GUAN-ZHOU'S UNIFIED VERSION OF OPTIMAL L^2 EXTENSION THEOREM ON WEAKLY PSEUDOCONVEX KÄHLER MANIFOLDS

QI'AN GUAN, ZHITONG MI, AND ZHENG YUAN

Abstract. In this note, we establish Guan-Zhou's unified version of optimal L^2 extension theorem for holomorphic vector bundles with smooth hermitian metrics on weakly pseudoconvex Kähler manifolds. Combining with previous work of Guan-Mi-Yuan, we generalize Guan-Zhou's unified version of optimal L^2 extension theorem to weakly pseudoconvex Kähler manifolds.

1. Introduction

We recall the L^2 extension problem (see [8], see also [27]) as follows: let Y be a complex subvariety of a complex manifold M; given a holomorphic object f on Y satisfying certain L^2 estimate on Y, finding a holomorphic extension F of f from Y to M, together with a good or even optimal L^2 estimate of F on M.

The existence part of L^2 extension problem was firstly solved by Ohsawa-Takegoshi [22] and their result is called the Ohsawa-Takegoshi L^2 extension theorem now. After the Ohsawa-Takegoshi L^2 extension theorem, many generalizations of L^2 extension theorem and applications of the theorem were established, e.g., see [2, 3, 6, 9, 10, 19, 23, 25]. Especially, L^2 extension theorems for different gain with non-optimal estimate were obtained by Berndtsson, Demailly, Ohsawa, Mcneal-Varolin, et al, see [1, 7, 21, 24, 26].

The second part of L^2 extension problem was called the L^2 extension problem with optimal estimate or sharp L^2 extension problem (see [27]). The method of undetermined functions was introduced to study the sharp L^2 extension problem by Guan-Zhou-Zhu [18, 30]. For bounded pseudoconvex domains in $\mathbb{C}^n$, Blocki [4] developed the equation in the method of undetermined functions of Guan-Zhou-Zhu [30], and obtained the optimal version of Ohsawa-Takegoshi's L^2 extension theorem in [22]. Using the method of undetermined functions, Guan-Zhou [16] (see also [15, 17]) established an L^2 extension theorem with optimal estimate on Stein manifolds for continuous gain, which implied various optimal versions of L^2 extension theorem. In [28] and [29], Zhou-Zhu proved optimal L^2 extension theorem for smooth gain on weakly pseudoconvex Kähler manifolds.

Recall that, in [16] and [29] (see also [28]), both the line bundle case (with singular metric) and the vector bundles case (with smooth metric) were considered respectively. Note that [29] (see also [28]) did not fully generalize the results in

Date: April 3, 2024.
2020 *Mathematics Subject Classification.* 14F18, 32D15, 32U05, 32Q15.
Key words and phrases. Holomorphic vector bundles, Optimal L^2 extension theorem, Weakly pseudoconvex manifolds.

[16]. It is natural to ask the following question (we posed following question for line bundles in [13]).

Question 1.1. *Can one give unified versions of the optimal L^2 extension theorems (for both line bundles with singular metrics and vector bundles with smooth metrics) of Guan-Zhou [16] and Zhou-Zhu [29] (see also [28]).*

In [13], we presented an optimal L^2 extension theorems for holomorphic vector bundles equipped with singular Nakano-semipositive metrics on weakly pseudoconvex Kähler manifolds, which can be viewed as a unified version of the line bundle case (with singular metrics) in [16] and [29] and hence answered Question 1.1 for line bundles (with singular metrics).

In this note, we give an affirmative answer to Question 1.1 for the vector bundle case (with smooth metric) by presenting an optimal L^2 extension theorem of holomorphic vector bundles with smooth hermitian metrics for continuous gain on weakly pseudoconvex Kähler manifolds, which is a unified version of the optimal L^2 extension theorems for vector bundles (with smooth metrics) of [16] and [29].

1.1. Main result.

Definition 1.2. *A function $\psi : M \to [-\infty, +\infty)$ on a complex manifold M is said to be quasi-plurisubharmonic if ψ is locally the sum of a plurisubharmonic function and a smooth function (or equivalently, if $i\partial\bar{\partial}\psi$ is locally bounded from below). In addition, we say that ψ has neat analytic singularities if every point $z \in M$ possesses an open neighborhood U on which ψ can be written as*

$$\psi = c \log \sum_{1 \le j \le N} |g_j|^2 + v,$$

where $c \ge 0$ is a constant, $g_j \in \mathcal{O}(U)$ and $v \in C^\infty(U)$.

Definition 1.3. *If ψ is a quasi-plurisubharmonic function on an n-dimensional complex manifold M, the multiplier ideal sheaf $\mathcal{I}(\psi)$ is the coherent analytic subsheaf of $\mathcal{O}_M$ defined by*

$$\mathcal{I}(\psi)_z = \left\{ f \in \mathcal{O}_{M,z} : \exists U \ni z, \int_U |f|^2 e^{-\psi} d\lambda < +\infty \right\},$$

where U is an open coordinate neighborhood of z and $d\lambda$ is the Lebesgue measure in the corresponding open chart of $\mathbb{C}^n$.

We say that the singularities of ψ are log canonical along the zero variety $Y = V(I(\psi))$ if $\mathcal{I}((1-\epsilon)\psi)|_Y = \mathcal{O}_M|_Y$ for any $\epsilon > 0$.

Let (M, ω) be an n-dimensional Kähler manifold, and let $dV_{M,\omega} = \frac{1}{n!}\omega^n$ be the corresponding Kähler volume element.

Definition 1.4. *Let ψ be a quasi-plurisubharmonic function on M with neat analytic singularities. Assume that the singularities of ψ are log canonical along the zero variety $Y = V(I(\psi))$. Denote $Y^0 = Y_{\mathrm{reg}}$ the regular point set of Y. If $g \in C_c(Y^0)$ and $\hat{g} \in C_c(M)$ satisfy $\hat{g}|_{Y^0} = g$ and $(supp\, \hat{g}) \cap Y = Y^0$, we set*

$$\int_{Y^0} g dV_{M,\omega}[\psi] = \limsup_{t \to +\infty} \int_{\{-t-1<\psi<-t\}} \hat{g} e^{-\psi} dV_{M,\omega}. \tag{1.1}$$

Remark 1.5 (see [9])**.** *By Hironaka's desingularization theorem, it is not hard to see that the limit in the right of equality* (1.1) *does not depend on the continuous extension $\hat{g}$ and $dV_{M,\omega}[\psi]$ is well defined on Y^0 .*

We recall the following class of functions, so called "gain".

Definition 1.6 (see [16]). *Let $T \in (-\infty, +\infty)$ and $\delta \in (0, +\infty)$. Let $\mathcal{G}_{T,\delta}$ be the class of functions $c(t)$ which satisfies the following statements,*
(1) $c(t)$ is a continuous positive function on $[T, +\infty)$,
(2) $\int_T^{+\infty} c(t)e^{-t}dt < +\infty$,
(3) for any $t > T$, the following inequality holds,

$$\begin{aligned}&\left(\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1\right)^2 > \\ &c(t)e^{-t}\left(\int_T^t (\frac{1}{\delta}c(T)e^{-T} + \int_T^{t_2} c(t_1)e^{-t_1}dt_1)dt_2 + \frac{1}{\delta^2}c(T)e^{-T}\right).\end{aligned} \tag{1.2}$$

Remark 1.7. *The number $-T$, $\frac{1}{\delta}$ and function $c(t)$ are equal to the number α_0, α_1 and function $\frac{1}{R(-t)e^{-t}}$ in [29]. We would like to use $-T$, $\frac{1}{\delta}$ and $c(t)$ in this note.*

In this note, we establish Guan-Zhou's unified version of optimal L^2 extension theorem for holomorphic vector bundles with smooth hermitian metrics on weakly pseudoconvex Kähler manifolds.

Theorem 1.8 (Main theorem). *Let $c(t) \in \mathcal{G}_{T,\delta}$, where $\delta < +\infty$. Let (M, ω) be a weakly pseudoconvex Kähler manifold. Let $\psi < -T$ be a quasi-plurisubharmonic function on M with neat analytic singularities. Let $Y := V(\mathcal{I}(\psi))$ and assume that ψ has log canonical singularities along Y. Let E be a rank r holomorphic vector bundle over M equipped with a smooth Hermitian metric h. Assume that*
(1) $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi \otimes Id_E \geq 0$ on $M\backslash\{\psi = -\infty\}$ in the sense of Nakano;
(2) $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi \otimes Id_E + \frac{1}{s(-\psi)}\sqrt{-1}\partial\bar{\partial}\psi \otimes Id_E \geq 0$ on $M\backslash\{\psi = -\infty\}$ in the sense of Nakano, where

$$s(t) := \frac{\int_T^t \left(\frac{1}{\delta}c(T)e^{-T} + \int_T^{t_2} c(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta^2}c(T)e^{-T}}{\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1}.$$

Then for every section $f \in H^0(Y^0, (K_M \otimes E)|_{Y^0})$ on $Y^0 = Y_{reg}$ such that

$$\int_{Y^0} |f|^2_{\omega,h} dV_{M,\omega}[\psi] < +\infty, \tag{1.3}$$

there exists a section $F \in H^0(M, K_M \otimes E)$ such that $F|_{Y_0} = f$ and

$$\int_M c(-\psi)|F|^2_{\omega,h}dV_{M,\omega} \leq \left(\frac{1}{\delta}c(T)e^{-T} + \int_T^{+\infty} c(t_1)e^{-t_1}dt_1\right)\int_{Y^0} |f|^2_{\omega,h}dV_{M,\omega}[\psi]. \tag{1.4}$$

Combining Theorem 1.8 with previous work of Guan-Mi-Yuan (see Theorem 1.13 in [13]), we generalize Guan-Zhou's unified version of optimal L^2 extension theorem to weakly pseudoconvex Kähler manifolds.

Remark 1.9. *When M is a Stein manifold, Theorem 1.8 was proved by Guan-Zhou [16]. When $c(t)$ is a smooth function, $\liminf_{t\to+\infty} c(t) > 0$ and $c(t)e^{-t}$ is decreasing with respect to t near $+\infty$, Theorem 1.8 was proved by Zhou-Zhu [29] (see also [28]).*

Remark 1.10. *Note that the curvature condition (1) and (2) in Theorem 1.8 require the Nakano positivity of metric $(E, he^{-\psi})$ and the smooth metric (E, h) itself can not be Nakano semi-positive in Theorem 1.8. In [13], the singular hermitian metric h on E must be singular Nakano semi-positive in the sense of Definition 1.5 in [13].*

2. Preparations

2.1. **Preparations for the proof of main theorem.** In this section, we make some preparations for the proof of main theorem.

We would like to recall some lemmas which will be used in this section.

Lemma 2.1 (Theorem 1.5 in [5]). *Let M be a Kähler manifold, and Z be an analytic subset of M. Assume that Ω is a relatively compact open subset of M possessing a complete Kähler metric. Then $\Omega\backslash Z$ carries a complete Kähler metric.*

Lemma 2.2 (Lemma 6.9 in [5]). *Let Ω be an open subset of $\mathbb{C}^n$ and Z be a complex analytic subset of Ω. Assume that v is a (p,q-1)-form with L^2_{loc} coefficients and h is a (p,q)-form with L^1_{loc} coefficients such that $\bar\partial v = h$ on $\Omega\backslash Z$ (in the sense of distribution theory). Then $\bar\partial v = h$ on Ω.*

Lemma 2.3 (Remark 3.2 in [7]). *Let (M, ω) be a complete Kähler manifold equipped with a (non-necessarily complete) Kähler metric ω, and let Q be a Hermitian vector bundle over M. Assume that η and g are smooth bounded positive functions on M and let $B := [\eta\sqrt{-1}\Theta_Q - \sqrt{-1}\partial\bar\partial\eta - \sqrt{-1}g\partial\eta\wedge\bar\partial\eta, \Lambda_\omega]$. Assume that $\delta \geq 0$ is a number such that $B + \delta I$ is semi-positive definite everywhere on $\wedge^{n,q}T^*M \otimes Q$ for some $q \geq 1$. Then given a form $v \in L^2(M, \wedge^{n,q}T^*M \otimes Q)$ such that $D''v = 0$ and $\int_M \langle (B+\delta I)^{-1}v, v\rangle_Q dV_M < +\infty$, there exists an approximate solution $u \in L^2(M, \wedge^{n,q-1}T^*M \otimes Q)$ and a correcting term $h \in L^2(M, \wedge^{n,q}T^*M \otimes Q)$ such that $D''u + \sqrt{\delta}h = v$ and*

$$\int_M (\eta + g^{-1})^{-1}|u|^2_Q dV_M + \int_M |h|^2_Q dV_M \leq \int_M \langle (B+\delta I)^{-1}v, v\rangle_Q dV_M. \tag{2.1}$$

Let M be a complex manifold. Let ω be a continuous hermitian metric on M. Let dV_M be a continuous volume form on M. We denote by $L^2_{p,q}(M, \omega, dV_M)$ the spaces of L^2 integrable (p,q)-forms over M with respect to ω and dV_M. It is known that $L^2_{p,q}(M, \omega, dV_M)$ is a Hilbert space.

Lemma 2.4 (see Lemma 9.1 in [12]). *Let $\{u_n\}_{n=1}^{+\infty}$ be a sequence of (p,q)-forms in $L^2_{p,q}(M, \omega, dV_M)$ which is weakly convergent to u. Let $\{v_n\}_{n=1}^{+\infty}$ be a sequence of Lebesgue measurable real functions on M which converges pointwisely to v. We assume that there exists a constant $C > 0$ such that $|v_n| \leq C$ for any n. Then $\{v_n u_n\}_{n=1}^{+\infty}$ weakly converges to vu in $L^2_{p,q}(M, \omega, dV_M)$.*

Let X be an $n-$dimensional complex manifold and ω be a hermitian metric on X. Let Q be a holomorphic vector bundle on X with rank r. Let $\{h_i\}_{i=1}^{+\infty}$ be a family of C^2 smooth hermitian metric on Q and h be a measurable metric on Q such that $\lim_{i\to+\infty} h_i = h$ almost everywhere on X. We assume that h_i is increasingly convergent to h as $i \to +\infty$.

The following optimal L^2 extension theorem for vector bundles with smooth hermitian metric on Stein manifolds will be used in our discussion.

Theorem 2.5 (see Theorem 2.1 in [16])**.** *Let $c(t) \in \mathcal{G}_{T,\delta}$ for some $T \in (-\infty, +\infty)$ and $0 < \delta < +\infty$. Let M be a Stein manifold and ω be a hermitian metric on M. Let h be a smooth hermitian metric on a holomorphic vector bundle E on M with rank r. Let $\psi < -T$ be a quasi-plurisubharmonic function on X with neat analytic singularities. Let $Y := V(\mathcal{I}(\psi))$ and assume that ψ has log canonical singularities along Y. Assume that*
(1) $\sqrt{-1}\Theta_h + \sqrt{-1}\partial\bar{\partial}\psi \otimes Id_E$ is Nakano semi-positive on $M\backslash\{\psi = -\infty\}$,
(2) there exists a continuous function $a(t)$ on $(T, +\infty]$ such that $0 < a(t) \le s(t)$ and $a(-\psi)\sqrt{-1}\Theta_{he^{-\psi}} + \sqrt{-1}\partial\bar{\partial}\psi \otimes Id_E$ is Nakano semi-positive on $M\backslash\{\psi = -\infty\}$, where

$$s(t) := \frac{\int_T^t \left(\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta^2}c(T)e^{-T}}{\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1}.$$

Then for any holomorphic section f of $K_M \otimes E|_Y$ on Y satisfying

$$\int_{Y_0} |f|^2_{\omega,h} dV_{M,\omega}[\psi] < +\infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F|_Y = f$ and

$$\int_M c(-\psi)|F|^2_{\omega,h} dV_{M,\omega} \le \left(\frac{1}{\delta}c(T)e^{-T} + \int_T^{+\infty} c(t_1)e^{-t_1}dt_1\right)\int_{Y_0} |f|^2_{\omega,h} dV_{M,\omega}[\psi].$$

The following lemma will be used in the proof of the main theorem.

Lemma 2.6 (see Theorem 4.4.2 in [20])**.** *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, and φ be a plurisubharmonic function on Ω. For any $w \in L^2_{p,q+1}(\Omega, e^{-\varphi})$ with $\bar{\partial}w = 0$, there exists a solution $s \in L^2_{p,q}(\Omega, e^{-\varphi})$ of the equation $\bar{\partial}s = w$ such that*

$$\int_\Omega \frac{|s|^2}{(1+|z|^2)^2} e^{-\varphi} d\lambda \le \int_\Omega |w|^2 e^{-\varphi} d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$.

Let $c(t)$ belong to class $\mathcal{G}_{T,\delta}$. Recall that

$$s(t) := \frac{\int_T^t \left(\frac{1}{\delta}c(T)e^{-T} + \int_T^{t_2} c(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta^2}c(T)e^{-T}}{\frac{1}{\delta}c(T)e^{-T} + \int_T^t c(t_1)e^{-t_1}dt_1}.$$

We have following regularization lemma for $c(t)$.

Lemma 2.7 (see Lemma 2.34 in [13])**.** *Let $c(t) \in \mathcal{G}_{T,\delta}$. Let $\{\beta_m < 1\}$ be a sequence of positive real numbers such that β_m decreasingly converges to 0 as $m \to +\infty$. Then there exists a sequence of positive functions $c_m(t)$ on $[T, +\infty)$, which satisfies:*
(1) $c_m(t) \in \mathcal{G}_{T,\delta}$;
(2) $c_m(t)e^{-t}$ is decreasing with respect to t near $+\infty$;
(3) $c_m(t)$ is smooth on $[T + 4\beta_m, +\infty)$;
(4) $c_m(t)$ are uniformly convergent to $c(t)$ on any compact subset of $(T, +\infty)$;
(5) $\frac{1}{\delta}c_m(T)e^{-T} + \int_T^{+\infty} c_m(t)e^{-t}dt$ converges to $\frac{1}{\delta}c(T)e^{-T} + \int_T^{+\infty} c(t)e^{-t}dt < +\infty$ as $m \to +\infty$;
(6) For each m, there exists $\kappa_m > 0$ such that

$$S_m(t) := \frac{\int_T^t \left(\frac{1}{\delta}c_m(T)e^{-T} + \int_T^{t_2} c_m(t_1)e^{-t_1}dt_1\right)dt_2 + \frac{1}{\delta^2}c_m(T)e^{-T} + \kappa_m}{\frac{1}{\delta}c_m(T)e^{-T} + \int_T^t c_m(t_1)e^{-t_1}dt_1} > s(t),$$

for any $t \geq T$ *and* $S'_m(t) > 0$ *on* $[T + \beta_m, +\infty)$.

The following Lemma will be used in the proof of the Theorem 1.8.

Lemma 2.8 (see Lemma 2.4 in [14]). *Let* M *be a complex manifold. Let* S *be an analytic subset of* M. *Let* $\{g_j\}_{j=1,2,...}$ *be a sequence of nonnegative Lebesgue measurable functions on* M, *which satisfies that* g_j *are almost everywhere convergent to* g *on* M *when* $j \to +\infty$, *where* g *is a nonnegative Lebesgue measurable function on* M. *Assume that for any compact subset* K *of* $M \backslash S$, *there exist* $s_K \in (0, +\infty)$ *and* $C_K \in (0, +\infty)$ *such that*

$$\int_K {g_j}^{-s_K} dV_M \leq C_K$$

for any j, *where* dV_M *is a continuous volume form on* M.

Let $\{F_j\}_{j=1,2,...}$ *be a sequence of holomorphic* $(n, 0)$ *form on* M. *Assume that* $\liminf_{j\to+\infty} \int_M |F_j|^2 g_j \leq C$, *where* C *is a positive constant. Then there exists a subsequence* $\{F_{j_l}\}_{l=1,2,...}$, *which satisfies that* $\{F_{j_l}\}$ *is uniformly convergent to a holomorphic* $(n, 0)$ *form* F *on* M *on any compact subset of* M *when* $l \to +\infty$, *such that*

$$\int_M |F|^2 g \leq C.$$

3. Proof of Theorem 1.8

In this section, we prove Theorem 1.8.

Proof. As M is weakly pseudoconvex, there exists a smooth plurisubharmonic exhaustion function P on M. Let $M_k := \{P < k\}$ $(k = 1, 2, ...,)$. We choose P such that $M_1 \neq \emptyset$.

Then M_k satisfies $M_k \Subset M_{k+1} \Subset ...M$ and $\cup_{k=1}^n M_k = M$. Each M_k is weakly pseudoconvex Kähler manifold with exhaustion plurisubharmonic function $P_k = 1/(k - P)$.

We will fix k *during our discussion until the last step.*

Step 1: regularization of $c(t)$**.**

As e^ψ is a smooth function on M and $\psi < -T$ on M, we know that

$$\sup_{M_k} \psi < -T - 8\epsilon_k,$$

where $\epsilon_k > 0$ is a real number depending on k.

It follows from $c(t)$ belongs to class $\mathcal{G}_{T,\delta}$, by Lemma 2.7, that we have a sequence of functions $\{c_k(t)\}_{k\in\mathbb{Z}^+}$ which satisfies $c_k(t)$ is continuous on $[T, +\infty)$, and smooth on $[T + 4\epsilon_k, +\infty)$ and other conditions in Lemma 2.7. Condition (6) of Lemma 2.7 tells that

$$S_k(t) := \frac{\int_T^t \left(\frac{1}{\delta} c_k(T)e^{-T} + \int_T^{t_2} c_k(t_1)e^{-t_1} dt_1 \right) dt_2 + \frac{1}{\delta^2} c_k(T)e^{-T} + \kappa_k}{\frac{1}{\delta} c_k(T)e^{-T} + \int_T^t c_k(t_1)e^{-t_1} dt_1} > s(t),$$

for any $t \geq T$ and $S'_k(t) > 0$ on $[T + \epsilon_k, +\infty)$.

As $S_k(t) > S(t)$ on $t \geq T$, we know that

$$S_k(-\psi)\big(\sqrt{-1}\partial\bar{\partial}\varphi + \sqrt{-1}\partial\bar{\partial}\psi \otimes \mathrm{Id}_E\big) + \sqrt{-1}\partial\bar{\partial}\psi \otimes \mathrm{Id}_E \geq 0$$

on $M\backslash\{\psi=-\infty\}$ in the sense of currents. Denote $u_k(t):=-\log(\frac{1}{\delta}c(T)e^{-T}+\int_T^t c_k(t_1)e^{-t_1}dt_1)$. We note that we still have $S_k'(t)-S_k(t)u_k'(t)=1$ and $(S_k(t)+\frac{S_k'^2(t)}{u_k''(t)S_k(t)-S_k''(t)})e^{u_k(t)-t}=\frac{1}{c_k(t)}$.

Step 2: construction of a family of smooth extensions $\tilde{f}$ of f to a neighborhood of Y to M.

Let $\mathcal{U}=\{U_i\}_{i\in I}$ be a locally finite covering of M such that each open subset U_i is Stein. Denote the singular part of Y by Y_{sing}. Note that $Y_{\text{sing}}\cap U_i$ is an analytic subset of $Y\cap U_i$ (see Theorem 4.31 of Chapter 2 in [11]) and U_i is Stein. We can find holomorphic functions $\{g_{i,j}\}_{j=1}^N$ for some $N\in\mathbb{Z}_{\ge1}$ on U_i such that $Y_{\text{sing}}\cap U_i\subset W_i:=\cap_{j=1}^N\{g_{i,j}=0\}$ and W_i does not contain any irreducible component of $Y\cap U_i$. Since U_i is Stein, we know that $U_i\backslash W_i$ is also Stein. Letting $c(t)\equiv1$, it follows from Theorem 2.5 that there exists a holomorphic section $\tilde{f}_i\in\Gamma\big(U_i\backslash W_i,\mathcal{O}_M(K_M\otimes E)\big)$ which is a holomorphic extension of f from $(U_i\backslash W_i)\cap Y$ to $U_i\backslash W_i$ and $\int_{U_i\backslash W_i}|\tilde{f}_i|^2_{\omega,h}dV_{U_i}<\infty$. It follows from $\int_{U_i\backslash W_i}|\tilde{f}_i|^2_{\omega,h}dV_{U_i}<\infty$ that $\tilde{f}_i$ can be extended to a holomorphic section f_i on U_i. Note that $f_i|_{(U_i\backslash W_i)\cap Y}=\tilde{f}_i|_{(U_i\backslash W_i)\cap Y}=f$. As W_i does not contain any irreducible component of $Y\cap U_i$, we know that $f_i|_{U_i\cap Y^0}=f$.

Thus there exists a holomorphic section $f_i\in\Gamma\big(U_i,\mathcal{O}_M(K_M\otimes E)\big)$ which is a holomorphic extension of f from $U_i\cap Y^0$ to U_i. Let $\{\xi_i\}_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$, and denote

$$\tilde{f}:=\sum_{i\in I}\xi_if_i.$$

Then $\tilde{f}$ is smooth on M, and we have

$$\begin{aligned}\bar\partial\tilde{f}|_{U_j}&=\bar\partial\tilde{f}-\bar\partial f_j\\&=\bar\partial(\sum_{i\in I}\xi_if_i)-\bar\partial(\sum_{i\in I}\xi_if_j)\\&=\sum_{i\in I}\bar\partial\xi_i\wedge(f_i-f_j),\ \text{for all } j\in I.\end{aligned}\tag{3.1}$$

Note that $f_i-f_j=0$ on $U_i\cap U_j\cap Y$. It follows from ψ has neat analytic singularities and is log canonical along the zero variety Y that we know that

$$f_i-f_j\in\Gamma\big(U_i\cap U_j,\mathcal{O}_M(K_M\otimes E)\otimes\mathcal{I}(\psi)\big).\tag{3.2}$$

Note that k is fixed until the last step and $M_k\Subset M$. We may assume that $M_k\subset\cup_{i=1}^NU_i$, where N is a positive integer.

Step 3: recall some notations.

Let $\epsilon\in(0,\frac{1}{8})$. Let $\{v_{t_0,\epsilon}\}_{\epsilon\in(0,\frac{1}{8})}$ be a family of smooth increasing convex functions on $\mathbb{R}$, such that:

(1) $v_{t_0,\epsilon}(t)=t$ for $t\ge-t_0-\epsilon$, $v_\epsilon(t)=\mathit{constant}$ for $t<-t_0-1+\epsilon$;

(2) $v_\epsilon''(t)$ are convergence pointwisely to $\mathbb{I}_{(-t_0-1,-t_0)}$,when $\epsilon\to0$, and $0\le v_\epsilon''(t)\le\frac{1}{1-4\epsilon}\mathbb{I}_{(-t_0-1+\epsilon,-t_0-\epsilon)}$ for ant $t\in\mathbb{R}$;

(3) ${v_\epsilon}'(t)$ are convergence pointwisely to $b(t)$ which is a continuous function on $\mathbb{R}$ when $\epsilon \to 0$ and $0 \le {v_\epsilon}'(t) \le 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_{t_0,\epsilon}\}_{\epsilon\in(0,\frac{1}{8})}$ by setting

$$\begin{aligned} v_{t_0,\epsilon}(t) :=& \int_{-\infty}^{t}\left(\int_{-\infty}^{t_1}(\frac{1}{1-4\epsilon}\mathbb{I}_{(-t_0-1+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon})(s)ds\right)dt_1 \\ &- \int_{-\infty}^{-t_0}\left(\int_{-\infty}^{t_1}(\frac{1}{1-4\epsilon}\mathbb{I}_{(-t_0-1+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon})(s)ds\right)dt_1 - t_0, \end{aligned}$$

where $\rho_{\frac{1}{4}\epsilon}$ is the kernel of convolution satisfying $\mathrm{supp}(\rho_{\frac{1}{4}\epsilon}) \subset (-\frac{1}{4}\epsilon, \frac{1}{4}\epsilon)$. Then it follows that

$$ {v_{t_0,\epsilon}}''(t) = \frac{1}{1-4\epsilon}\mathbb{I}_{(-t_0-1+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}(t), $$

and

$$ {v_{t_0,\epsilon}}'(t) = \int_{-\infty}^{t}\left(\frac{1}{1-4\epsilon}\mathbb{I}_{(-t_0-1+2\epsilon,-t_0-2\epsilon)} * \rho_{\frac{1}{4}\epsilon}\right)(s)ds. $$

Note that $\mathrm{supp}\, {v_{t_0,\epsilon}}''(t) \Subset (-t_0-1+\epsilon, -t_0-\epsilon)$ and $\mathrm{supp}\big(1-{v_{t_0,\epsilon}}'(t)\big) \Subset (-\infty, -t_0-\epsilon)$

We also note that $S_k \in C^\infty([T+4\epsilon_k, +\infty))$ satisfies $S_k' > 0$ on $[T+\epsilon_k, +\infty)$ and $u_k \in C^\infty([T+4\epsilon_k, +\infty))$ satisfies $\lim_{t\to+\infty} u_k(t) = -\log(\frac{1}{\delta}c_k(T)e^{-T} + \int_T^{+\infty} c_k(t_1)e^{-t_1}dt_1)$ and $u_k' < 0$. Recall that $u_k(t)$ and $S_k(t)$ satisfy

$$ S_k'(t) - S_k(t)u_k'(t) = 1 $$

and

$$ (S_k(t) + \frac{S_k'^2(t)}{u_k''(t)S_k(t) - S_k''(t)})e^{u_k(t)-t} = \frac{1}{c_k(t)}. $$

Note that $u_k''S_k - S_k'' = -S_k'u_k' > 0$ on $[T+2\epsilon_k, +\infty)$. Denote $\tilde{g}_k(t) := \frac{u_k''S_k - S_k''}{S_k'^2}(t)$, then $\tilde{g}_k(t)$ is a positive smooth function on $[T+4\epsilon_k, +\infty)$.

Denote $\eta := S_k(-v_{t_0,\epsilon}(\psi))$, $\phi := u_k(-v_{t_0,\epsilon}(\psi))$ and $g := \tilde{g}_k(-v_{t_0,\epsilon}(\psi))$. Then η and g are smooth bounded positive functions on M_k such that $\eta + g^{-1}$ is a smooth bounded positive function on M_k.

Denote $\sum := \{\psi = -\infty\}$. As ψ has neat analytic singularities, we know that $\sum$ is an analytic subset of M and ψ is smooth on $M\backslash\sum$. Note that, by Lemma 2.1, $M_k\backslash\sum$ carries a complete Kähler metric. Denote $\tilde{h} := he^{-\psi}e^{-\phi}$ on $M_k\backslash\sum$.

Step 4: some calculations.

We set $B = [\eta\sqrt{-1}\Theta_{\tilde{h}} - \sqrt{-1}\partial\bar{\partial}\eta \otimes \mathrm{Id}_E - \sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\otimes\mathrm{Id}_E, \Lambda_\omega]$ on $M_k\backslash\sum$. Direct calculation shows that

$$\begin{aligned} \partial\bar{\partial}\eta =& -S_k'\big(-v_{t_0,\epsilon}(\psi)\big)\partial\bar{\partial}\big(v_{t_0,\epsilon}(\psi)\big) + S_k''\big(-v_{t_0,\epsilon}(\psi)\big)\partial\big(v_{t_0,\epsilon}(\psi)\big)\wedge\bar{\partial}\big(v_{t_0,\epsilon}(\psi)\big), \\ \eta\Theta_{\tilde{h}} =& \eta\partial\bar{\partial}\phi\otimes\mathrm{Id}_E + \eta\Theta_h + \eta\partial\bar{\partial}\psi\otimes\mathrm{Id}_E \\ =& S_k u_k''\big(-v_{t_0,\epsilon}(\psi)\big)\partial\big(v_{t_0,\epsilon}(\psi)\big)\wedge\bar{\partial}\big(v_{t_0,\epsilon}(\psi)\big)\otimes\mathrm{Id}_E - S_k u_k'\big(-v_{t_0,\epsilon}(\psi)\big)\partial\bar{\partial}\big(v_{t_0,\epsilon}(\psi)\big)\otimes\mathrm{Id}_E \\ &+ S_k\Theta_h + S_k\partial\bar{\partial}\psi\otimes\mathrm{Id}_E. \end{aligned}$$

Hence

$$\begin{aligned}&\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta\otimes \mathrm{Id}_E-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\otimes \mathrm{Id}_E\\ =&S_k\sqrt{-1}\Theta_h+S_k\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E\\ &+\big(S_k'-S_ku_k'\big)\big(v_{t_0,\epsilon}'(\psi)\sqrt{-1}\partial\bar{\partial}(\psi)+v_{t_0,\epsilon}''(\psi)\sqrt{-1}\partial(\psi)\wedge\bar{\partial}(\psi)\big)\otimes \mathrm{Id}_E\\ &+[\big(u_k''S_k-S_k''\big)-\tilde{g}_kS_k'^2]\sqrt{-1}\partial\big(v_{t_0,\epsilon}(\psi)\big)\wedge\bar{\partial}\big(v_{t_0,\epsilon}(\psi)\big)\otimes \mathrm{Id}_E,\end{aligned}$$

where we omit the term $-v_{t_0,\epsilon}(\psi)$ in $(S_k'-S_ku_k')\big(-v_{t_0,\epsilon}(\psi)\big)$ and $[(u_k''S_k-S_k'')-\tilde{g}_kS_k'^2]\big(-v_{t_0,\epsilon}(\psi)\big)$ for simplicity. Note that $S_k'(t)-S_k(t)u_k'(t)=1$, $\frac{u_k''(t)S_k(t)-S_k''(t)}{S_k'^2(t)}-\tilde{g}_k(t)=0$. We have

$$\begin{aligned}&\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta\otimes \mathrm{Id}_E-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\otimes \mathrm{Id}_E\\ =&S_k\sqrt{-1}\Theta_h+S_k\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E \qquad (3.3)\\ &+\big(v_{t_0,\epsilon}'(\psi)\sqrt{-1}\partial\bar{\partial}(\psi)+v_{t_0,\epsilon}''(\psi)\sqrt{-1}\partial(\psi)\wedge\bar{\partial}(\psi)\big)\otimes \mathrm{Id}_E.\end{aligned}$$

We would like to discuss a property of $S_k(t)$.

Lemma 3.1 (see [13])**.** *For large enough t_0, and for any $\varepsilon\in(0,1/4)$, the inequality*

$$S_k(-v_{t_0,\varepsilon}(t))\geq S_k(-t)v_{t_0,\varepsilon}'(t) \qquad (3.4)$$

holds for any $t\in(-\infty,-T)$.

It follows from curvature condition of Theorem 1.8, equality (3.3) and Lemma 3.1 that we have

$$\begin{aligned}&\eta\sqrt{-1}\Theta_{\tilde{h}}-\sqrt{-1}\partial\bar{\partial}\eta\otimes \mathrm{Id}_E-\sqrt{-1}g\partial\eta\wedge\bar{\partial}\eta\otimes \mathrm{Id}_E\\ =&S_k(-v_{t_0,\varepsilon}(t))\sqrt{-1}\Theta_h+S_k(-v_{t_0,\varepsilon}(t))\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E\\ &+\big(v_{t_0,\epsilon}'(\psi)\sqrt{-1}\partial\bar{\partial}(\psi)+v_{t_0,\epsilon}''(\psi)\sqrt{-1}\partial(\psi)\wedge\bar{\partial}(\psi)\big)\otimes \mathrm{Id}_E\\ \geq&S_k\big(-\psi\big)v_{t_0,\epsilon}'(\psi)\Big(\sqrt{-1}\Theta_h+\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E\Big)\\ &+v_{t_0,\epsilon}'(\psi)S_k(-\psi)\frac{1}{S_k(-\psi)}\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E+v_{t_0,\epsilon}''(\psi)\sqrt{-1}\big(\partial\psi\wedge\bar{\partial}\psi\big)\otimes \mathrm{Id}_E\\ =&S_k\big(-\psi\big)v_{t_0,\epsilon}'(\psi)\Big(\sqrt{-1}\Theta_h+\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E+\frac{1}{S_k(-\psi)}\sqrt{-1}\partial\bar{\partial}\psi\otimes \mathrm{Id}_E\Big)\\ &+v_{t_0,\epsilon}''(\psi)\sqrt{-1}\big(\partial\psi\wedge\bar{\partial}\psi\big)\otimes \mathrm{Id}_E\\ \geq&v_{t_0,\epsilon}''(\psi)\sqrt{-1}\big(\partial\psi\wedge\bar{\partial}\psi\big)\otimes \mathrm{Id}_E. \qquad (3.5)\end{aligned}$$

Then by (3.5), we have

$$B\geq v_{t_0,\epsilon}''(\psi)[\sqrt{-1}\big(\partial\psi\wedge\bar{\partial}\psi\big)\otimes \mathrm{Id}_E,\Lambda_\omega] \qquad (3.6)$$

holds on $M_k\backslash\sum$.

Let $\lambda_{t_0}:=D''[(1-v_{t_0,\epsilon}'(\psi))\tilde{f}]$. Then we know that λ_{t_0} is well defined on M_k, $D''\lambda_{t_0}=0$ and

$$\begin{aligned}\lambda_{t_0}&=-v_{t_0,\epsilon}''(\psi)\bar{\partial}\psi\wedge\tilde{f}+\big(1-v_{t_0,\epsilon}'(\psi)\big)D''\tilde{f}\\ &=\lambda_{1,t_0}+\lambda_{2,t_0},\end{aligned}$$

where $\lambda_{1,t_0}:=-v_{t_0,\epsilon}''(\psi)\bar{\partial}\psi\wedge\tilde{f}$ and $\lambda_{2,t_0}:=\big(1-v_{t_0,\epsilon}'(\psi)\big)D''\tilde{f}$. Note that

$$\mathrm{supp}\lambda_{1,t_0}\subset\{-t_0-1+\epsilon<\psi<-t_0-\epsilon\}$$

and

$$\mathrm{supp}\lambda_{2,t_0} \subset \{\psi < -t_0 - \epsilon\}.$$

It follows from inequality (3.6) that we have

$$\begin{aligned}&\langle B^{-1}\lambda_{1,t_0}, \lambda_{1,t_0}\rangle_{\omega,\tilde{h}}|_{M_k\setminus\sum}\\ \le& v''_{t_0,\epsilon}(\psi)|\tilde{f}|^2_{\omega,h}e^{-\psi-\phi}.\end{aligned}$$

Then we know that

$$\begin{aligned}&\int_{M_k\setminus\sum}\langle B^{-1}\lambda_{1,t_0}, \lambda_{1,t_0}\rangle_{\tilde{h}} dV_{M,\omega}\\ \le&\int_{M_k\setminus\sum} v''_{t_0,\epsilon}(\psi)|\tilde{f}|^2_{\omega,h}e^{-\psi-\phi}dV_{M,\omega}\\ \le& I_{1,t_0,\epsilon} := \sup_{M_k} e^{-\phi}\int_{M_k} v''_{t_0,\epsilon}(\psi)|\tilde{f}|^2_{\omega,h}e^{-\psi}dV_{M,\omega}.\end{aligned}\tag{3.7}$$

Denote

$$I_{1,t_0} := (\sup_{t\ge t_0} e^{-u_k(t)})\int_{M_k\cap\{-t_0-1<\psi<-t_0\}}|\tilde{f}|^2_{\omega,h}e^{-\psi}dV_{M,\omega}.$$

It follows from $\tilde{f}$ is a smooth extension of f from Y_0 to M and the definition of $dV_M[\psi]$ and $u(t)$ that we know

$$\begin{aligned}&\limsup_{t_0\to+\infty} I_{1,t_0}\\ \le&\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\limsup_{t_0\to+\infty}\int_{M_k\cap\{-t_0-1<\psi<-t_0\}}|\tilde{f}|^2_{\omega,h}e^{-\psi}dV_{M,\omega}\\ \le&\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0\cap M_k}|f|^2_{\omega,h}dV_{M,\omega}[\psi]\\ \le&\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}\tag{3.8}$$

By inequalities (3.7) and (3.8), we have

$$\begin{aligned}&\limsup_{t_0\to+\infty} I_{1,t_0,\epsilon}\\ \le&\frac{1}{1-4\epsilon}\limsup_{t_0\to+\infty} I_{1,t_0}\\ \le&\frac{1}{1-4\epsilon}\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}\tag{3.9}$$

Denote

$$\begin{aligned}I_{2,t_0} :=&\int_{M_k\setminus\sum}\langle\lambda_{2,t_0}, \lambda_{2,t_0}\rangle_{\tilde{h}}dV_{M,\omega}\\ \le&\int_{M_k\cap\{\psi<-t_0-\epsilon\}}|D''\tilde{f}_{t_0}|^2_{\omega,h}e^{-\psi-\phi}dV_{M,\omega}\\ \le&\big(\sup_{t\ge t_0}e^{-u(t)}\big)\int_{M_k\cap\{\psi<-t_0\}}|D''\tilde{f}_{t_0}|^2_{\omega,h}e^{-\psi}dV_{M,\omega}.\end{aligned}\tag{3.10}$$

It follows from equality (3.1) and Cauchy-Schwarz inequality that when t_0 is big enough,

$$I_{2,t_0} \leq C_8 \sum_{1\leq i,j\leq N} \int_{U_i\cap U_j\cap\{\psi<-t_0\}} |\tilde{f}_i - \tilde{f}_j|^2_{\omega,h} e^{-\psi} dV_{M,\omega}, \tag{3.11}$$

where $C_8 > 0$ is a real number independent of t_0.

For any $1 \leq i,j \leq N$, we denote

$$I_{i,j,t_0} := \int_{U_i\cap U_j\cap\{\psi<-t_0\}} |\tilde{f}_i - \tilde{f}_j|^2_{\omega,h} e^{-\psi} dV_{M,\omega}.$$

It follows from equality (3.2) and dominated convergence theorem that we have

$$\lim_{t_0\to+\infty} I_{i,j,t_0} = \lim_{t_0\to+\infty} \int_{U_i\cap U_j\cap\{\psi<-t_0\}} |\tilde{f}_i - \tilde{f}_j|^2_{\omega,h} e^{-\psi} dV_{M,\omega} = 0.$$

Hence we have

$$\lim_{t_0\to+\infty} I_{2,t_0} = 0. \tag{3.12}$$

Step 5: solving $\bar{\partial}-$equation with error term.

Given $\tau > 0$, note that

$$\langle a_1 + a_2, a_1 + a_2\rangle \leq (1+\tau)\langle a_1, a_1\rangle + (1+\frac{1}{\tau})\langle a_2, a_2\rangle$$

holds for any a_1, a_2 in an inner product space $(H, \langle\cdot,\cdot\rangle)$. It follows from inequality (3.6) that on $M_k\backslash\sum$, for any $\tau > 0$, we have

$$\begin{aligned}
&\int_{M_k\backslash\sum} \langle\big(B + \sqrt{I_{2,t_0}}\mathrm{Id}_E\big)^{-1}\lambda_{t_0}, \lambda_{t_0}\rangle_{\omega,\tilde{h}} dV_{M,\omega} \\
\leq &\int_{M_k\backslash\sum} (1+\tau)\langle\big(B + \sqrt{I_{2,t_0}}\mathrm{Id}_E\big)^{-1}\lambda_{1,t_0}, \lambda_{1,t_0}\rangle_{\omega,\tilde{h}} dV_{M,\omega} \\
&+ \int_{M_k\backslash\sum} (1+\frac{1}{\tau})\langle\big(B + \sqrt{I_{2,t_0}}\mathrm{Id}_E\big)^{-1}\lambda_{2,t_0}, \lambda_{2,t_0}\rangle_{\omega,\tilde{h}} dV_{M,\omega} \\
\leq &(1+\tau)\int_{M_k\backslash\sum} \langle B^{-1}\lambda_{1,t_0}, \lambda_{1,t_0}\rangle_{\omega,\tilde{h}} dV_{M,\omega} \\
&+(1+\frac{1}{\tau})\int_{M_k\backslash\sum} \langle\frac{1}{\sqrt{I_{2,t_0}}}\lambda_{2,t_0}, \lambda_{2,t_0}\rangle_{\omega,\tilde{h}} dV_{M,\omega} \\
= &(1+\tau)I_{1,t_0,\epsilon} + (1+\frac{1}{\tau})\frac{1}{\sqrt{I_{2,t_0}}} I_{2,t_0} \\
\leq &(1+\tau)I_{1,t_0,\epsilon} + (1+\frac{1}{\tau})\sqrt{I_{2,t_0}},
\end{aligned} \tag{3.13}$$

By inequalities (3.9) and (3.12), we know that $(1+\tau)I_{1,t_0,\epsilon} + (1+\frac{1}{\tau})\sqrt{I_{2,t_0}}$ is finite.

From now on, we fix some $\epsilon \in (0, \frac{1}{8})$. Then by Lemma 2.3, there exist $u_{k,t_0,\epsilon} \in L^2(M_k\backslash\sum, K_M\otimes E, \omega\otimes\tilde{h})$ and $\eta_{k,t_0,\epsilon} \in L^2(M_k\backslash\sum, \wedge^{n,1}T^*M\otimes E, \omega\otimes\tilde{h})$ such that

$$D'' u_{k,t_0,\epsilon} + (I_{2,t_0})^{\frac{1}{4}}\eta_{k,t_0,\epsilon} = \lambda_{t_0} \tag{3.14}$$

and

$$\begin{aligned}&\int_{M_k\backslash\sum}(\eta+g^{-1})^{-1}|u_{k,t_0,\epsilon}|^2_{\omega,\tilde{h}}dV_{M,\omega}+\int_{M_k\backslash\sum}|\eta_{k,t_0,\epsilon}|^2_{\omega,\tilde{h}}dV_{M,\omega}\\ \le&(1+\tau)I_{1,t_0,\epsilon}+(1+\frac{1}{\tau})\sqrt{I_{2,t_0}}<+\infty.\end{aligned}\tag{3.15}$$

By definition, $(\eta+g^{-1})^{-1}=c_k(-v_{t_0,\epsilon}(\psi))e^{v_{t_0,\epsilon}(\psi)}e^{\phi}$. It follows from inequality (3.15) that

$$\begin{aligned}&\int_{M_k\backslash\sum}c_k\big(-v_{t_0,\epsilon}(\psi)\big)e^{v_{t_0,\epsilon}(\psi)-\psi}|u_{k,t_0,\epsilon}|^2_{\omega,h}dV_{M,\omega}\\ \le&(1+\tau)I_{1,t_0,\epsilon}+(1+\frac{1}{\tau})\sqrt{I_{2,t_0}},\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&\int_{M_k\backslash\sum}|\eta_{k,t_0,\epsilon}|^2_{\omega,h}e^{-\psi-\phi}dV_{M,\omega}\\ \le&(1+\tau)I_{1,t_0,\epsilon}+(1+\frac{1}{\tau})\sqrt{I_{2,t_0}}.\end{aligned}\tag{3.17}$$

Note that $v_{t_0,\epsilon}(\psi)$ is bounded on M_k and $c_k(t)e^{-t}$ is decreasing near $+\infty$. We know that $c(-v_{t_0,\epsilon}(\psi))e^{v_{t_0,\epsilon}(\psi)}$ has positive lower bound on M_k. We also have $e^{-\phi}=e^{-u(-v_{t_0,\epsilon}(\psi))}$ has positive lower bound on M_k. As ψ is upper-bounded on M_k, $e^{-\psi}$ also have positive lower bound on M_k. By inequalities (3.16) and (3.17), we know that

$$u_{k,t_0,\epsilon}\in L^2(M_k\backslash\sum,K_M\otimes E,\omega\otimes h)$$

and

$$\eta_{k,t_0,\epsilon}\in L^2(M_k\backslash\sum,\wedge^{n,1}T^*M\otimes E,\omega\otimes h).$$

By Lemma 2.2 and equality (3.14), we know that

$$D''u_{k,t_0,\epsilon}+(I_{2,t_0})^{\frac{1}{4}}\eta_{k,t_0,\epsilon}=\lambda_{t_0}\tag{3.18}$$

holds on M_k.

Recall that $\lambda_{t_0}:=D''[(1-v'_{t_0,\epsilon}(\psi))\tilde{f}]$. Denote $F_{k,t_0,\epsilon}:=\lambda_{t_0}-u_{k,t_0,\epsilon}$. Then we have

$$D''F_{k,t_0,\epsilon}=(I_{2,t_0})^{\frac{1}{4}}\eta_{k,t_0,\epsilon}\tag{3.19}$$

holds on M_k. It follows from $\sum$ is a set of measure zero and inequalities (3.16) and (3.17) that we have

$$\begin{aligned}&\int_{M_k}c_k\big(-v_{t_0,\epsilon}(\psi)\big)e^{v_{t_0,\epsilon}(\psi)-\psi}|u_{k,t_0,\epsilon}|^2_{\omega,h}dV_{M,\omega}\\ =&\int_{M_k}c_k\big(-v_{t_0,\epsilon}(\psi)\big)e^{v_{t_0,\epsilon}(\psi)-\psi}|F_{k,t_0,\epsilon}-\big((1-v'_{t_0,\epsilon}(\psi))\tilde{f}\big)|^2_{\omega,h}dV_{M,\omega}\\ \le&(1+\tau)I_{1,t_0,\epsilon}+(1+\frac{1}{\tau})\sqrt{I_{2,t_0}},\end{aligned}\tag{3.20}$$

and

$$\begin{aligned}&\int_{M_k}|\eta_{k,t_0,\epsilon}|^2_{\omega,h}e^{-\psi-\phi}dV_{M,\omega}\\ \le&(1+\tau)I_{1,t_0,\epsilon}+(1+\frac{1}{\tau})\sqrt{I_{2,t_0}}.\end{aligned}\tag{3.21}$$

Step 6: when $t_0 \to +\infty$.

We note that k is fixed in this step. It follows from inequality (3.20), $c_k(t)e^{-t}$ is decreasing with respect to t near $+\infty$ and $v_{t_0,\epsilon}(\psi) \geq \psi$ that, when t_0 is big enough, we have

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_{k,t_0,\epsilon} - \big((1-v'_{t_0,\epsilon}(\psi))\tilde{f}\big)|^2_{\omega,h} dV_{M,\omega}\\ \leq&(1+\tau)I_{1,t_0,\epsilon} + (1+\frac{1}{\tau})\sqrt{I_{2,t_0}}.\end{aligned} \tag{3.22}$$

Then we have

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_{k,t_0,\epsilon}|^2_{\omega,h} dV_{M,\omega}\\ \leq&(1+\tau)\int_{M_k} c_k(-\psi)|F_{k,t_0,\epsilon} - \big((1-v'_{t_0,\epsilon}(\psi))\tilde{f}\big)|^2_{\omega,h} dV_{M,\omega}\\ &+(1+\frac{1}{\tau})\int_{M_k} c_k(-\psi)|\big((1-v'_{t_0,\epsilon}(\psi))\tilde{f}\big)|^2_{\omega,h} dV_{M,\omega}\\ \leq&(1+\tau)^2 I_{1,t_0,\epsilon} + (1+\tau)(1+\frac{1}{\tau})\sqrt{I_{2,t_0}}\\ &+(1+\frac{1}{\tau})C_{t_0},\end{aligned} \tag{3.23}$$

where $C_{t_0} := \int_{M_k} c_k(-\psi)|\big((1-v'_{t_0,\epsilon}(\psi))\tilde{f}\big)|^2_{\omega,h} dV_{M,\omega}$. Note that $(1-v'_{t_0,\epsilon}(\psi)) \leq \mathbb{I}_{\{\psi<-t_0\}}$. We have

$$\begin{aligned}C_{t_0} =&\int_{M_k} c_k(-\psi)|\big((1-v'_{t_0,\epsilon}(\psi))\tilde{f}\big)|^2_{\omega,h} dV_{M,\omega}\\ \leq&\int_{M_k} c_k(-\psi)\mathbb{I}_{\{\psi<-t_0\}}|\tilde{f}|^2_{\omega,h} dV_{M,\omega}\\ =&\sum_{j=0}^{+\infty}\int_{M_k} c_k(-\psi)\mathbb{I}_{\{-t_0-1-j\leq\psi<-t_0-j\}}|\tilde{f}|^2_{\omega,h} dV_{M,\omega}.\end{aligned} \tag{3.24}$$

As $c_k(t)e^{-t}$ is decreasing with respect to t near $+\infty$, we take t_0 big enough, such that $c_k(t)e^{-t}$ is decreasing with respect to t on $[t_0-1,+\infty)$. Then on $\{t_0+j < -\psi \leq t_0+1+j\}$, we have $c_k(-\psi)e^{\psi} \leq c(t_0+j)e^{-t_0-j}$. Combining with inequality (3.24), we have

$$\begin{aligned}C_{t_0} \leq&\sum_{j=0}^{+\infty}\int_{M_k} c_k(-\psi)\mathbb{I}_{\{-t_0-1-j\leq\psi<-t_0-j\}}|\tilde{f}|^2_{\omega,h} dV_{M,\omega}\\ \leq&\sum_{j=0}^{+\infty}\int_{M_k} c_k(t_0+j)e^{-t_0-j}e^{-\psi}\mathbb{I}_{\{-t_0-1-j\leq\psi<-t_0-j\}}|\tilde{f}|^2_{\omega,h} dV_{M,\omega}\\ =&\sum_{j=0}^{+\infty} c_k(t_0+j)e^{-t_0-j}\int_{M_k}\mathbb{I}_{\{-t_0-1-j\leq\psi<-t_0-j\}}|\tilde{f}|^2_{\omega,h}e^{-\psi} dV_{M,\omega}.\end{aligned} \tag{3.25}$$

It follows from condition (1.3) that

$$\limsup_{t\to+\infty}\int_{\{-t-1<\psi<-t\}}|\tilde{f}|^2_{\omega,h}e^{-\psi}dV_{M,\omega}<+\infty.$$

Hence, when t_0 is big enough, we have

$$\sup_{j\geq 0}\int_{M_k}\mathbb{I}_{\{-t_0-1-j\leq\psi<-t_0-j\}}|\tilde{f}|^2_{\omega,h}e^{-\psi}dV_{M,\omega}\leq C_1,$$

for some constant $C_1>0$ independent of j. Then we know

$$C_{t_0}\leq C_1\sum_{j=0}^{+\infty}c_k(t_0+j)e^{-t_0-j}\leq C_1\int_{t_0-1}^{+\infty}c_k(t)e^{-t}dt,$$

and we have

$$\lim_{t_0\to+\infty}C_{t_0}=0. \tag{3.26}$$

By inequalities (3.23), (3.9), (3.12) and (3.26), when t_0 is big enough, we have,

$$\sup_{t_0}\int_{M_k}c_k(-\psi)|F_{k,t_0,\epsilon}|^2_{\omega,h}dV_{M,\omega}<+\infty.$$

Since the closed unit ball of Hilbert space is weakly compact, we know that there exists a subsequence of $\{F_{k,t_0,\epsilon}\}_{t_0}$ (also denoted by $\{F_{k,t_0,\epsilon}\}$) weakly convergent to some $F_{k,\epsilon}$ in $L^2(M_k,K_M\otimes E,c_k(-\psi)\omega\otimes h)$ as $t_0\to+\infty$. Again by inequalities (3.23), (3.9), (3.12) and (3.26), we have

$$\begin{aligned}
&\int_{M_k}c_k(-\psi)|F_{k,\epsilon}|^2_{\omega,h}dV_{M,\omega}\\
\leq&\liminf_{t_0\to+\infty}\int_{M_k}c_k(-\psi)|F_{k,t_0,\epsilon}|^2_{\omega,h}dV_{M,\omega}\\
\leq&\limsup_{t_0\to+\infty}\left((1+\tau)^2I_{1,t_0,\epsilon}+(1+\tau)(1+\frac{1}{\tau})\sqrt{I_{2,t_0}}+(1+\frac{1}{\tau})C_{t_0}\right)\\
=&\frac{(1+\tau)^2}{1-4\epsilon}\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].
\end{aligned}$$

By the construction of $e^{-\phi}$ and $\sup_{M_k}\psi<-T-8\epsilon_k$, we know that $e^{-\phi}=e^{-u_k(-v_{t_0,\epsilon}(\psi))}$ has a uniformly positive lower bounded with respect to t_0. It follows from ψ is upper bounded on M_k, inequalities (3.21), (3.9) and (3.12) that we have

$$\sup_{t_0}\int_{M_k}|\eta_{k,t_0,\epsilon}|^2_{\omega,h}dV_{M,\omega}<+\infty.$$

Since the closed unit ball of Hilbert space is weakly compact, we can extract a subsequence of $\{\eta_{k,t_0,\epsilon}\}_{t_0}$ (also denoted by $\{v_{k,t_0,\epsilon}\}_{t_0}$) weakly convergent to $v_{k,\epsilon}$ in $L^2(M_k,\wedge^{n,1}T^*M\otimes E,\omega\otimes h)$ as $t_0\to+\infty$. As $\lim_{t_0\to+\infty}I_{2,t_0}=0$, we know I_{2,t_0} is uniformly upper bounded with respect to t_0. It follows from Lemma 2.4 that $(I_{2,t_0})^{\frac{1}{4}}\eta_{k,t_0,\epsilon}$ is weakly convergent to 0 in $L^2(M_k,\wedge^{n,1}T^*M\otimes E,\omega\otimes h)$ as $t_0\to+\infty$. Hence $(I_{2,t_0})^{\frac{1}{4}}\eta_{k,t_0,\epsilon}$ is weakly convergent to 0 in $L^2_{\mathrm{loc}}(M_k,\wedge^{n,1}T^*M\otimes E,\omega\otimes h)$ as $t_0\to+\infty$

It follows from ψ is smooth on $M_k\backslash\sum$, $c_k(t)$ is smooth function on $[T+4\epsilon_k,+\infty)$ and $\{F_{k,t_0,\epsilon}\}$ weakly converges to $\{F_{k,\epsilon}\}$ in $L^2(M_k,K_M\otimes E,c_k(-\psi)\omega\otimes h)$ as $t_0\to$

$+\infty$ that we have $\{F_{k,t_0,\epsilon}\}$ also weakly converges to $\{F_{k,\epsilon}\}$ in $L^2_{\mathrm{loc}}(M_k\backslash\sum, K_M\otimes E,\omega\otimes h)$ as $t_0\to+\infty$.

Then it follows from equality (3.19) that, when $t_0\to+\infty$, we have

$$D''F_{k,\epsilon}=0 \text{ holds on } M_k\backslash\sum. \tag{3.27}$$

Hence $F_{k,\epsilon}$ is an E-valued holomorphic $(n,0)$-form on $M_k\backslash\sum$ which satisfies

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_{k,\epsilon}|^2_{\omega,h}dV_{M,\omega}\\ \le&\frac{(1+\tau)^2}{1-4\epsilon}\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}\tag{3.28}$$

Step 7: solving $\bar\partial$-equation locally.

In this step, we prove that $F_{k,\epsilon}$ is actually a holomorphic extension of f from $Y^0\cap M_k$ to M_k.

Let $x\in M_k\cap Y^0$ be any point. Let $\tilde U_x\Subset M_k$ be a local coordinate ball which is centered at x. We assume that $E|_{\tilde U_x}$ is trivial vector bundle.

Note that we have

$$D''u_{k,t_0,\epsilon}+(I_{2,t_0})^{\frac14}\eta_{k,t_0,\epsilon}=D''[\big(1-v'_{t_0,\epsilon}(\psi)\big)\tilde f].$$

It follows from inequality (3.21) and the construction of $e^{-\phi}$ that we have

$$\int_{\tilde U_x}|\eta_{k,t_0,\epsilon}|^2_h e^{-\psi}\le C_2,\tag{3.29}$$

where $C_2>0$ is a positive number independent of t_0.

Note that $D''\big((I_{2,t_0})^{\frac14}\eta_{k,t_0,\epsilon}\big)=0$. It follows from Lemma 2.6 that there exists an E-valued $(n,0)$-form $s_{k,t_0,\epsilon}\in L^2(\tilde U_x, K_M\otimes E, he^{-\psi})$ such that $D''s_{k,t_0,\epsilon}=(I_{2,t_0})^{\frac14}\eta_{k,t_0,\epsilon}$ and

$$\int_{\tilde U_x}|s_{k,t_0,\epsilon}|^2_h e^{-\psi}\le C_3\int_{\tilde U_x}|(I_{2,t_0})^{\frac14}\eta_{k,t_0,\epsilon}|^2_h e^{-\psi}\le C_3C_2\sqrt{I_{2,t_0}},\tag{3.30}$$

where $C_3>0$ is a positive number independent of t_0. Hence we have

$$\int_{\tilde U_x}|s_{k,t_0,\epsilon}|^2_h\le C_4\sqrt{I_{2,t_0}},\tag{3.31}$$

where $C_4>0$ is a positive number independent of t_0.

Now define $G_{k,t_0,\epsilon}:=-u_{k,t_0,\epsilon}-s_{k,t_0,\epsilon}+\big(1-v'_{t_0,\epsilon}(\psi)\big)\tilde f$ on $\tilde U_x$. Then we know that $G_{k,t_0,\epsilon}=F_{k,t_0,\epsilon}-s_{k,t_0,\epsilon}$ and $D''G_{k,t_0,\epsilon}=0$. Hence $G_{k,t_0,\epsilon}$ is holomorphic on $\tilde U_x$ and we know that $u_{k,t_0,\epsilon}+s_{k,t_0,\epsilon}$ is smooth on $\tilde U_x$.

It follows from $c_k(t)e^{-t}$ is decreasing with respect to t near $+\infty$ and inequalities (3.23) and (3.30) that we have

$$\begin{aligned}&\int_{\tilde U_x}c_k(-\psi)|G_{k,t_0,\epsilon}|^2_h\\ \le&2\tilde C_5\int_{\tilde U_x}c_k(-\psi)|F_{k,t_0,\epsilon}|^2_h+2\tilde C_5\int_{\tilde U_x}|s_{k,t_0,\epsilon}|^2_h e^{-\psi}\le C_5,\end{aligned}\tag{3.32}$$

where $\tilde C_5, C_5>0$ are positive numbers independent of t_0.

It follows from inequality (3.20), the construction of $v_{t_0,\epsilon}(\psi)$ and $c_k(t)e^{-t}$ is decreasing with respect to t near $+\infty$ that we have

$$\int_{\tilde{U}_x} |u_{k,t_0,\epsilon}|^2_h e^{-\psi} \le C_{6,t_0}, \tag{3.33}$$

where $C_{6,t_0} > 0$ is a sequence of positive number depending on t_0. Then, by inequalities (3.30) and (3.33), we have

$$\int_{\tilde{U}_x} |u_{k,t_0,\epsilon} + s_{k,t_0,\epsilon}|^2_{\tilde{h}} e^{-\psi} \le 2C_{6,t_0} + 2C_3C_2\sqrt{I_{2,t_0}}. \tag{3.34}$$

Note that $e^{-\psi}$ is not integrable along Y and $u_{k,t_0,\epsilon} + s_{k,t_0,\epsilon}$ is smooth on $\tilde{U}_x$. By (3.34), we know that $u_{k,t_0,\epsilon}+s_{k,t_0,\epsilon} = 0$ on $\tilde{U}_x \cap Y$ for any t_0. Hence $G_{k,t_0,\epsilon} = \tilde{f}_{t_0} = f$ on $\tilde{U}_x \cap Y_0$ for any t_0.

It follows from inequality (3.31) that there exists a subsequence of $\{s_{k,t_0,\epsilon}\}$ $\big($also denoted by $\{s_{k,t_0,\epsilon}\}\big)$ weakly convergent to 0 in $L^2(\tilde{U}_x, K_M \otimes E, h)$ as $t_0 \to +\infty$. Note that $\{F_{k,t_0,\epsilon}\}$ weakly converges to $F_{k,\epsilon}$ in $L^2_{\mathrm{loc}}(\tilde{U}_x\backslash \sum, K_M \otimes E, h)$ as $t_0 \to +\infty$. Hence we know that $\{G_{k,t_0,\epsilon}\}$ weakly converges to $F_{k,\epsilon}$ in $L^2_{\mathrm{loc}}(\tilde{U}_x\backslash \sum, K_M \otimes E, h)$as $t_0 \to +\infty$.

It follows from inequality (3.32) and Lemma 2.8 that we know there exists a subsequence of $\{G_{k,t_0,\epsilon}\}$ $\big($also denoted by $\{G_{k,t_0,\epsilon}\}\big)$ compactly convergent to an E-valued holomorphic $(n,0)$-form $G_{k,\epsilon}$ on $\tilde{U}_x$ as $t_0 \to +\infty$. As $G_{k,t_0,\epsilon} = f$ on $\tilde{U}_x \cap Y_0$ for any t_0, we know that $G_{k,\epsilon} = f$ on $\tilde{U}_x \cap Y_0$.

As $\{G_{k,t_0,\epsilon}\}$ compactly converges to $G_{k,\epsilon}$ on $\tilde{U}_x$ as $t_0 \to +\infty$ and $\{G_{k,t_0,\epsilon}\}$ weakly converges to $F_{k,\epsilon}$ in $L^2_{\mathrm{loc}}(\tilde{U}_x\backslash \sum, K_M \otimes E, h)$ as $t_0 \to +\infty$, by the uniqueness of weak limit, we know that $G_{k,\epsilon} = F_{k,\epsilon}$ on any relatively compact open subset of $\tilde{U}_x$. Note that $G_{k,,\epsilon}$ is holomorphic on $\tilde{U}_x$ and $F_{k,\epsilon}$ is holomorphic on $\tilde{U}_x\backslash \sum$, we have $F_{k,\epsilon} \equiv G_{k,\epsilon}$ on $\tilde{U}_x\backslash \sum$, and we know that $F_{k,\epsilon}$ can extended to an E-valued holomorphic $(n,0)$-form on $\tilde{U}_x$ which equals to $G_{k,\epsilon}$. As $G_{k,\epsilon} = f$ on $\tilde{U}_x \cap Y_0$, we know that $F_{k,\epsilon} = f$ on $\tilde{U}_x \cap Y_0$. Since x is arbitrarily chosen, we know that $F_{k,\epsilon}$ is holomorphic on M_k and $F_{k,\epsilon} = f$ on $M_k \cap Y_0$.

Step 8: end of the proof.

Now we have a family of E-valued holomorphic $(n,0)$-forms $F_{k,\epsilon}$ on M_k such that $F_{k,\epsilon} = f$ on $M_k \cap Y_0$ and

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_{k,\epsilon}|^2_{\omega,h} dV_{M,\omega}\\ \le&\frac{(1+\tau)^2}{1-4\epsilon}\Big(\frac{1}{\delta}c_k(T)e^{-T} + \int_T^{+\infty} c_k(t_1)e^{-t_1}dt_1\Big)\int_{Y^0} |f|^2_{\omega,h} dV_{M,\omega}[\psi].\end{aligned} \tag{3.35}$$

Recall that $\epsilon \in (0, \frac{1}{8})$. By inequality (3.35), we have

$$\sup_{\epsilon\in(0,\frac{1}{8})} \int_{M_k} c_k(-\psi)|F_{k,\epsilon}|^2_{\omega,h} dV_{M,\omega} < +\infty. \tag{3.36}$$

For any compact subset $K \subset M_k\backslash \sum = \{\psi = -\infty\}$, as ψ is smooth on $M_k\backslash \sum$, we know that ψ is upper and lower bounded on K. As $c_k(t)$ is continuous on $[T, +\infty)$, we have $c_k(-\psi)$ is uniformly lower bounded on K. It follows from Lemma

2.8 and inequality (3.36) that there exists a subsequence of $\{F_{k,\epsilon}\}_\epsilon$ (also denoted by $\{F_{k,\epsilon}\}_\epsilon$) compactly convergent to an E-valued holomorphic $(n,0)$-form F_k on M_k as $\epsilon \to 0$. It follows from Fatou's lemma (let $\epsilon \to 0$) and inequality (3.35) that we have

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_k|^2_{\omega,h}dV_{M,\omega}\\ \le&\liminf_{\epsilon\to 0}\int_{M_k} c_k(-\psi)|F_{k,\epsilon}|^2_{\omega,h}dV_{M,\omega}\\ \le&\liminf_{\epsilon\to 0}\frac{(1+\tau)^2}{1-4\epsilon}\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi]\\ \le&(1+\tau)^2\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}$$

Hence there exists a family of E-valued holomorphic $(n,0)$-forms F_k on M_k such that $F_k = f$ on $M_k \cap Y_0$ and

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_k|^2_{\omega,h}dV_{M,\omega}\\ \le&(1+\tau)^2\big(\frac{1}{\delta}c_k(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}$$

Since $\tau > 0$ is arbitrarily chosen and $c_k(T)e^{-T} = c(T)e^{-T}$, we have

$$\begin{aligned}&\int_{M_k} c_k(-\psi)|F_k|^2_{\omega,h}dV_{M,\omega}\\ \le&\big(\frac{1}{\delta}c(T)e^{-T}+\int_T^{+\infty}c_k(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}\tag{3.37}$$

Let $k_1 > k$ be big enough. It follows from inequality (3.37), $M_k \Subset M_{k_1}$ and $\frac{1}{\delta}c(T)e^{-T}+\int_T^{+\infty}c_k(t)e^{-t}dt$ converges to $\frac{1}{\delta}c(T)e^{-T}+\int_T^{+\infty}c(t)e^{-t}dt < +\infty$ as $k \to +\infty$ that we have

$$\sup_{k_1}\int_{M_k} c_{k_1}(-\psi)|F_{k_1}|^2_{\omega,h}dV_{M,\omega} < +\infty. \tag{3.38}$$

For any compact subset $K \subset M_k\backslash \sum = \{\psi = -\infty\}$, as ψ is smooth on $M_k\backslash\sum$, we know that ψ is upper and lower bounded on K. It follows from $c_{k_1}(t)$ are uniformly convergent to $c(t)$ on any compact subset of $(T,+\infty)$ and $c(t)$ is a positive continuous function on $[T,+\infty)$ that we know $c_{k_1}(-\psi)$ is uniformly lower bounded on K. By Lemma 2.8 and inequality (3.38), we know that there exists a subsequence of $\{F_{k_1}\}_{k_1\in\mathbb{Z}^+}$ (also denoted by $\{F_{k_1}\}_{k_1\in\mathbb{Z}^+}$) compactly convergent to an E-valued holomorphic $(n,0)$-form $\tilde{F}_k$ on M_k. It follows from Fatou's lemma (let $k_1 \to +\infty$) and inequality (3.37) that we have

$$\begin{aligned}&\int_{M_k} c(-\psi)|\tilde{F}_k|^2_{\omega,h}dV_{M,\omega}\\ \le&\big(\frac{1}{\delta}c(T)e^{-T}+\int_T^{+\infty}c(t_1)e^{-t_1}dt_1\big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}\tag{3.39}$$

As $\{F_{k_1}\}_{k_1\in\mathbb{Z}^+}$ compactly convergent to an E-valued holomorphic $(n,0)$-form $\tilde{F}_k$ on M, we know that $\tilde{F}_k = f$ on $M_k \cap Y_0$.

Again for any compact subset $K \subset M\backslash Y$, as ψ is smooth on $M\backslash Y$, we know that ψ is upper and lower bounded on K. It follows from $c(t)$ is a positive continuous function on $[T, +\infty)$ that we know $c(-\psi)$ is uniformly lower bounded on K. By Lemma 2.8 and inequality (3.39), we know that there exists a subsequence of $\{\tilde{F}_k\}_{k\in\mathbb{Z}^+}$ (also denoted by $\{\tilde{F}_k\}_{k\in\mathbb{Z}^+}$) compactly convergent to an E-valued holomorphic $(n,0)$-form F on M. It follows from Fatou's lemma (let $k \to +\infty$) and inequality (3.39) that we have

$$\begin{aligned}&\int_{M_k} c(-\psi)|F|^2_{\omega,h}dV_{M,\omega}\\ \le&\Big(\frac{1}{\delta}c(T)e^{-T}+\int_T^{+\infty}c(t_1)e^{-t_1}dt_1\Big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}$$

Letting $k \to +\infty$, by monotone convergence theorem, we have

$$\begin{aligned}&\int_{M} c(-\psi)|F|^2_{\omega,h}dV_{M,\omega}\\ \le&\Big(\frac{1}{\delta}c(T)e^{-T}+\int_T^{+\infty}c(t_1)e^{-t_1}dt_1\Big)\int_{Y^0}|f|^2_{\omega,h}dV_{M,\omega}[\psi].\end{aligned}$$

As $\{\tilde{F}_k\}_{k\in\mathbb{Z}^+}$ is compactly convergent to an E-valued holomorphic $(n,0)$-form F on M, we know that $F = f$ on $M \cap Y_0$.

Theorem 1.8 has been proved. □

Acknowledgements. The authors would like to thank Professor Xiangyu Zhou for his encouragement. The authors would like to thank Dr. Shijie Bao for checking the manuscript.

The first author and the second author were supported by National Key R&D Program of China 2021YFA1003100. The first author was supported by NSFC-11825101, NSFC-11522101 and NSFC-11431013. The second author was supported by the Talent Fund of Beijing Jiaotong University. The third author was supported by China Postdoctoral Science Foundation BX20230402 and 2023M743719.

References

[1] B. Berndtsson, The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman, Ann. L'Inst. Fourier (Grenoble) 46 (1996), no. 4, 1083–1094.

[2] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, Annals of Math, 169 (2009), 531–560.

[3] B. Berndtsson and M. Păun. Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J., 145(2):341–378, 2008.

[4] Z. Blocki, Suita conjecture and the Ohsawa-Takegoshi extension theorem, Invent Math., 193 (2013), 149-158.

[5] J.-P. Demailly, Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète.(French) L^2 *estimates for the $\bar{\partial}$-operator of a semipositive holomorphic vector bundle over a complete Kähler manifold*, Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 3, 457-511.

[6] J.-P. Demailly, Singular Hermitian metrics on positive line bundles. Complex algebraic varieties. Springer, Berlin, Heidelberg, 1992: 87-104.

[7] J.-P. Demailly, On the Ohsawa-Takegoshi-Manivel L^2 extension theorem. In: Proceedings of the Conference in Honour of the 85th Birthday of Pierre Lelong, Paris, Sept 1997. Progress in Mathematics. Birkhäuser, Basel (2000).
[8] J.-P. Demailly, Analytic Methods in Algebraic Geometry, Higher Education Press, Beijing, 2010.
[9] J.-P. Demailly, Extension of holomorphic functions defined on non reduced analytic subvarieties, pp. 191–222 in The legacy of Bernhard Riemann after one hundred and fifty years, Vol. I, edited by L. Ji et al., Adv. Lect. Math. (ALM) 35, International Press, Somerville, MA, 2016.
[10] J.-P. Demailly, C.D. Hacon and M. Păun, Extension theorems, non-vanishing and the existence of good minimal models. Acta mathematica, 2013, 210(2): 203-259.
[11] J.-P. Demailly, Complex analytic and differential geometry, electronically accessible at http://www-fourier.ujf-grenoble.fr/~demailly/books.html
[12] Q.A. Guan, Z.T. Mi and Z. Yuan, Boundary points, minimal L^2 integrals and concavity property V—vector bundles. J. Geom. Anal. 33 (2023), no. 9, Paper No. 305, 86 pp.
[13] Q.A. Guan, Z.T. Mi and Z. Yuan, Optimal L^2 extension for holomorphic vector bundles with singular hermitian metrics, arXiv:2210.06026.
[14] Q.A. Guan and Z. Yuan, Concavity property of minimal L^2 integrals with Lebesgue measurable gain. Nagoya Math. J. 252 (2023), 842–905.
[15] Q.A. Guan and X.Y. Zhou. Optimal constant problem in the L^2 extension theorem. C. R. Math. Acad. Sci. Paris Ser I, 2012, 350: 753–756.
[16] Q.A. Guan and X.Y. Zhou, A solution of an L^2 extension problem with an optimal estimate and applications, Ann. of Math. (2) 181 (2015), no. 3, 1139–1208.
[17] Q.A. Guan and X.Y. Zhou, Optimal constant in an L^2 extension problem and a proof of a conjecture of Ohsawa, Sci. China Math., 2015, 58(1):35-59.
[18] Q.A. Guan, X.Y. Zhou and L.F. Zhu, On the Ohsawa-Takegoshi L^2 extension theorem and the twisted Bochner-Kodaira identity. C. R. Math. Acad. Sci. Paris 349(13–14), 797–800 (2011).
[19] C. Hacon, M. Popa and C. Schnell, Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun, Local and global methods in algebraic geometry, 143–195, Contemp. Math., 712, Amer. Math. Soc. [2018].
[20] L. Hörmander, An introduction to complex analysis in several complex variables, third edition, NorthHolland Mathematical Library, 7, North-Holland Publishing Co., Amsterdam, 1990.
[21] J. D. McNeal and D. Varolin, "Analytic inversion of adjunction: L^2 extension theorems with gain", Ann. Inst. Fourier 57 (2007), no. 3, p. 703-718.
[22] T. Ohsawa and K. Takegoshi, On the extension of L^2 holomorphic functions. Math. Z. 195 (1987), no. 2, 197–204.
[23] T. Ohsawa, On the extension of L^2 holomorphic functions. II. Publ. Res. Inst. Math. Sci. 24 (1988), no. 2, 265–275.
[24] T. Ohsawa, On the extension of L^2 holomorphic functions. III. negligible weights, Math. Z. 219 (1995), no. 2, 215–225.
[25] T. Ohsawa, On the extension of L^2 holomorphic functions. IV. A new density concept. In: Geometry and Analysis on Complex Manifolds, pp. 157-170. World Scientific, River Edge (1994)

[26] T. Ohsawa, On the extension of L^2 holomorphic functions. V. Effects of generalization, Nagoya Math. J. 161 (2001), 1-21. Erratum to: "On the extension of L^2 holomorphic functions. V. Effects of generalization" [Nagoya Math. J. 161 (2001), 1-21]. Nagoya Math.J. 163 (2001), 229.

[27] X.Y. Zhou, A survey on L^2 extension problem. Complex geometry and dynamics, 291-309, Abel Symp., 10, Springer, Cham, 2015.

[28] X.Y. Zhou and L.F. Zhu, An optimal L^2 extension theorem on weakly pseudoconvex Kähler manifolds. (English summary) J. Differential Geom. 110 (2018), no. 1, 135-186.

[29] X.Y. Zhou and L.F. Zhu, Optimal L^2 extension of sections from subvarieties in weakly pseudoconvex manifolds. Pacific J. Math. 309 (2020), no. 2, 475-510.

[30] L.F. Zhu, Q.A. Guan and X.Y. Zhou, On the Ohsawa–Takegoshi L^2 extension theorem and the Bochner–Kodaira identity with non-smooth twist factor, J. Math. Pures Appl. 97 (2012), 579–601, MR 2921602, Zbl 1244.32005.

Qi'an Guan: School of Mathematical Sciences, Peking University, Beijing 100871, China.

Email address: guanqian@math.pku.edu.cn

Zhitong Mi: School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China.

Email address: zhitongmi@amss.ac.cn

Zheng Yuan: Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.

Email address: yuanzheng@amss.ac.cn