POLARIZED K3 SURFACES WITH AN AUTOMORPHISM OF ORDER 3 AND LOW PICARD NUMBER

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Abstract. In this paper, for each $d > 0$, we study the minimum integer $h_{3,2d} \in \mathbb{N}$ for which there exists a complex polarized K3 surface (X, H) of degree $H^2 = 2d$ and Picard number $\rho(X) := \operatorname{rank} \operatorname{Pic} X = h_{3,2d}$ admitting an automorphism of order 3. We show that $h_{3,2} \in \{4, 6\}$ and $h_{3,2d} = 2$ for $d > 1$. Analogously, we study the minimum integer $h^*_{3,2d} \in \mathbb{N}$ for which there exists a complex polarized K3 surface (X, H) as above plus the extra condition that the automorphism acts as the identity on the Picard lattice of X. We show that $h^*_{3,2d}$ is equal to 2 if $d > 1$ and equal to 6 if $d = 1$. We provide explicit examples of K3 surfaces defined over $\mathbb{Q}$ realizing these bounds.

1. Introduction

The study of automorphisms of K3 surfaces has seen a very intense activity in the last 40 years. In [8, 10] Nikulin and Stark proved that a group acting purely non-symplectically on an algebraic K3 is cyclic and finite. More in particular, Nikulin proves that such a group can have order at most 66; if the group has prime order, then its maximal order is 19, [9, Theorem 0.1.c), Corollary 3.2]. In these notes we consider non-symplectic automorphisms of order 3, a topic extensively treated in [1, 11]. In particular, we focus on the interplay between the existence of non-symplectical automorphism of order 3, a polarization of given degree, and the Picard number of the surface, as already done in [3] for non-symplectic involutions.

More precisely, let (X, H) denote a complex polarized K3 surface of degree $2d$, that is, H is an ample divisor of X and $H^2 = 2d$. Assume that X admits an automorphism $\alpha \in \operatorname{Aut} X$ of prime order p. Then α induces an action α^* on $H^{2,0}(X) = \langle \omega \rangle$, and hence $\alpha^* = \zeta\omega$, with $\zeta^p = 1$. In this paper we focus on the case $p = 3$. In this case, if $\zeta = 1$ then α is called *symplectic* and $\rho(X) \geq 13$, see [9, §10]; if $\zeta \neq 1$, that is, ζ is a primitive 3-rd root of unity then α is called *non-symplectic* and $\rho(X) \geq 2$, see [1, 11]. As done in [3], one may ask when is this lower bound realized depending on the degree of the polarization.

Definition 1.1. Let

$$\mathcal{H}_{3,2d} := \{(X, H, \alpha)\}$$

denote the set of complex polarized K3 surfaces (X, H) such that $H^2 = 2d > 0$ and X admits an automorphism α of order 3, one can then define

$$h_{3,2d} = \min_{X \in \mathcal{H}_{3,2d}} \{\rho(X)\} .$$

In this work we prove the following result.

Theorem 1.2. *If $d > 1$, then $h_{3,2d} = 2$. For $d = 1$, we have $h_{3,2d} = h_{3,2} \in \{4, 6\}$.*

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To prove that $h_{3,2d} = 2$ for every $d > 1$ we will only consider *non-symplectic* automorphisms because, as noted above, symplectic automorphisms force the Picard number to be higher than desired. We first show that if X is a K3 surface of degree $2d$ with a (non-symplectic) automorphism of order 3, then $\rho(X) \geq 2$. Then we complete the proof by providing an explicit example of a K3 surface with an automorphism of order 3, a polarization of degree $2d > 2$ and Picard number 2. To provide such example for every $d > 1$, it is enough to consider a single K3 surface with an automoprhism of order 3 and Picard lattice isometric to $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, because this lattice admits ample divisors of degree $2d$ for every $d > 1$.

The above argument leaves open the case for $d = 1$. Indeed for $h_{3,2}$ we only have a partial answer. The partiality of this result is due to the wide range of possibilities that arise when studying the Picard lattice of a K3 surface with a non-symplectic automorphism of order 3 and a polarization of degree 2. In this case, on the one hand it is easy to show that the Picard number has to be larger than 2 and that there are examples of such surfaces with Picard number 6; on the other hand, it is hard to control the ample cone of all the possible Picard lattices with rank 4 and we could neither find in the literature nor construct examples of K3 surfaces of degree 2 and Picard number 4 admitting an automorphism of order 3. See Remark 3.4 for more details.

The impasse can be overcome if we allow for one extra hypothesis, in the spirit of [11]: we assume that α acts as the identity on the Picard lattice. As we will see in §4, this is equivalent to considering 'generic' K3 surfaces with the desired properties. This extra assumption leads to the following definitions.

Definition 1.3. We define

$$\mathcal{H}^*_{3,2d} := \{(X, H, \alpha) \in \mathcal{H}_{3,2d} \mid \alpha^*_{|\operatorname{Pic} X} = \mathrm{id}\}$$

and

$$h^*_{3,2d} = \min_{X \in \mathcal{H}^*_{3,2d}} \{\rho(X)\} .$$

Clearly $h^*_{3,2d} \geq h_{3,2d}$. We then prove the following result.

Theorem 1.4. *The following equalities hold:*

$$h^*_{3,2d} = \begin{cases} 6 & \text{if } d = 1 , \\ 2 & \text{if } d > 1 . \end{cases}$$

The first equality follows almost immediately from the first statement of Theorem 1.2; the second equality builds upon the second statement: in this case we only have two possible lattices of rank 4 and we show that none of them admits a polarization of degree 2. The crucial ingredient for the proofs of all the above results is the classification of the fixed locus of an automorphism of order 3, provided by Artebani and Sarti in [1], see Theorem 2.6.

Remark 1.5. The problem treated in this paper naturally generalizes to any prime order p: for any prime p one can define $\mathcal{H}_{p,2d}$ and $h_{p,2d}$ substitutig 3 with p in Definition 1.1. In [9], Nikulin proves that if α is a non-symplectic automorphism of prime order p on a K3 surface, then $p \leq 19$, see [9, Theorem 3.1(c)]. Using the classification of non-symplectic automorphisms of prime order by Artebani, Sarti, Taki [2], one might try to compute $h_{p,2d}$ for every prime $p \leq 19$ and for every $d \geq 1$. This is indeed a joint work in progress with Wim Nijgh and Pablo Quezada Mora.

The paper is structured as follows: in §2 we briefly review the background of complex K3 surfaces with an automorphism of order 3; in §3 we prove Theorem 1.2; finally, Theorem 1.4 is proved in §4.

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2. Projective K3 surfaces with a non-symplectic automorphism of order 3

There are several works on complex K3 surfaces with an automorphism of order 3, in these notes we will mostly use the results presented in [1].

Let X be a complex projective K3 surface and assume it admits an automorphism $\alpha \in \operatorname{Aut} X$ of order 3. Also assume that α is non-symplectic. Hence $\alpha^3 = 1$ and $\alpha^*(\omega) = \zeta\omega$, where ω is the class generating $H^{2,0}(X)$ and ζ is a primitive third root of unity. In what follows, ζ will always denote a primitive third root of unity. Recall that if L is a lattice, we denote by $L^* := \operatorname{Hom}(L, \mathbb{Z})$ its *dual lattice* and by $A_L = L^*/L$ its *discriminant group*. Notice that α induces an isometry of the lattice $H^2(X, \mathbb{Z})$, which we will denote by α^*. As $\operatorname{Pic} X$ can be viewed as a sublattice of $H^2(X, \mathbb{Z})$, we will denote by α^* also the isometry of $\operatorname{Pic} X$ induced by α.

Definition 2.1. We define $N(\alpha) := (H^2(X, \mathbb{Z}))^{\alpha^*}$, the sublattice of $H^2(X, \mathbb{Z})$ fixed by α^*.

Definition 2.2. Let $\mathcal{E} = \mathbb{Z}[\zeta]$ denote the ring of Eisenstein integers. A *$\mathcal{E}$-lattice* is a couple (L, σ) where L is a lattice and σ is a fixed-point-free isometry of order 3 on L. If σ acts as the identity on A_L, then (L, σ) is called an *$\mathcal{E}^*$-lattice*.

Proposition 2.3. *Let (X, α) be a complex K3 surface with a non-symplectic automorphism of order 3. Then*

(1) $N(\alpha)$ is a primitive 3-elementary sublattice of $\operatorname{Pic} X$;
(2) $(N(\alpha)^\perp, \alpha^*)$ is a $\mathcal{E}^*$-lattice, where $N(\alpha)^\perp$ is the orthogonal complement of $N(\alpha)$ inside $H^2(X, \mathbb{Z})$;
(3) (T_X, α^*) is a $\mathcal{E}$-sublattice of $N(\alpha)^\perp$, where T_X denotes the transcendental lattice of X.

Proof. This is the reformulation of [8, Theorem 0.1] and [7, Lemma 1.1] as in [1, Theorem 1.4]. □

Lemma 2.4. *The following statements hold:*

(1) Any $\mathcal{E}$-lattice has even rank;
(2) Any $\mathcal{E}^*$-lattice is 3-elementary.

Proof. This is [1, Lemma 1.3]. □

Corollary 2.5. *Let (X, α) be a complex K3 surface with a non-symplectic automorphism of order 3. Then $\rho(X)$ and $\operatorname{rk} N(\alpha)$ are even.*

Proof. By Proposition 2.3 we have that (T_X, α^*) is a $\mathcal{E}$-lattice. Then from Lemma 2.4 it follows that $\operatorname{rk} T_X$ is even. As $\rho(X) = 22 - \operatorname{rk} T_X$, we conclude the argument.

The same argument applied to $N(\alpha)^\perp$ shows that $\operatorname{rk} N(\alpha)$ is also even. □

The main result of [1] is the complete classification of the K3 surfaces (X, α) in terms of the fixed loci $\operatorname{Fix} \alpha \subset X$ and $N(\alpha) \subset \operatorname{Pic} X$. Moreover, for each case they also provide a projective model realizing it. Their results can be summarized as follows.

Theorem 2.6. [1, Theorems 2.8 and 3.4] *Let (X, α) be a complex K3 surface with an automorphism α of order 3. Then $\operatorname{Fix}\alpha$ consists of $n \leq 9$ points and $k \leq 6$ curves. The couple (n, k) uniquely determines $N(\alpha)$. All the possible triples $(n, k, N(\alpha))$ are listed in* [1, Table 2].

Conversely, for every triple $(n, k, N(n, k))$ in [1, Table 2] *there exists a complex K3 surface $X_{n,k}$ with a non-symplectic automorphism α of order 3 such that $\operatorname{Fix}\alpha$ consists of n points and k curves and $\operatorname{Pic} X_{n,k} = N(\alpha) \cong N(n, k)$. For each triple $(n, k, N(n, k))$ a projective model of such $X_{n,k}$ is given.*

As we are only interested in K3 surfaces with low Picard number, in Table 1 we only include the first entries of [1, Table 2], omitting the transcendental lattice and indicating the type of projective model provided by Artebani and Sarti.

n	k	N	model for $X_{n,k}$
0	1	$U(3)$	Quadric $\cap$ cubic $\subset \mathbb{P}^4$
	2	U	Weierstrass model
1	1	$U(3) \oplus A_2$	Quartic in $\mathbb{P}^3$
	2	$U \oplus A_2$	Weierstrass model
2	1	$U(3) \oplus A_2^{\oplus 2}$	Double cover of $\mathbb{P}^2$
	2	$U \oplus A_2^{\oplus 2}$	Weierstrass model

Table 1. Table of possible cases of $(n, k, N(\alpha))$ for (X, α) with $\operatorname{rk} N(\alpha) \leq 6$. In the last column we indicate the type of projective model provided in [1].

This result is very convenient because it tells us where to look in order to find polarized K3 surfaces of any degree admitting an automorphism of order 3, as shown in the following sections.

Remark 2.7. The K3 surfaces with a given marking and an automorphism of order 3 form a *subfamily* of K3 surfaces with the same marking. To see this, let (X, α) be a very generic complex K3 surface with a non-symplectic automorphism α of order 3.

Let V denote the $\mathbb{C}$-vector space given by $H^2(X, \mathbb{Z}) \otimes \mathbb{C}$. Then α^* acts on V and its action induces an orthogonal decomposition of V into eigenspaces:

$$V = V_1 \oplus V_\zeta \oplus V_{\zeta^2}\ .$$

As α is non-symplectic we can assume that $H^{0,2}(X) \subseteq V_{\zeta^2}$. We know that $N(\alpha) \subseteq \operatorname{Pic} X$ by Proposition 2.3.(1); as we assumed X to be very generic, we have that $\operatorname{Pic} X = N(\alpha) = V_1$ and hence $T_X \otimes \mathbb{C} = V_\zeta \oplus V_{\zeta^2}$. As V_ζ and V_{ζ^2} are swapped by α^*, they have the same dimension, and so we conclude that

$$\operatorname{rk} T_X = \dim(T_X \otimes \mathbb{C}) = 2 \dim V_\zeta.$$

This means that if (X, α) is a K3 surface with an automorphism of order 3, its period ω lies in $\mathbb{P}(V_\zeta)$ which has dimension

$$\dim \mathbb{P}(V_\zeta) = (\operatorname{rk} T_X)/2 - 1 = 10 - \rho(X)/2.$$

On the other hand, if we just consider a marked K3 surface X with $\operatorname{Pic} X \cong L$, without any other assumption, then the period of X will lie in

$$\{\omega \in \mathbb{P}(T_X \otimes \mathbb{C}) \ :\ \langle \omega, \omega \rangle = 0\,,\ \langle \omega, \bar{\omega} \rangle > 0\,\},$$

which has dimension $\operatorname{rk} T_X - 2 = 20 - \rho(X)$.

3. Proof of the first theorem

Throughout this section, let X be the complex projective elliptic K3 surface defined by

$$X\colon y^2 = x^3 + p(t)\ , \tag{1}$$

with $p(t)$ a polynomial of degree 12 with only simple roots. It is easy to see that the map $\alpha\colon (x, y, t) \mapsto (\zeta x, y, t)$ defines an automorphism of order 3 on X. We also assume $p(t)$, and hence X, to be generic.

Lemma 3.1. *(1) X admits an elliptic fibration;*

(2) $\operatorname{Pic} X = \langle F, O\rangle \cong U$, where F is the class of the fiber of the elliptic fibration and O is the class of the unique section.

(3) $\alpha^*(F) = F$ and $\alpha^*(O) = O$, that is, $N(\alpha) = \operatorname{Pic} X$.

Proof. The existence of the elliptic fibration is immediate from (1). It is also easy to see that the Gram matrix of $\langle F, O\rangle$ is

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix},$$

and hence $U \cong \langle F, O\rangle \subseteq \operatorname{Pic} X$. As X is assumed to be generic, we also have that $\langle F, O\rangle = \operatorname{Pic} X$, see [1, Proposition 4.2, Theorem 5.6]. The third statement is then just an immediate consequence of the second, as by [1, Proposition 4.2] we have that $N(\alpha) \cong U \cong \operatorname{Pic} X$. □

Proposition 3.2. *Let e, f be the generators of $U \cong \operatorname{Pic} X$ such that $e^2 = f^2 = 0$ and $e.f = 1$. Then, up to a choice of signs, the ample cone of $\operatorname{Pic} X$ is given by the divisors $D = xe + yf$ such that $y > x > 0$.*

Proof. Notice that in U there are only two -2-classes: $\pm(e - f)$. Assume $O = (e - f)$ is effective. Hence O is the only effective -2-curve of S. The positive cone of X is given by divisors $xe + yf$ such that $xy > 0$. Hence the ample cone is given by divisors $D = xe + yf$ such that $xy > 0$ and $D.(e - f) > 0$. As $D.(e - f) = -x + y$, we obtain the desired statement. □

We are now ready to prove the first theorem.

Proof of Theorem 1.2. To prove the first statement, consider X defined in (1). Let e, f denote two generators of $U \cong \operatorname{Pic} X$ as in Proposition 3.2, and consider the divisor $D = e + df$. Then, by Proposition 3.2, $D \in \operatorname{Pic} X$ is ample because $d > 1$. As $D^2 = 2d$ and X has an automorphism of order 3 then $X \in \mathcal{H}_{3,2d}$ and hence $h_{3,2d} \leq 2$. As in general $h_{3,2d} \geq 2$ (it follows immediately from Theorem 2.6), we conclude that $h_{3,2d} = 2$.

To prove the second statement, consider $(Y, H, \alpha) \in \mathcal{H}_{3,2}$ and let $N(\alpha) \subset \operatorname{Pic} Y$ be the fixed locus of α^*. As the the fixed locus of α is not empty [1, Theorem 2.2] we have that $\rho(Y) \geq \operatorname{rk} N(\alpha) \geq 2$. If $\rho(Y) = 2$ it follows that $\operatorname{Pic} Y = N(\alpha) = U$ or $U(3)$ (Theorem 2.6). In both cases $\operatorname{Pic} Y$ does not admit an ample divisor of degree 2, hence a contradiction with the hypothesis that $H^2 = 2$. From this it follows that $h_{3,2} > 2$.

On the other hand, from Table 1 we see that there exists a complex K3 surface Y with an automorphism σ of order 3 and $\operatorname{Pic} Y = N(\sigma) \cong U(3) \oplus A_2^{\oplus 2}$ and this Y can be realized as double cover of $\mathbb{P}^2$, i.e., it admits a polarization of degree 2. From this it follows that $h_{3,2} \leq 6$.

As the Picard lattice of a K3 surface with a non-symplectic automorphism of order 3 is always even (Corollary 2.5), we conclude that $h_{3,2} \in \{4, 6\}$, proving the second statement. □

Remark 3.3. We are left to show that, for every integer $d > 1$, the bound $h_{3,2d} = 2$ can be attained over $\mathbb{Q}$. In fact, a priori X is only defined over $\mathbb{C}$. One might expect that for a random choice of rational coefficients of $p(t)$ one still obtains a K3 surface with Picard number 2. A practical problem arises though: computing the Picard number of a K3 surface given in its Weierstrass form is not easy.

Luckily we can use the beautiful K3 surface

$$X_{66} \colon y^2 = x^3 + t(t^{11} - 1)$$

considered by Kondo in [5], that is defined over $\mathbb{Q}$. This surface has the remarkable property of being the *unique* K3 surfaces (up to isomorphism) to admit a $\mathbb{Z}/66\mathbb{Z}$-action, hence admitting an automorphism of order 3. The Picard lattice of X_{66} is indeed U [5, Example 3.0.1].

Remark 3.4. Assume (Y, H, α) is in $\mathcal{H}_{3,2}$. As we will see in the next section, if

$$\operatorname{Pic} Y = N(\alpha)$$

then $\rho(Y) \geq 6$. Unfortunately, the above equality does not need to always hold, for instance see Example 3.5. Therefore at the moment we cannot exclude that $\rho(Y) = 4$. In this case, $\operatorname{Pic} Y$ is a finite index overlattice of $N(\alpha) \oplus L$, where L is the orthogonal complement of $N(\alpha)$ in $\operatorname{Pic} Y$. From Table 1, we know that $N(\alpha)$ is either U or $U(3)$. Notice that if $N(\alpha) = U$, then $\operatorname{Pic} Y = N(\alpha) \oplus L$, see [4, Example 14.0.6]. Moreover, as L is not contained in $N(\alpha)$, the isometry α^* of $\operatorname{Pic} Y$ induces an isometry of order 3 on L. Hence L is a negative definite lattice of rank 2 with an isometry of order 3: by [6, Lemma 6.11], it follows that $L \cong A_2(j)$ for some $j \geq 1$. This means that $\operatorname{Pic} Y$ is either $U \oplus A_2(j)$ or an overlattice of $U(3) \oplus A_2(j)$, for some value of $j \geq 1$. It is possible to show that for some values of j, the above lattices do not admit 2-divisors, but we have been unable to reach a general understanding of the existence of ample 2-divisors for all the values of j.

Example 3.5. It is not hard to construct an example of a K3 surface with an automorphism of order 3 acting non-trivially on the Picard lattice. For example, consider an elliptic K3 surface Z defined as in (1), with $p(t)$ with only simple roots and equal to $f_6^2 - g_4^3$, for some f_6 and g_4 polynomials in t of degree 6 and 4, respectively. Then $N(\alpha) \cong U$ and one can easily check that Z has at least two sections, namely (g_4, f_6, t) and $(\zeta_3 g_4, f_6, t)$. We then conclude that $\rho(Z) \geq 4$ and $N(\alpha) = U \subsetneq \operatorname{Pic} Z$.

4. The proof of the second theorem

In this section assume that (X, H, α) is in $\mathcal{H}^*_{3,2d}$, that is, X is a projective K3 surface with a polarization H of degree $2d$ and an automorphism α of order p acting as the identity on the whole $\operatorname{Pic} X$. Using the notation introduced in §2, this means that $N(\alpha) = \operatorname{Pic} X$. It also means that (X, α) is generic in the moduli space of K3 surface with an automorphism σ of order 3 and fixed locus of σ^* equal to $N(\alpha)$, see [1, Theorem 5.6]. The classification of the fixed locus of an order 3 non-symplectic automorphism given in [1, 11] is the key in establishing $h^*_{3,2}$.

Lemma 4.1. *For every $d > 1$, one has $h^*_{3,2d} = 2$.*

Proof. Recalling that $h^*_{3,2d} \geq h_{3,2d}$, Theorem 1.2 implies that $h^*_{3,2d} \geq 2$. On the other hand, Lemma 3.1 shows that the surface defined in (1) is in $\mathcal{H}^*_{3,2d}$ and has Picard number 2, concluding the proof. □

Remark 4.2. Kondo's surface mentioned in Remark 3.3 shows that also the bound $h^*_{3,2d}$ can be attained over $\mathbb{Q}$, for every $d > 1$.

We are left with case for $d=1$.

Lemma 4.3. *If $X \in \mathcal{H}^*_{3,2d}$ has Picard number 4, then $d > 1$.*

Proof. From the hypothesis, using Table 1, it immediately follows that $\operatorname{Pic} X$ is either $U \oplus A_2$ or $U(3) \oplus A_2$.

First we show that $U(3) \oplus A_2$ does not admit 2-divisors at all. Let u_1, u_2 and a_1, a_2 denote the generators of $U(3)$ and A_2, respectively and let

$$D := x_1u_1 + x_2u_2 + y_1a_1 + y_2a_2$$

a 2-divisor, that is, $D^2 = 2$. Dividing by 2, we get the following equality:

$$3x_1x_2 - y_1^2 + y_1y_2 - y_2^2 = 1 \ . \tag{2}$$

This can be rewritten as

$$3x_1x_2 - 1 = y_1^2 - y_1y_2 + y_2^2. \tag{3}$$

Reducing modulo 3, (3) induces the following equation:

$$y_1^2 - y_1y_2 + y_2^2 \equiv 2 \mod 3 \ . \tag{4}$$

It easy to see by direct computations that (4) has no solutions in $\mathbb{Z}/3\mathbb{Z}$, proving the claim.

Assume then that $\operatorname{Pic} X \cong U \oplus A_2$. This implies that $U \hookrightarrow N(\alpha) = \operatorname{Pic} X$ and hence X is elliptic and can be described by the following Weierstrass equation [1, Proposition 4.2]:

$$y^2 = x^3 + p(t),$$

where p is a polynomial of degree 12. As $\operatorname{Pic} X \cong U \oplus A_2$ we see that X has only one reducible singular fiber, of Kodaira type IV. This implies that $\operatorname{Pic} X$ is generated by the class of the fiber F, the class of the section O and the two components E_1, E_2 of the reducible fiber not meeting O. Using these four generators, the Gram matrix of $\operatorname{Pic} X$ is the following.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

Let us write $H = aO + fF + e_1E_1 + e_2E_2$ and notice that the third component of the singular fiber can be written as $E_3 = F - E_1 - E_2$. Then $H^2 = 2$ implies

$$af + e_1e_2 = e_1^2 + e_2^2 + a^2 + 1 \ . \tag{5}$$

As H is ample, its intersection with all the -2-curves is positive, that is,

$$\begin{cases} H.O & = f - 2a > 0 \ , \\ H.E_1 & = -2e_1 + e_2 > 0 \ , \\ H.E_2 & = e_1 - 2e_2 > 0 \ , \\ H.E_3 & = a + e_1 + e_2 > 0 \ . \end{cases}$$

From the above inequalities we deduce that

$$\begin{cases} 0 < 2a < f \ , \\ e_1, e_2 < 0 \ , \\ 0 < -e_1 - e_2 < a \ . \end{cases}$$

Consider then the quantity $2a^2+1$. As $2a < f$ and $e_1e_2 \geq 1$ we can write

$$2a^2+1 < af + e_1e_2 < af + 3e_1e_2 \, .$$

Using (5), we can substitute $af + e_1e_2$, hence obtaining

$$2a^2+1 < e_1^2 + e_2^2 + a^2 + 1 + 2e_1e_2 = (-e_1-e_2)^2 + a^2 + 1 < 2a^2+1$$

as $-e_1-e_2$ is strictly smaller than a. In this way we get

$$2a^2+1 < 2a^2+1 \, ,$$

a contradiction, proving that $U \oplus A_2$ admits no ample 2-divisors. This concludes the proof. □

Lemma 4.4. $h^*_{3,2} = 6$.

Proof. As $h^*_{3,2} \geq h_{3,2}$, from Theorem 1.2 it follows that $h^*_{3,2} > 2$. From Lemma 4.3 we also know that $h^*_{3,2} \neq 4$, hence $h^*_{3,2} \geq 6$. As already noted in the proof of the second statement of Theorem 1.2, from Table 1 we see that there exists a complex K3 surface Y with an automorphism σ of order 3 and $\operatorname{Pic} Y = N(\sigma) \cong U(3) \oplus A_2^{\oplus 2}$ and this Y can be realized as double cover of $\mathbb{P}^2$. From this it follows that $h^*_{3,2} = 6$. □

Now we have everything we need to prove the second theorem.

Proof of Theorem 1.4. The first equality is Lemma 4.4; the second is Lemma 4.1. □

Remark 4.5. It is easy to show that the bound $h^*_{3,2}$ can be attained over $\mathbb{Q}$. Indeed the paper [1] tells us how to find explicit examples of K3 surfaces of degree 2 with Picard number equal to 6, just by considering a surface as [1, Proposition 4.11] that is generic enough. For example, consider the K3 surface $X_{2,1}$ given by the double cover of $\mathbb{P}^2$ branched along the curve

$$B\colon F_6(x_0,x_1) + F_3(x_0,x_1)x_2^3 + bx_2^6$$

with

$$\begin{aligned} F_6 &:= -x_0^6 + 2x_0^5x_1 - x_0^4x_1^2 - 2x_0^3x_1^3 - x_0^2x_1^4 + x_0x_1^5 - x_1^6 \, , \\ F_3 &:= 2x_0^2x_1 - x_1^3 \, , \\ b &:= 2 \, . \end{aligned}$$

From [1, Proposition 4.11] we know that $\rho(X_{2,1}) \geq 6$. By reducing modulo a prime of good reduction, e.g. 11, one can see that $\rho(X_{2,1}) = 6$ and hence $\operatorname{Pic} X_{2,1} \cong U \oplus A_2^{\oplus 2}$.

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