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Gambaudo–Ghys construction on bounded cohomology

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Abstract. We consider a generalized Gambaudo–Ghys construction on bounded cohomology and prove its injectivity. As a corollary, we prove that the third bounded cohomology of the group of area-preserving diffeomorphisms on the 2-disk is infinite-dimensional. Additionally, we establish similar results for the 2-sphere, the 2-torus, and the annulus.

1. Introduction

Let $\mathbb{D}$ denote the unit 2-disk equipped with an area form, and $\mathcal{G}$ denote the group $\mathrm{Diff}(\mathbb{D}, \partial\mathbb{D}, \mathrm{area})$ of area-preserving diffeomorphisms on $\mathbb{D}$ that are the identity near the boundary $\partial\mathbb{D}$. In [**8**], Gambaudo and Ghys gave a construction of quasimorphisms on $\mathcal{G}$ using the signature of braids. By generalizing their method, Brandenbursky [**2**] defined a linear map $\Gamma\colon Q(P_m) \to Q(\mathcal{G})$, where $Q(G)$ denotes the space of homogeneous quasimorphisms on a group G, and P_m denotes the pure braid group on m strands. Let B_m be the braid group on m strands, and $i\colon P_m \to B_m$ the standard inclusion. In [**13**], Ishida proved that the composition map $\Gamma \circ i^*\colon Q(B_m) \to Q(\mathcal{G})$ is injective. He also proved that the map $\overline{EH}_b^2(B_m) \to \overline{EH}_b^2(\mathcal{G})$ induced by $\Gamma \circ i^*$ is injective, where $\overline{EH}_b^n(G)$ denotes the reduced exact bounded cohomology of G (with coefficients in $\mathbf{R}$). For definitions on the cohomology and bounded cohomology of groups, see Section 2.1. In this paper, we generalize Ishida's result to higher-dimensional bounded cohomology for the case of three strands. We define a map $\overline{E\Gamma}_b\colon \overline{EH}_b^n(B_m) \to \overline{EH}_b^n(\mathcal{G})$ that generalizes the Gambaudo–Ghys construction and prove the following theorem.

Theorem 1.1. *For $n \geq 2$, the map $\overline{E\Gamma}_b\colon \overline{EH}_b^n(B_3) \to \overline{EH}_b^n(\mathcal{G})$ is injective.*

As a corollary, we obtain the following result.

Corollary 1.2. *The dimension of $\overline{EH}_b^3(\mathcal{G})$ is uncountably infinite.*

Our work is inspired by the work of Brandenbursky and Marcinkowski [**5**]; for a complete Riemannian manifold M of finite volume, under a certain condition on $\pi_1(M)$, they proved that the third bounded cohomology $\overline{EH}_b^3(\mathrm{Diff}_0(M, \mathrm{vol}))$ of the identity component $\mathrm{Diff}_0(M, \mathrm{vol})$ of the volume-preserving diffeomorphism group on M is uncountably infinite-dimensional. Note that their result does not yield Corollary 1.2 since $\pi_1(\mathbb{D})$ is

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trivial. We also note that Nitsche [**17**] has generalized the work of Brandenbursky and Marcinkowski and ours to higher degrees.

We also prove similar results for compact surfaces Σ with non-negative Euler characteristic $\chi(\Sigma) \geq 0$. Let $B_m(\Sigma)$ and $P_m(\Sigma)$ denote the braid group and the pure braid group on a surface Σ, respectively. Let $\mathcal{G}_\Sigma$ denote the identity component of the group $\mathrm{Diff}_0(\Sigma, \partial\Sigma, \mathrm{area})$ of area-preserving diffeomorphisms on Σ that are the identity near the boundary $\partial\Sigma$. For $m \in \mathbf{N}$, let $K(\Sigma, m)$ denote the kernel of the forgetful map $\mathrm{MCG}(\Sigma, m) \to \mathrm{MCG}(\Sigma)$ (see Section 2.2). We also define the map $\overline{E\Gamma}_b^\Sigma \colon \overline{EH}_b^n(K(\Sigma, m)) \to \overline{EH}_b^n(\mathcal{G}_\Sigma)$ prove the following.

THEOREM 1.3. *Let Σ be a compact oriented surface such that $\chi(\Sigma) \geq 0$ and set $m = 2 + \chi(\Sigma)$. For $n \geq 2$, the map $\overline{E\Gamma}_b^\Sigma \colon \overline{EH}_b^n(K(\Sigma, m)) \to \overline{EH}_b^n(\mathcal{G}_\Sigma)$ is injective.*

Note that the case $n = 2$ is proved by Brandenbursky and Marcinkowski [**4**, Theorem 2.5]. We also note that the case $\Sigma = \mathbb{D}$ corresponds to Theorem 1.1 since $K(\mathbb{D}, m)$ is isomorphic to the braid group B_m. Similarly to Corollary 1.2, we obtain the following result.

COROLLARY 1.4. *Let Σ be a compact oriented surface such that $\chi(\Sigma) \geq 0$. The dimension of $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is uncountably infinite.*

We note that Corollary 1.4 is not deduced from the result of Brandenbursky and Marcinkowski [**5**]. On the other hand, their result covers the case of surfaces with negative Euler characteristics. Therefore, in some sense, our results and theirs are complementary to each other in the case of 2-manifolds. Namely, we have the following theorem.

THEOREM 1.5. *For any compact oriented surface Σ, the dimension of $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is uncountably infinite.*

2. Preliminary

2.1. (Bounded) cohomology of groups

We review the definitions on (bounded) cohomology of groups. Let G be a group. A (homogeneous) *n-cochain* on G is a function $c \colon G^{n+1} \to \mathbf{R}$ such that $c(g_0 h, \dots, g_n h) = c(g_0, \dots, g_n)$ for any $g_0, \dots, g_n, h \in G$. Let $C^n(G)$ denote the set of homogeneous n-cochains. We define the coboundary map $\delta \colon C^{n-1}(G) \to C^n(G)$ by

$$\delta c(g_0, \dots, g_n) = \sum_{i=0}^{n} (-1)^i c(g_0, \dots, \widehat{g_i}, \dots, g_n)$$

for $c \in C^{n-1}(G)$. Here, the symbol $\widehat{g}$ means that we omit the entry g. The cochain complex $(C^\bullet(G), \delta)$ defines the *group cohomology* $H^\bullet(G)$ of G. For a cochain $c \in C^n(G)$, we define $\|c\|_\infty \in [0, \infty]$ by

$$\|c\|_\infty = \sup_{g_0, \dots, g_n \in G} |c(g_0, \dots, g_n)|.$$

We say that a cochain $c \in C^n(G)$ is *bounded* if $\|c\|_\infty \in [0, \infty)$. Let $C_b^n(G)$ denote the set of bounded n-cochains. The cochain complex $(C_b^\bullet(G), \delta)$ defines the *bounded cohomology* $H_b^\bullet(G)$ of G. The inclusion $C_b^n(G) \to C^n(G)$ induces a homomorphism $H_b^n(G) \to H^n(G)$, which is called the *comparison map*. The kernel of the comparison map $H_b^n(G) \to H^n(G)$ is called the *exact bounded cohomology* and is denoted by $EH_b^n(G)$. The norm $\|\cdot\|_\infty$ on $C_b^n(G)$ induces the canonical semi-norm $\|\cdot\|$ on $H_b^n(G)$ defined by

$$\|u\| = \inf_{[c]=u} \|c\|_\infty$$

for $u \in H_b^n(G)$. The quotient space of $EH_b^n(G)$ by its norm zero subspace is called the *reduced exact bounded cohomology* and is denoted by $\overline{EH}_b^n(G)$.

We summarize several facts which we use later.

THEOREM 2.1 ([**10**, p.39]). *Let G be a group, H a normal subgroup of G, and $i\colon H \to G$ the inclusion map. If G/H is amenable, then the induced map $i^*\colon H_b^n(G) \to H_b^n(H)$ is injective and isometric for every $n \geq 1$.*

The following theorem is known as the mapping theorem (for groups).

THEOREM 2.2 ([**10**, p.40]). *If $\phi\colon G_1 \to G_2$ is a surjective group homomorphism with an amenable kernel, then the induced map $\phi^*\colon H_b^n(G_2) \to H_b^n(G_1)$ is an isometric isomorphism for every $n \geq 1$.*

If G is an amenable group, then its bounded cohomology $H_b^n(G)$ vanishes for every $n \geq 1$. On the other hand, the bounded cohomology of non-positive curvature groups tends to be highly non-trivial. For example, the following theorem is known.

THEOREM 2.3 ([**7**, Corollary 6.5]). *If G is an acylindrically hyperbolic group, then the dimension of $\overline{EH}_b^3(G)$ is uncountably infinite.*

Examples of acylindrically hyperbolic groups include: non-elementary hyperbolic groups, relatively hyperbolic groups, mapping class groups of hyperbolic surfaces, outer automorphism groups of non-abelian free groups, and most 3-manifold groups (see [**18**] for more information on acylindrically hyperbolic groups).

2.2. Braid groups

Let M be a compact connected oriented manifold. Let $X_m(M)$ denote the (*orderd*) *configuration space* of m points in M, i.e.,

$$X_n(M) = \{(x_1, \ldots, x_m) \in M^m \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Note that $X_m(M)$ is a codimension 0 submanifold of M^m. The fundamental group of $X_m(M)$ is called the *pure braid group on m strands* on M and denoted by $P_m(M)$. If $\dim M \geq 3$, it is known that the inclusion $X_m(M) \to M^m$ induces an isomorphism $P_m(M) \to \pi_1(M^m) \cong \pi_1(M) \times \cdots \times \pi_1(M)$ [**1**, Theorem 1.5]. Hence, we are especially interested in the case of $\dim M = 2$.

Let Σ be a compact connected oriented 2-dimensional manifold. The action of the symmetric group defines the quotient space $X_m(\Sigma)/\mathfrak{S}_m$, which is called the *unordered configuration space*. The fundamental group of $X_m(\Sigma)/\mathfrak{S}_m$ is called the *braid group on* m *strands* on Σ and denoted by $B_m(\Sigma)$. There exists a short exact sequence

$$1 \to P_m(\Sigma) \to B_m(\Sigma) \to \mathfrak{S}_m \to 1.$$

Thus, we regard $P_m(\Sigma)$ as a normal subgroup of $B_m(\Sigma)$. Note that $B_m(\mathbb{D})$ is the ordinary Artin braid group B_m and $P_m(\mathbb{D})$ is the pure braid group P_m.

Fix a base point $\bar{z}$ of the unordered configuration space $X_m(\Sigma)/\mathfrak{S}_m$. Let $P = \{x_1, \ldots, x_m\}$ be a set of distinct m points in Σ. We define the evaluation map $\mathrm{ev}_{\bar{z}}\colon \mathrm{Homeo}(\Sigma) \to X_m(\Sigma)/\mathfrak{S}_m$ by $\mathrm{ev}_{\bar{z}}(g) = g \cdot \bar{z}$, where the action of $\mathrm{Homeo}(\Sigma)$ on $X_m(\Sigma)/\mathfrak{S}_m$ is induced by the diagonal action. It is known that $\mathrm{ev}_{\bar{z}}$ is a locally trivial fibration with fiber $\mathrm{Homeo}(\Sigma - P)$ (see [**14**, Lemma 1.35]). Thus, this fibration induces the long exact sequence

$$\cdots \pi_1(\mathrm{Homeo}(\Sigma), \mathrm{id}_\Sigma) \xrightarrow{\mathrm{ev}_{\bar{z}}^*} B_m(\Sigma) \xrightarrow{Push} \mathrm{MCG}(\Sigma, m) \xrightarrow{Forget} \mathrm{MCG}(\Sigma) \to 1$$

and the induced map $Forget\colon \mathrm{MCG}(\Sigma, m) \to \mathrm{MCG}(\Sigma)$ is called the *forgetful map*, where $\mathrm{MCG}(\Sigma, m)$ and $\mathrm{MCG}(\Sigma)$ are the mapping class groups $\pi_0(\mathrm{Homeo}(\Sigma - P))$ and $\pi_0(\mathrm{Homeo}(\Sigma))$, respectively. Let $K(\Sigma, m)$ denote the kernel of the forgetful map. The map $Push\colon B_m(\Sigma) \to \mathrm{MCG}(\Sigma, m)$ is called the *push map*. Note that $K(\Sigma, m) = \mathrm{Ker}(Forget) = \mathrm{Im}(Push)$.

If Σ has non-empty boundary, then $\mathrm{Homeo}(\Sigma)$ is locally contractible [**12**], and thus $K(\Sigma, m)$ is isomorphic to $B_m(\Sigma)$. For the case where Σ is closed, the following result is known.

Theorem 2.4 ([**1**, Theorem 4.3]). *Let Σ be a closed oriented surface of genus g. If $g \geq 2$, then $K(\Sigma, m)$ is isomorphic to $B_m(\Sigma)$. If $g \geq 1, m \geq 2$ or $g = 0, m \geq 3$, then $K(\Sigma, m)$ is isomorphic to the central quotient $B_m(\Sigma)/Z(B_m(\Sigma))$.*

3. Generalized Gambaudo–Ghys construction

In this section, we discuss a generalized Gambaudo–Ghys construction.

3.1. The braid γ

Set $\mathcal{G} = \mathrm{Diff}(\mathbb{D}, \partial\mathbb{D}, \mathrm{area})$ and fix a base point $\bar{z} = (z_1, \ldots, z_m) \in X_m(\mathbb{D})$. For simplicity, we assume that $\mathbb{D}$ is equipped with the standard Euclidean area form. For every $g \in \mathcal{G}$ and almost every $\bar{x} = (x_1, \ldots, x_m) \in X_m(\mathbb{D})$, we define a pure braid $\gamma(g, \bar{x}) \in P_m$ as follows. We take an isotopy $\{g_t\}_{0 \leq t \leq 1}$ of g such that $g_0 = \mathrm{id}_{\mathbb{D}}$, $g_1 = g$, and $g_t \in \mathcal{G}$ for every $t \in [0, 1]$. We define a loop $l(\{g_t\}, \bar{x})\colon [0, 1] \to X_m(\mathbb{D})$ in $X_m(\mathbb{D})$ by

$$l(\{g_t\}, \bar{x})(t) = \begin{cases} \{(1-3t)z_i + 3tx_i\}_{i=1,\ldots,m} & \text{if } 0 \leq t \leq 1/3, \\ \{g_{3t-1}(x_i)\}_{i=1,\ldots,m} & \text{if } 1/3 \leq t \leq 2/3, \\ \{(3-3t)g(x_i) + (3t-2)z_i\}_{i=1,\ldots,m} & \text{if } 2/3 \leq t \leq 1. \end{cases}$$

The braid $\gamma(g,\bar{x}) \in P_m$ is defined as the element of $\pi_1(X_m(\mathbb{D}),\bar{z})$ represented by the loop $l(\{g_t\},\bar{x})$. Although $\gamma(g,\bar{x})$ is not defined for every $\bar{x} \in X_m(\mathbb{D})$, there exists a full measure subspace Ω_m of $X_m(\mathbb{D})$ with the following property: the braid $\gamma(g,\bar{x})$ is defined if and only if both $\bar{x}$ and $g \cdot \bar{x}$ belong to Ω_m [**9**, Section 3.2].

As is well known, $\mathcal{G}$ is contractible (since $\mathcal{G}$ is homotopy equivalent to $\mathrm{Diff}(\mathbb{D},\partial\mathbb{D})$, which is contractible [**19**]). Therefore, the above definition of $\gamma(g,\bar{x})$ does not depend on the choice of an isotopy $\{g_t\}_{0\leq t\leq 1}$.

3.2. The maps Γ_b and Γ

For $c \in C_b^n(B_m)$, we define a map $\widehat{\Gamma}_b(c)\colon \mathcal{G}^{n+1} \to \mathbf{R}$ by

$$\widehat{\Gamma}_b(c)(g_0,\ldots,g_n) = \int_{\bar{x}\in X_m(\mathbb{D})} c(\gamma(g_0,\bar{x}),\ldots,\gamma(g_n,\bar{x}))d\bar{x} \tag{3.1}$$

for $g_0,\ldots,g_n \in \mathcal{G}$. Since c is bounded and the map $\bar{x} \mapsto c(\gamma(g_0,\bar{x}),\ldots,\gamma(g_n,\bar{x}))$ is defined on a full measure subset

$$\{\bar{x} \in X_m(\mathbb{D}) \mid \bar{x}, g_0\cdot\bar{x},\ldots,g_m\cdot\bar{x} \in \Omega_m\} = \Omega \cap g_0^{-1}(\Omega_m) \cap\cdots\cap g_n^{-1}(\Omega_m).$$

of $X_m(\mathbb{D})$, the map $\widehat{\Gamma}_b(c)$ is well-defined.

LEMMA 3.1. *For every $c \in C_b^n(B_m)$, $\widehat{\Gamma}_b(c)$ is a bounded homogeneous cochain. Moreover, the map $\widehat{\Gamma}_b\colon C_b^n(B_m) \to C_b^n(\mathcal{G})$ is a cochain map.*

PROOF. Since

$$|\widehat{\Gamma}_b(c)(g_0,\ldots,g_n)| \leq \mathrm{vol}(X_m(\mathbb{D}))\cdot\|c\|_\infty$$

for every $g_0,\ldots,g_n \in \mathcal{G}$, $\widehat{\Gamma}_b(c)$ is bounded. Note that $\gamma(gh,\bar{x}) = \gamma(g,h\cdot\bar{x})\gamma(h,\bar{x})$ for $g,h \in \mathcal{G}$, where $\mathcal{G}$ acts diagonally on $X_m(\mathbb{D})$. Thus,

$$\begin{aligned}\widehat{\Gamma}_b(c)(g_0h,\ldots,g_nh) &= \int_{\bar{x}\in X_m(\mathbb{D})} c(\gamma(g_0h,\bar{x}),\ldots,\gamma(g_nh,\bar{x}))d\bar{x}\\ &= \int_{\bar{x}\in X_m(\mathbb{D})} c(\gamma(g_0,h\cdot\bar{x})\gamma(h,\bar{x}),\ldots,\gamma(g_n,h\cdot\bar{x})\gamma(h,\bar{x}))d\bar{x}\\ &= \int_{\bar{x}\in X_m(\mathbb{D})} c(\gamma(g_0,h\cdot\bar{x}),\ldots,\gamma(g_n,h\cdot\bar{x}))d\bar{x}.\end{aligned}$$

Since the action of h preserves the volume form, $\widehat{\Gamma}(c)(g_0h,\ldots,g_nh) = \widehat{\Gamma}(c)(g_0,\ldots,g_n)$ and hence $\widehat{\Gamma}(c)$ is homogenous. By definition, the map $\widehat{\Gamma}$ and the coboundary map δ commute. Thus, $\widehat{\Gamma}$ is a cochain map. □

By Lemma 3.1, the map $\widehat{\Gamma}_b\colon C_b^n(B_m) \to C_b^n(\mathcal{G})$ induces the homomorphism $\Gamma_b\colon H_b^n(B_m) \to H_b^n(\mathcal{G})$.

We also define a map $\widehat{\Gamma}\colon C^n(B_m) \to C^n(\mathcal{G})$ on the ordinary cochain complex as in equation (3.1). The well-definedness of the map $\widehat{\Gamma}(c)\colon \mathcal{G}^{n+1} \to \mathbf{R}$ is not obvious since $c \in C^n(B_m)$ is not necessarily bounded, but the map $\widehat{\Gamma}(c)$ is well-defined since the map $\gamma(g,\cdot)\colon X_m(\mathbb{D}) \to B_m$ has essentially finite image (i.e., there exists a full measure subset of $X_m(\mathbb{D})$ whose image by the map is a finite subset in B_m) [**4**, Lemma 2.1]. Let $\Gamma\colon H^n(B_m) \to H^n(\mathcal{G})$, $E\Gamma_b\colon EH_b^n(B_m) \to EH_b^n(\mathcal{G})$ and $\overline{E\Gamma}_b\colon \overline{EH}_b^n(B_m) \to \overline{EH}_b^n(\mathcal{G})$ be the maps induced by $\widehat{\Gamma}$.

4. Proof of main result

In this section, we prove Theorem 1.1. We reduce Theorem 1.1 to the following key lemma, which corresponds to [**5**, Lemma 4.1]. Recall that $i\colon P_3 \to B_3$ denotes the inclusion map.

LEMMA 4.1. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon\colon P_3 \to \mathcal{G}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon\to 0} \|\rho_\epsilon^*(\overline{E\Gamma}_b(u)) - \Lambda \cdot i^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(B_3)$, where $\rho_\epsilon^*\colon \overline{EH}_b^n(\mathcal{G}) \to \overline{EH}_b^n(P_3)$ is the map induced by ρ_ϵ.

Before we prove Lemma 4.1, we give the proof of Theorem 1.1 from Lemma 4.1.

PROOF OF THEOREM 1.1. Let $u \in \overline{EH}_b^n(B_3)$ be a non-trivial class. It means that $\|u\| > 0$, and thus $\|i^*(u)\| > 0$ by Theorem 2.1. Therefore, by Lemma 4.1, we can see that $\|\rho_\epsilon^*(\overline{E\Gamma}_b(u))\| > 0$ for sufficiently small $\epsilon > 0$. It means that $\overline{E\Gamma}_b(u)$ is non-trivial and hence $\overline{E\Gamma}_b$ is injective. □

Corollary 1.2 is deduced from Theorem 1.1 as follows.

PROOF OF COROLLARY 1.2. By Theorem 2.3, the dimension of $\overline{EH}_b^3(B_3/Z(B_3))$ is uncountably infinite since $B_3/Z(B_3) \cong PSL(2,\mathbf{Z})$ is non-elementary hyperbolic. The quotient map $B_3 \to B_3/Z(B_3)$ induces an isomorphism $H_b^n(B_3) \to H_b^n(B_3/Z(B_3))$ by Theorem 2.2. Since $H^3(B_3) = 0$ (see [**20**, Chapter I] for example) and $H^3(PSL(2,\mathbf{Z})) \cong H^3(\mathbf{Z}/2\mathbf{Z}) \oplus H^3(\mathbf{Z}/3\mathbf{Z}) = 0$, $EH_b^3(B_3)$ and $EH_b^3(B_3/Z(B_3))$ are also isomorphic. Therefore, by Theorem 1.1, $\overline{EH}_b^3(\mathrm{Diff}(\mathbb{D},\mathrm{area}))$ is also uncountably infinite-dimensional. □

In the rest of this section, we prepare for the proof of Lemma 4.1 in Sections 4.1 and 4.2, and prove Lemma 4.1 in Section 4.3. The strategy of our proof comes from the work of Brandenbursky and Marcinkowski [**5**], and the method is inspired by the work of Ishida [**13**].

4.1. Construction of ρ_ϵ

For each ϵ with $0 < \epsilon < 1$, we construct a homomorphism $\rho_\epsilon\colon P_3 \to \mathcal{G}$. Recall that $\bar{z} = (z_1, z_2, z_3)$ denotes the base point of $X_3(\mathbb{D})$. For each $i = 1, 2, 3$, we take an open neighborhood U_i^ϵ of z_i in $\mathbb{D}$ such that

- $U_i^\epsilon \cap U_j^\epsilon = \emptyset$ if $i \neq j$, and
- $\mathrm{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon \cup U_3^\epsilon$.

We take open subsets W_{12}^ϵ and V_{12}^ϵ of $\mathbb{D}$ which are diffeomorphic to a disk such that

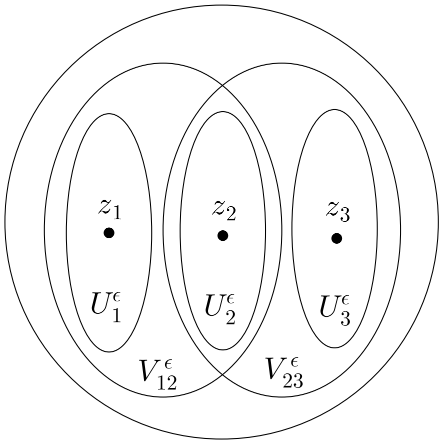

Figure 1. Open subsets in $\mathbb{D}$

- $U_1^\epsilon \cup U_2^\epsilon \subset \overline{W_{12}^\epsilon} \subset V_{12}^\epsilon$ and
- $V_{12}^\epsilon \cap U_3^\epsilon = \emptyset$.

Here, $\overline{W_{12}^\epsilon}$ denotes the closure of W_{12}^ϵ in $\mathbb{D}$. We also take W_{23}^ϵ and V_{23}^ϵ similarly (see Figure 1). Finally, we take open disks W_{123}^ϵ and V_{123}^ϵ to be $V_{12}^\epsilon \cup V_{23}^\epsilon \subset \overline{W_{123}^\epsilon} \subset V_{123}^\epsilon$.

We define $\rho_\epsilon\colon P_3 \to \mathcal{G}$ as follows. Set $a_1 = (\sigma_1)^2$, $a_2 = (\sigma_2)^2$, and $a_3 = \Delta^2$. Here, $\Delta^2 = (\sigma_1\sigma_2)^3$ is the full twist. Then P_3 has a presentation

$$P_3 = \langle a_1, a_2, a_3 \mid a_1a_3 = a_3a_1, a_2a_3 = a_3a_2 \rangle \cong F_2 \times \mathbf{Z}.$$

For open disks V and W such that $\overline{W} \subset V$, let $g_{V,W} \in \mathcal{G}$ denote a diffeomorphism that rotates W once such that $\mathrm{supp}(g_{V,W}) \subset V$. We define $\rho_\epsilon\colon P_3 \to \mathcal{G}$ by $\rho_\epsilon(a_1) = g_{V_{12}^\epsilon, W_{12}^\epsilon}$, $\rho_\epsilon(a_2) = g_{V_{23}^\epsilon, W_{23}^\epsilon}$ and $\rho_\epsilon(a_3) = g_{V_{123}^\epsilon, W_{123}^\epsilon}$. Note that $\rho_\epsilon(a_3)|_{W_{123}^\epsilon} = \mathrm{id}_{W_{123}^\epsilon}$. Since $\mathrm{supp}(\rho_\epsilon(a_1)) \subset V_{12}^\epsilon \subset W_{123}^\epsilon$, $\rho_\epsilon(a_1)$ and $\rho_\epsilon(a_3)$ commute. Similarly, $\rho_\epsilon(a_2)$ and $\rho_\epsilon(a_3)$ are also commutative. Thus ρ_ϵ is well-defined.

4.2. Calculation of $\gamma(\rho_\epsilon(\alpha), \bar{x})$

For $u = [c] \in \overline{EH}_b^n(B_3)$, $\rho_\epsilon^*(\overline{E\Gamma}_b(u)) \in \overline{EH}_b^n(P_3)$ is the cohomology class of the cochain defined by

$$(\alpha_0, \dots, \alpha_n) \mapsto \int_{\bar{x} \in X_3(\mathbb{D})} c(\gamma(\rho_\epsilon(\alpha_0), \bar{x}), \dots, \gamma(\rho_\epsilon(\alpha_n), \bar{x}))d\bar{x}$$

for $\alpha_0, \dots, \alpha_n \in P_3$. We calculate $\gamma(\rho_\epsilon(\alpha), \bar{x}) \in P_3$ for $\alpha \in P_3$ and $\bar{x} = (x_1, x_2, x_3) \in X_3(\mathbb{D})$. To describe it, we prepare several notions. We say that $x \in X_3(\mathbb{D})$ is an *ϵ-good point* if all of x_1, x_2, and x_3 are in U^ϵ. Otherwise, we say that $\bar{x}$ is an *ϵ-bad point.* We say that an ϵ-good point $\bar{x}$ is *of type* (p, q, r) if

$$\#(U_1^\epsilon \cap \{x_1, x_2, x_3\}) = p, \quad \#(U_2^\epsilon \cap \{x_1, x_2, x_3\}) = q, \quad \#(U_3^\epsilon \cap \{x_1, x_2, x_3\}) = r.$$

We define homomorphisms $s_i\colon P_3 \to \mathbf{Z}$ $(i = 1, 2, 3)$ by $s_i(a_j) = \delta_{ij}$ for $1 \le i, j \le 3$, where δ_{ij} is the Kronecker delta. For each type (p, q, r), we define a homomorphism $\phi_{pqr}\colon P_3 \to P_3$ by

$$\phi_{pqr}(\alpha) = \begin{cases} \alpha & \text{type } (1,1,1), \\ a_3^{s_1(\alpha)+s_3(\alpha)} & \text{type } (3,0,0) \text{ or } (2,1,0), \\ a_3^{s_2(\alpha)+s_3(\alpha)} & \text{type } (0,0,3) \text{ or } (0,1,2), \\ a_3^{s_1(\alpha)+s_2(\alpha)+s_3(\alpha)} & \text{type } (0,3,0), \\ a_1^{s_1(\alpha)}a_3^{s_3(\alpha)} & \text{type } (2,0,1), \\ a_2^{s_2(\alpha)}a_3^{s_3(\alpha)} & \text{type } (1,0,2), \\ a_1^{s_1(\alpha)}a_3^{s_2(\alpha)+s_3(\alpha)} & \text{type } (0,2,1), \\ a_2^{s_2(\alpha)}a_3^{s_1(\alpha)+s_3(\alpha)} & \text{type } (1,2,0). \end{cases} \tag{4.1}$$

The following is the key to the proof of Lemma 4.1 (compare with Ishida [**13**, Theorem 1.2]).

LEMMA 4.2. *For almost every ϵ-good point $\bar{x} \in X_m(\mathbb{D})$ of type (p, q, r), there exists a braid $\beta(\bar{x}) \in B_3$ such that*

$$\gamma(\rho_\epsilon(\alpha, \bar{x})) = \beta(\bar{x})\phi_{pqr}(\alpha)\beta(\bar{x})^{-1}$$

for every $\alpha \in P_3$.

PROOF. Let $\bar{x} = (x_1, x_2, x_3)$ be of type (p, q, r). Assume that $x_{i_1}, \dots, x_{i_p} \in U_1^\epsilon$, $x_{j_1}, \dots, x_{j_q} \in U_2^\epsilon$, and $x_{k_1}, \dots, x_{k_r} \in U_3^\epsilon$. The subscript i is defined so that $i_1 < \cdots < i_p$; the same applies to j and k. We define $\sigma_{\bar{x}} \in \mathfrak{S}_3$ by

$$\sigma_{\bar{x}} = \begin{pmatrix} i_1 & \cdots & i_p & j_1 & \cdots & j_q & k_1 & \cdots & k_r \\ 1 & \cdots & p & p+1 & \cdots & p+q & p+q+1 & \cdots & p+q+r \end{pmatrix}.$$

For example, if $x_1, x_3 \in U_2^\epsilon$ and $x_2 \in U_3^\epsilon$, then $j_1 = 1$, $j_2 = 3$, $k_1 = 2$ and thus $\sigma_{\bar{x}} = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$. Note that $\sigma_{\bar{x}} = e$ if $\bar{x}$ is of type $(0, 0, 3)$, $(0, 3, 0)$ or $(3, 0, 0)$.

We define $\beta(\bar{x}) \in B_3$ as the element of $\pi_1(X_3(\mathbb{D})/\mathfrak{S}_3, \bar{z})$ represented by the loop $l\colon [0, 1] \to X_3(\mathbb{D})/\mathfrak{S}_3$ defined by

$$l(t) = \begin{cases} \{(1-2t)z_i + 2tx_i\}_{i=1,2,3} & \text{if} \quad 0 \le t \le 1/2, \\ \{(2-2t)x_i + (2t-1)z_{\sigma_{\bar{x}}(i)}\}_{i=1,2,3} & \text{if} \quad 1/2 \le t \le 1. \end{cases}$$

The braid $\beta(\bar{x})$ is defined for almost every ϵ-good point $\bar{x}$. Note that the projection $B_3 \to \mathfrak{S}_3$ maps $\beta(\bar{x})$ to $\sigma_{\bar{x}}$. Then, the calculation of $\gamma(\rho_\epsilon(a_i), \bar{x})$ for the generators a_1, a_2, a_3 of P_3 is as follows (see also Figure 2):

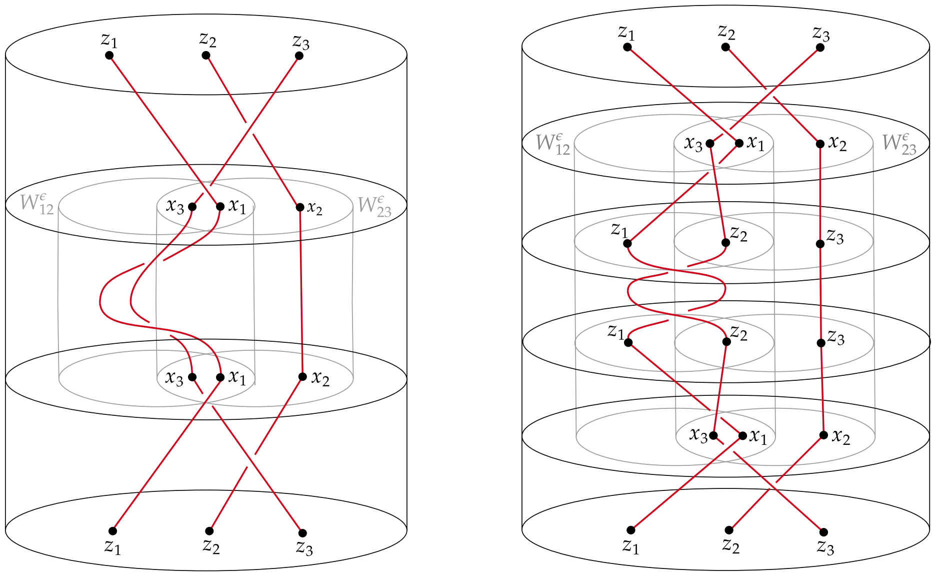

Figure 2. Braids $\gamma(\rho_\epsilon(a_1),\bar{x})$ and $\beta(\bar{x})a_1\beta(\bar{x})^{-1}$ for $\bar{x}$ is of type (0,1,2)

- $\gamma(\rho_\epsilon(a_1),\bar{x}) = \begin{cases} e & \text{if} \quad p+q \le 1, \\ \beta(\bar{x})a_1\beta(\bar{x})^{-1} & \text{if} \quad p+q=2, \\ \beta(\bar{x})a_3\beta(\bar{x})^{-1} & \text{if} \quad p+q=3. \end{cases}$
- $\gamma(\rho_\epsilon(a_2),\bar{x}) = \begin{cases} e & \text{if} \quad q+r \le 1, \\ \beta(\bar{x})a_2\beta(\bar{x})^{-1} & \text{if} \quad q+r=2, \\ \beta(\bar{x})a_3\beta(\bar{x})^{-1} & \text{if} \quad q+r=3. \end{cases}$
- $\gamma(\rho_\epsilon(a_3),\bar{x}) = \beta(\bar{x})a_3\beta(\bar{x})^{-1}$

If $(p,q,r)=(1,1,1)$, since $\gamma(\rho_\epsilon(a_i),\bar{x}) = \beta(\bar{x})a_i\beta(\bar{x})^{-1}$ for $i=1,2,3$, it follows that $\gamma(\rho_\epsilon(\alpha),\bar{x}) = \beta(\bar{x})\alpha\beta(\bar{x})^{-1}$ for every $\alpha \in P_3$. If $(p,q,r) \neq (1,1,1)$, noting that a_3 commutes with any braid, we obtain the assertion. ☐

4.3. Proof of the key lemma

Now we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. For each ϵ with $0<\epsilon<1$, we take an open neighborhood U_i^ϵ of z_i in $\mathbb{D}$ for $i=1,2,3$, and construct the homomorphism $\rho_\epsilon\colon P_3 \to \mathcal{G}$ as in Section 4.1. Let X^ϵ_{pqr} denote the set of ϵ-good points in $X_3(\mathbb{D})$ of type (p,q,r) and Y^ϵ denote the set of ϵ-bad points. We define cochains $c^\epsilon_{pqr}, c^\epsilon_Y \in C^n_b(P_3)$ by

$$c^\epsilon_{pqr}(\alpha_0,\dots,\alpha_n) = \int_{\bar{x}\in X^\epsilon_{pqr}} c(\gamma(\rho_\epsilon(\alpha_0),\bar{x}),\dots,\gamma(\rho_\epsilon(\alpha_n),\bar{x}))d\bar{x},$$

$$c_Y^\epsilon(\alpha_0,\dots,\alpha_n) = \int_{\bar{x}\in Y^\epsilon} c(\gamma(\rho_\epsilon(\alpha_0),\bar{x}),\dots,\gamma(\rho_\epsilon(\alpha_n),\bar{x}))d\bar{x}$$

for $\alpha_0,\dots,\alpha_n \in P_3$. Note that

$$\rho_\epsilon^*(\overline{E\Gamma}_b(u)) = \sum_{p,q,r}[c_{pqr}^\epsilon] + [c_Y^\epsilon] \in \overline{EH}_b^n(P_3).$$

For $c \in C^n(B_3)$ and $\beta \in B_3$, let $\beta\cdot c \in C^n(B_3)$ denote the cochain defined by

$$(\beta\cdot c)(\gamma_0,\dots,\gamma_n) = c(\beta\gamma_0\beta^{-1},\dots,\beta\gamma_n\beta^{-1}).$$

for $\gamma_0,\dots,\gamma_n \in B_3$. By Lemma 4.2, c_{pqr}^ϵ satisfies that

$$\begin{aligned} c_{pqr}^\epsilon(\alpha_0,\dots,\alpha_n) &= \int_{\bar{x}\in X_{pqr}^\epsilon} c(\beta(\bar{x})\phi_{pqr}(\alpha_0)\beta(\bar{x})^{-1},\dots,\beta(\bar{x})\phi_{pqr}(\alpha_n)\beta(\bar{x})^{-1})d\bar{x} \\ &= \sum_{\beta\in B_3} \mathrm{vol}\left(\{\bar{x}\in X_{pqr}^\epsilon \mid \beta(\bar{x}) = \beta\}\right)(\beta\cdot c)(\phi_{pqr}(\alpha_0),\dots,\phi_{pqr}(\alpha_n)) \end{aligned}$$

for any $\alpha_0,\dots,\alpha_n \in P_3$. Since $[\beta\cdot c] = [c] = u$ for any $\beta \in B_3$, it holds that

$$[c_{pqr}^\epsilon] = \mathrm{vol}(X_{pqr}^\epsilon)\cdot\phi_{pqr}^*(i^*(u)). \tag{4.2}$$

If $(p,q,r) = (1,1,1)$, since $\phi_{111} = \mathrm{id}$ and by equation (4.2),

$$[c_{111}^\epsilon] = \mathrm{vol}(X_{111}^\epsilon)\cdot i^*(u) = 3!\cdot\mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)\,\mathrm{area}(U_3^\epsilon)\cdot i^*(u).$$

If $(p,q,r) \neq (1,1,1)$, by equation (4.1), the homomorphism ϕ_{pqr} factors through the abelian subgroup $\langle a_i, a_3\rangle \cong \mathbf{Z}^2$ of P_3, where $i = 1$ or $i = 2$. Since $\mathbf{Z}^2$ is amenable, $\overline{EH}_b^n(\mathbf{Z}^2) = 0$. Thus $\phi_{pqr}^* = 0$ and hence $[c_{pqr}^\epsilon] = 0$ by equation (4.2).

By the definition of c_Y^ϵ,

$$|c_Y^\epsilon(\alpha_0,\dots,\alpha_n)| \le \mathrm{vol}(Y^\epsilon)\|c\|_\infty.$$

Since $\mathrm{vol}(Y^\epsilon) = \mathrm{vol}(X_3(\mathbb{D})) - \mathrm{vol}(U^\epsilon\times U^\epsilon\times U^\epsilon) = 1-(1-\epsilon)^3$, $\lim_{\epsilon\to 0}\|[c_Y^\epsilon]\| = 0$.

Therefore, by setting $\Lambda = \lim_{\epsilon\to 0} 3!\cdot\mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)\,\mathrm{area}(U_3^\epsilon)$,

$$\lim_{\epsilon\to 0}\|\rho_\epsilon^*(\overline{E\Gamma}_b(u)) - \Lambda\cdot i^*(u)\| = 0. \qquad \square$$

5. The case of other surfaces

In this section, we apply the argument from the previous section to the case of other surfaces and prove Theorem 1.3. Let Σ be a compact surface with an area form. We set $\mathcal{G}_\Sigma = \mathrm{Diff}_0(\Sigma,\partial\Sigma,\mathrm{area})$.

We provide a generalized Gambaudo–Ghys construction on surfaces (see also [**4**, Section 2]). Take a continuous map $\iota\colon \mathbb{D}\to\Sigma$ such that $\iota|_{\mathbb{D}\setminus\partial\mathbb{D}}$ is injective and $\iota(\mathbb{D}\setminus\partial\mathbb{D})$ is of full measure in Σ. Take a base point $\bar{z} = (z_1,\dots,z_m)$ of $X_m(\Sigma)$ so that $z_i \in \iota(\mathbb{D}\setminus\partial\mathbb{D})$ for each i. Let $g \in \mathcal{G}_\Sigma$ and fix an isotopy $\{g_t\}_{0\le t\le 1}$ of g. For $\bar{x} = (x_1,\dots,x_m) \in \iota_*(\Omega_m)$, we define the loop $l(\{g_t\},\bar{x})\colon [0,1]\to X_m(\Sigma)$ by

$$l(\{g_t\},\bar{x})(t)=\begin{cases}\Big\{\iota\Big((1-3t)\cdot\iota^{-1}(z_i)+3t\cdot\iota^{-1}(x_i)\Big)\Big\}_{i=1,\dots,m} & \text{if } 0\le t\le 1/3,\\ \{g_{3t-1}(x_i)\}_{i=1,\dots,m} & \text{if } 1/3\le t\le 2/3,\\ \Big\{\iota\Big((3-3t)\cdot\iota^{-1}(g(x_i))+(3t-2)\cdot\iota^{-1}(z_i)\Big)\Big\}_{i=1,\dots,m} & \text{if } 2/3\le t\le 1.\end{cases}$$

Let $\gamma(\{g_t\},\bar{x})$ denote an element of $\pi_1(X_m(\Sigma),\bar{z})\cong P_m(\Sigma)$ represented by the loop $l(\{g_t\},\bar{x})$. In general, $\gamma(\{g_t\},\bar{x})$ depends on the choice of an isotopy $\{g_t\}$ but $Push(\gamma(\{g_t\},\bar{x}))\in \mathrm{MCG}(\Sigma,m)$ does not: if $\{g'_t\}_{0\le t\le 1}$ is another isotopy of g, then $\gamma(\{g_t\},\bar{x})\gamma(\{g'_t\},\bar{x})^{-1}\in \mathrm{Im}(ev_z^*)=\mathrm{Ker}(Push)$. Thus, $\gamma(\{g_t\},\bar{x})$ defines an element of $K(\Sigma,m)$ and we write this element as $\gamma(g,\bar{x})$.

In this way, we can define the map $\widehat{\Gamma}_b^{\Sigma}\colon C_b^n(K(\Sigma,m))\to C_b^n(\mathcal{G}_\Sigma)$ in the same way as in Section 3.2 since $\iota_*(\Omega_m)$ is a full measure subset of $X_m(\Sigma)$. Here, $\iota_*\colon X_m(\mathbb{D})\to X_m(\Sigma)$ is the map induced by ι. This map $\widehat{\Gamma}_b^{\Sigma}$ is a cochain map by the same arguments as in Lemma 3.1, and induces the map $\Gamma_b^{\Sigma}\colon H_b^n(K(\Sigma,m))\to H_b^n(\mathcal{G}_\Sigma)$. Moreover, the map $\widehat{\Gamma}^{\Sigma}\colon C^n(K(\Sigma,m))\to C^n(\mathcal{G}_\Sigma)$ defined by

$$\widehat{\Gamma}^{\Sigma}(c)(g_0,\dots,g_n)=\int_{\bar{x}\in X_m(\Sigma)} c(\gamma(g_0,\bar{x}),\dots,\gamma(g_n,\bar{x}))d\bar{x}$$

is well-defined since the map $\gamma(g,\cdot)\colon X_m(\Sigma)\to MCG(\Sigma,m)$ has essentially finite image [**4**, Lemma 2.1]. Then $\widehat{\Gamma}^{\Sigma}$ induces the map $\Gamma^{\Sigma}\colon H^n(K(\Sigma,m))\to H^n(\mathcal{G}_\Sigma)$ and hence induces the map $\overline{E\Gamma}_b^{\Sigma}\colon \overline{EH}_b^n(K(\Sigma,m))\to \overline{EH}_b^n(\mathcal{G}_\Sigma)$.

5.1. The case of an annulus

Let $\mathbb{A}$ denote an annulus $S^1\times[0,1]$. The braid group $B_m(\mathbb{A})$ on $\mathbb{A}$ is isomorphic to the inverse image $\pi^{-1}(\mathfrak{S}_m)$ of the subgroup $\mathfrak{S}_m\subset\mathfrak{S}_{m+1}$ of $\mathfrak{S}_{m+1}$ by the projection $\pi\colon B_{m+1}\to\mathfrak{S}_{m+1}$ [**15**, Theorem 2]. This is because a "pillar" in $\mathbb{A}\times[0,1]$ can be seen as a "fixed" strand (Figure 3, see also [**15**, Section 2]). Namely, the pure braid group $P_m(\mathbb{A})$ on $\mathbb{A}$ is isomorphic to the ordinary pure braid group P_{m+1}, thus we identity them. Let $i\colon P_m(\mathbb{A})\to B_m(\mathbb{A})$ be the inclusion map and set $j=Push\circ i\colon P_m(\mathbb{A})\to K(\mathbb{A},m)$.

Lemma 5.1. *There exist a constant $\Lambda>0$ and a family of homomorphisms $\{\rho_\epsilon\colon P_2(\mathbb{A})\to\mathcal{G}_{\mathbb{A}}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon\to 0}\|\rho_\epsilon^*(\overline{E\Gamma}_b^{\mathbb{A}}(u))-\Lambda\cdot j^*(u)\|=0$$

for any $u\in\overline{EH}_b^n(K(\mathbb{A},2))$.

Proof. For each ϵ, we take open neighborhood U_i^ϵ of $z_i\in\mathbb{A}$ $(i=1,2)$ so that

- $U_1^\epsilon\cap U_2^\epsilon=\emptyset$ and
- $\mathrm{area}(U^\epsilon)=1-\epsilon$, where $U^\epsilon=U_1^\epsilon\cup U_2^\epsilon$.

Moreover, we take open subsets W_1^ϵ and V_1^ϵ of $\mathbb{A}$ that are diffeomorphic to an annulus so that

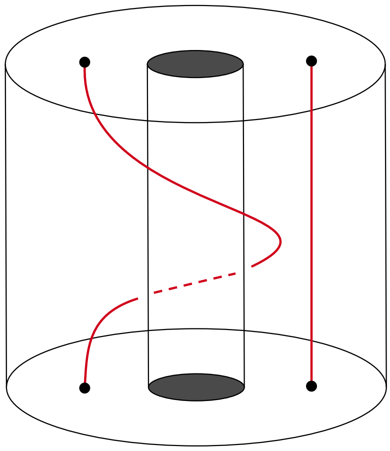

Figure 3. The 2-braid $(\sigma_1)^2$ on $\mathbb{A}$

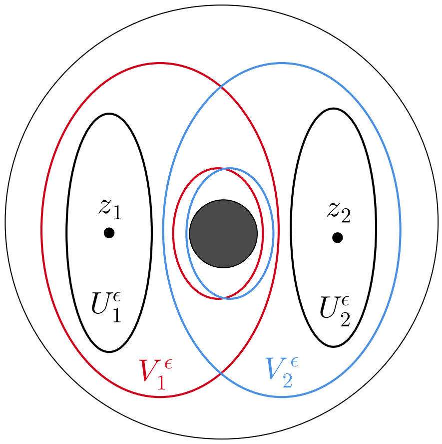

Figure 4. Open subsets in $\mathbb{A}$

- $U_1^\epsilon \subset \overline{W_1^\epsilon} \subset V_1^\epsilon$,
- $V_1^\epsilon \cap U_2^\epsilon = \emptyset$ and
- the inclusion map $W_1^\epsilon \to \mathbb{A}$ induces an isomorphism $\pi_1(W_1^\epsilon) \to \pi_1(\mathbb{A})$.

We also take W_2^ϵ and V_2^ϵ in a similar way (Figure 4). Finally, we take open annulus W_{12}^ϵ and V_{12}^ϵ to be $V_1^\epsilon \cup V_2^\epsilon \subset \overline{W_{12}^\epsilon} \subset V_{12}^\epsilon$.

We define $\rho_\epsilon \colon P_2(\mathbb{A}) \to \mathcal{G}_{\mathbb{A}}$ as follows. Recall that $P_2(\mathbb{A}) \cong P_3$ has a presentation

$$P_3 = \langle a_1, a_2, a_3 \mid a_1 a_3 = a_3 a_1, a_2 a_2 = a_3 a_2 \rangle \cong F_2 \times \mathbf{Z},$$

where $a_1 = (\sigma_1)^2$, $a_2 = (\sigma_2)^2$, and $a_3 = \Delta^2 = (\sigma_1\sigma_2)^3$. For open annuli V and W such that $\overline{W} \subset V$, let $g_{V,W} \in \mathcal{G}_{\mathbb{A}}$ denote a diffeomorphism which rotates W once such that $\mathrm{supp}(g_{V,W}) \subset V$. We define $\rho_\epsilon \colon P_2(\mathbb{A}) \to \mathcal{G}_{\mathbb{A}}$ by $\rho_\epsilon(a_1) = g_{V_1^\epsilon, W_1^\epsilon}$, $\rho_\epsilon(a_2) = g_{V_2^\epsilon, W_2^\epsilon}$ and $\rho_\epsilon(a_3) = g_{V_{12}^\epsilon, W_{12}^\epsilon}$.

We say that $\bar{x} = (x_1, x_2) \in X_2(\mathbb{A})$ is an *ϵ-good point* if both x_1 and x_2 are in U^ϵ. We say that an ϵ-good point $\bar{x}$ is *of type* (p,q) if

$$\#(U_1^\epsilon \cap \{x_1, x_2\}) = p, \quad \#(U_2^\epsilon \cap \{x_1, x_2\}) = q.$$

Let $\bar{x} \in X_2(\mathbb{A})$ be an ϵ-good point of type (p,q). If $(p,q) \neq (1,1)$, we can see that $\gamma(\rho_\epsilon(\alpha), \bar{x}) \in Z(P_2(\mathbb{A})) = \langle \Delta^2 \rangle$ for any $\alpha \in P_2(\mathbb{A})$. By an argument similar to the proof of Lemma 4.1, we can prove that

$$\lim_{\epsilon \to 0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^{\mathbb{A}}(u)) - \Lambda \cdot j^*(u)\| = 0$$

by setting $\Lambda = \lim_{\epsilon \to 0} 2! \cdot \mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)$. $\square$

5.2. The case of a sphere

Let $\mathbb{S}$ denote the 2-sphere. We summarize several facts on the braid group on $\mathbb{S}$ (see [**11**, Section 4.1] for example).

An inclusion $\mathbb{D} \to \mathbb{S}$ induces the projection $B_m \to B_m(\mathbb{S})$, and let δ_i denote the image of σ_i by this projection. It is known that the kernel of this projection is normally generated by $\sigma_1\sigma_2\cdots\sigma_{m-2}\sigma_{m-1}^2\sigma_{m-2}\cdots\sigma_2\sigma_1$. The natural map $X_{m-1}(\mathbb{D}) \to X_m(\mathbb{S})$ induces the map $P_{m-1} \to P_m(\mathbb{S})$ and it is known to be surjective. If $m \geq 4$, $Z(P_m(\mathbb{S})) = Z(B_m(\mathbb{S}))$ is generated by the full twist $\xi^2 = (\delta_1\delta_2\cdots\delta_{m-1})^m$ and ξ^2 has order two.

We consider in particular the case $m = 4$. Then $P_4(\mathbb{S})$ has a presentation

$$P_4(\mathbb{S}) = \langle a_1, a_2, a_3 \mid a_1a_3 = a_3a_1, a_2a_3 = a_3a_2, (a_3)^2 = e\rangle \cong F_2 \times \mathbf{Z}/2\mathbf{Z},$$

where $a_1 = (\xi_1)^2$, $a_2 = (\xi_2)^2$, and $a_3 = \xi^2$. Let $i\colon P_4(\mathbb{S}) \to B_4(\mathbb{S})$ be the inclusion. By Theorem 2.4 and since $Z(P_4(\mathbb{S})) = Z(B_4(\mathbb{S}))$, the map $Push \circ i\colon P_4(\mathbb{S}) \to K(\mathbb{S}, 4)$ induces the map $j\colon P_4(\mathbb{S})/Z(P_4(\mathbb{S})) \to K(\mathbb{S}, 4)$. For an element $\alpha \in G$ of a group G, let $\overline{\alpha} \in G/Z(G)$ denote the equivalence class of α. We regard the group $P_4(\mathbb{S})/Z(P_4(\mathbb{S}))$ as a free group F_2 of rank 2 generated by $\overline{a_1}$ and $\overline{a_2}$.

Lemma 5.2. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon\colon F_2 \to \mathcal{G}_{\mathbb{S}}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon\to 0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^{\mathbb{S}}(u)) - \Lambda \cdot j^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(K(\mathbb{S}, 4))$.

Proof. For each ϵ, we take open neighborhoods U_i^ϵ of $z_i \in \mathbb{S}$ $(i = 1, 2, 3, 4)$ so that

- $U_i^\epsilon \cap U_j^\epsilon = \emptyset$ if $i \neq j$ and
- $\mathrm{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon \cup U_3^\epsilon \cup U_4^\epsilon$.

Moreover, we take open subsets W_{12}^ϵ and V_{12}^ϵ of $\mathbb{S}$ which are diffeomorphic to a disk so that

- $U_1^\epsilon \cup U_2^\epsilon \subset \overline{W_{12}^\epsilon} \subset V_{12}^\epsilon$,
- $V_{12}^\epsilon \cap U_3^\epsilon = \emptyset$ and
- $V_{12}^\epsilon \cap U_4^\epsilon = \emptyset$.

We also take W_{23}^ϵ and V_{23}^ϵ similarly (see Figure 5). We define $\rho_\epsilon\colon F_2 \to \mathcal{G}_{\mathbb{S}}$ as in the case of the disk. We define $s_i\colon F_2 \to \mathbf{Z}$ by $s_i(\overline{a_j}) = \delta_{ij}$. We calculate $\gamma(\rho_\epsilon(\alpha), \bar{x}) \in K(\mathbb{S}, 4)$ for $\alpha \in F_2$ and $\bar{x} \in X_4(\mathbb{S})$. We say that $\bar{x} = (x_1, x_2, x_3, x_4) \in X_4(\mathbb{S})$ is an *ϵ-good point* if all of x_1, x_2, x_3, and x_4 are in U^ϵ. We say that an ϵ-good point $\bar{x}$ is *of type* (p, q, r, s) if

$$\#(U_1^\epsilon \cap \{x_1, x_2, x_3, x_4\}) = p, \quad \#(U_2^\epsilon \cap \{x_1, x_2, x_3, x_4\}) = q,$$
$$\#(U_3^\epsilon \cap \{x_1, x_2, x_3, x_4\}) = r, \quad \#(U_4^\epsilon \cap \{x_1, x_2, x_3, x_4\}) = s.$$

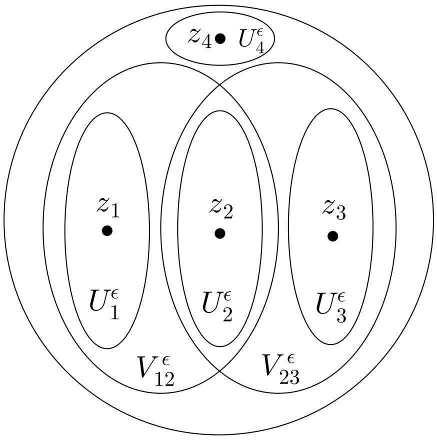

Figure 5. Open subsets in $\mathbb{S}$

Let X^ϵ_{pqrs} denote the set of ϵ-good points $\bar{x}$ is of type (p, q, r, s). We define a cochain $c^\epsilon_{pqrs} \in C^n_b(F_2)$ by

$$c^\epsilon_{pqrs}(\alpha_0, \dots, \alpha_n) = \int_{X^\epsilon_{pqrs}} c(\gamma(\rho_\epsilon(\alpha_0), \bar{x}), \dots, \gamma(\rho_\epsilon(\alpha_n), \bar{x}))d\bar{x}$$

for $\alpha_0, \dots, \alpha_n \in F_2$. By an argument similar to the proof of Lemma 4.1, for $[c^\epsilon_{pqrs}]$ to be non-zero, both W^ϵ_{12} and W^ϵ_{23} must contain exactly two points, since the full twist of three or four strands is in the center $Z(P_4(\mathbb{S}))$. Thus, if (p, q, r, s) is not $(1, 1, 1, 1)$, $(0, 2, 0, 2)$ or $(2, 0, 2, 0)$, then $[c^\epsilon_{pqrs}] = 0$.

Let $\bar{x} \in X_4(\mathbb{S})$ be an ϵ-good point of type $(1, 1, 1, 1)$, $(0, 2, 0, 2)$ or $(2, 0, 2, 0)$. Similarly to Lemma 4.2, for almost every $\bar{x} \in X_4(\mathbb{S})$, there exists $\beta(\bar{x}) \in K(\mathbb{S}, 4) = Push(B_4(\mathbb{S}))$ such that

$$\beta(\bar{x})^{-1}\gamma(\rho_\epsilon(\alpha), \bar{x})\beta(\bar{x}) = \begin{cases} \alpha & \text{type } (1,1,1,1), \\ (\overline{a_1})^{s_1(\alpha)+s_2(\alpha)} & \text{type } (0,2,0,2), \\ (\overline{a_1})^{s_1(\alpha)}(\overline{a_3})^{s_2(\alpha)} & \text{type } (2,0,2,0), \end{cases}$$

for $\alpha \in F_2$. Hence, we can prove $[c^\epsilon_{0202}] = [c^\epsilon_{2020}] = 0$ and

$$[c^\epsilon_{1111}] = \mathrm{vol}(X^\epsilon_{1111}) \cdot j^*(u)$$

by an argument similar to the proof of Lemma 4.1. Therefore,

$$\lim_{\epsilon \to 0} \|\rho^*_\epsilon(\overline{E\Gamma}^{\mathbb{S}}_b(u)) - \Lambda \cdot j^*(u)\| = 0$$

by setting

$$\Lambda = \lim_{\epsilon \to 0} 4! \cdot \mathrm{area}(U_1^\epsilon)\,\mathrm{area}(U_2^\epsilon)\,\mathrm{area}(U_3^\epsilon)\,\mathrm{area}(U_4^\epsilon). \qquad \square$$

5.3. The case of a torus

Let $\mathbb{T}$ denote the 2-torus. We summarize several facts on $B_2(\mathbb{T})$ (see [**3**, Section 2.2] for example). Recall that $\bar{z} = (z_1, z_2)$ denotes a base point of $X_2(\mathbb{T})$. We define a braid a_1 (resp. b_1) so that it moves z_1 to the meridian (resp. longitude) direction and rotates once and does not move z_2. We define a_2 and b_2 similarly by exchanging the role of z_1 and z_2. It is known that $P_2(\mathbb{T}) \cong F_2 \times \mathbf{Z}^2$. Namely, the set $\{\overline{a_1}, \overline{b_1}\}$ generates $P_2(\mathbb{T})/Z(P_2(\mathbb{T})) \cong F_2$ and $\{a_1a_2, b_1b_2\}$ generates $Z(P_2(\mathbb{T})) \cong \mathbf{Z}^2$. As in Section 5.2, we obtain the map $j\colon F_2 \to K(\mathbb{T}, 2)$ induced by $Push \circ i\colon P_2(\mathbb{T}) \to K(\mathbb{T}, 2)$. Here, we consider $P_2(\mathbb{T})/Z(P_2(\mathbb{T}))$ as $F_2 = \langle \overline{a_1}, \overline{b_1} \rangle$.

Lemma 5.3. *There exist a constant $\Lambda > 0$ and a family of homomorphisms $\{\rho_\epsilon\colon F_2 \to \mathcal{G}_{\mathbb{T}}\}_{0<\epsilon<1}$ such that*

$$\lim_{\epsilon \to 0} \|\rho_\epsilon^*(\overline{E\Gamma}_b^{\mathbb{T}}(u)) - \Lambda \cdot j^*(u)\| = 0$$

for any $u \in \overline{EH}_b^n(K(\mathbb{T}), 2)$.

Proof. For each ϵ, we take open neighborhoods U_i^ϵ of $z_i \in \mathbb{T}$ $(i = 1, 2)$ so that

- $U_1^\epsilon \cap U_2^\epsilon = \emptyset$ and
- $\mathrm{area}(U^\epsilon) = 1 - \epsilon$, where $U^\epsilon = U_1^\epsilon \cup U_2^\epsilon$.

Moreover, we take open subsets W_a^ϵ and V_a^ϵ of $\mathbb{T}$ that are diffeomorphic to an annulus so that

- $U_1^\epsilon \subset \overline{W_a^\epsilon} \subset V_a^\epsilon$,
- $V_a^\epsilon \cap U_2^\epsilon = \emptyset$ and
- W_a^ϵ and V_a^ϵ contain a meridian.

We also take W_b^ϵ and V_b^ϵ similarly but to contain a longitude (see Figure 6).

We define $\rho_\epsilon\colon F_2 \to \mathcal{G}_{\mathbb{T}}$ as follows. We take an isotopy $\{(g_a)_t\}$ that rotates W_a^ϵ once and whose support is contained in V_a^ϵ. For the generator $a_1 \in F_2$, we define $\rho_\epsilon(a_1) = (g_a)_1$. We also define $\rho_\epsilon(b_1)$ similarly.

We say that $\bar{x} = (x_1, x_2) \in X_2(\mathbb{T})$ is an *ϵ-good point* if both x_1 and x_2 are in U^ϵ. We say that an ϵ-good point $\bar{x}$ is *of type* (p, q) if

$$\#(U_1^\epsilon \cap \{x_1, x_2\}) = p, \quad \#(U_2^\epsilon \cap \{x_1, x_2\}) = q.$$

Let $\bar{x} \in X_2(\mathbb{T})$ be an ϵ-good point of type (p, q). We take an isotopy $\{(g_a)_t\}$ defined above. Then, similar to Lemma 4.2 again, for almost every $\bar{x} \in X_2(\mathbb{T})$, there exists $\beta'(\bar{x}) \in B_2(\mathbb{T})$ such that

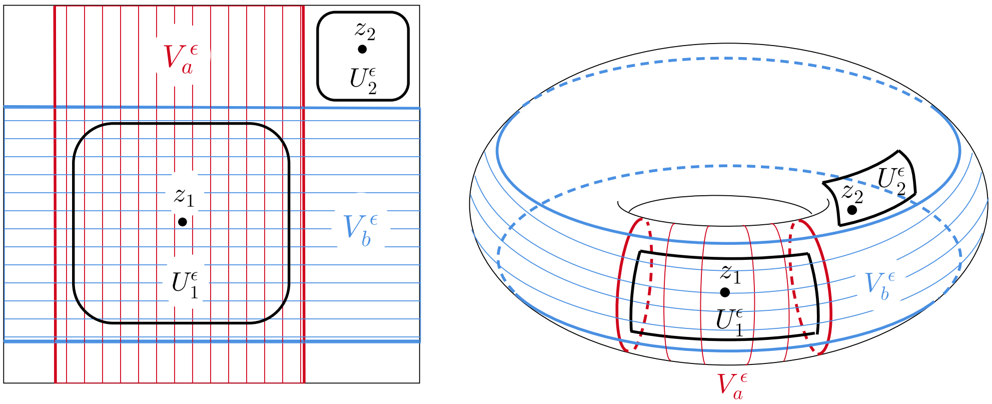

Figure 6. Open subsets in $\mathbb{T}$

$$\gamma(\{(g_a)_t\}, \bar{x}) = \begin{cases} e & (p = 0), \\ \beta'(\bar{x}) a_1 \beta'(\bar{x})^{-1} & (p = 1), \\ \beta'(\bar{x}) a_1 a_2 \beta'(\bar{x})^{-1} & (p = 2). \end{cases}$$

Thus, we obtain

$$\gamma(\rho_\epsilon(\overline{a_1}), \bar{x}) = \begin{cases} \beta(\bar{x}) \overline{a_1} \beta(\bar{x})^{-1} & (p = 1), \\ e & (\text{otherwise}), \end{cases}$$

where $\beta(\bar{x}) = Push(\beta'(\bar{x})) \in K(\mathbb{T}, 2)$. Similarly, we can see that

$$\gamma(\rho_\epsilon(\overline{b_1}), \bar{x}) = \begin{cases} \beta(\bar{x}) \overline{b_1} \beta(\bar{x})^{-1} & (q = 1), \\ e & (\text{otherwise}). \end{cases}$$

Hence, for $\alpha \in F_2$, $\gamma(\rho_\epsilon(\alpha), \bar{x}) = \beta(\bar{x}) \alpha \beta(\bar{x})^{-1}$ if $\bar{x}$ is of type $(1, 1)$. By the argument similar to the proof of Lemma 4.1, we can prove that

$$\lim_{\epsilon \to 0} \| \rho_\epsilon^* (\overline{E\Gamma}_b^{\mathbb{T}}(u)) - \Lambda \cdot j^*(u) \| = 0$$

by setting $\Lambda = \lim_{\epsilon \to 0} 2! \cdot \operatorname{area}(U_1^\epsilon) \operatorname{area}(U_2^\epsilon)$. □

5.4. Remaining proofs

We complete the proof of Theorems 1.3, 1.5 and Corollary 1.4.

Proof of Theorem 1.3. By Lemmas 5.1, 5.2, and 5.3, we can prove Theorem 1.3 by the same argument as in the proof of Theorem 1.1. □

Proof of Corollary 1.4. Since $B_2(\mathbb{A}) \cong K(\mathbb{A}, 2)$ is a finite index subgroup of B_3, the inclusion map $K(\mathbb{A}, 2) \to B_3$ induces an isometric injective map $EH_b^3(B_3) \to EH_b^3(K(\mathbb{A}, 2))$ by Theorem 2.1. As we saw in the proof of Theorem 1.1, $\overline{EH}_b^3(B_3)$ is uncountably infinite-dimensional, and thus $\overline{EH}_b^3(K(\mathbb{A},2))$ is also uncountably infinite-dimensional.

It is known that $\mathrm{MCG}(\mathbb{S},4)$ surjects onto $PSL(2,\mathbf{Z})$ and its kernel is $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ (see [**6**, Proposition 2.7]). Thus $\mathrm{MCG}(\mathbb{S},4)$ is quasi-isometric to $PSL(2,\mathbf{Z})$. Since $PSL(2,\mathbf{Z})$ is non-elementary hyperbolic, $\mathrm{MCG}(\mathbb{S},4)$ is also. Hence, by Theorems 2.3 and 2.4, $\overline{EH}_b^3(K(\mathbb{S},4))\cong\overline{EH}_b^3(\mathrm{MCG}(\mathbb{S},4))$ is also uncountably infinite-dimensional.

Set $G=B_2(\mathbb{T})/Z(B_2(\mathbb{T}))$. Then G has a presentation

$$G=\langle a,b,c \mid a^2=b^2=c^2=1\rangle$$

[**16**, Exercise 6.3]. Thus G is isomorphic to $\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}*\mathbf{Z}/2\mathbf{Z}$ and hence it is non-elementary hyperbolic (because it is virtually free). Hence, by Theorems 2.3 and 2.4, $\overline{EH}_b^3(G)\cong\overline{EH}_b^3(K(\mathbb{T},2))$ is uncountably infinite-dimensional.

Therefore, we obtain the conclusion by Theorem 1.3. $\square$

Proof of Theorem 1.5. If $\chi(\Sigma)\geq 0$ then, by Corollary 1.4, $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is infinite-dimensional. If $\chi(\Sigma)<0$, $\pi_1(\Sigma)$ is a non-elementary hyperbolic group. Therefore, by the result of Brandenbursky and Marcinkowski [**5**], $\overline{EH}_b^3(\mathcal{G}_\Sigma)$ is infinite-dimensional. $\square$

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