Submitted to
*Journal of the Mathematical Society of Japan*

# Density Results on Hyperharmonic Integers

By Doğa Can SERTBAŞ

(Received Nov. 99, 20XX)
(Revised Jan. 99, 20XX)

**Abstract.** It was conjectured that there are no hyperharmonic integers $h_n^{(r)}$ except 1. In 2020, a disproof of this conjecture was given by showing the existence of infinitely many hyperharmonic integers. However, the corresponding proof does not give any general density results related to hyperharmonic integers. In this paper, we first get better error estimates for the counting function of the pairs $(n, r)$ that correspond to non-integer hyperharmonic numbers using sums on gaps between consecutive prime numbers. Then, based on a plausible assumption on prime powers with restricted digits, we show that there exist positive integers $n$ such that the set of positive integers $r$ where $h_n^{(r)} \in \mathbb{Z}$ has positive density. Apart from that, we also obtain exact densities of sets $\{r \in \mathbb{Z}_{>0} : h_{33}^{(r)} \in \mathbb{Z}\}$ and $\{r \in \mathbb{Z}_{>0} : h_{39}^{(r)} \in \mathbb{Z}\}$. Finally, we give the smallest hyperharmonic integer $h_n^{(r)}$ greater than 1, which is obtained when $n = 33$ and $r = 10\,667\,968$.

## 1. Introduction

Any partial sum of the harmonic series is called a harmonic number. More precisely, the *n-th harmonic number* is the sum of the reciprocals of the first $n$ positive integers, namely

$$h_n := \sum_{k=1}^{n} \frac{1}{k}.$$

In 1915, Theisinger [32] proved that $h_n$ is never an integer, when $n > 1$. Moreover, Kürschák [23] deduced that for any different positive integers $m, n \geq 1$, the corresponding difference of harmonic numbers $h_m - h_n$ is also a non-integer rational number.

A generalization of harmonic numbers was introduced by Conway and Guy in [8]. They defined the *n-th hyperharmonic number of order r* as

$$h_n^{(r)} := \sum_{k=1}^{n} h_k^{(r-1)}, \tag{1}$$

for given natural numbers $n, r \geq 1$, with the initial case $h_n^{(1)} = h_n$. In the same book, they also showed that hyperharmonic numbers satisfy the following equality.

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}). \tag{2}$$

2020 *Mathematics Subject Classification.* Primary 11B83, 05A10; Secondary 11B75.
*Key Words and Phrases.* hyperharmonic numbers, integerness property, prime numbers.

These numbers share lots of analytic and arithmetic properties. For instance, it can be obtained using (2) that $h_n^{(r)} \ll_r n^{r-1}\log n$, when $r$ is fixed. A finer estimate was given in [17] and it was shown that for any given natural numbers $r$ and $\ell$ and sufficiently large $n$, we have

$$h_n^{(r)} = \frac{n^{r-1}\log n}{(r-1)!} + \frac{(\gamma - h_{r-1})n^{r-1}}{(r-1)!} + \sum_{k=0}^{r-2}\left(a_{r,k}n^k \log n + b_{r,k}n^k\right), + \sum_{j=1}^{\ell-1}\frac{c_{r,j}}{n^j} + \mathcal{O}_{r,\ell}\left(\frac{1}{n^\ell}\right)$$

where $\gamma$ denotes the Euler-gamma constant and $a_{r,k}$, $b_{r,k}$, $c_{r,j}$ are explicitly computable constants for $0 \le k \le r-2$ and $1 \le j \le \ell - 1$. On the arithmetic side, Mező [26] investigated the integrality property of hyperharmonic numbers. In particular, he showed that $h_n^{(r)}$ is not an integer for $n > 1$ and $r \in \{2, 3\}$. This result of Mező was extended in [3, 4, 14] and it was proved that almost all hyperharmonic numbers are not integers. Namely, if $S(x)$ denotes the number of $(n, r)$ tuples with $h_n^{(r)}$ is not an integer where $1 \le n, r \le x$, then $S(x) \sim x^2$. Recently, this type of density result was improved in [1] and the current best known estimate is

$$S(x) = x^2 + \mathcal{O}_A\left(\frac{x^{\frac{80}{59}}}{(\log x)^A}\right), \tag{3}$$

for any $A > 0$. Here, the implied constant in the error term depends only on $A$. Moreover, using some certain conjectures related to primes in short intervals, one can also obtain better error terms in (3). For instance, it was shown in [1] that the estimate

$$S(x) = x^2 + \mathcal{O}\left(x \log^3 x\right) \tag{4}$$

holds under Crámer's conjecture which states that there is an absolute positive constant $c$ such that the interval $(x - c\log^2 x, x]$ contains a prime number, for all sufficiently large $x$.

In this note, we give better estimations on the number of exceptional $(n, r)$ tuples for which the corresponding hyperharmonic number $h_n^{(r)}$ is an integer. We improve the previous error term in (3) unconditionally. Also, to obtain a better conditional result, we use a slightly weaker version of the Riemann hypothesis which is called the Lindelöf hypothesis. It asserts that for any given $\varepsilon > 0$ the Riemann zeta function satisfies $\zeta(1/2 + it) = \mathcal{O}_\varepsilon(t^\varepsilon)$ (see for instance [27, Chapter 10]). Furthermore, we obtain the same error term in (4) under the Riemann hypothesis.

THEOREM 1.1. *Let* $S(x) = |\{(n, r) \in [1, x] \times [1, x] \, : \, h_n^{(r)} \notin \mathbb{Z}\}|$. *For any* $\varepsilon > 0$ *we have*

$$S(x) = x^2 + \mathcal{O}_\varepsilon\left(x^{\frac{631}{531}+\varepsilon}\right)$$

*unconditionally. Additionally, if we assume the Lindelöf hypothesis, we get*

$$S(x) = x^2 + \mathcal{O}_\varepsilon\left(x^{1+\varepsilon}\right).$$

*Moreover, conditionally on the Riemann hypothesis, we have that*

$$S(x) = x^2 + \mathcal{O}\left(x \log^3 x\right).$$

Note that the order of the magnitude in the error term is

$$\frac{631}{531} \approx 1.18833 < 1.35594 \approx \frac{80}{59}$$

which can be seen as a considerable improvement in the exponent. According to previously mentioned results, it seemed unlikely to have any hyperharmonic integers. However, it was already shown in [31] that there are infinitely many hyperharmonic integers when $n = 33$. Here, we extend this result and find sets of $r$ values with $h_n^{(r)} \in \mathbb{Z}$ such that their densities are positive when $n = 3P^\eta$ where $\eta \in \{1,2,3\}$ and $P$ is a prime of the form $\pm 1 + 12\ell$ for some $\ell \in \mathbb{Z}_{>0}$ such that the ternary representation of $P^\eta$ does not contain any twos in it. It is better to note that these types of integers are studied extensively [5, 9–13, 19, 20, 24, 25]. Especially, the work of Maynard [24] contains estimates on the set of prime numbers with restricted digits when written in base $q$ for any sufficiently large $q$. However, the existence of infinitely many primes with missing digits is still an open problem when these primes are written in ternary.

Theorem 1.2. *Let $P$ be a prime number such that $P \equiv \pm 1 \pmod{12}$. Also, assume that*

$$P^\eta = \sum_{i=0}^{\beta} P_i^{(\eta)} 3^i \in (3^\beta, 8 \cdot 3^{\beta-1}), \quad \begin{array}{l} \text{with } \beta \geq 2,\ \eta \in \{1,2,3\} \text{ and} \\ P_i^{(\eta)} \in \{0,1\} \text{ when } i \in \mathbb{Z} \cap [1, \beta-2]. \end{array} \tag{$\star$}$$

*Then, there exists an integer $r \in [1, (3P^\eta)!]$ such that $R \equiv r \pmod{(3P^\eta)!}$ implies that $h_{3P^\eta}^{(R)} \in \mathbb{Z}$. Moreover, let $\mathcal{R}_n$ denote the set of $r$ values where $h_n^{(r)} \in \mathbb{Z}$ for a given positive integer $n$. Then, for any $n = 3P^\eta$ for which $P^\eta$ satisfies the condition ($\star$) for some $\eta \in \{1,2,3\}$, the set $\mathcal{R}_n$ has density which is greater than or equal to $\frac{1}{n!}$.*

As a corollary of Theorem 1.2, we get that the densities of the sets $\mathcal{R}_{33}$ and $\mathcal{R}_{39}$ are at least

$$\frac{1}{33!} \approx 1.1516 \cdot 10^{-37} \quad \text{and} \quad \frac{1}{39!} \approx 4.90247 \cdot 10^{-47}, \tag{5}$$

respectively, since the primes 11 and 13 satisfy condition ($\star$). Besides, doing some computations and using the prescribed techniques in this note, we obtain exact densities of these sets. In fact, these computations yield the structure of the smallest hyperharmonic integer.

Theorem 1.3. *The smallest hyperharmonic integer $h_n^{(r)}$ greater than 1 is equal to*

3009287172811364404984125778708627188141159718559721813893101188860332195034648261187262268354557608058585276611064376994779359430272826344742020434523221316911052660055855963776173497027117, *and it is obtained when* $n = 33$ *and* $r = 10\,667\,968$.

Before proving our results, we mention some of the known facts related to non-integer hyperharmonic numbers which can be obtained from [14, Theorems 1, 2 and 4].

Fact 1.4. Let $n > 1$ and $\alpha, r \geq 1$ be integers. If

1. $r \leq 20\,001$, or
2. $n \leq 32$, or
3. $n$ is even, or
4. $r$ is odd, or
5. $n$ is a prime power, or
6. $n$ is of the form $3p^\alpha$, for some prime number $p \equiv \pm 5 \pmod{12}$, or
7. $n$ is of the form $5p^\alpha$, for some prime number $p$ which is congruent to one of the following

$$\begin{aligned}&7,11,13,14,19,21,22,23,26,28,31,33,38,39,41,42,\\&44,46,52,53,56,57,61,62,63,66,67,69,76,78,79,82,\\&83,84,88,89,92,93,99,101,103,104,106,107,112,\\&114,117,119,122,123,124,126,131,132,134,138\end{aligned} \pmod{145},$$

then the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Moreover, there is a constant $C \in \left(0, \frac{1}{2}\right)$ such that for any sufficiently large integer $n$ and for any $r \leq Cn^{1.475}$ we have $h_n^{(r)} \notin \mathbb{Z}$.

In this note, we deal with the values of $n$ which do not satisfy the conditions of Fact 1.4. We denote by $\mathfrak{N}$ the set of all such exceptional natural numbers. Thanks to SageMath [29], the first few values of $\mathfrak{N}$ are

$$\mathfrak{N} = \{1, 33, 39, 45, 63, 69, 77, 85, 91, 99, 105, 111, 117, 119, 133, 135, 141, 143, 145, \ldots\}. \quad (6)$$

The structure of this paper can be explained as follows. In Section 2, we prove some general facts related to hyperharmonic numbers. After that, we give a proof of Theorem 1.1 using certain type of sums on the differences of consecutive prime numbers in Section 3. Also, for some specific integers $n \in \mathbb{Z}_{>0}$, a set of positive density that consists of positive integers $r$ which lead to integer values of $h_n^{(r)}$ will be introduced in Section 4. Moreover, we do the corresponding computations to find exact densities of sets $\{r \in \mathbb{Z}_{>0} : h_{33}^{(r)} \in \mathbb{Z}\}$ and $\{r \in \mathbb{Z}_{>0} : h_{39}^{(r)} \in \mathbb{Z}\}$, and this will be mentioned in Section 5. Finally, we determine the smallest hyperharmonic integer and its $(n, r)$ value.

### 1.1. Notation

In this note, $p$ always denotes a prime number unless specified. Usually, the $p$-ary representation of a natural number $n$ is given as $(n_{m-1}, \ldots, n_0)_p$ where $m = \lceil \log_p n \rceil$.

Also, we use the $\nu_p(n)$ notation to represent the $p$-adic valuation of a natural number $n$, that is, for a given $n \in \mathbb{Z}$ we denote

$$\nu_p(n) := \begin{cases} m, \text{ if } p^m \mid n, \text{ but } p^{m+1} \nmid n, \\ \infty, \text{ if } n = 0. \end{cases}$$

Next, we extend this notation to a rational number $q = a/b$ by setting $\nu_p(q) = \nu_p(a) - \nu_p(b)$ where $a, b \in \mathbb{Z}$. Moreover, $\mathbb{F}_q$ always denotes the finite field with $q$ elements. Apart from that, let $g : \mathbb{R} \to \mathbb{R}$ be a function and suppose that $g(x) > 0$ for $x \geq d$, where $d$ is a real number. We write $f(x) = \mathcal{O}(g(x))$, or $f(x) \ll g(x)$, if there exists a constant $c > 0$ such that

$$|f(x)| \leq cg(x), \quad \text{for all } x \geq d. \tag{7}$$

Also, the notation $f(x) = \mathcal{O}_\ell(g(x))$ indicates that the constant $c > 0$ given in (7) may depend on $\ell$. Moreover, we say that $f(x)$ is asymptotic to $g(x)$, denoted by $f(x) \sim g(x)$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. Furthermore, we follow the same notation given in [14, Sections 1 and 3]: let $I(n, r) = \{r, \ldots, n + r - 1\}$. For any prime $p$ and a finite set $\mathcal{S}$ in $\mathbb{Z}$, define

$$\mu_p(\mathcal{S}) := \max\{\nu_p(a) : a \in \mathcal{S}\} \quad \text{and} \quad \mathcal{M}_p(\mathcal{S}) := \left|\mathcal{S} \cap p^{\mu_p(\mathcal{S})}\mathbb{Z}\right|. \tag{8}$$

Finally, for any set $\mathcal{S}$ in $\mathbb{Z}$ and any natural number $n > 1$ we write

$$\mathfrak{R}_{\mathcal{S}}(n) := \{r \in \mathbb{Z}_{>0} : \nu_p\left(h_n^{(r)}\right) \geq 0 \ \text{ for any } \ p \in \mathbb{P} \cap \mathcal{S}\},$$

where $\mathbb{P}$ denotes the set of all prime numbers. If we let $\mathfrak{R}_p(n) := \mathfrak{R}_{\{p\}}(n)$, then we see that

$$\mathfrak{R}_{\mathcal{S}}(n) = \bigcap_{p\in\mathcal{S}} \mathfrak{R}_p(n).$$

## 2. General Facts Related To Hyperharmonic Numbers

Before getting into details, we give some of the general facts related to hyperharmonic numbers. For this purpose, we first mention an easy consequence of (2) which is also proved in [2, Lemma 2.1].

LEMMA 2.1. *Let $n$ be a positive integer and*

$$f_n(x) = \prod_{i=0}^{n-1}(x + i). \tag{9}$$

*Then, for any positive integer $r$ we have*

$$h_n^{(r)} = \frac{f_n'(r)}{n!},$$

*where $f_n'(x)$ denotes the derivative of the polynomial $f_n(x)$.*

PROOF. Observe that $\log f_n(x) = \sum_{i=0}^{n-1} \log(x+i)$. Taking the derivative of both sides yields

$$\frac{f_n'(x)}{f_n(x)} = \sum_{i=0}^{n-1} \frac{1}{x+i}.$$

By (2), we conclude that

$$\begin{aligned} h_n^{(r)} &= \binom{n+r-1}{r-1} \left( \frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n+r-1} \right) \\ &= \frac{f_n(r)}{n!} \cdot \frac{f_n'(r)}{f_n(r)} = \frac{f_n'(r)}{n!}, \end{aligned}$$

as desired. □

REMARK 2.2. By [6, Theorem 2], one can see that $f_n'(r) = \begin{bmatrix} n+r \\ r+1 \end{bmatrix}_r$, where $\begin{bmatrix} a \\ b \end{bmatrix}_r$ denotes the $r$-Stirling number of the first kind which is given in [6, Definition 2].

As a result, we get the following consequence of Lemma 2.1 which can be also found in [2, Proposition 4.1].

COROLLARY 2.3. *Let $n$ and $r$ be positive integers. For any $k \in \mathbb{Z}_{\geq 0}$, the difference $h_n^{(r+k\cdot n!)} - h_n^{(r)}$ is an integer. In particular, $h_n^{(r)}$ is an integer if and only if $h_n^{(r+k\cdot n!)} \in \mathbb{Z}$ for any non-negative integer $k$.*

PROOF. Let $f_n(x) \in \mathbb{Z}[x]$ be the polynomial that is introduced in (9). Also, write $f_n'(x) = \sum_{i=0}^{n-1} a_i x^i$ for some $a_i \in \mathbb{Z}$. Then by Lemma 2.1, for any $k \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} h_n^{(r+k\cdot n!)} - h_n^{(r)} &= \frac{f_n'(r+kn!) - f_n'(r)}{n!} \\ &= \frac{1}{n!} \cdot \sum_{i=0}^{n-1} a_i \left( (r+kn!)^i - r^i \right) \\ &= \frac{1}{n!} \cdot \sum_{i=0}^{n-1} a_i kn! \left( (r+kn!)^{i-1} + \cdots + r^{i-1} \right) \\ &= \sum_{i=0}^{n-1} a_i k \left( (r+kn!)^{i-1} + \cdots + r^{i-1} \right), \end{aligned}$$

which indiates that $h_n^{(r+k\cdot n!)} - h_n^{(r)} \in \mathbb{Z}$. The second part follows immediately. □

To obtain a non-negative $p$-adic valuation, we will use the following fact which is already given in [31, Lemma 3].

FACT 2.4. For any given positive integers $n$ and $r$, the corresponding hyperharmonic number $h_n^{(r)}$ is an integer if and only if $\nu_p\left(h_n^{(r)}\right) \geq 0$ for all prime numbers $p \leq n$.

Remark 2.5. Using Fact 2.4, we see that, for a fixed $n$, there exists $r$ with $h_n^{(r)} \in \mathbb{Z}$ if and only if $\bigcap_{p\le n} \mathfrak{R}_p(n) \neq \varnothing$.

We will give a consequence of [14, Lemma 12] which will be useful to obtain equivalent conditions for a non-negative $p$-adic valuation of a hyperharmonic number when a certain prime number $p$ is specified.

Corollary 2.6. *Let $n$ and $r$ be given positive integers. For any prime number $p$, we have*

$$|I(n,r)\cap p\mathbb{Z}| \in \left\{ \left\lfloor \frac{n}{p} \right\rfloor, \left\lfloor \frac{n}{p} \right\rfloor + 1 \right\}.$$

*Moreover, if $p$ divides $n$, then $|I(n,r)\cap p\mathbb{Z}| = \left\lfloor \frac{n}{p} \right\rfloor$.*

Now, we can state the necessary and the sufficient conditions to obtain $\nu_p\left(h_n^{(r)}\right) \ge 0$ for any prime number $p$ and positive integers $n, r$.

Lemma 2.7. *Assume that $n$ and $r$ are given positive integers. Let $k = \left\lfloor \frac{n}{p} \right\rfloor$ and $c = \left\lceil \frac{r}{p} \right\rceil$ where $p$ is a given prime.*

(i) *If $|I(n,r)\cap p\mathbb{Z}| = k$, then $\nu_p\left(h_n^{(r)}\right) \ge 0$ if and only if $\nu_p\left(h_k^{(c)}\right) \ge 1$.*

(ii) *If $|I(n,r)\cap p\mathbb{Z}| = k+1$, then $\nu_p\left(h_n^{(r)}\right) \ge 1$ if and only if $\nu_p\left((k+1)h_{k+1}^{(c)}\right) \ge 1$.*

*Moreover, $\nu_p\left(h_n^{(r)}\right) \ge 0$ is equivalent to $\nu_p\left((c+k)h_k^{(c)}\right) \ge 0$ when $|I(n,r)\cap p\mathbb{Z}| = k+1$.*

Proof. For any $n \in \mathbb{Z}_{>0}$, let $f_n(x)$ be the polynomial defined in (9). By Corollary 2.6, we know that either $|I(n,r)\cap p\mathbb{Z}| = k$, or $|I(n,r)\cap p\mathbb{Z}| = k+1$ holds. In the former case, we have

$$\begin{aligned} h_n^{(r)} &= \frac{A}{B} \cdot \frac{cp\cdot(c+1)p\cdots(c+k-1)p}{p\cdot 2p\cdots kp}\left(\frac{1}{p}\sum_{i=0}^{k-1}\frac{1}{c+i} + q\right) \qquad (10)\\ &= \frac{A}{B}\cdot\frac{f_k(c)}{k!}\left(\frac{1}{p}\cdot\frac{f_k'(c)}{f_k(c)} + q\right) = \frac{A}{B}\cdot\frac{f_k'(c)}{p\cdot k!} + \frac{A}{B}\cdot\frac{f_k(c)}{k!}\cdot q, \end{aligned}$$

where $q \in \mathbb{Q}$, $A, B \in \mathbb{Z}$ and $\nu_p(A) = \nu_p(B) = 0 \le \nu_p(q)$. By the non-Archimedean property and Lemma 2.1, we see that

$$\nu_p\left(h_n^{(r)}\right) \ge \min\left\{\nu_p\left(\frac{f_k'(c)}{p\cdot k!}\right), \nu_p\left(\frac{f_k(c)}{k!}\cdot q\right)\right\} = \min\left\{\nu_p\left(h_k^{(c)}\right) - 1, \frac{f_k(c)}{k!}\cdot q\right\},$$

and the equality holds when the $p$-adic valuations on the right hand side are different. Observe that $\nu_p\left(\frac{f_k(c)}{k!}\cdot q\right) \ge 0$ as $\frac{f_k(c)}{k!} = \binom{c+k-1}{k} \in \mathbb{Z}$. Hence, the necessary and the sufficient condition to obtain a non-negative $p$-adic valuation for $h_n^{(r)}$ is to have

$\nu_p\left(h_k^{(c)}\right) \geq 1$, when $|I(n,r) \cap p\mathbb{Z}| = k$.

Similarly, if $|I(n,r) \cap p\mathbb{Z}| = k+1$, then we have

$$\begin{aligned} h_n^{(r)} &= \frac{A}{B} \cdot \frac{cp \cdot (c+1)p \cdots (c+k)p}{p \cdot 2p \cdots kp} \left( \frac{1}{p} \sum_{i=0}^{k} \frac{1}{c+i} + q \right) \\ &= \frac{A}{B} \cdot \frac{p \cdot f_{k+1}(c)}{k!} \left( \frac{1}{p} \cdot \frac{f'_{k+1}(c)}{f_{k+1}(c)} + q \right) = \frac{A}{B} \cdot \frac{f'_{k+1}(c)}{k!} + \frac{A}{B} \cdot \frac{p f_{k+1}(c)}{k!} \cdot q \end{aligned}$$

for some $q \in \mathbb{Q}$ and $A, B \in \mathbb{Z}$, where $\nu_p(A) = \nu_p(B) = 0 \leq \nu_p(q)$. This indicates that

$$\begin{aligned} h_n^{(r)} &= \frac{A}{B} \cdot \frac{(k+1) f'_{k+1}(c)}{(k+1)!} + \frac{A}{B} \cdot \frac{p(k+1) f_{k+1}(c)}{(k+1)!} \cdot q \\ &= \frac{A}{B} \cdot (k+1) h_{k+1}^{(c)} + p(k+1) \cdot \binom{c+k}{k+1} q. \end{aligned}$$

Again, by the non-Archimedean property, we see that

$$\nu_p\left(h_n^{(r)}\right) \geq \min\left\{ \nu_p\left((k+1) h_{k+1}^{(c)}\right), \nu_p\left((k+1) \cdot \binom{c+k}{k+1} q\right) + 1 \right\}. \tag{11}$$

Note that $\nu_p\left((k+1) \cdot \binom{c+k}{k+1} q\right) + 1 \geq 1$, as $(k+1)\binom{c+k}{k+1} \in \mathbb{Z}$ and $\nu_p(q) \geq 0$. Hence, by (11), we conclude that $\nu_p\left(h_n^{(r)}\right) \geq 1$ if and only if $\nu_p\left((k+1) h_{k+1}^{(c)}\right) \geq 1$ when $|I(n,r) \cap p\mathbb{Z}| = k+1$. Similarly, it can be seen that

$$\nu_p\left(h_n^{(r)}\right) \geq 0 \text{ is equivalent to } \nu_p\left((k+1) h_{k+1}^{(c)}\right) \geq 0. \tag{12}$$

To see the last part, observe that

$$\begin{aligned} f'_{k+1}(x) &= \left( (x+k) \cdot \prod_{i=0}^{k-1} (x+i) \right)' = \prod_{i=0}^{k-1} (x+i) + (x+k) \cdot \left( \prod_{i=0}^{k-1} (x+i) \right)' \\ &= f_k(x) + (x+k) \cdot f'_k(x) \end{aligned}$$

holds, by the definition of $f_n(x)$. This yields

$$\begin{aligned} (k+1) h_{k+1}^{(c)} &= \frac{f'_{k+1}(c)}{k!} = \frac{f_k(c)}{k!} + \frac{(c+k) f'_k(c)}{k!} \\ &= \binom{c+k-1}{k} + (c+k) h_k^{(c)}. \end{aligned}$$

Similar arguments show that the necessary and the sufficient condition to obtain a non-negative $p$-adic valuation for $(k+1) h_{k+1}^{(c)}$ is to have $\nu_p\left((c+k) h_k^{(c)}\right) \geq 0$, as $\binom{c+k-1}{k} \in \mathbb{Z}$. □

It is better to note that a part of Lemma 2.7 is also mentioned in [14, Section 5]. In particular, it is shown that the condition $|I(n,r) \cap p\mathbb{Z}| = k+1$ together with $k < p$

implies that $\nu_p\left(h_n^{(r)}\right) \geq 0$ where $k$ is defined as in Lemma 2.7. Here, we give a criterion which also covers this fact in a more general setting.

Remark 2.8. Notice that when $\nu_p\left(h_k^{(c)}\right) \geq 1$, we have $\nu_p\left(h_n^{(r)}\right) \geq 0$. To see this, first assume that $|I(n,r) \cap p\mathbb{Z}| = k+1$. In that case, $\nu_p\left((c+k)h_k^{(c)}\right) \geq 1$, as $c+k \in \mathbb{Z}$. This shows that $\nu_p\left(h_n^{(r)}\right) \geq 0$ when $|I(n,r) \cap p\mathbb{Z}| = k+1$, and the other case follows immediately from Lemma 2.7.

Now, we give a generalization of Babbage's and Wolstenholme's theorems which can be found in [7], [15, Lemma 2.2] and [16, Lemma 2.1].

Lemma 2.9. *For any prime number $p \geq 3$, let $\{a_1, a_2, \ldots, a_{p-1}\}$ be a set such that $a_i \equiv i \pmod p$ for each $i \in \{1, 2, \ldots, p-1\}$. Then, the sum $\sum_{i=1}^{p-1} \frac{1}{a_i}$ has a positive p-adic valuation. Moreover, for any non-negative integer $a \geq 0$ and prime $p \geq 5$, we have $\nu_p\left(\sum_{i=1}^{p-1} \frac{1}{ap+i}\right) \geq 2$.*

Proof. Note that the sum

$$\sum_{i=1}^{p-1} \frac{1}{a_i} \equiv \sum_{i=1}^{p-1} {a_i}^{-1} \pmod p.$$

Since $a_i \equiv i \pmod p$ for each $i \in \{1, 2, \ldots, p-1\}$, this sum is also congruent to

$$\sum_{i=1}^{p-1} \frac{1}{a_i} \equiv \frac{(p-1)p}{2} \pmod p,$$

which indicates that $\nu_p\left(\sum_{i=1}^{p-1} \frac{1}{a_i}\right) \geq 1$. The second part follows from [16, Lemma 2.1]. □

Next, we obtain the following corollary which is a combination of Lemmas 2.7 and 2.9.

Corollary 2.10. *For a given prime number $p \geq 3$, let $n = kp$ and $r = (c-1)p+j$ for some $c \in \mathbb{Z}_{>0}$ and $j \in \{1, 2, \ldots, p\}$. Then, $\nu_p\left(h_n^{(r)}\right) \geq \alpha$ if and only if $\nu_p\left(h_k^{(c)}\right) \geq \alpha+1$, for any $\alpha \in \{0, 1\}$. Also, if $p \geq 5$ and $j \in \{1, p\}$, then $\nu_p\left(h_n^{(r)}\right) \geq \beta$ if and only if $\nu_p\left(h_k^{(c)}\right) \geq \beta+1$, for any $\beta \in \{0, 1, 2\}$.*

Proof. First, observe that $c = \left\lceil \frac{r}{p} \right\rceil$. Since $p \mid n$, we have $|I(n,r) \cap p\mathbb{Z}| = k$ by Corollary

2.6. Similar to (10) in the proof of Lemma 2.7, we see that

$$h_n^{(r)} = \frac{A}{B}\binom{c+k-1}{k}\Bigg(\sum_{s=j}^{p-1}\frac{1}{(c-1)p+s}+\frac{1}{p}\sum_{i=0}^{k-1}\frac{1}{c+i}+\sum_{i=0}^{k-1}\sum_{\ell=1}^{p-1}\frac{1}{(c+i)p+\ell} + \sum_{t=1}^{j-1}\frac{1}{(c+k-1)p+t}\Bigg), \tag{13}$$

where $\nu_p(A) = \nu_p(B) = 0$. By Lemma 2.9, we obtain that

$$\nu_p\left(\sum_{i=0}^{k-1}\sum_{\ell=1}^{p-1}\frac{1}{(c+i)p+\ell}\right) \geq 1, \tag{14}$$

for any odd prime $p$, and

$$\nu_p\left(\sum_{i=0}^{k-1}\sum_{\ell=1}^{p-1}\frac{1}{(c+i)p+\ell}\right) \geq 2, \tag{15}$$

when $p \geq 5$. Also, the same lemma implies

$$\nu_p\left(\sum_{s=j}^{p-1}\frac{1}{(c-1)p+s}+\sum_{t=1}^{j-1}\frac{1}{(c+k-1)p+t}\right) \geq 1. \tag{16}$$

Note that (16) becomes

$$\nu_p\left(\sum_{s=j}^{p-1}\frac{1}{(c-1)p+s}+\sum_{t=1}^{j-1}\frac{1}{(c+k-1)p+t}\right) \geq 2, \tag{17}$$

when $j \in \{1, p\}$ and $p \geq 5$, by Lemma 2.9. Hence, if we rewrite (13), we get that

$$h_n^{(r)} = \frac{A}{B}\binom{c+k-1}{k}\cdot\left(\frac{1}{p}\frac{f_k'(c)}{f_k(c)}+q\right) = \frac{A}{B}\cdot\frac{h_k^{(c)}}{p}+\frac{A}{B}\binom{c+k-1}{k}\cdot q, \tag{18}$$

where

$$q = \sum_{s=j}^{p-1}\frac{1}{(c-1)p+s}+\sum_{t=1}^{j-1}\frac{1}{(c+k-1)p+t}+\sum_{i=0}^{k-1}\sum_{\ell=1}^{p-1}\frac{1}{(c+i)p+\ell}.$$

Combining (14) and (16), we deduce that $\nu_p(q) \geq 1$ by the non-Archimedean property of the $p$-adic valuation. In that case, for any $\alpha \in \{0, 1\}$ we have $\nu_p\left(h_n^{(r)}\right) \geq \alpha$ if and only if $\nu_p\left(h_k^{(c)}\right) \geq \alpha+1$ by (18), as $\nu_p(A) = \nu_p(B) = 0$ and $\binom{c+k-1}{k} \in \mathbb{Z}$. Similarly, if $j \in \{1, p\}$ and $p \geq 5$, then $\nu_p(q) \geq 2$ by (15) and (17). Thus, for any $\beta \in \{0, 1, 2\}$, the necessary and the sufficient condition to obtain $\nu_p\left(h_n^{(r)}\right) \geq \beta$ is to have $\nu_p\left(h_k^{(c)}\right) \geq \beta+1$, by equation (18). □

REMARK 2.11. Note that we cannot take $\beta > 2$ in Corollary 2.10, as

$\nu_p\left(\binom{c+k-1}{k}q\right) = 2$ may hold.

Using Lemma 2.7, we get necessary congruences for $r$ to obtain a non-negative $p$-adic valuation for $h_n^{(r)}$, when $\sqrt{n} < p < n$.

Proposition 2.12. *For any positive integer $n$, let $p$ be a prime number such that $\sqrt{n} < p < n$. Define $k = \left\lfloor \frac{n}{p} \right\rfloor$. Assume that $f_k(x)$ is defined as in (9). Then $\nu_p\left(h_n^{(r)}\right) \geq 0$ if and only if*

*(2.12.1)* $r \equiv 1-b \pmod p$ *for some* $b \in \{1,\ldots,n_p\}$ *where* $n \equiv n_p \pmod p$ *and* $0 < n_p < p$, *or*

*(2.12.2)* $r \equiv (s-1)p+j \pmod{p^2}$ *where $s$ is a root of $f_k'(x)$ modulo $p$ and $j \in \{1,\ldots,p\}$*

*holds.*

Proof. Take any positive integer $n$ and a prime number $p$ such that $\sqrt{n} < p < n$ holds. Then, observe that $k = \left\lfloor \frac{n}{p} \right\rfloor \leq \frac{n}{p} < p$. This indicates that for any $z \in \mathbb{Z}_{\geq 1}$ we have

$$\nu_p\left(h_k^{(z)}\right) \geq 0, \tag{19}$$

as $h_k^{(z)} = \frac{f_k'(z)}{k!}$ by Lemma 2.1, and $\nu_p(k!) = 0$.

Now, suppose that $\nu_p\left(h_n^{(r)}\right) \geq 0$. Recall by Corollary 2.6 that there are two possibilities for $|I(n,r) \cap p\mathbb{Z}|$: it is either equal to $k$ or $k+1$. We first assume that $|I(n,r) \cap p\mathbb{Z}| = k$. Let $c = \left\lceil \frac{r}{p} \right\rceil$. By Lemma 2.7, we know that $\nu_p\left(h_k^{(c)}\right) \geq 1$. In that case, $\nu_p(f_k'(c)) \geq 1 + \nu_p(k!)$ where $f_k'(x)$ is the derivative of the polynomial $f_k(x) = x(x+1)\cdots(x+k-1)$. As $k < p$ for $p \in (\sqrt{n}, n)$, we see that $\nu_p(f_k'(c)) \geq 1$. This implies that there is a root $s$ of $f_k'(x)$ modulo $p$ such that $c = \left\lceil \frac{r}{p} \right\rceil \equiv s \pmod p$. In other words, there exists an integer $\ell$ such that

$$s - 1 + \ell p < \frac{r}{p} \leq s + \ell p. \tag{20}$$

This is also equivalent to the fact $(s-1)p + \ell p^2 + 1 \leq r \leq sp + \ell p^2$, as $r \in \mathbb{Z}$. Hence, $r \equiv (s-1)p + j \pmod{p^2}$ for some $j \in \{1,\ldots,p\}$.

Next, suppose that $\nu_p\left(h_n^{(r)}\right) \geq 0$ and $|I(n,r) \cap p\mathbb{Z}| = k+1$. By Corollary 2.6, we see that $p \nmid n$. So in that case, we can say that $n = kp + n_p$ for some $0 < n_p < p$. Also, let $r = cp + (1-b)$ where $0 \leq b - 1 < p$. Observe that if $|I(n,r) \cap p\mathbb{Z}| = k+1$, then $r \leq cp \leq (c+k)p \leq n + r - 1 = (c+k)p + (n_p - b)$. The last inequality yields $b \leq n_p$. Moreover, we have $1 \leq b$. Thus, we deduce that $r \equiv 1 - b \pmod p$ for some $b \in \{1,\ldots,n_p\}$, where $n \equiv n_p \pmod p$.

For the other direction, assume that (2.12.1) or (2.12.2) holds. In the latter case, we see that $r \equiv (s-1)p + j + \ell p^2$ for some $j \in \{1, \ldots, p\}$ and $\ell \in \mathbb{Z}$. This indicates that the inequalities in (20) hold, which is also equivalent to have $c = \left\lceil \frac{r}{p} \right\rceil \equiv s \pmod{p}$. Hence, we obtain that $\nu_p\left(f_k'(c)\right) \geq 1$ where $f_k(x) = x(x+1)\cdots(x+k-1)$. Note that $\nu_p(k!) = 0$, as $k < p$. Gathering this with Lemma 2.1 yields $\nu_p\left(h_k^{(c)}\right) = \nu_p\left(f_k'(c)\right) - \nu_p(k!) \geq 1$. As a result, we deduce $\nu_p\left(h_n^{(r)}\right) \geq 0$ by Remark 2.8.

Finally, if $r \equiv 1 - b \pmod{p}$ for some $b \in \{1, \ldots, n_p\}$ where $n \equiv n_p \pmod{p}$ and $0 < n_p < p$, then we can write $n = kp + n_p$. This implies that $0 \geq 1-b \geq 1-n_p > -(p-1)$. If $r = zp + (1-b)$ for some $z \in \mathbb{Z}$, then we get that $zp - (p-1) < r \leq zp$. This indicates that $z = \left\lceil \frac{r}{p} \right\rceil = c$. So consider $n + r - 1 = (c+k)p + (n_p - b) \geq (c+k)p > cp \geq r$, as $k \geq 1$. Hence, we have $|I(n,r) \cap p\mathbb{Z}| = k+1$. By (19) we know that $\nu_p\left(h_k^{(c)}\right) \geq 0$, as $k < p$. Using this and Lemma 2.7, we conclude that $\nu_p\left(h_n^{(r)}\right) \geq 0$, as $c + k \in \mathbb{Z}$. $\square$

REMARK 2.13. For any $n \in \mathfrak{N}$ where $\mathfrak{N}$ is defined in (6), the condition $k < p$ is equivalent to $\sqrt{n} < p$ for any $p \in \mathbb{P}$. To see this, first take any prime $p > \sqrt{n}$. Clearly, we have $k = \left\lfloor \frac{n}{p} \right\rfloor \leq \frac{n}{p} < p$. Also, if $k = \left\lfloor \frac{n}{p} \right\rfloor < p$, then $\frac{n}{p} \leq p$, as $p \in \mathbb{Z}$. Hence, $n \leq p^2$, and since $n \in \mathfrak{N}$ we have $n \neq p^2$. Thus, $\sqrt{n} < p$ is equivalent to the fact that $k < p$ for $n \in \mathfrak{N}$.

REMARK 2.14. Observe that if we take a prime $p$ satisfying $\frac{n}{2} < p < n$, then it is enough to check the condition (2.12.1). To see this, first note that $k = 1$ as $1 < \frac{n}{p} < 2$. Therefore, $p \nmid n$ and $f_k'(x) = 1$. This indicates that $f_k'(x)$ cannot have any roots modulo $p$. Hence, we get that

$$\mathfrak{R}_{\left(\frac{n}{2},n\right)}(n) = \left\{ r \in \mathbb{N}_{>0} : r \equiv 1 - b \pmod{p},\, b \in \{1, \ldots, n_p\},\, p \in \mathbb{P} \cap \left(\frac{n}{2}, n\right) \right\}.$$

REMARK 2.15. If $p \mid n$ and $\sqrt{n} < p < n$, then it is enough to consider the case (2.12.2), since $|I(n,r) \cap p\mathbb{Z}|$ is always equal to $k$ as it is given in Corollary 2.6. Note that there can be at most one such prime number $p \mid n$. Therefore, if $P$ denotes this prime, then we deduce that

$$\mathfrak{R}_P(n) = \left\{ r \in \mathbb{N}_{>0} \;:\; \begin{array}{c} r \equiv (s-1)P + j \pmod{P^2}, \\ f_k'(s) \equiv 0 \pmod{P},\, j \in \{1, \ldots, P\} \end{array} \right\},$$

by Proposition 2.12.

Finally, we mention the following consequence of Proposition 2.12 which is also given in [2, Remark 3.8].

COROLLARY 2.16. *For any given integer $n \geq 4$, let $p^{(n)}$ denote the greatest prime that is less than $n$. Then, $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$ if and only if there is an integer $t$ such that $r \in ((t+1)p^{(n)} - n, tp^{(n)}]$.*

Proof. By the well known Bertrand's Postulate, we see that $\frac{n}{2} < p^{(n)} < n$ for any $n \geq 4$. So by Remark 2.14, we obtain that $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$ if and only if $r \equiv 1 - b \pmod{p^{(n)}}$ for some $b \in \{1, \ldots, n_{p^{(n)}}\}$ where $n \equiv n_{p^{(n)}} \pmod{p^{(n)}}$ and $0 < n_{p^{(n)}} < p^{(n)}$. This is also equivalent to the fact that there exists an integer $t$ such that $r = 1 - b + tp^{(n)}$ where $b \in \{1, \ldots, n - p^{(n)}\}$, as $p^{(n)} \in \left(\frac{n}{2}, n\right)$ and $n_{p^{(n)}} = n - p^{(n)}$. In other words, $(t+1)p^{(n)} - n < r \leq tp^{(n)}$, since $t$ and $p^{(n)}$ are integers. Thus, we prove the corollary. □

## 3. Upper Bounds on the Number of Hyperharmonic Integers

In order to obtain a better error term for $S(x)$, it is enough to estimate an upper bound on the possible number of hyperharmonic integers $h_n^{(r)}$ where $(n, r) \in [1, x] \times [1, x]$. To do this, we will use [18, Theorem 1] and [33] where the latter one depends on the Lindelöf Hypothesis. In addition, using a celebrated result of Selberg [30, Theorem 3] together with Bertrand's Postulate we get the following fact.

Fact 3.1. Let $\varepsilon > 0$ be given. If $p_k$ denotes the $k$-th prime number, then we have

$$\sum_{p_k \leq x} (p_{k+1} - p_k)^2 \ll_\varepsilon x^{\frac{23}{18}+\varepsilon}. \tag{21}$$

Moreover, the Lindelöf Hypothesis implies that

$$\sum_{p_k \leq x} (p_{k+1} - p_k)^2 \ll_\varepsilon x^{1+\varepsilon}. \tag{22}$$

Finally, if we assume the Riemann Hypothesis, then we obtain that

$$\sum_{p_k \leq x} \frac{(p_{k+1} - p_k)^2}{p_k} \ll \log^3 x. \tag{23}$$

Now, we prove Theorem 1.1 from the introduction.

*Proof of Theorem 1.1.* For any $n \leq x$, define the exceptional set of $r \leq x$ values as $E_n(x) = |\{r \leq x \,:\, h_n^{(r)} \in \mathbb{Z}\}|$. Observe that $E_n(x) = 0$ when $n \notin \mathfrak{N}$, by Fact 1.4. So, the definition of $S(x)$ implies that

$$S(x) + \sum_{\substack{n \leq x \\ n \in \mathfrak{N}}} E_n(x) = \lfloor x \rfloor^2 = x^2 + \mathcal{O}(x). \tag{24}$$

This indicates that

$$S(x) = x^2 + \mathcal{O}\left( x + \sum_{\substack{N < n \leq x \\ n \in \mathfrak{N}}} E_n(x) \right),$$

for any positive integer $N$. By Fact 1.4, we know that there is a constant $C \in \left(0, \frac{1}{2}\right)$ such that for any sufficiently large integer $n$ and any $r \leq Cn^{1.475}$ the corresponding hyperharmonic number $h_n^{(r)}$ is not an integer. Therefore, take $N$ sufficiently large so that $h_n^{(r)} \notin \mathbb{Z}$ for any $n > N$ and $r \leq Cn^{1.475}$. Choose $X = C_0 x^{1/1.475}$ where $C_0 = C^{-1/1.475} > 0$. Then, $n \geq X$ yields $h_n^{(r)} \notin \mathbb{Z}$, as $r \leq x \leq Cn^{1.475}$. Hence,

$$S(x) = x^2 + \mathcal{O}\left( x + \sum_{\substack{N<n<X \\ n\in\mathfrak{N}}} E_n(x) \right). \tag{25}$$

Denote the greatest prime that is less than or equal to $n$ as $p^{(n)}$. Note that the inequality

$$E_n(x) \leq \left|\{r \leq x \,:\, \nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0\}\right|$$

is satisfied. By Corollary 2.16 we obtain that $\nu_{p^{(n)}}\left(h_n^{(r)}\right) \geq 0$ if and only if there exists an integer $t$ such that $r \in ((t+1)p^{(n)} - n, tp^{(n)}]$. Hence, we see that

$$E_n(x) \leq \sum_{\substack{r\leq x \\ \nu_{p^{(n)}}\left(h_n^{(r)}\right)\geq 0}} 1 \leq \sum_{t=1}^{\lfloor x/p^{(n)}\rfloor+1} \sum_{r=(t+1)p^{(n)}-n+1}^{tp^{(n)}} 1. \tag{26}$$

Since $t \in \mathbb{Z}$, the set $\mathbb{Z} \cap ((t+1)p^{(n)} - n, tp^{(n)}]$ contains exactly $n - p^{(n)}$ many elements. Define $\Delta(n) := n - p^{(n)}$. Observe that $\Delta(n) \neq 0$, as $n \in \mathfrak{N}$. Therefore, by inequality (26) we have

$$\begin{aligned} \sum_{\substack{N<n<X \\ n\in\mathfrak{N}}} E_n(x) &= \mathcal{O}\left( \sum_{n\leq X} \sum_{t=1}^{\lfloor x/p^{(n)}\rfloor+1} \Delta(n) \right) \\ &= \mathcal{O}\left( \sum_{n\leq X} \frac{x\Delta(n)}{p^{(n)}} \right) = \mathcal{O}\left( x \sum_{n\leq X} \frac{n-p^{(n)}}{p^{(n)}} \right), \end{aligned} \tag{27}$$

as $p^{(n)} < n \leq X < x$ for sufficiently large $x$. Now, observe that

$$\sum_{n\leq X} \frac{n-p^{(n)}}{p^{(n)}} \leq \sum_{2\leq p_k\leq X} \sum_{p_k<n\leq p_{k+1}} \frac{n-p_k}{p_k} \leq \sum_{2\leq p_k\leq X} \frac{(p_{k+1}-p_k)^2}{p_k}, \tag{28}$$

where $p_k$ denotes the $k$-th prime number. If we let $T(X) = \sum_{p_k\leq X} (p_{k+1}-p_k)^2$, then by the partial summation we get that

$$\sum_{2\leq p_k\leq X} \frac{(p_{k+1}-p_k)^2}{p_k} = \frac{T(X)}{X} + \int_2^X \frac{T(y)}{y^2} dy + \mathcal{O}(1). \tag{29}$$

By (21) in Fact 3.1, we know that $T(X) \ll_\varepsilon X^{\frac{23}{18}+\varepsilon}$ for any $\varepsilon > 0$. In that case, (29) becomes

$$\sum_{2\leq p_k\leq X} \frac{(p_{k+1}-p_k)^2}{p_k} \ll_\varepsilon X^{\frac{5}{18}+\varepsilon} + \int_Y^X y^{\varepsilon-\frac{13}{18}} dy \ll_\varepsilon X^{\frac{5}{18}+\varepsilon}, \tag{30}$$

for any fixed $Y \in [2, X]$. Assembling (28) and (30), we obtain

$$\sum_{n \le X} \frac{n - p^{(n)}}{p^{(n)}} \ll_\varepsilon X^{\frac{5}{18}+\varepsilon}. \tag{31}$$

Since $X = C_0 x^{1/1.475}$ for some $C_0 > 0$, we get that

$$\sum_{\substack{N<n<X \\ n \notin \mathbb{P}}} E_n(x) \ll_\varepsilon x \cdot x^{\frac{5}{18}\cdot\frac{40}{59}+\varepsilon}$$

by feeding (31) into (27). Thus, the first result of the theorem follows by (25). Similarly, when we assume the Lindelöf Hypothesis, we have $T(x) \ll_\varepsilon x^{1+\varepsilon}$ by (22) in Fact 3.1. Using (29), we observe that

$$\sum_{2 \le p_k \le X} \frac{(p_{k+1} - p_k)^2}{p_k} \ll_\varepsilon X^\varepsilon + \int_Y^X y^{\varepsilon-1} dy \ll_\varepsilon X^\varepsilon, \tag{32}$$

where $Y \in [2, X]$ is fixed. Gathering (32) together with (27) and (28) indicates that

$$\sum_{\substack{N<n<X \\ n \notin \mathbb{P}}} E_n(x) \ll_\varepsilon x^{1+\varepsilon}.$$

Hence, we deduce the second part of the theorem by (25). Finally, suppose the Riemann hypothesis holds. Combining (25), (27) and (28) together with (23) in Fact 3.1 yields the last part of the theorem. □

## 4. A Lower Density Result related to Hyperharmonic Integers

In this section, we deal with the positive integer values of $r$ for which $\nu_p\left(h_n^{(r)}\right) \ge 0$ when $n \in \mathfrak{N}$ and $p \in \mathbb{P}$ are fixed. For this purpose, we first recall some basic facts related to $p$-adic valuations of the binomial coefficients.

Remark 4.1. By (8), it can be deduced that

$$\lfloor \log_p n \rfloor \le \mu_p(I(n,r)) \le \lfloor \log_p (n+r-1) \rfloor \tag{33}$$

This fact is also mentioned in [14, Lemma 15]. Also by [14, Proposition 17], we know that for any prime number $p$ we have

$$\mu_p(I(n,r)) = \max\{i \in \{0,1,\dots,u\} : s_i > r'_i\}, \tag{34}$$

where $n = (n_{m-1}, \dots, n_0)_p$, $r - 1 = (r'_{v-1}, r'_{v-2}, \dots, r'_0)_p$ and $n + r - 1 = (s_u, \dots, s_0)_p$ when $u = \max\{m, v\}$. Moreover, it is known by the same proposition that

$$\nu_p\left(\binom{n+r-1}{r-1}\right) \le \mu_p(I(n,r)). \tag{35}$$

As it is given the proof of [14, Proposition 17] the equality in (35) holds if and only if we obtain a carry at each step $i \in \{0, 1, \dots, \mu_p(I(n,r)) - 1\}$ in the addition of $n =$

$(n_{m-1},\ldots,n_0)_p$ and $r-1=(r'_{v-1},r'_{v-2},\ldots,r'_0)_p$. This fact can be obtained by either [21] or [22, p. 116] where it is proved that the $p$-adic valuation of the binomial coefficient given in (35) is equal to the number of carries that occur in the addition of $n$ and $r-1$ when they are in their $p$-ary representations.

### 4.1. The necessary and sufficient condition for $\nu_2\left(h_n^{(r)}\right)\geq 0$ when $n$ is odd

Before obtaining the exact form of $r$ for which $\nu_p\left(h_n^{(r)}\right)\geq 0$, when $p=2$ and $n$ is odd, we prove the following lemma.

LEMMA 4.2. *For any positive integers $n$ and $r$, and a prime number $p$, the necessary and the sufficient condition to obtain*

$$\nu_p\left(\binom{n+r-1}{r-1}\right)=\mu_p\left(I(n,r)\right) \tag{36}$$

*is to have a carry at each step $i=0,1,\ldots,m-2$ after the addition of $n$ and $r-1$ where $n=(n_{m-1},\ldots,n_0)_p$ and $r-1=(r'_{v-1},r'_{v-2},\ldots,r'_0)_p$.*

PROOF. Let $p\in\mathbb{P}$ be given. Assume that $n=(n_{m-1},\ldots,n_0)_p$ and $r-1=(r'_{v-1},\ldots,r'_0)_p$. By Remark 4.1, we see that equality (36) holds if and only if the number of carries that occur in the addition of $n$ and $r-1$ is equal to $\mu_p\left(I(n,r)\right)$. Recall that the number of carries that can occur in the addition of $n$ and $r-1$ is at most $\mu_p\left(I(n,r)\right)$ by (35) in Remark 4.1. Also, we know by (34) that $\mu_p\left(I(n,r)\right)=\max\ \{i\in\{0,1,\ldots,u\}\,:\,s_i>r'_i\}$ where $n+r-1=(s_u,\ldots,s_0)_p$ and $u=\max\{m,v\}$. These facts indicate that a carry must occur after the addition of the digits $n_i$ and $r'_i$ for $i\in\{0,1,\ldots,\mu_p\left(I(n,r)\right)-1\}$, as (36) holds. Therefore, the sufficiency part is trivial by (33) in Remark 4.1, as $m-1=\lfloor\log_p n\rfloor\leq\mu_p\left(I(n,r)\right)$. For the other direction, suppose that a carry occurs after the addition of each step $i\in\{0,1,\ldots,m-2\}$, but there are at most $\mu_p\left(I(n,r)\right)-1$ carries in the addition of $n$ and $r-1$. Observe that $\mu_p\left(I(n,r)\right)\geq m$, since the number of carries that occured in the addition is at least $m-1$ by the assumption. Moreover, by the assumption, there is a $w\in\{m-1,\ldots,\mu_p\left(I(n,r)\right)-1\}$ such that a carry does not occur after the addition $r'_w$ and $n_w$. Note that a carry cannot occur in the $k$-th entry when $k\geq\mu_p\left(I(n,r)\right)$ due to (34) in Remark 4.1. Now, take the smallest such $w\in\{m-1,\ldots,\mu_p\left(I(n,r)\right)-1\}$ and consider $j\in\{w+1,\ldots,\mu_p\left(I(n,r)\right)-1\}$, if $\mathbb{Z}\cap[w+1,\mu_p\left(I(n,r)\right)-1]$ is non-empty. Since $n_j=0$ for all $j\in\{w+1,\ldots,\mu_p\left(I(n,r)\right)-1\}\subseteq\{m,\ldots,\mu_p\left(I(n,r)\right)-1\}$ and a carry does not occur at step $w$, we see that $s_j=r'_j$ for $j\in\{w+1,\ldots,\mu_p\left(I(n,r)\right)-1\}$ where $n+r-1=(s_u,\ldots,s_0)_p$. Also, notice that $s_w>r'_w$ holds by the choice of $w$, since a carry occurs in the $(w-1)$-st step. By (34) in Remark 4.1, we deduce that $w=\mu_p\left(I(n,r)\right)$ which is impossible, as $w\in\{m-1,\ldots,\mu_p\left(I(n,r)\right)-1\}$. Thus, we conclude that if a carry occurs at each $i$-th step where $i\in\{0,1,\ldots,m-2\}$, then there are $\mu_p\left(I(n,r)\right)$ many carries occur in the addition of $n$ and $r-1$. □

Now, we can get the exact structure of $r$ which gives $\nu_2\left(h_n^{(r)}\right)\geq 0$ when an odd natural number $n>2$ is given.

PROPOSITION 4.3. *Let $n>2$ be an odd natural number. Also, assume that $n=$*

$(n_{m-1},\ldots,n_0)_2$ *is the binary representation of* $n$ *where* $m=\lceil \log_2 n\rceil$. *Define the set*

$$\mathfrak{E}_2(n) := \left\{ 0 < e \leq 2^m \,:\, \begin{array}{l} e-1=(e_{m-1},\ldots,e_0)_2,\ e_0=1, \\ \text{and } e_i = 1 \text{ when } n_i = 0 \end{array} \right\} \cup \{0\}. \tag{37}$$

*For any* $A\geq 0$ *and* $e\in\mathfrak{E}_2(n)$ *with* $(A,e)\neq(0,0)$, *we have* $\nu_2\left(h_n^{(r)}\right)\geq 0$ *if and only if* $r=A\cdot 2^m+e$. *In particular, if* $r\equiv 0 \pmod{2^{m-1}}$, *then* $\nu_2\left(h_n^{(r)}\right)\geq 0$.

Proof. We will follow the same idea in the proof of [31, Proposition 4]. By [16, Corollary 3.7], we know that $\nu_2\left(h_n^{(r)}\right)\leq 0$. Also, by equation (2) we observe that $\nu_2\left(h_n^{(r)}\right)=\nu_2\left(\binom{n+r-1}{r-1}\right)+\nu_2\left(h_{n+r-1}-h_{r-1}\right)$. So, $\nu_2\left(h_n^{(r)}\right)\geq 0$ is satisfied if and only if

$$\nu_2\left(\binom{n+r-1}{r-1}\right) = -\nu_2\left(h_{n+r-1}-h_{r-1}\right).$$

Assume that $\theta=\mu_2(I(n,r))$. This indicates that there is an odd integer $c$ such that $c\cdot 2^\theta\in I(n,r)$. Therefore, $(c-1)2^\theta,(c+1)2^\theta\notin I(n,r)$ holds, since $(c-1)$ and $(c+1)$ are even. This implies that $\mathcal{M}_2(I(n,r))=1$. Hence, $h_{n+r-1}-h_{r-1}=\dfrac{1}{c2^\theta}+Q$ is satisfied for some $Q\in\mathbb{Q}$ where $\nu_2(Q)\geq -(\theta-1)$. The non-Archimedian property of 2-adic valuation shows that $\nu_2(h_{n+r-1}-h_{r-1})=-\theta$. Thus, the inequality $\nu_2\left(h_n^{(r)}\right)\geq 0$ holds if and only if $\nu_2\left(\binom{n+r-1}{r-1}\right)=\theta=\mu_2(I(n,r))$. By Lemma 4.2, note that the latter fact is also equivalent to obtaining a carry at each step $i\in\{0,1,\ldots,m-2\}$ in the binary addition of $n$ and $r-1$. We now show that letting $r=A\cdot 2^m+e$, for any $A\geq 0$ and $e\in\mathfrak{E}_2(n)$ that gives $r>0$, is necessary and sufficient to obtain $m-1$ carries after the addition of the first $m-1$ digits.

So first, we assume that $r=A\cdot 2^m+e>0$ for some $A\geq 0$ and $e\in\mathfrak{E}_2(n)$. If $e=0$, then $A\neq 0$. In that case, $r-1=(A-1)2^m+(2^m-1)$. Also, suppose that $A-1=(A'_{\ell-1},\ldots,A'_0)_2$ is the binary representation of $A-1$, for some $\ell\in\mathbb{Z}_{>0}$. In that case, we have

$$\begin{aligned} r-1 &= (A'_{\ell-1},\ldots,A'_0,1,\ \ 1\ \ ,\ldots,\ 1,1)_2 \\ n &= (\ \ 0\ \ ,\ldots,\ 0\ ,1,n_{m-2},\ldots,n_1,1)_2. \end{aligned}$$

Observe that a carry occurs after the addition of $r'_i$ and $n_i$ for every $i\in\{0,1,\ldots,m-2\}$. By Lemma 4.2, we get that $\nu_2\left(\binom{n+r-1}{r-1}\right)=\mu_2(I(n,r))$. This yields $\nu_2\left(h_n^{(r)}\right)\geq 0$.

Also, when $e>0$ and $r=A\cdot 2^m+e$, we have $r=A\cdot 2^m+(e-1)$. If $A=(A_{\ell-1},\ldots,A_0)_2$ for some positive integer $\ell$, then

$$\begin{aligned} r-1 &= (A_{\ell-1},\ldots,A_0,e_{m-1},e_{m-2},\ \ldots,e_1,1)_2 \\ n &= (\ \ 0\ \ ,\ldots,\ 0\ ,\ \ 1\ \ ,n_{m-2},\ldots,n_1,1)_2. \end{aligned}$$

Observe that we obtain a carry after adding the very first digits $n_0 = 1$ and $e_0 = 1$. Take any $i \in \{1, \dots, m-2\}$. Assume that there is a carry after the addition of $j$-th digits for any $j \in \{0, \dots, i-1\}$ and consider the $i$-th digit. If $n_i = 0$, then by the definition of the set $\mathfrak{E}_2(n)$ we know that $e_i = 1$ as $e \in \mathfrak{E}_2(n)$. Since there is a carry that occurs in the $(i-1)$-st digit, we see that a carry also occurs in the $i$-th digit. If $n_i = 1$, then we do not have any condition on $e_i \in \{0, 1\}$. Again, since a carry occurs in the $(i-1)$-st digit, we obtain a carry from the addition of $n_i$ and $e_i$ as $n_i = 1$. Hence, we deduce that we obtain carries from the first $m-1$ digits, and by Lemma 4.2 we deduce that $\nu_2\left(h_n^{(r)}\right) \geq 0$.

For the sufficiency part, assume that $r$ is not of the given form, namely $r = A \cdot 2^m + B$ where $A \in \mathbb{Z}_{\geq 0}$ and $B \notin \mathfrak{E}_2(n)$ for some $0 < B < 2^m$. If $B - 1 = (B_{m-1}, \dots, B_0)_2$ then either $B_0 = 0$, or there exists an $i \in \{1, \dots, m-2\}$ such that $n_i = B_i = 0$. In either case, we cannot obtain any carry after the addition of $n_i$ and $B_i$, where $i \in \{0, \dots, m-2\}$. Thus, we conclude by Remark 4.1 and Lemma 4.2 that $\nu_2\left(\binom{n+r-1}{r-1}\right) < \mu_2(I(n,r))$; and hence $\nu_2\left(h_n^{(r)}\right) < 0$.

Finally, we note that $2^{m-1}, 2^m \in \mathfrak{E}_2(n)$ for any odd natural number $n$, where $m = \lceil \log_2 n \rceil$; and this gives the last part of the proposition. □

### 4.2. Sufficient conditions to obtain $\boldsymbol{\nu_p\left(h_n^{(r)}\right) \geq 0}$ where $\boldsymbol{n}$ is not a multiple of $\boldsymbol{p}$

As we know from Fact 2.4, it is enough to obtain non-negative $p$-adic valuations for all primes $p \leq n$. So, we now introduce a sufficient condition to have $\nu_p\left(h_n^{(r)}\right) \geq 0$ for all primes $p \nmid n$.

LEMMA 4.4. *For any positive integers $n$ and $r$, and a prime number $p$, we have*

$$\nu_p\left(h_n^{(r)}\right) \geq \min\left\{\nu_p(r) - \nu_p(n!), -\nu_p(n)\right\}.$$

*In particular, $\nu_p\left(h_n^{(r)}\right) \geq 0$ if and only if $\nu_p(r) \geq \nu_p((n-1)!)$ when $p \nmid n$.*

PROOF. By the definition of $f_n(x)$, we see that

$$\begin{aligned} f_n'(x) &= \left(x \cdot \prod_{i=1}^{n-1}(x+i)\right)' = \prod_{i=1}^{n-1}(x+i) + x \cdot \left(\prod_{i=1}^{n-1}(x+i)\right)' \\ &= f_{n-1}(x+1) + x f_{n-1}'(x+1). \end{aligned}$$

Note that the constant term of $f_n'(x)$ is $(n-1)!$. This implies that $r \mid f_n'(r) - (n-1)!$. Also by Lemma 2.1, we deduce that

$$\begin{aligned} \nu_p\left(h_n^{(r)}\right) &= \nu_p\left(\frac{f_n'(r)}{n!}\right) = \nu_p\left(\frac{f_n'(r) - (n-1)!}{n!} + \frac{1}{n}\right) \\ &\geq \min\left\{\nu_p(r) - \nu_p(n!), -\nu_p(n)\right\}. \end{aligned} \tag{38}$$

The last part can be obtained by inequality (38) unless $p \mid n$. □

Remark 4.5. Let $n \in \mathfrak{N}$ and $p \in (\sqrt{n}, n)$ with $p \nmid n$. Instead of taking $\nu_p(r) \geq \left\lfloor \frac{n}{p} \right\rfloor = \nu_p(n!)$, it is enough to have $r \equiv 0 \pmod{p}$ by Proposition 2.12 in order to obtain $\nu_p\left(h_n^{(r)}\right) \geq 0$.

### 4.3. A non-negativity criterion for $\nu_3\left(h_{3N}^{(r)}\right)$ when $3 \nmid N$

To show Theorem 1.2, we first need an auxiliary lemma.

Lemma 4.6. *Let $n \in \{4,5,6,7,8\}$ and*

$$\mathfrak{T} = \left\{ \begin{array}{c} (4{,}3), (4,12), (4,21), (5,2), (5,3), (5,11), (5,12), (5,20), \\ (5,21), (6,10), (6,11), (6,12), (7,1), (7,20), (8,10) \end{array} \right\}. \tag{39}$$

*Then, for any positive integer $r$, we have $\nu_3\left(h_n^{(r)}\right) \geq 1$ if and only if $r \equiv t_n \pmod{27}$ for some $(n, t_n) \in \mathfrak{T}$.*

Proof. Let $n \in \{4,5\}$ be given. By SageMath [29], we observe that the only root of $f_4'(x)$ in modulo 9 is congruent to $3 \pmod 9$, where $f_n(x)$ is given in (9). Similarly, we get that 2 and 3 are the only roots of the polynomial $f_5'(x)$ modulo 9. Note that for any $n \in \{4,5\}$ and $(n, t_n) \in \mathfrak{T}$ we have either $t_4 \equiv 3 \pmod 9$ or $t_5 \equiv a \pmod 9$ where $a \in \{2,3\}$. This indicates that the set $\{t_n \pmod 9 : (n, t_n) \in \mathfrak{T}\}$ contains only the roots of the polynomial $f_n'(x)$ modulo 9, for any given $n \in \{4,5\}$. Since $f_n'(x)$ is a polynomial with integer coefficients, we see that the congruence $r \equiv t_n \pmod 9$ is equivalent to have $\nu_3(f_n'(r)) \geq 2$. Also, note that $\nu_3(n!) = 1$, when $n \in \{4,5\}$. Therefore, we get by Lemma 2.1 that $\nu_3\left(h_n^{(r)}\right) = \nu_3(f_n'(r)) - \nu_3(n!) \geq 1$.

For the second case, take $n \in \{6,7,8\}$. Computations show that the set $\{t_n \pmod{27} : (n, t_n) \in \mathfrak{T}\}$ is the set of all roots of the polynomial $f_n'(x)$ modulo 27, where $f_n(x) = x(x+1)\cdots(x+n-1)$. Similar to the previous case, $\nu_3(f_n'(r)) \geq 3$ holds if and only if $r \equiv t_n \pmod{27}$, as $f_n'(x) \in \mathbb{Z}[x]$. Combining this with Lemma 2.1 yields $\nu_3\left(h_n^{(r)}\right) = \nu_3(f_n'(r)) - \nu_3(n!) \geq 1$, as $\nu_3(n!) = 2$ for $n \in \{6,7,8\}$. □

Remark 4.7. One can observe by SageMath [29] that for any $n \in \{4,5\}$ we have $f_n'(t_n) \equiv 0 \pmod{27}$ if and only if $(n, t_n) \in \{(4,12), (5,3), (5,20)\}$, where $f_n(x) = x(x+1)\cdots(x+n-1)$. In that case, the condition $r \equiv t_n \pmod{27}$ is equivalent to have $\nu_3(f_n'(r)) \geq 3$, as the derivative of $f_n(x)$ is a polynomial with integer coefficients. This fact will be useful to obtain the exact density of the set $\{r \in \mathbb{Z}_{>0} : h_{39}^{(r)} \in \mathbb{Z}\}$.

Next, we give a result related to integers with missing digits in order to obtain a non-negative 3-adic valuation for some certain hyperharmonic numbers.

Proposition 4.8. *Let $\beta \geq 2$ be an integer. Define the set*

$$\mathcal{N}_\beta := \left\{ N = (N_\beta, \ldots, N_0)_3 \ : \begin{array}{c} 3N_\beta + N_{\beta-1} \in \{3,4,5,6,7\}, \\ \forall i \in \mathbb{Z} \cap [1, \beta-2] \ N_i \neq 2 \end{array} \right\}, \tag{40}$$

*where $N = (N_\beta, \ldots, N_0)_3$ denotes the ternary representation of the positive integer $N$.*

*If $3 \nmid N$ such that $N = (N_\beta, \dots, N_0)_3$ and $N \in \mathcal{N}_\beta$, then there exists a positive integer $r$ depending on $n = 3N_\beta + N_{\beta-1} + 1$ such that $r \equiv 3^\beta t_n \pmod{3^{\beta+3}}$ implies that $\nu_3\left(h_{3N}^{(r)}\right) \geq 0$ for some $(n, t_n) \in \mathfrak{T}$, where $\mathfrak{T}$ is given in (39).*

PROOF. Assume that $N \in \mathcal{N}_\beta$ for some fixed $\beta \geq 2$ and $3 \nmid N$. In that case, $N$ can be written as $N = (N_\beta, \dots, N_0)_3$ where $N_0 \neq 0$, and $N_i \neq 2$ for all $i \in \{1, \dots, \beta - 2\}$. Here, the set $\{1, \dots, \beta - 2\}$ is empty when $\beta = 2$. Also, let $n = 3N_\beta + N_{\beta-1} + 1$, which implies that $n \in \{4, 5, 6, 7, 8\}$. Now, take the corresponding $t_n$ where $(n, t_n) \in \mathfrak{T}$, and choose $r = 3^\beta t_n + 3^{\beta+3}\ell$ for some $\ell \in \mathbb{Z}_{\geq 0}$. For any $i \in \{1, \dots, \beta - 1\}$ define

$$n_1 := N, \qquad n_{i+1} := \left\lfloor \frac{n_i}{3} \right\rfloor + 1,$$
$$r_1 := \left\lceil \frac{r}{3} \right\rceil, \qquad r_{i+1} := \left\lceil \frac{r_i}{3} \right\rceil.$$

Since $N_i \neq 2$ for any $i \in \{1, \dots, \beta - 2\}$, we see that

$$n_i = (N_\beta, N_{\beta-1}, \dots, N_i + 1)_3 = \sum_{j=i+1}^{\beta} N_j 3^{j-i} + N_i + 1, \tag{41}$$

when $i \in \{1, \dots, \beta - 1\}$. Also, we have $N_i + 1 \in \{1, 2\}$. This indicates that $3 \nmid n_i$ for any $i \in \{1, 2, \dots, \beta - 1\}$. Moreover, we get

$$n_\beta = 3N_\beta + N_{\beta-1} + 1 = n. \tag{42}$$

Apart from that, we obtain

$$r_i = 3^{\beta-i} t_n + 3^{\beta-i+3}\ell, \tag{43}$$

for any $i \in \{1, \dots, \beta - 1\}$, and

$$r_\beta = t_n + 27\ell. \tag{44}$$

By (41) and (43), note that for any $i \in \{1, \dots, \beta - 1\}$ the cardinality

$$\begin{aligned} |I(n_i, r_i) \cap 3\mathbb{Z}| &= \left| \left\{ 3^{\beta-i}(t_n + 3^3\ell), \dots, \sum_{j=i+1}^{\beta} N_j 3^{j-i} + N_i + 3^{\beta-i}(t_n + 3^3\ell) \right\} \cap 3\mathbb{Z} \right| \\ &= \left| \left\{ r_i, r_i + 3, \dots, r_i + 3 \sum_{j=i+1}^{\beta} N_j 3^{j-i-1} \right\} \right| = \sum_{j=i+1}^{\beta} N_j 3^{j-i-1} + 1 \\ &= \sum_{j=i+2}^{\beta} N_j 3^{j-(i+1)} + N_{i+1} + 1 = n_{i+1}, \end{aligned} \tag{45}$$

as $N_i \in \{0, 1\}$ and $3 \mid r_i$. Using (45) together with Lemma 2.7, we derive that $\nu_3\left(h_{n_i}^{(r_i)}\right) \geq 1$ if and only if $\nu_3\left(n_{i+1} \cdot h_{n_{i+1}}^{(r_{i+1})}\right) \geq 1$ for any $i \in \{1, \dots, \beta - 1\}$, as $n_{i+1} = \left\lfloor \frac{n_i}{3} \right\rfloor + 1$ and $r_{i+1} = \left\lceil \frac{r_i}{3} \right\rceil$. Also, since $3 \nmid n_i$ for any $i \in \{1, \dots, \beta - 1\}$, it follows that $\nu_3\left(h_{n_1}^{(r_1)}\right) =$

$\nu_3\left(h_N^{(r_1)}\right) \geq 1$, as $\nu_3\left(h_{n_\beta}^{(r_\beta)}\right) \geq 1$, by Lemma 4.6, (42) and (44). Finally, observe by Corollary 2.6 that $|I(3N,r)\cap 3\mathbb{Z}| = \left\lfloor \frac{3N}{3} \right\rfloor = N$. Thus, we deduce by Lemma 2.7 that $\nu_3\left(h_{3N}^{(r)}\right) \geq 0$, since $\nu_3\left(h_N^{(r_1)}\right) \geq 1$ and $r_1 = \left\lceil \frac{r}{3} \right\rceil$. □

We conclude this subsection with several remarks related to Proposition 4.8.

Remark 4.9. Let $\beta \geq 2$ and $N = (N_\beta, \ldots, N_0)_3$ where $N_i \neq 2$ for all $i \in \mathbb{Z}\cap[1,\beta-3]$. If $9N_\beta+3N_{\beta-1}+N_{\beta-2} = 24$, we can also find a value of $r$ such that $\nu_3\left(h_{3N}^{(r)}\right) \geq 0$. In fact, the same idea in the proof of Proposition 4.8 follows when we take $t_8 \equiv 37 \pmod{81}$. Then, one gets $\nu_3\left(h_{24}^{(3t_8)}\right) \geq 1$ using Corollary 2.10, as $\nu_3\left(h_8^{(t_8)}\right) \geq 2$.

Remark 4.10. It can be observed that the statement of Proposition 4.8 can be modified with respect to some general prime $q \geq 3$. To this end, let $\beta \geq 1$ and

$$\mathcal{N}_\beta^{(q,t)} := \left\{ N = (N_\beta, \ldots, N_0)_q \ : \begin{array}{l} N_\beta + 1 \in J_q^{(t)} \cap [1, q-1], \\ \forall i \in \mathbb{Z}\cap[1,\beta-1]\ N_i \neq q-1 \end{array} \right\}$$

for some fixed $t \in \mathbb{Z}_{>0}$, where $J_q^{(t)} = \left\{ n \in \mathbb{Z}_{>0} : \nu_q\left(h_n^{(t)}\right) \geq 1 \right\}$ which is also defined in [16]. If $J_q^{(t)}$ is non-empty and $N$ is a positive integer which is not a multiple of $q$ such that $N = (N_\beta, \ldots, N_0)_q$ and $N \in \mathcal{N}_\beta^{(q,t)}$, then there exists a positive integer $r$ depending on $t$ such that $r \equiv q^{\beta+1}t \pmod{q^{\beta+3}}$ implies that $\nu_q\left(h_{qN}^{(r)}\right) \geq 0$. The same idea in the proof of Proposition 4.8 works for the proof of this statement. In fact, one can choose $t = 1$ and $N_\beta = q-2$ in the definition of $\mathcal{N}_\beta^{(q,t)}$ in order to obtain a non-empty $J_q^{(t)}$, as $\nu_q\left(h_{q-1}^{(1)}\right) = \nu_q\left(h_{q-1}\right) \geq 1$ for $q \geq 3$ by Babbage's theorem. Similar to Proposition 4.8 and Remark 4.9, one can also set conditions on $qN_\beta + N_{\beta-1} + 1$ to extend the possible choices for $N = (N_\beta, \ldots, N_0)_q$, but we skip the details of such cases here.

Remark 4.11. Note that there are infinitely many positive integers that belong to the set $\mathcal{N}_\beta$ for some $\beta \geq 2$, where $\mathcal{N}_\beta$ is given in (40). To see this, let $\mathcal{N} = \bigcup_{\beta\geq 2} \mathcal{N}_\beta$. Take $x \in [3^{\alpha+1}, 3^{\alpha+2})$ sufficiently large so that $\alpha \geq 3$. To give a lower bound on the number of elements in the set $\mathcal{N} \cap (a+12\mathbb{Z}) \cap [1,x]$ for some fixed $a \in \{0,\ldots,11\}$, we first count the possible number of digits for any $N \in \mathcal{N}_\beta \cap (a+12\mathbb{Z})$ for some fixed $\beta \in \mathbb{Z}\cap[3,\alpha]$, when $N = (N_\beta, \ldots, N_0)_3$ is written in its ternary representation. Observe that for any $a \in \{0,\ldots,11\}$, there exist unique $a_3 \in \{0,1,2\}$ and $a_4 \in \{0,1,2,3\}$ such that $a \equiv a_3 \pmod 3$ and $a \equiv a_4 \pmod 4$. Hence, the last digit of any element $N \in \mathcal{N}_\beta \cap (a+12\mathbb{Z})$ is determined by $a_3$, namely $N_0 = a_3$. Also, there are 2 possible choices for each of the values $N_i$, as $N_i \in \{0,1\}$ when $i \in \mathbb{Z}\cap[1,\beta-2]$. After determining each $N_i$ for $i \in \{0,1,\ldots,\beta-2\}$, one can choose $y \in \{3,4,5,6\}$ so that the congruence

$$N = \sum_{i=0}^{\beta} N_i 3^i = a_3 + \sum_{i=1}^{\beta-2} N_i 3^i + 3^{\beta-1}y \equiv a_4 \pmod 4$$

is satisfied. Therefore, there are at least $2^{\beta-2}$ many $N$ values such that $N \in \mathcal{N}_\beta \cap (a+$

$12\mathbb{Z})$. This indicates that the number of elements in $\mathcal{N} \cap (a+12\mathbb{Z}) \cap [1,x]$ is at least

$$\sum_{\beta=3}^{\alpha} 2^{\beta-2} = 2^{\alpha-1} - 2 \gg x^{\log_3 2},$$

as $\alpha = \lfloor \log_3 x \rfloor - 1$. The result of this type is also considered in [11] where the basis of the representation and the corresponding modulus are relatively prime to each other.

### 4.4. The case $n = 3P^\eta$ where $P \equiv \pm 1 \pmod{12}$ is prime

From now on, we only consider the certain type of integers in $\mathfrak{N}$. In particular, we take the integers of the form $n = 3P^\eta$ where $\eta$ is a positive integer, $P$ is a prime number and $P \equiv \pm 1 \pmod{12}$. Before dealing with the $P$-adic valuation of $h_n^{(r)}$, we first recall the definition of $f_n(x)$ by (9):

$$f_n(x) = \prod_{i=0}^{n-1} (x+i). \tag{46}$$

It is mentioned in the proof of [14, Corollary 24] that the polynomial

$$f_3'(x) = 3x^2 + 6x + 2 = 3(x+1)^2 - 1 \tag{47}$$

has a root modulo $p$ if and only if 3 is a square modulo $p$. This is also equivalent to have $p \equiv \pm 1 \pmod{12}$ which can be seen by the quadratic reciprocity law. Moreover, note that the only root of $f_3''(x) = 6x + 6$ is $-1$ in $\mathbb{F}_p$ which is not a root of $f_3'(x)$ modulo $p$. Hence, by the well known Hensel's Lemma (see for instance [28, Theorem 2.23]) we obtain that the polynomial $f_3'(x)$ has a root modulo $p^\alpha$ for each $\alpha \in \mathbb{Z}_{>0}$, when $p \equiv \pm 1 \pmod{12}$. Using this fact together with Corollary 2.10, we extend Lemma 2.7 and Proposition 2.12 simultaneously, when $n = 3P^\eta$ and $p = P$.

COROLLARY 4.12. *Let $n = 3P^\eta$ for some $\eta \in \{1,2,3\}$ and $P \equiv \pm 1 \pmod{12}$. Also, let $f_3'(x)$ be the polynomial given in (47). Assume that $s_\alpha$ denotes the root of $f_3'(x)$ modulo $P^\alpha$ for any $\alpha \in \mathbb{Z}_{>0}$. If $\eta \in \{1,2\}$ and $r \equiv j + (s_\eta - 1)P^\eta \pmod{P^{2\eta}}$ for some $j \in \{1,2,\ldots,P\}$, then $\nu_P\left(h_n^{(r)}\right) \geq 0$. Moreover, if $r \equiv 1 + (s_3 - 1)P^\eta \pmod{P^{2\eta}}$, then $\nu_P\left(h_{3P^\eta}^{(r)}\right) \geq 0$ for any $\eta \in \{1,2,3\}$.*

PROOF. First of all, note that $P \geq 11$. Therefore, $\eta = 1$ implies that $P \in (\sqrt{n}, n)$. By Remark 2.15, we get that $\nu_P\left(h_n^{(r)}\right) \geq 0$ if and only if $r$ is of the form $j + (s_1 - 1)P + \ell P^2$ for some $j \in \{1,2,\ldots,P\}$ and $\ell \in \mathbb{Z}_{\geq 0}$. So, let $\eta = 2$. Choose $r = j + (s_2 - 1)P^2 + \ell P^4$ for some $\ell \in \mathbb{Z}_{\geq 0}$ and $j \in \{1,2,\ldots,P\}$. Define

$$c_1 := \left\lceil \frac{r}{P} \right\rceil \quad \text{and} \quad c_2 := \left\lceil \frac{c_1}{P} \right\rceil. \tag{48}$$

In that case, we have $c_1 = 1 + (s_2 - 1)P + \ell P^3$ and $c_2 = s_2 + \ell P^2$. This indicates that $c_2 \equiv s_2 \pmod{P^2}$. Hence, we get

$$\nu_P\left(h_3^{(c_2)}\right) = \nu_P\left(\frac{f_3'(c_2)}{3!}\right) = \nu_P\left(\frac{3c_2^2 + 6c_2 + 2}{6}\right) \geq 2, \tag{49}$$

as $P > 6$. Also, since $c_1 = (c_2 - 1)P + 1$, we see that $\nu_P\left(h_{3P}^{(c_1)}\right) \geq 1$ by Corollary 2.10. Similarly, Corollary 2.10 implies that $\nu_P\left(h_n^{(r)}\right) \geq 0$, as $r = (c_1 - 1)P + j$ for some $j \in \{1, 2, \ldots, P\}$.

For the last part of the corollary, observe that the congruence $r \equiv 1 + (s_\eta - 1)P^\eta \pmod{P^{2\eta}}$ is satisfied for any $\eta \in \{1, 2\}$, as $s_3 \equiv s_\eta \pmod{P^\eta}$ by Hensel's Lemma. Therefore, we conclude by the first part of this corollary that $\nu_P\left(h_n^{(r)}\right) \geq 0$ when $n = 3P^\eta$ and $\eta \in \{1, 2\}$. So, assume that $\eta = 3$. Take $r = 1 + (s_3 - 1)P^3 + \ell P^6$ for some non-negative integer $\ell$. In addition to definitions of $c_1$ and $c_2$ given in (48), define $c_3 := \left\lceil \frac{c_2}{P} \right\rceil$. Using these definitions, we get

$$\begin{aligned}
c_1 &= \left\lceil \frac{r}{P} \right\rceil = 1 + (s_3 - 1)P^2 + \ell P^5, \\
c_2 &= \left\lceil \frac{c_1}{P} \right\rceil = 1 + (s_3 - 1)P + \ell P^4, \\
c_3 &= \left\lceil \frac{c_2}{P} \right\rceil = s_3 + \ell P^3.
\end{aligned}$$

This shows that $c_3 \equiv s_3 \pmod{P^3}$, and similar to (49), we obtain that

$$\nu_P\left(h_3^{(c_3)}\right) = \nu_P\left(\frac{f_3'(c_3)}{3!}\right) = \nu_P\left(\frac{3c_3^2 + 6c_3 + 2}{6}\right) \geq 3,$$

as $P > 6 \geq 5$. Since $c_2 = (c_3 - 1)P + 1$, $c_1 = (c_2 - 1)P + 1$ and $r = (c_1 - 1)P + 1$, we deduce by Corollary 2.10 that $\nu_P\left(h_{3P}^{(c_2)}\right) \geq 2$, $\nu_P\left(h_{3P^2}^{(c_1)}\right) \geq 1$ and $\nu_P\left(h_{3P^3}^{(r)}\right) \geq 0$, respectively. □

Remark 4.13. Observe that there are other values of $r$ such that the corresponding $P$-adic valuation of $h_{3P^\eta}^{(r)}$ is non-negative. For instance, one may take $r = s_3 P^\eta + \ell P^{2\eta}$ for some $\ell \in \mathbb{Z}_{>0}$ to obtain $\nu_P\left(h_{3P^\eta}^{(r)}\right) \geq 0$, when $\eta \in \{1, 2, 3\}$. However, for our purposes, it will be enough to deal with positive integer values of $r$ where $r \equiv 1 + (s_3 - 1)P^\eta \pmod{P^{2\eta}}$.

So far, we found some certain conditions for $r$ in order to obtain $\nu_p\left(h_n^{(r)}\right) \geq 0$ for any $p \in \mathbb{P}_{\leq n}$, where $n = 3P^\eta$ for a prime $P \equiv \pm 1 \pmod{12}$ and $\eta \in \{1, 2, 3\}$ such that $P^\eta \in \mathcal{N}_\beta$ and $\mathcal{N}_\beta$ is given in (40) for some $\beta \geq 2$. This fact will be useful to prove Theorem 1.2 from the introduction.

*Proof of Theorem 1.2.* Let $n = 3P^\eta$ for some $\eta \in \{1, 2, 3\}$ and a prime $P \equiv \pm 1 \pmod{12}$. Also, assume that $P^\eta = (P_\beta^{(\eta)}, \ldots, P_0^{(\eta)})_3 \in (3^\beta, 8 \cdot 3^{\beta-1})$ for $\beta \geq 2$, where $(P_\beta^{(\eta)}, \ldots, P_0^{(\eta)})_3$ denotes the ternary representation of $P^\eta$. Note that, $3P_\beta^{(\eta)} + P_{\beta-1}^{(\eta)} \in \{3, 4, 5, 6, 7\}$. Moreover, suppose that

$$\begin{aligned}
Q &= 2^{m-1} \cdot \prod_{\substack{2<p<\sqrt{n} \\ p \nmid n}} p^{\nu_p((n-1)!)} \cdot \prod_{\substack{\sqrt{n}<p<n \\ p \nmid n}} p, \\
r &\equiv 0 \pmod{Q}, \qquad (50)
\end{aligned}$$

$$r \equiv 3^{\beta} t_{n_\beta} \pmod{3^{\beta+3}}, \tag{51}$$

$$r \equiv 1 + (s_3 - 1)P^{\eta} \pmod{P^{2\eta}}, \tag{52}$$

where $s_3$ is a root of $f_3'(x)$ modulo $P^3$, $n_\beta = 3P_\beta^{(\eta)} + P_{\beta-1}^{(\eta)} + 1$, $t_{n_\beta}$ is given in Proposition 4.8 and $m = \lceil \log_2 n \rceil$. By the Chinese Remainder Theorem, we have a unique common solution modulo $3^{\beta+3}QP^{2\eta}$ that satisfies congruences (50), (51) and (52), as $3 \nmid PQ$ and $\gcd(P, Q) = 1$. Call that solution $S \pmod{3^{\beta+3}QP^{2\eta}}$. If $r \equiv S \pmod{3^{\beta+3}QP^{2\eta}}$, then by Proposition 4.3, Lemma 4.4, Remark 4.5, Proposition 4.8 and Corollary 4.12 we deduce that there exists a value of $r$ such that $\nu_p\left(h_n^{(r)}\right) \geq 0$ for any $p \in \mathbb{P}_{\leq n}$. This yields by Fact 2.4 that $h_n^{(r)}$ is an integer. Hence, Corollary 2.3 indicates $h_n^{(r+k\cdot n!)}$ is an integer for all values of $k$ where $r + k \cdot n! > 0$. Thus, choosing $r \in [1, (3P^{\eta})!]$ and $R = r + k \cdot (3P^{\eta})!$ for any $k \geq 0$ gives the first part of the theorem, as $n = 3P^{\eta}$.

For the last part of the theorem, we know that there exists an $r_0 \in [1, n!]$ such that $h_n^{(r_0)} \in \mathbb{Z}$ when $n$ is of the form $3P^{\eta}$, where $P \equiv \pm 1 \pmod{12}$ is a prime number satisfying ($\star$) for some $\eta \in \{1, 2, 3\}$. Now, define

$$\overline{\mathcal{R}}_n = \{r \in [1, n!] : h_n^{(r)} \in \mathbb{Z}\}. \tag{53}$$

Observe that $\overline{\mathcal{R}}_n$ is non-empty. In that case,

$$\mathcal{R}_n \cap [1, x] = \bigcup_{j=0}^{k-1} (\mathcal{R}_n \cap (j \cdot n!, (j+1) \cdot n!]) \cup (\mathcal{R}_n \cap (k \cdot n!, x]),$$

where $k = \left\lfloor \frac{x}{n!} \right\rfloor$. For any $j \in \{0, 1, \ldots, k-1\}$, since $h_n^{(r_0 + j\cdot n!)}$ is also an integer by Corollary 2.3, we see that the number of elements in the set $\mathcal{R}_n \cap (j \cdot n!, (j+1) \cdot n!]$ is equal to $|\overline{\mathcal{R}}_n|$. This indicates that $|\mathcal{R}_n \cap [1, x]| = k \cdot |\overline{\mathcal{R}}_n| + |\mathcal{R}_n \cap (k \cdot n!, x]| \geq k|\overline{\mathcal{R}}_n|$ and $|\mathcal{R}_n \cap [1, x]| \leq k|\overline{\mathcal{R}}_n| + (x - kn!)$. Using these inequalities, we get

$$\begin{aligned}
\frac{|\overline{\mathcal{R}}_n|}{n!} - \frac{|\overline{\mathcal{R}}_n|}{x} &= \left(\frac{x}{n!} - 1\right) \cdot \frac{|\overline{\mathcal{R}}_n|}{x} \\
&< \left\lfloor \frac{x}{n!} \right\rfloor \cdot \frac{|\overline{\mathcal{R}}_n|}{x} \\
&\leq \frac{|\mathcal{R}_n \cap [1, x]|}{x} \\
&\leq \left\lfloor \frac{x}{n!} \right\rfloor \cdot \frac{|\overline{\mathcal{R}}_n|}{x} + \frac{1}{x} \cdot \left(x - \left\lfloor \frac{x}{n!} \right\rfloor \cdot n!\right) \\
&< \frac{|\overline{\mathcal{R}}_n|}{n!} + \frac{1}{x}\left(x + \left(1 - \frac{x}{n!}\right) n!\right) = \frac{|\overline{\mathcal{R}}_n|}{n!} + \frac{n!}{x}.
\end{aligned}$$

Since $n$ is fixed, the set $\mathcal{R}_n$ has density and it is equal to

$$\frac{|\overline{\mathcal{R}}_n|}{n!}. \tag{54}$$

Thus, we conclude that the density of the set $\mathcal{R}_n$ is at least $\frac{1}{n!}$, as $\overline{\mathcal{R}}_n$ is non-empty. □

Remark 4.14. As one may observe that the last part of Theorem 1.2 is related with [14, Proposition 9] where it was shown that

$$\lim_{n\to\infty}\frac{|\{r\le x : h_n^{(r)}\notin\mathbb{Z}\}|}{x}=1,$$

when $n=o(x)$. However, it should be noted that this fact does not contradict with Theorem 1.2, as $n$ is fixed in Theorem 1.2.

## 5. Computations & The Smallest Hyperharmonic Integer

In this final section, we first obtain subsets of $\mathcal{R}_{33}$ and $\mathcal{R}_{39}$ whose densities are bigger than the ones given in (5), where $\mathcal{R}_n=\{r\in\mathbb{N} : h_n^{(r)}\in\mathbb{Z}\}$. Note by Fact 1.4 that these are the only $n$ values up to 40 for which the corresponding hyperharmonic number $h_n^{(r)}$ may be an integer.

Proposition 5.1. *The density of the set* $\mathcal{R}_{33}=\{r\in\mathbb{Z}_{>0} : h_{33}^{(r)}\in\mathbb{Z}\}$ *is equal to*

$$\frac{271\,666\,053\,120}{2\,168\,219\,346\,697\,404\,000}\approx 1.25294543\cdot 10^{-7}.$$

Proof. Let $n=33$. In order to find a bigger set of $r$ values, we follow the same idea in the proof of Theorem 1.2. That is, we find the necessary modular equivalences for the values of $r$ such that $\nu_p\left(h_{33}^{(r)}\right)\ge 0$ for each $p\le 33$. So first of all, we deal with the case $p=3$. Define $r_1=\left\lceil\frac{r}{3}\right\rceil$. Since 3 divides 33, $\nu_3\left(h_{33}^{(r)}\right)\ge 0$ if and only if $\nu_3\left(h_{11}^{(r_1)}\right)\ge 1$ by Corollary 2.10. Using Corollary 2.6 yields $|I(11,r_1)\cap 3\mathbb{Z}|\in\{3,4\}$. Note that $|I(11,r_1)\cap 3\mathbb{Z}|=3$ if and only if $r_1\equiv 1 \pmod 3$. In this case, by equation (2), we have

$$\begin{aligned} h_{11}^{(r_1)} &= \frac{A}{B}\cdot\frac{3c\cdot 3(c+1)\cdot 3(c+2)}{3\cdot 6\cdot 9}\left(\frac{1}{3}\sum_{i=0}^{2}\frac{1}{c+i}+q\right)\\ &= \frac{A}{2B}\cdot\frac{c(c+1)(c+2)}{9}\cdot\frac{f_3'(c)}{f_3(c)}+\frac{Ac(c+1)(c+2)}{6B}\cdot q, \end{aligned} \tag{55}$$

where $c=\left\lceil\frac{r_1}{3}\right\rceil$ and $\nu_3(A)=\nu_3(B)=0\le\nu_3(q)$ for some $A,B\in\mathbb{Z}_{>0}$ and $q\in\mathbb{Q}$. Observe that $f_3(x)$ and $f_3'(x)$ can be obtained from (46) and (47), respectively. Note that the polynomial $f_3'(x)=3x^2+6x+2$ does not have any roots in $\mathbb{F}_3$. Therefore, $\nu_3\left(h_{11}^{(r_1)}\right)=-2$ as the 3-adic valuation of the second summand in (55) is greater than or equal to 0. Thus, we conclude that $\nu_3\left(h_{33}^{(r)}\right)<0$, if $r_1\equiv 1 \pmod 3$. Secondly, $r_1\equiv 0,2 \pmod 3$ is equivalent to the fact that $|I(11,r_1)\cap 3\mathbb{Z}|=4$. In that case, $\nu_3\left(h_{11}^{(r_1)}\right)\ge 1$ if and only if $\nu_3\left(4h_4^{(r_2)}\right)=\nu_3\left(h_4^{(r_2)}\right)\ge 1$ by Lemma 2.7, where $r_2=\left\lceil\frac{r_1}{3}\right\rceil$. Recall by Lemma 4.6 that $\nu_3\left(h_4^{(r_2)}\right)\ge 1$ can be obtained if and only if $r_2\equiv 3 \pmod 9$. Note that this can be satisfied if and only if $r_1\equiv 7,8,9 \pmod{27}$, as $r_2=\left\lceil\frac{r_1}{3}\right\rceil$. If we combine this fact with $r_1\equiv 0,2 \pmod 3$, we deduce that

$r_1 \equiv 8, 9 \pmod{27}$. Since $r_1 = \left\lceil \frac{r}{3} \right\rceil$, we conclude that the necessary and the sufficient condition to obtain $\nu_3\left(h_{33}^{(r)}\right) \geq 0$ is to have $r \equiv 22, 23, 24, 25, 26, 27 \pmod{81}$.

Next, we check the congruences for $r$ to get $\nu_5\left(h_{33}^{(r)}\right) \geq 0$. Recall by Corollary 2.6 that there are two options for the size of the set $I(n,r) \cap 5\mathbb{Z}$: it is either $|I(n,r) \cap 5\mathbb{Z}| = 6$, or $|I(n,r) \cap 5\mathbb{Z}| = 7$. Observe that the former case occurs if and only if

$$r \equiv 1, 2 \pmod{5}. \tag{56}$$

This can be seen by simply counting the elements in $I(n,r)$ which are multiples of 5 when we choose a value for $r$ in modulo 5. So, if $r \equiv 1, 2 \pmod 5$, then by Lemma 2.7 we see that $\nu_5\left(h_{33}^{(r)}\right) \geq 0$ holds if and only if $\nu_5\left(h_6^{(r_1)}\right) \geq 1$, where $r_1 = \left\lceil \frac{r}{5} \right\rceil$. Note that this is equivalent to have

$$\nu_5\left(\frac{f_6'(r_1)}{6!}\right) \geq 1 \tag{57}$$

by Lemma 2.1. In other words, (57) holds if and only if $f_6'(r_1) \equiv 0 \pmod{25}$. By SageMath [29], we get that the only root of the polynomial $f_6'(x)$ in modulo 25 is congruent to $10 \pmod{25}$. This indicates that $r \equiv 46, 47, 48, 49, 50 \pmod{125}$ is sufficient to have $\nu_5\left(h_6^{(r_1)}\right) \geq 1$, as $r_1 = \left\lceil \frac{r}{5} \right\rceil$. Combining this together with (56), we deduce that $r \equiv 46, 47 \pmod{125}$ implies that $\nu_5\left(h_{33}^{(r)}\right) \geq 0$. Similarly, we have $|I(n,r) \cap 5\mathbb{Z}| = 7$ if and only if $r \equiv 0, 3, 4 \pmod 5$ by (56). By (12) in the proof of Lemma 2.7, we see that $\nu_5\left(h_{33}^{(r)}\right) \geq 0$ holds if and only if $\nu_5\left(7h_7^{(r_1)}\right) = \nu_5\left(h_7^{(r_1)}\right) \geq 0$, where $r_1 = \left\lceil \frac{r}{5} \right\rceil$. In that case, we can use Remark 2.14 to see that $\nu_5\left(h_{33}^{(r)}\right) \geq 0$ is equivalent to $r_1 \equiv 0, 4 \pmod{5}$. The latter case is satisfied if and only if $r \equiv 0, 16, 17, 18, 19, 20, 21, 22, 23, 24 \pmod{25}$. Combining this fact together with $r \equiv 0, 3, 4 \pmod 5$, we conclude that $r \equiv 0, 18, 19, 20, 23, 24 \pmod{25}$ implies $\nu_5\left(h_{33}^{(r)}\right) \geq 0$. Thus, there are exactly 32 different values in modulo 125 which leads to a non-negative 5-adic valuation for $h_{33}^{(r)}$.

For the case $p = 2$, we see that $\mathfrak{E}_2(33) = \{32, 64\}$ by the definition of $\mathfrak{E}_2(n)$ given in (37). This implies by Proposition 4.3 that $r \equiv 0 \pmod{32}$ if and only if $\nu_2\left(h_{33}^{(r)}\right) \geq 0$. Finally, we use Proposition 2.12 and Remark 2.14 to obtain the condition on $r$ for the inequality $\nu_p\left(h_{33}^{(r)}\right) \geq 0$, for $p \in \mathbb{P} \cap (5, 33]$. Thanks to SageMath [29], we get possible choices for $r$ modulo either $p$ or $p^2$, for each prime $p \in (\sqrt{n}, n)$ when $n = 33$. Table 1 contains the number of these congruence classes.

Combining the values in Table 1 together with Proposition 4.3 and the Chinese Remainder Theorem, we see that there are exactly 271 666 053 120 many possible choices for $r$ in modulo

$$M = 2^5 \cdot 3^4 \cdot 5^3 \cdot \prod_{\sqrt{n}<p<\frac{n}{2}} p^2 \cdot \prod_{\frac{n}{2}<p<n} p = 2\,168\,219\,346\,697\,404\,000$$

Table 1. The number of congruences together with the corresponding moduli for each $p \in \mathbb{P} \cap [2, 33]$, when $n = 33$.

| The prime $p$ | The number of possible congruence classes for $r$ | Mod |
|---|---|---|
| 2 | 1 | 32 |
| 3 | 6 | 81 |
| 5 | 32 | 125 |
| 7 | 37 | 49 |
| 11 | 22 | 121 |
| 13 | 97 | 169 |
| 17 | 16 | 17 |
| 19 | 14 | 19 |
| 23 | 10 | 23 |
| 29 | 4 | 29 |
| 31 | 2 | 31 |

for $n = 33$. Since $M$ divides $33! = 2^{31} \cdot 3^{15} \cdot 5^7 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31$, there are exactly $271\,666\,053\,120 \cdot (33!/M)$ many possible choices for $r$ in modulo $33!$. This indicates that $\left|\overline{\mathcal{R}}_{33}\right| = 271\,666\,053\,120 \cdot (33!/M)$ where $\overline{\mathcal{R}}_n$ is defined in (53). By (54) in the proof of Theorem 1.2, we conclude that the density of the set $\mathcal{R}_{33}$ is equal to

$$\frac{271\,666\,053\,120 \cdot 33!}{M \cdot 33!} \approx 1.25294543 \cdot 10^{-7},$$

as desired. $\square$

A similar result can also be obtained for $n = 39$, since it is of the form $n = 3p$ where $p \equiv \pm 1 \pmod{12}$.

Proposition 5.2. *The density of the set* $\mathcal{R}_{39} = \{r \in \mathbb{Z}_{>0} : h_{39}^{(r)} \in \mathbb{Z}\}$ *is equal to*

$$\frac{3\,025\,668\,318\,167\,040}{77\,737\,168\,237\,142\,025\,612\,000} \approx 3.89217717 \cdot 10^{-8}.$$

Proof. Assume that $n = 39$. We apply the same technique in the proof of Proposition 5.1. So, take $p = 3$. Let $r_1 = \left\lceil \frac{r}{3} \right\rceil$. Then, Corollary 2.10 yields $\nu_3\left(h_{39}^{(r)}\right) \geq 0$ if and only if $\nu_3\left(h_{13}^{(r_1)}\right) \geq 1$, as $3 \mid 39$. Also, by Corollary 2.6 we have $|I(13, r_1) \cap 3\mathbb{Z}| \in \{4, 5\}$. Observe that $|I(13, r_1) \cap 3\mathbb{Z}| = 4$ if and only if $r_1 \equiv 1, 2 \pmod 3$. Using equation (2), we get

$$\begin{aligned} h_{13}^{(r_1)} &= \frac{A}{B} \cdot \frac{3c \cdot 3(c+1) \cdot 3(c+2) \cdot 3(c+3)}{3 \cdot 6 \cdot 9 \cdot 12} \left( \frac{1}{3} \sum_{i=0}^{3} \frac{1}{c+i} + q \right) \\ &= \frac{A}{8B} \cdot \frac{f_4'(c)}{9} + \frac{A}{B} \cdot \binom{c+3}{4} \cdot q, \end{aligned} \tag{58}$$

where $f_n(x)$ is given in (46), $c = \left\lceil \frac{r_1}{3} \right\rceil$ and $\nu_3(A) = \nu_3(B) = 0 \leq \nu_3(q)$ for some $A, B \in \mathbb{Z}_{>0}$ and $q \in \mathbb{Q}$. By Remark 4.7, we know that $\nu_3(f_4'(c)) \geq 3$ if and only if

$c \equiv 12 \pmod{27}$. In that case, $\nu_3\left(\left(\binom{c+3}{4}\right)\right) \geq 1$ as $c \equiv 0 \pmod 3$ and so $\nu_3\left(h_{13}^{(r_1)}\right) \geq 1$ by equation (58). On the other hand, if $c \not\equiv 12 \pmod{27}$, then either $\nu_3\left(f_4'(c)\right) = 2$, or $\nu_3\left(f_4'(c)\right) \leq 1$ holds. Note by Lemma 4.6 and Remark 4.7 that the former case holds if and only if $c \equiv 3, 21 \pmod{27}$. This implies that $\nu_3\left(\left(\binom{c+3}{4}\right)\right) \geq 1$ as $c \equiv 0 \pmod 3$. Hence, we deduce that $\nu_3\left(h_{13}^{(r_1)}\right) = 0$, by equation (58) and the non-Archimedean property of the 3-adic valuation. Similarly, if $c \not\equiv 12 \pmod{27}$ and $\nu_3\left(f_4'(c)\right) \leq 1$, then $\nu_3\left(h_{13}^{(r_1)}\right) < 0$ by equation (58), as $\nu_3\left(\left(\binom{c+3}{4}\right)\right) \geq 0$ for all $c \in \mathbb{Z}_{\geq 0}$. Thus, we see that $c \equiv 12 \pmod{27}$ is equivalent to have $\nu_3\left(h_{13}^{(r_1)}\right) \geq 1$ when $r_1 \equiv 1, 2 \pmod 3$. Note that $c \equiv 12 \pmod{27}$ if and only if $r_1 \equiv 34, 35, 36 \pmod{81}$. Combining this together with the condition $r_1 \equiv 1, 2 \pmod 3$, we deduce that $r_1 \equiv 34, 35 \pmod{81}$ yields a positive 3-adic valuation for $h_{13}^{(r_1)}$. Moreover, letting $r_1 \equiv 0 \pmod 3$ leads to $|I(13, r_1) \cap 3\mathbb{Z}| = 5$. By Lemma 2.7, we see that $\nu_3\left(h_{13}^{(r_1)}\right) \geq 1$ if and only if $\nu_3\left(5h_5^{(r_2)}\right) = \nu_3\left(h_5^{(r_2)}\right) \geq 1$ where $r_2 = \left\lceil \frac{r_1}{3} \right\rceil$. Also by Lemma 4.6, we observe that the congruence $r_2 \equiv 2, 3 \pmod 9$ is equivalent to have $\nu_3\left(h_5^{(r_2)}\right) \geq 1$. Note that the case $r_2 \equiv 2, 3 \pmod 9$ can be satisfied as long as $r_1 \equiv 4, 5, 6, 7, 8, 9 \pmod{27}$. Since we choose $r_1 \equiv 0 \pmod 3$ for this case earlier, we get that $r_1 \equiv 6, 9 \pmod{27}$ indicates a positive 3-adic valuation for $h_{13}^{(r_1)}$. Combining this congruence class with the previous one, namely $r_1 \equiv 34, 35 \pmod{81}$, yields that $\nu_3\left(h_{13}^{(r_1)}\right) \geq 1$, when $r_1 \equiv 6, 9, 33, 34, 35, 36, 60, 63 \pmod{81}$. In other words, there are exactly 24 possible choices for $r$ modulo 243 for which the corresponding 3-adic valuation of $h_{39}^{(r)}$ is non-negative, as $r_1 = \left\lceil \frac{r}{3} \right\rceil$.

Now, we consider the case $p = 5$. Recall by Corollary 2.6 that $|I(39, r) \cap 5\mathbb{Z}| \in \{7, 8\}$. Observe that the case $|I(39, r) \cap 5\mathbb{Z}| = 7$ is equivalent to have $r \equiv 1 \pmod 5$. By Lemma 2.7, we know that $\nu_5\left(h_{39}^{(r)}\right) \geq 0$ holds if and only if $\nu_5\left(h_7^{(c)}\right) \geq 1$, where $c = \left\lceil \frac{r}{5} \right\rceil$. In that case, we see that $\nu_5\left(h_7^{(c)}\right) = \nu_5\left(\frac{f_7'(c)}{7!}\right) \geq 1$ by Lemma 2.1. Note that the latter inequality holds if and only if $c$ is a root of $f_7'(x)$ modulo 25. Using equation (46) and SageMath [29], we deduce that 9 and 10 are the only roots of the polynomial $f_7'(x)$ modulo 25. Hence, taking $r \equiv 41, 42, 43, 44, 45, 46, 47, 48, 49, 50 \pmod{125}$ yields $\nu_5\left(h_7^{(c)}\right) \geq 1$, as $c = \left\lceil \frac{r}{5} \right\rceil$. Combining this together with the fact $r \equiv 1 \pmod 5$, we get that $r \equiv 41, 46 \pmod{125}$ implies that $\nu_5\left(h_{39}^{(r)}\right) \geq 0$. Moreover, if $r \not\equiv 1 \pmod 5$ we have $|I(39, r) \cap 5\mathbb{Z}| = 8$. This indicates by (12) in the proof of Lemma 2.7 that $\nu_5\left(h_{39}^{(r)}\right) \geq 0$ if and only if $\nu_5\left(8h_8^{(r_1)}\right) = \nu_5\left(h_8^{(r_1)}\right) \geq 0$, where $r_1 = \left\lceil \frac{r}{5} \right\rceil$. Using Remark 2.14, we obtain that $\nu_5\left(h_8^{(r_1)}\right) \geq 0$ holds if and only if $r_1 \equiv 0, 3, 4 \pmod 5$, as $4 < p < 8$ when $p = 5$. This fact together with the condition $r \not\equiv 1 \pmod 5$ implies that $r \equiv 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 25 \pmod{25}$. In that case, we get $\nu_5\left(h_{39}^{(r)}\right) \geq 0$. Combining previous congruence classes with $r \equiv 41, 46 \pmod{125}$ yields that there are exactly 62 values modulo 125 such that the corresponding hyperharmonic number $h_{39}^{(r)}$ has a non-negative 5-adic valuation.

Next, we see that $\mathfrak{E}_2(39) = \{26, 28, 30, 32, 58, 60, 62, 64\}$ by (37) in Proposition 4.3. This indicates by Proposition 4.3 that there are 4 different possible choices for $r$ in modulo 32. Lastly, we use Proposition 2.12 and Remark 2.14 to get $\nu_p\left(h_{39}^{(r)}\right) \geq 0$ for $p \in \mathbb{P} \cap (5, 39]$, as we did in the proof of Proposition 5.1. By SageMath [29], we obtain the corresponding number of congruences that is given in Table 2.

Table 2. The number of congruences together with the corresponding moduli for each $p \in \mathbb{P} \cap [2, 39]$, when $n = 39$.

| The prime $p$ | The number of possible congruence classes for $r$ | Mod |
|---|---|---|
| 2 | 4 | 32 |
| 3 | 24 | 243 |
| 5 | 62 | 125 |
| 7 | 28 | 49 |
| 11 | 76 | 121 |
| 13 | 26 | 169 |
| 17 | 97 | 289 |
| 19 | 37 | 361 |
| 23 | 16 | 23 |
| 29 | 10 | 29 |
| 31 | 8 | 31 |
| 37 | 2 | 37 |

Using Table 2 and the Chinese Remainder Theorem, we see that there are 3 025 668 318 167 040 many possible choices for $r$ in modulo

$$M = 2^5 \cdot 3^5 \cdot 5^3 \cdot \prod_{\sqrt{n}<p<\frac{n}{2}} p^2 \cdot \prod_{\frac{n}{2}<p<n} p = 77\,737\,168\,237\,142\,025\,612\,000$$

for $n = 39$. Similar to the proof of Proposition 5.1, there are $3\,025\,668\,318\,167\,040 \cdot (39!/M)$ many possible choices for $r$ in modulo 39!, as $M$ divides $39! = 2^{35} \cdot 3^{18} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37$. This shows that $\left|\overline{\mathcal{R}}_{39}\right| = 3\,025\,668\,318\,167\,040 \cdot (39!/M)$, where $\overline{\mathcal{R}}_n$ is given in (53). Thus, we conclude that the density of the set $\mathcal{R}_{39}$ is equal to

$$\frac{3\,025\,668\,318\,167\,040 \cdot 39!}{M \cdot 39!} \approx 3.89217717 \cdot 10^{-8},$$

by (54) in the proof of Theorem 1.2. □

The smallest hyperharmonic integer $h_n^{(r)}$ which we found during our computation is

3009287172811364404984125778708627188141159718559721813893101188860332195034648261187262268354557608058585276611064376994779359430272826344742020434523221316911052660055855963776173497027117

which is a 190-digit natural number. In order to obtain this hyperharmonic integer, we take

$$n = 33 \text{ and } r = 10\,667\,968. \tag{59}$$

This result was derived from a partial list of congruence classes for $r$ which are obtained in the proofs of Propositions 5.1 and 5.2. Note that it can be used to find the smallest hyperharmonic integer which is given in Theorem 1.3 from the introduction.

*Proof of Theorem 1.3.* Let $R = 10\,667\,968$ and $h_{n_0}^{(r_0)}$ be the smallest hyperharmonic integer that is greater than 1. Observe by (59) and the remark before it, we get that $h_{33}^{(R)} \in \mathbb{Z}$ and $h_{33}^{(R)} < 10^{190}$. In order to prove the theorem, we first show that $r_0 \leq R$. So, assume not. Then, by Fact 1.4, we see that $n_0 \geq 33$ and $r_0 - 2 \geq R$. Using the definition of hyperharmonic numbers given in (1), we get that

$$h_{n_0}^{(r_0)} = \sum_{k=1}^{n_0} h_k^{(r_0-1)} > h_{33}^{(r_0-1)} = \sum_{k=1}^{33} h_k^{(r_0-2)} \geq h_{33}^{(R)},$$

which is impossible by (59). Hence, we have $r_0 \leq R$. Also, we know by Fact 1.4 that $r_0 \geq 20\,002$. Now, using (1) we get

$$h_{n_0}^{(20\,002)} \leq h_{n_0}^{(r_0)} \leq h_{33}^{(R)} < 10^{190}.$$

This indicates by (2) that

$$\binom{n_0 + 20\,001}{20\,001} \cdot (h_{n_0+20\,001} - h_{20\,001}) < 10^{190}.$$

Combining this together with Fact 1.4 implies

$$\binom{n_0 + 20\,001}{20\,001} (h_{20\,034} - h_{20\,001}) < 10^{190},$$

as $n_0 \geq 33$. If we denote $\alpha = h_{20\,034} - h_{20\,001}$, we obtain

$$\binom{n_0 + 20\,001}{20\,001} = \binom{n_0 + 20\,001}{n_0} < \frac{10^{190}}{\alpha}. \tag{60}$$

Using lower bounds on binomial coefficients, we see that

$$\left(\frac{n_0 + 20\,001}{20\,001}\right)^{20\,001} < \frac{10^{190}}{\alpha}.$$

Computations yield that $n_0 \leq 448$. Moreover, we know by (60) that

$$\frac{20002^{n_0}}{n_0!} < \binom{n_0 + 20\,001}{n_0} < \frac{10^{190}}{\alpha}, \tag{61}$$

since $n_0 \geq 33$. Using SageMath [29] we obtain that $n_0 \leq 66$, as $n_0 \leq 448$. This indicates by (6) that $n_0 \in \{33, 39, 45, 63\}$. To find the smallest hyperharmonic integer which is greater than 1, it is enough to check the corresponding values of $h_n^{(r)}$ for tuples

$$(n, r) \in \{33, 39, 45, 63\} \times [20\,002, 10\,667\,968]$$

since $r_0 \leq R = 10\,667\,968$. Using Fact 1.4, Remark 2.14 and SageMath [29], we conclude that the smallest hyperharmonic integer $h_{n_0}^{(r_0)}$ greater than 1 is equal to

$$\begin{aligned}&3009287172811364404984125778708627188141159718559721813893101188860332195 03\\&4648261187262268354557608058585276611064376994779359430272826344742020434 52\\&322131691105266005585596377617349 7027117,\end{aligned}$$

and it is obtained when $n_0 = 33$ and $r_0 = R = 10\,667\,968$. □


## Acknowledgements

The author is grateful to the anonymous referee and the editor for their comments which drastically improved the quality and the presentation of the paper.


## References


[1] E. Alkan, H. Göral, D. C. Sertbaş, Hyperharmonic numbers can rarely be integers, INTEGERS, **18** (2018), Paper No. A43, 1–15.

[2] Ç. Altuntaş, H. Göral, D. C. Sertbaş, The Difference of Hyperharmonic Numbers via Geometric and Analytic Methods, J. Korean Math. Soc., **59** (2022), no. 6, 1103–1137.

[3] R. A. Amrane, H. Belbachir, Non-integerness of class of hyperharmonic numbers, Ann. Math. Inform., **37** (2010), 7–10.

[4] R. A. Amrane, H. Belbachir, Are the hyperharmonics integral? A partial answer via the small intervals containing primes, C. R. Math. Acad. Sci. Paris, **349** (2011), 115–117.

[5] W. D. Banks, I. E. Shparlinski, Arithmetic properties of numbers with restricted digits, Acta Arith. (4), **112** (2004), 313–332.

[6] A. T. Benjamin, D. Gaebler, R. Gaebler, A combinatorial approach to hyperharmonic numbers, INTEGERS, **3** (2003), Paper No. A15, 1–9.

[7] L. Carlitz, A note on Wolstenholme's theorem, Amer. Math. Monthly, **61** (1954), 174–176.

[8] J. H. Conway, R. K. Guy, The Book of Numbers, Springer-Verlag, New York, 1996.

[9] C. Dartyge, C. Mauduit, Nombres presque premiers dont l'écriture en base $r$ ne comporte pas certains chiffres, J. Number Theory (2), **81** (2000), 270–291.

[10] C. Dartyge, C. Mauduit, Ensembles de densité nulle contenant des entiers possédant au plus deux facteurs premiers, J. Number Theory (2), **91** (2001), 230–255.

[11] P. Erdős, C. Mauduit, A. Sárközy, On Arithmetic Properties of Integers with Missing Digits I: Distribution in Residue Classes, J. Number Theory (2), **70**(1998), 99–120.

[12] P. Erdős, C. Mauduit, A. Sárközy, On arithmetic properties of integers with missing digits. II: Prime factors, Discrete Math. (1–3), **200** (1999), 149–164.

[13] M. Filaseta, S. Konyagin, Squarefree values of polynomials all of whose coefficients are 0 and 1, Acta Arith., **74** (1996), 191–205.

[14] H. Göral, D. C. Sertbaş, Almost all Hyperharmonic Numbers are not Integers, J. Number Theory, **171** (2017), 495–526.

[15] H. Göral, D. C. Sertbaş, A congruence for some generalized harmonic type sums, Int. J. Number Theory, **14** (2018), no. 04, 1033–1046.

[16] H. Göral, D. C. Sertbaş, Divisibility Properties of Hyperharmonic Numbers, Acta Math. Hungar., **154** (2018), 147–186.

[17] H. Göral, D. C. Sertbaş, Some Results on Harmonic Type Sums, Düzce Uni. J. Sci. and Tech., **8** (2020), 642–653.

[18] D. R. Heath-Brown, The Differences Between Consecutive Primes, III, J. London Math. Soc., **2-20** (1979), no. 2, 177–178.

[19] S. Konyagin, Arithmetic properties of integers with missing digits: distribution in residue classes,

Period. Math. Hungar. (1–2), **42** (2001), 145–162.
[20] S. Konyagin, C. Mauduit, A. Sárközy, On the Number of Prime Factors of Integers Characterized by Digit Properties, Period. Math. Hungar. (1), **40** (2000), 37–52.
[21] D. E. Knuth, H. S. Wilf, The power of a prime that divides a generalized binomial coefficient, J. reine angew. Math., **396** (1989), 212–219.
[22] E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, J. reine angew. Math., **44** (1852), 93–146.
[23] J. Kürschák, A harmonikus sorról (On the harmonic series, in Hungarian), Mat. Fiz. Lapok, **27** (1918), 299–300.
[24] J. Maynard, Primes with restricted digits, Invent. Math., **217** (2019), 127–218.
[25] J. Maynard, Primes and Polynomials With Restricted Digits, Int. Math. Res. Not. IMRN, **14** (2022), 10626–10648.
[26] I. Mező, About the non-integer property of hyperharmonic numbers, Annales Univ. Sci. Budapest, Sect. Math., **50** (2007), 13–20.
[27] H. Montgomery, R. Vaughan (2006), Multiplicative Number Theory I: Classical Theory (Cambridge Studies in Advanced Mathematics), Cambridge: Cambridge University Press.
[28] I. M. Niven, H. S. Zuckerman, H. L. Montgomery, An Introduction to the Theory of Numbers, 5th ed. New York: Wiley, 1991.
[29] SageMath, the Sage Mathematics Software System (Version 9.5), The Sage Developers, 2022,
[30] A. Selberg, On the normal density of primes in small intervals, and the difference between consecutive primes, Arch. Math. Naturvid., **47** (1943), no. 6, 87–105.
[31] D. C. Sertbaş, Hyperharmonic integers exist, C. R. Math. Acad. Sci. Paris, **358** (2020), no. 11-12, 1179–1185.
[32] L. Theisinger, Bemerkung über die harmonische reihe, Monatsh. Math. Phys., **26** (1915), 132–134.
[33] G. Yu, The differences between consecutive primes, Bull. London Math. Soc., **28** (1996), 242–248.

Doğa Can Sertbaş
Department of Mathematics, Faculty of Engineering and Natural Sciences, Istinye University, 34396 Sariyer, Istanbul, TURKEY.
E-mail: dogacan.sertbas@gmail.com