) $0 < \theta^*_{N,1}$ and $\theta^*_{N',m-1} < \pi$.

Proof. The proof is identical to the proof of Proposition 9.3. □

9.3. **The case $\varepsilon(f') \neq \varepsilon(f)$.** This case is similar to the preceding sections.

Lemma 9.7. *Suppose $m > 2\pi/\sqrt{N}$. Then:*

(1) for all $1 \le \ell \le m-1$, $\theta_{N,\ell} - \phi_{N',\ell} \ge \frac{\pi}{2\left(m+2\pi/\sqrt{N}\right)}$.
(2) if $N' > N \ge 16$, then for all $0 \le \ell \le m-2$, $\phi_{N',\ell+1} - \theta_{N,\ell} \ge \frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}}$.
(3) if $N' > N \ge 16$, then for all $1 \le \ell \le m-1$, $\theta_{N',\ell} - \phi_{N,\ell} \ge \frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}}$.
(4) for all $0 \le \ell \le m-2$, $\phi_{N,\ell+1} - \theta_{N',\ell} \ge \frac{\pi}{2(m+2\pi/\sqrt{N})}$.

Proof. (1) Suppose on the contrary that $\phi_{N',\ell} \geq \theta_{N,\ell}$. Taking the difference of Definitions 1.3, 1.4, and applying Lemma 3.1,

$$\begin{aligned} m\left(\theta_{N,\ell} - \phi_{N',\ell}\right) &= \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}} \sin\theta_{N,\ell} - \frac{2\pi}{\sqrt{N'}} \sin\phi_{N',\ell} \\ &\geq \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}} (\theta_{N,\ell} - \phi_{N',\ell}). \end{aligned}$$

This leads to a contradiction since $m > 2\pi/\sqrt{N}$. Thus $\theta_{N,\ell} > \phi_{N',\ell}$ and

$$m\left(\theta_{N,\ell} - \phi_{N',\ell}\right) \geq \frac{\pi}{2} + \frac{2\pi}{\sqrt{N}} (\phi_{N',\ell} - \theta_{N,\ell}),$$

implying that

$$\theta_{N,\ell} - \phi_{N',\ell} \geq \frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)}.$$

(2) Taking the difference of Definitions 1.3 and 1.4,

$$\begin{aligned} m(\phi_{N',\ell+1} - \theta_{N,\ell}) &= \frac{\pi}{2} + \frac{2\pi}{\sqrt{N'}} \sin\phi_{N',\ell+1} - \frac{2\pi}{\sqrt{N}} \sin\theta_{N,\ell} \\ &> \frac{\pi}{2} - \frac{2\pi}{\sqrt{N}}. \end{aligned}$$

Since $N \geq 16$, we get $\phi_{N',\ell+1} > \theta_{N,\ell}$ and $\phi_{N',\ell+1} - \theta_{N,\ell} \geq \frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}}$.

Parts (3) and (4) are similar to parts (2) and (1), respectively. ☐

The next lemma is similar to Lemma 9.2 and is proved similarly.

Lemma 9.8. *Suppose $m > 2\pi/\sqrt{N}$. Then*

(1) if $\frac{\pi}{2(m+2\pi/\sqrt{N})} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$, for all $1 \leq \ell \leq m-1$, $\theta^*_{N,\ell} > \phi^*_{N',\ell}$.
(2) if $\frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$, for all $0 \leq \ell \leq m-2$, $\phi^*_{N',\ell+1} > \theta^*_{N,\ell}$.
(3) if $\frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$, for all $1 \leq \ell \leq m-1$, $\theta^*_{N',\ell} > \phi^*_{N,\ell}$.
(4) if $\frac{\pi}{2(m+2\pi/\sqrt{N})} > \frac{C(k,N)}{2^m\sqrt{N}} + \frac{C(k,N')}{2^m\sqrt{N'}}$, for all $0 \leq \ell \leq m-2$, $\phi^*_{N,\ell+1} > \theta^*_{N',\ell}$.

The following proposition establishes interlacing between $A_{f'}$ and A_f when $\varepsilon(f') \neq \varepsilon(f)$.

Proposition 9.9. *Suppose $N' > N \geq 17$ and $k \geq 32$. Then*

(1) for all $0 \leq \ell \leq m-1$, $\theta^*_{N,\ell} > \phi^*_{N',\ell}$.
(2) for all $0 \leq \ell \leq m-2$, $\phi^*_{N',\ell+1} > \theta^*_{N,\ell}$.
(3) for all $0 \leq \ell \leq m-1$, $\theta^*_{N',\ell} > \phi^*_{N,\ell}$.
(4) for all $0 \leq \ell \leq m-2$, $\phi^*_{N,\ell+1} > \theta^*_{N',\ell}$.
(5) for all $0 \leq \ell \leq m-2$, $\theta^*_{N,\ell+1} > \theta^*_{N,\ell}$ and $\theta^*_{N',\ell+1} > \theta^*_{N',\ell}$.
(6) for all $0 \leq \ell \leq m-2$, $\phi^*_{N,\ell+1} > \phi^*_{N,\ell}$ and $\phi^*_{N',\ell+1} > \phi^*_{N',\ell}$.
(7) $0 < \phi^*_{N',1}, \phi^*_{N,1}$ and $\theta^*_{N,m-1}, \theta^*_{N,m-1} < \pi$.

Proof. (1) By [9, zgap.sage], we may bound $C(32, 17) < 141.671$. We first wish to show

$$\frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)} > \frac{2C(32, 17)}{2^m\sqrt{N}}.$$

When $m = 15$ and $N = 17$, we have $0.095\ldots > 0.002\ldots$, and as N increases, the left hand side increases while the right hand side decreases. Additionally for a fixed N, as m increases it is clear this inequality will continue to hold. Thus, by Lemma 2.5,

$$\begin{aligned}\frac{\pi}{2\left(m + \frac{2\pi}{\sqrt{N}}\right)} &> \frac{2C(32, 17)}{2^m\sqrt{N}} \\ &> \frac{C(k, N)}{2^m\sqrt{N}} + \frac{C(k, N')}{2^m\sqrt{N'}},\end{aligned} \tag{9.2}$$

so that by Lemma 9.8 part (1), we have $\theta^*_{N,\ell} > \phi^*_{N',\ell}$ for all $0 \leq \ell \leq m - 1$.

(2) Here we want to show

$$\frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}} > \frac{2C(32, 17)}{2^m\sqrt{N}}.$$

When $m = 15$ and $N = 17$, we have $0.003\ldots > 0.002\ldots$. Again, as N increases, the left hand side increases while the right hand side decreases, and for a fixed N, as m increases it is clear this inequality will continue to hold. Using Lemma 2.5,

$$\frac{\pi}{2m} - \frac{2\pi}{m\sqrt{N}} > \frac{C(k, N)}{2^m\sqrt{N}} + \frac{C(k, N')}{2^m\sqrt{N'}}.$$

Parts (3) and (4) follow from (9.2) and (9.2) using Lemma 9.8 parts (3) and (4).

For the remaining parts, by (9.2) and Lemma 2.5,

$$\frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{\pi}{m + \frac{2\pi}{\sqrt{N}}} > \frac{2C(k, N)}{2^m\sqrt{N}} > \frac{2C(k, N')}{2^m\sqrt{N'}}.$$

Thus by Lemmas 3.10 parts (1) and (2) and Lemma 3.11 parts (2) and (4), the remaining statements follow. ☐

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References

1. Archer Clayton, Helen Dai, Tianyu Ni, Hui Xue, and Jake Zummo, *Nonvanishing of second coefficients of Hecke polynomials*, revision submitted to J. Number Theory, 2024.
2. John Brian Conrey, David W. Farmer, and Özlem Imamoglu, *The nontrivial zeros of period polynomials of modular forms Lie on the unit circle*, Int. Math. Res. Not. IMRN (2013), no. 20, 4758–4771. MR 3118875
3. Fred Diamond and Jerry Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005. MR 2112196
4. Ahmad El-Guindy and Wissam Raji, *Unimodularity of zeros of period polynomials of Hecke eigenforms*, Bull. Lond. Math. Soc. **46** (2014), no. 3, 528–536. MR 3210708
5. William Frendreiss, Jennifer Gao, Austin Lei, Amy Woodall, Hui Xue, and Daozhou Zhu, *Stieltjes interlacing of the zeros of j_n*, Canad. Math. Bull. **65** (2022), no. 4, 976–993. MR 4521147

6. ______, *A Stieltjes separation property of zeros of Eisenstein series*, Kyushu J. Math. **76** (2022), no. 2, 407–439. MR 4495568
7. Trevor Griffin, Nathan Kenshur, Abigail Price, Bradshaw Vandenberg-Daves, Hui Xue, and Daozhou Zhu, *Interlacing of zeros of Eisenstein series*, Kyushu J. Math. **75** (2021), no. 2, 249–272. MR 4323910
8. Seokho Jin, Wenjun Ma, Ken Ono, and Kannan Soundararajan, *Riemann hypothesis for period polynomials of modular forms*, Proc. Natl. Acad. Sci. USA **113** (2016), no. 10, 2603–2608. MR 3482847
9. Kevin Huu Le and Daozhou Zhu, *Bounds of zeros*, 07 2023, https://doi.org/10.5281/zenodo.8327538.
10. Wen Ch'ing Winnie Li, *Newforms and functional equations*, Math. Ann. **212** (1975), 285–315. MR 369263
11. Hiroshi Nozaki, *A separation property of the zeros of Eisenstein series for* $\mathrm{SL}(2,\mathbb{Z})$, Bull. Lond. Math. Soc. **40** (2008), no. 1, 26–36. MR 2409175
12. Goro Shimura, *Introduction to the arithmetic theory of automorphic functions.*, Iwanami Shoten Publishers, Tokyo; Princeton University Press, Princeton, N.J.,,, 1971, Publications of the Mathematical Society of Japan, No. 11. MR 314766
13. Gábor Szegő, *Orthogonal polynomials*, fourth ed., American Mathematical Society Colloquium Publications, Vol. XXIII, American Mathematical Society, Providence, R.I., 1975. MR 0372517

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