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Hadamard variation of eigenvalues with respect to general domain perturbations

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Abstract. We study Hadamard variation of eigenvalues of Laplacian with respect to general domain perturbations. We show their existence up to the second order rigorously and characterize the derivatives, using associated eigenvalue problems in finite dimensional spaces. Then smooth rearrangement of multiple eigenvalues is explicitly given. This result follows from an abstract theory, applicable to general perturbations of symmetric bilinear forms.

1. Introduction

Our purpose is to study Hadamard variation of eigenvalues of the Laplace operator with mixed boundary conditions. We characterize the first and the second derivatives, using associated finite dimensional eigenvalue problems, particularly, for multiple eigenvalues.

Hadamard's variational formulae are used to provide effective numerical schemes for shape optimization and free boundary problems. In [12, 13] we have introduced a new variational formulation to filtration problem and applied the Hadamard variation. We have also derived Hadamard's variational formulae for Green's functions of the Poisson equation, extending Liouville's volume and area formulae up to the second order [14, 15]. This paper is devoted to the eigenvalue problem concerning general perturbation of Lipschitz domains. Meanwhile we develop abstract theory applicable to other problems.

So far, C^1 and analytic rearrangements of multiple eigenvalues of self-adjoint operator have been discussed, for example, pp. 44–52 of [10] and p.487 of [1]. There are, however, several comments on the complexity of the proof on C^1 category, as in pp. 122–123 of [5] and p. 490 of [1]. In this paper we examine this process of rearrangement in details, using the above described characterization of the derivatives, to ensure their efficiencies in C^2 categories.

Let Ω be a bounded Lipschitz domain in n-dimensional Euclidean space $\mathbb{R}^n$ for $n \geq 2$. Suppose that its boundary $\partial\Omega$ is divided into two relatively open sets γ_0 and γ_1, satisfying

$$\overline{\gamma_0} \cup \overline{\gamma_1} = \partial\Omega, \quad \overline{\gamma_0} \cap \overline{\gamma_1} = \varnothing. \tag{1}$$

As in [15], we thus avoid the technical difficulty described in p. 272 of [2], when γ_i, $i = 0, 1$ have their boundaries on $\partial\Omega$.

We study the eigenvalue problem of the Poisson equation with mixed boundary condition,

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \gamma_0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \gamma_1, \tag{2}$$

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where

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

is the Laplacian and ν denotes the outer unit normal vector on $\partial\Omega$. This problem takes the weak form, finding u satisfying

$$u \in V,\ B(u,u) = 1, \quad A(u,v) = \lambda B(u,v),\ \forall v \in V \tag{3}$$

defined for

$$A(u,v) = \int_\Omega \nabla u \cdot \nabla v \ dx, \quad B(u,v) = \int_\Omega uv \ dx$$

and

$$V = \{v \in H^1(\Omega) \mid v|_{\gamma_0} = 0\}. \tag{4}$$

This V is a closed subspace of $H^1(\Omega)$ under the norm

$$\|v\|_V = \sqrt{\|\nabla v\|_2^2 + \|v\|_2^2}.$$

To justify the above weak formulation, we confirm several fundamental facts on the Lipschitz domain [6]. First, the trace operator

$$v \in C^\infty(\overline{\Omega}) \mapsto v|_{\partial\Omega} \in C(\partial\Omega)$$

defined for

$$C^\infty(\overline{\Omega}) = \{v \in \overline{\Omega} \to \mathbb{R} \mid \exists \tilde{v} \in C_0^\infty(\mathbb{R}^n),\ \ \tilde{v}|_{\overline{\Omega}} = v\}$$

is so extended as a bounded linear operator

$$v \in H^1(\Omega) \ \mapsto \ v|_{\partial\Omega} \in H^{1/2}(\partial\Omega),$$

and there arises the isomorphism

$$v \in V/H_0^1(\Omega) \ \mapsto v|_{\gamma_1} \in H^{1/2}(\gamma_1).$$

Second, if $\Delta v \in V'$ is satisfied in the sense of distributions in Ω, the normal derivative of $v \in V$ on γ_1 is defined as in

$$\frac{\partial v}{\partial \nu} \in H^{-1/2}(\gamma_1) = (H^{1/2}(\gamma_1))',$$

and it holds that

$$\left\langle \varphi, \frac{\partial v}{\partial \nu} \right\rangle_{H^{1/2}(\gamma_1), H^{-1/2}(\gamma_1)} = (\nabla v, \nabla \varphi)_{L^2(\Omega)} + \langle \varphi, \Delta v \rangle_{V,V'}, \quad \forall \varphi \in V,$$

where $\langle \cdot, \cdot \rangle_{Y,Y'}$ denotes the paring between the Banach space Y and its dual space Y'.

See Theorem 2 of [15].

To confirm the well-posedness of (3), we note, first, that if $\gamma_0 \neq \emptyset$, there is coercivity of $A : V \times V \to \mathbb{R}$, which means the existence of $\delta > 0$ such that

$$A(v, v) \geq \delta \|v\|_V^2, \quad \forall v \in V. \tag{5}$$

If $\gamma_0 = \emptyset$ we replace A by $A + B$, denoted by $\tilde{A}$. Then this $\tilde{A} : V \times V \to \mathbb{R}$ is coercive, and the eigenvalue problem

$$u \in V, \ B(u, u) = 1, \quad \tilde{A}(u, v) = \tilde{\lambda} B(u, v), \ \forall v \in V,$$

is equivalent to (3) by $\tilde{\lambda} = \lambda + 1$. Henceforth, we assume (5), using this reduction if it is necessary.

Second, we note that $A : V \times V \to \mathbf{R}$ and $B : X \times X \to \mathbf{R}$ are bounded, coercive, and symmetric bilinear forms for $X = L^2(\Omega)$. Since $V \hookrightarrow X$ is compact, there is a sequence of eigenvalues to (3), denoted by

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty.$$

The associated eigenfunctions, $u_1, u_2, \cdots$, furthermore, form a complete ortho-normal system in X, provided with the inner product induced by $B = B(\cdot, \cdot)$:

$$B(u_i, u_j) = \delta_{ij}, \quad A(u_j, v) = \lambda_j B(u_j, v), \ \forall v \in V, \quad i, j = 1, 2, \cdots.$$

The j-th eigenvalue of (3) is given by the mini-max principle

$$\lambda_j = \min_{L_j} \max_{v \in L_j \setminus \{0\}} R[v] = \max_{W_j} \min_{v \in W_j \setminus \{0\}} R[v], \tag{6}$$

where

$$R[v] = \frac{A(v, v)}{B(v, v)}$$

is the Rayleigh quotient, and $\{L_j\}$ and $\{W_j\}$ denote the families of all subspaces of V with dimension and codimension j and $j-1$, respectively. See [11], for example, for these fundamental facts.

Let

$$T_t : \Omega \to \Omega_t = T_t(\Omega), \quad |t| < \varepsilon_0 \tag{7}$$

be a family of bi-Lipschitz homeomorphisms for $\varepsilon_0 > 0$, satisfying $T_0 = I$, the identity mapping. We assume that $T_t x$ is continuous in t uniformly in $x \in \Omega$, and recall the following definition used in [15].

Definition 1. *The family $\{T_t\}$ of bi-Lipschitz homeomorphisms is said to be p-differentiable in t for $p \geq 1$, if $T_t x$ is p-times differentiable in t for any $x \in \Omega$ and the mappings*

$$\frac{\partial^\ell}{\partial t^\ell} DT_t, \ \frac{\partial^\ell}{\partial t^\ell} (DT_t)^{-1} : \Omega \to M_n(\mathbb{R}), \quad 0 \leq \ell \leq p$$

are uniformly bounded in $(x,t) \in \Omega \times (-\varepsilon_0, \varepsilon_0)$, *where* DT_t *denotes the Jacobi matrix of* $T_t : \Omega \to \Omega_t$ *and* $M_n(\mathbb{R})$ *stands for the set of real* $n \times n$ *matrices. This* $\{T_t\}$ *is said to be continuously p-differentiable in t if it is p-differentiable and the mappings*

$$t \in (-\varepsilon_0, \varepsilon_0) \mapsto \frac{\partial^\ell}{\partial t^\ell} DT_t,\ \frac{\partial^\ell}{\partial t^\ell}(DT_t)^{-1} \in L^\infty(\Omega \to M_n(\mathbb{R})), \quad 0 \le \ell \le p$$

are continuous.

Putting

$$T_t(\gamma_i) = \gamma_{it}, \quad i = 0, 1, \tag{8}$$

we introduce the other eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \Omega_t, \quad u = 0 \text{ on } \gamma_{0t}, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \gamma_{1t}, \tag{9}$$

which is reduced to finding

$$u \in V_t,\ \int_{\Omega_t} u^2\ dx = 1, \quad \int_{\Omega_t} \nabla u \cdot \nabla v\ dx = \lambda \int_{\Omega_t} uv\ dx,\ \forall v \in V_t \tag{10}$$

for

$$V_t = \{v \in H^1(\Omega_t) \mid v|_{\gamma_{0t}} = 0\}. \tag{11}$$

Let $\lambda_j(t)$ be the j-th eigenvalue of the eigenvalue problem (9). In Section 4 we confirm that the eigenvalue problem (10)-(11) is reduced to

$$u \in V,\ B_t(u,u) = 1, \quad A_t(u,v) = \lambda B_t(u,v),\ \forall v \in V \tag{12}$$

by the transformation of variables $y = T_t x$ for $V \subset H^1(\Omega)$ defined by (4), where

$$B_t(u,v) = \int_\Omega uva_t\ dx, \quad A_t(u,v) = \int_\Omega Q_t[\nabla u, \nabla v] a_t\ dx, \tag{13}$$

and

$$a_t = \det DT_t, \quad Q_t = (DT_t)^{-1}(DT_t)^{-1T}. \tag{14}$$

Several representation formulae for

$$\lambda_j'(t) = \lim_{h \to 0} \frac{1}{h}(\lambda_j(t+h) - \lambda_j(t))$$

and

$$\lambda_j''(t) = \lim_{h \to 0} \frac{1}{h}(\lambda_j'(t+h) - \lambda_j'(t))$$

have been derived in [3]. Here we characterize these derivatives, using associated finite dimensional eigenvalue problems (Theorems 1 and 4). We prove also the existence of derivatives, particularly, the second derivatives of multiple eigenvalues. These results follow from the differentiability of A_t and B_t in t, defined by

$$\dot{A}_t(u,v) = \frac{d}{dt}A_t(u,v),\ \dot{B}_t(u,v) = \frac{d}{dt}B_t(u,v), \quad u,v \in V \tag{15}$$

and

$$\ddot{A}_t(u,v) = \frac{d^2}{dt^2}A_t(u,v),\ \ddot{B}_t(u,v) = \frac{d^2}{dt^2}B_t(u,v), \quad u,v \in V. \tag{16}$$

Then we show C^1 and C^2 rearrangements of eigenvalues, if these bilinear forms are continuous in t (Theorems 3 and 6). We give the algorithm explicitly, as the *transversal rearrangement* in Definition 3. Consequently, no rearrangment is necessary to confirm C^1 or C^2 smoothness of eigenvalues, if their multiplicities are constant. Also elementary symmetric functions made by possible multiple eigenvalues are C^1 or C^2. These properties are noticed by [7] and [8] in the real analytic category.

In [4], unilateral derivatives of the first order of the eigenvalue of the Stokes operator are calculated. It characterizes the derivatives using boundary integrals when the deformation of the domain is of normal direction. Our abstract theory reproduces and extends this result by the use of Piola transformation described in [9].

Our argument is executed in the H^1 category without requiring any further elliptic regularities. Hence the Lipschitz continuity of $\partial\Omega$ is sufficient to ensure all the results on (9) under the general perturbation of domains. We recall, in this context, that numerical computations on partial differential equations are mostly executed on Lipschitz domains. There, perturbation of the domain using Lipschitz continuous vector fields is often applied. This is the method of trial domains, efficient even to the case that *normal perturabation* of domains, called in [12], does not work because of the presence of corners on the boundary [12, 13].

2. Summary

As is noted in the previous section, C^1 and analytic categories for the smoothness of $\lambda_j(t)$ in t have been discussed. Since each eigenvalue λ of (12) is isolated, we can reduce this problem to a finite dimensional eigenvalue problem by the Lyapunov-Schmidt reduction as in p.486 of [1]. Rellich [10] in p.45 showed in this case that the continuous differentiablity in t of A_t and B_t implies that of $\lambda_j(t)$ under a suitable change of its order. Kato [5] in p.123 then provided an alternative proof of this C^1 rearrangement.

The other category of analyticity is studied in Chapter II of [10] and p.370 of [5]. If A_t and B_t are analytic in t, the eigenvlue problem is reduced to an algebraic equation with analytic coefficients in t. The Puiseux expansion of $\lambda_j(t)$ at $t = 0$ is described in p.31 of [10] and Chapter 2, Section 1 of [5]. Hence this $\lambda_j(t)$ is realized as an analytic function defined on a Riemann surface.

We study C^2 category. To begin with, we confirm Rellich's theorem on C^1 category, the continuous differentiablity of rearranged eigenvalues. Here we show this rearrangement explicitly, to reach existence, characterization, and continuity of the second derivatives (Definition 3). In more details, first, the existence of $\dot{A}_t$ and $\dot{B}_t$ in (15) implies that of the first unilateral derivatives

$$\dot{\lambda}_j^{\pm}(t) = \lim_{h\to\pm 0} \frac{1}{h}(\lambda_j(t+h) - \lambda_j(t)) \tag{17}$$

for each j and t. These derivatives, furthermore, are characterized by the other finite dimensional eigenvalue problems in accordance with the multiplicity of $\lambda_j(t)$ (Theorem 12). Second, if the above $\dot{A}_t$ and $\dot{B}_t$ are continuous in t, and if

$$\lambda_{k-1}(t) < \lambda_k(t) \le \cdots \le \lambda_{k+m-1}(t) < \lambda_{k+m}(t) \tag{18}$$

holds for $t \in I = (-\varepsilon_0, \varepsilon_0)$, there are C^1 curves

$$\tilde{C}_j, \ k \le j \le k+m-1,$$

made by at most countably many rearrangements of the C^0 curves

$$C_j = \{\lambda_j(t) \mid t \in I\}, \quad k \le j \le k+m-1.$$

(Theorem 14).

In this paper, we notice two properties of the unilateral derivatives $\dot{\lambda}_j^{\pm}$, for the proof of this Rellich's theorem. First, there arises that

$$\dot{\lambda}_j^{+}(t) = \dot{\lambda}_{2\ell+n-1-j}^{-}(t), \quad \ell \le j \le \ell+n-1$$

if

$$\lambda_{\ell-1}(t) < \lambda_\ell(t) = \cdots = \lambda_{\ell+n-1}(t) < \lambda_{\ell+n}(t) \tag{19}$$

holds for $\ell \ge k$, $\ell+n \le m$ (Theorem 12). Second, the unilateral derivative $\dot{\lambda}_j^{\pm}$ are provided with the unilateral continuity as in

$$\lim_{h\to\pm 0} \dot{\lambda}_j^{\pm}(t+h) = \dot{\lambda}_j^{\pm}(t)$$

if both $\dot{A}_t$ and $\dot{B}_t$ are continuous in $t \in I$ (Theorem 13).

As for the second derivatives of eigenvalues, we assume the existence of $\ddot{A}_t$ and $\ddot{B}_t$ in (16) besides $\dot{A}_t$ and $\dot{B}_t$. Then each $j = 1, 2, \cdots$ admits the existence of

$$\ddot{\lambda}_j^{\pm}(t) = \lim_{h\to\pm 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda_j(t) - h\dot{\lambda}_j^{\pm}(t)). \tag{20}$$

These limits are again unilateral and characterized by the other eigenvalue problem in a finite dimensional space (Theorem 15). The unilateral continuity of these $\ddot{\lambda}_j^{\pm}(t)$ is then assured under the continuity of $\ddot{A}_t$ and $\ddot{B}_t$ in t, similarly. These properties induce C^2 smoothness of $\tilde{C}_j$, $k \le j \le k+m-1$, once their C^1 smoothness is achieved (Theorem 24).

This paper is composed of eight sections. The results on the Hadamard variation to (9) are described in the next section. In Section 4 we use the transformation of variables to introduce an abstract setting of the problem. Section 5 is concerned on the continuity of eigenvalues and eigenspaces. We study the first derivative of eigenvalues in Section 6, and then its C^1 rearrangement in Section 7. Finally, Section 8 is devoted to the second derivatives.

3. Hadamard variation

Here we state the results on the first and the second derivatives of $\lambda_j = \lambda_j(t)$ in (12). Fix $t \in I$, and assume (19) for $\ell = k$ and $n = m$ for $k, m = 1, 2, \cdots$ with the convention $\lambda_0(t) = -\infty$. Put

$$\lambda \equiv \lambda_k(t) = \cdots = \lambda_{k+m-1}(t), \tag{21}$$

and let Y_t^λ, $\dim Y_t^\lambda = m$, be the eigenspace corresponding to this eigenvalue λ.

THEOREM 1. *Assume the above situation, and let $\{T_{t'}\}$ be 1-differentiable at $t' = t$. Then, there exist the unilateral limits*

$$\dot{\lambda}_j^{\pm} = \lim_{h \to \pm 0} \frac{1}{h}(\lambda_j(t+h) - \lambda), \quad k \le j \le k+m-1,$$

which satisfies

$$\nu_j \equiv \dot{\lambda}_j^{+} = \dot{\lambda}_{2k+m-1-j}^{-}, \quad k \le j \le k+m-1. \tag{22}$$

This ν_j is the q-th eigenvalue of the matrix

$$G_t^\lambda = \left(E_t^\lambda(\tilde{\phi}_i, \tilde{\phi}_j) \right)_{1 \le i,j \le m} \tag{23}$$

for $q = j - k + 1$, where $\{\tilde{\phi}_j \mid 1 \le j \le m\}$ is a basis of Y_t^λ, satisfying

$$B_t(\tilde{\phi}_i, \tilde{\phi}_j) = \delta_{ij}, \quad 1 \le i, j \le m \tag{24}$$

and

$$E_t^\lambda = \dot{A}_t - \lambda \dot{B}_t$$

for $\dot{A}_t$ and $\dot{B}_t$ defined by (13), (14), and (15).

REMARK 1. *If $\langle \phi_j \mid 1 \le j \le m \rangle$ is the other basis of Y_t^λ satisfying*

$$B_t(\phi_i, \phi_j) = \delta_{ij}, \quad 1 \le i, j \le m, \tag{25}$$

it holds that

$$\phi_j = \sum_{i=1}^m q_j^i \tilde{\phi}_i, \ 1 \le j \le m$$

with the orthogonal $m \times m$ matrix $Q = (q_j^i)$. Hence ν_j, $k \le j \le k+m-1$, in (22) is determined, indpendent of the choice of $\langle \tilde{\phi}_j \mid 1 \le j \le m \rangle$.

Remark 2. *Under (21), it holds that*

$$\dot\lambda_k^+(t) \le \cdots \le \dot\lambda_{k+m-1}^+(t), \quad \dot\lambda_k^-(t) \ge \cdots \ge \dot\lambda_{k+m-1}^-(t).$$

A direct consequnece of this theorem is the existence of the unilateral derivatives $\dot\lambda_j^\pm(t)$ in (17) for any $t \in I$ and $j = 1, 2, \cdots$, if $\{T_t\}$ is 1-differentiable in I. Then we obtain the following theorem.

Theorem 2. *If $\{T_t\}$ is continuously 1-differentiable in $t \in I$, the functions $\dot\lambda_j^\pm = \dot\lambda_j^\pm(t)$ are unilaterally continuous, so that it holds that*

$$\lim_{h\to\pm 0} \dot\lambda_j^\pm(t+h) = \dot\lambda_j^\pm(t)$$

for any t and j.

This fact ensures the following theorem of Rellich.

Theorem 3. *Let $\{T_t\}$ be continuously 1-differentiable in t, and assume*

$$\lambda_{k-1}(t) < \lambda_k(t) \le \cdots \le \lambda_{k+m-1}(t) < \lambda_{k+m}(t), \quad \forall t \in I$$

for some $k, m = 1, 2, \cdots$. Let

$$C_j = \{\lambda_j(t) \mid t \in I\}, \quad k \le j \le k+m-1$$

be C^0 curves. Then, there exist C^1 curves denoted by $\tilde C_i$, $k \le i \le k+m-1$, made by a rearrangement of

$$\{C_j \mid k \le j \le k+m-1\}$$

at most countably many times.

Turning to the second derivatives, we fix $t \in I$ again, and assume that $\{T_{t'}\}$ is twice differentiable at $t' = t$. Suppose (19) for $\ell = k$ and $n = m$, put λ as in (21), and let $k \le \ell < r \le k+m$ be such that

$$\dot\lambda_{\ell-1}^+ < \lambda' \equiv \dot\lambda_\ell^+ = \cdots = \dot\lambda_{r-1}^+ < \dot\lambda_r^+ \tag{26}$$

in Theorem 1. To state the finite dimensonal eigenvalue problem characterizing the second derivatives of $\lambda_j(t')$ at $t' = t$ for $\ell \le j \le r-1$, we introduce the following definition.

Definition 2. *Let $R : X = L^2(\Omega) \to Y_t^\lambda$ be the orthogonal projection with respect to $B_t(\cdot,\cdot)$ and $P = I - R$, where $I : X \to X$ is the identity operator. Let, furthermore, A_t, B_t, $\dot A_t$, $\dot B_t$, $\ddot A_t$, and $\ddot B_t$ be as in (13), (14), (15), and (16). Then we define $w = \gamma(u) \in PV$ for $u \in V$ by*

$$C_t(w, v) = -\dot C_t^{\lambda,\lambda'}(u, v), \quad \forall v \in PV, \tag{27}$$

where

$$C_t = A_t - \lambda B_t, \quad \dot{C}_t^{\lambda,\lambda'} = \dot{A}_t - \lambda \dot{B}_t - \lambda' B_t.$$

We put also

$$F_t^{\lambda,\lambda'}(u,v) = (\ddot{A}_t - \lambda \ddot{B}_t - 2\lambda' \dot{B}_t)(u,v) - 2C_t(\gamma(u),\gamma(v)), \quad u,v \in V.$$

Remark 3. *To confirm the unique solvability of $w = \gamma(u)$, let $Q : L^2(\Omega) \to Z_t^k$ be the orthogonal projection with respect to $B_t(\cdot,\cdot)$, where Z_t^k denotes the finite dimensional space of $L^2(\Omega)$ generated by the eigenfunctions corresponding to the eigenvalues $\lambda_1(t), \cdots, \lambda_{k-1}(t)$. Let, furthermore, $V_0 = QV$ and $V_1 = (I-Q)RV$. First, there is a unique $w_0 \in V_0$ satisfying*

$$C_t(w_0, v) = -\dot{C}_t^{\lambda,\lambda'}(u,v), \quad \forall v \in V_0 \tag{28}$$

because C_t is negative definite on $V_0 \times V_0$. Second, there is also a unique $w_1 \in V_1$ satisfying

$$C_t(w_1, v) = -\dot{C}_t^{\lambda,\lambda'}(u,v), \quad \forall v \in V_1 \tag{29}$$

because C_t is positive definite on $V_1 \times V_1$. Then we obtain (27) for $w = w_0 + w_1$, because

$$C_t(w_0,v) = C_t(w_0,Qv), \; C_t(w_1,v) = C_t(w_1,(I-Q)v), \quad \forall v \in RV,$$

and hence

$$\begin{aligned} C_t(w,v) &= C_t(w_0,v) + C_t(w_1,v) = C_t(w_0,Qv) + C_t(w_1,(I-Q)v) \\ &= -\dot{C}_t^{\lambda,\lambda'}(u,Qv) - \dot{C}_t^{\lambda,\lambda'}(u,(I-Q)v) = -\dot{C}_t(u,v), \quad \forall v \in RV. \end{aligned}$$

We thus obtain $w \in RV$ satisfying (27). If $w \in RV$ is a solution to (27), conversely, then $w_0 = Qw \in V_0$ and $w_1 = (I-Q)w \in V_1$ solve (28) and (29), respectively. Then there arises the uniquess of such $w = w_0 + w_1$ because these $w_0 \in V_0$ and $w_1 \in V_1$ are unique.

Recall

$$Y_t^\lambda = \langle \tilde{\phi}_j \mid k \le j \le k+m-1 \rangle$$

with (24).

Theorem 4. *Under the above situation of (19) and (26), there exist*

$$\lambda_j'' = \lim_{h \to +0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda - h\lambda'), \quad \ell \le j \le r-1. \tag{30}$$

This λ_j'' is the q-th eigenvalue of the matrix

$$H_t^{\lambda,\lambda'} = \left(F_t^{\lambda,\lambda'}(\tilde{\phi}_i, \tilde{\phi}_j) \right)_{\ell \le i,j \le r-1} \tag{31}$$

for $q = j - \ell + 1$. *If (26) is replaced by*

$$\dot{\lambda}^-_{\ell-1}(t) > \lambda' \equiv \dot{\lambda}^-_\ell(t) = \cdots = \dot{\lambda}^-_{r-1}(t) > \dot{\lambda}^-_r(t),$$

there arises that

$$\lambda''_j(t) = \lim_{h\to -0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda - h\lambda'), \quad \ell \leq j \leq r-1.$$

As in Remark 1 on the first derivative, the above λ''_j, $\ell \leq j \leq r-1$, is determined independent of the choice of

$$\langle \tilde{\phi}_j \mid \ell \leq j \leq r-1 \rangle.$$

Theorems 1, 4 imply also the existence of the unilateral limits $\ddot{\lambda}^\pm_j(t)$ in (20) for any t and j if $\{T_t\}$ is 2-differentiable in $t \in I$, that is,

$$\ddot{\lambda}^\pm_j(t) = \lim_{h\to\pm 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda_j(t) - h\dot{\lambda}^\pm_j(t)).$$

Remark 4. *By Liouville's theorem on general perturbation of domains studied in* [15]*, the matrix* G^λ_t *in (23) is represented by the surface integrals of* $\tilde{\phi}_j$*,* $1 \leq j \leq m$*. This property is confirmed by* [4] *for the Stokes operator with Dirichlet condition under a special perturbation of domains, called the normal perturbation in* [14]*. Similarly, the matrix* $H^{\lambda,\lambda'}_t$ *in (31) is represented by the surface integrals of* $\tilde{\phi}_j$ *and* $\gamma(\tilde{\phi}_j)$*,* $1 \leq j \leq m$.

Then we obtain the following theorems.

Theorem 5. *If* $\{T_t\}$ *is continuously* 2*-differentiable in* t*, then* $\ddot{\lambda}^\pm_j = \ddot{\lambda}^\pm_j(t)$ *are unilaterally continuous, so that it holds that*

$$\lim_{h\to\pm 0} \ddot{\lambda}^\pm_j(t+h) = \ddot{\lambda}^\pm_j(t)$$

for any $t \in I$ *and* $j = 1, 2, \cdots$.

Theorem 6. *If* $\{T_t\}$ *is continuously* 2*-differentiable in* t*, the* C^1 *curves* $\tilde{C}_j$*,* $k \leq j \leq k+m-1$ *in Theorem 3 is* C^2.

Although the above theorems are to be extended to $p \geq 3$ of Definition 1, we restrict ourselves to $p = 1, 2$ in this paper. Yet these results on $p = 2$ are efficient to examine the harmonic concavity of $\lambda_j(t)$ in t studied by [3] for the first eigenvalue to (2) with $\gamma_1 = \emptyset$. We emphasize also that the treatment of the second derivatives is rather different from that of the first ones. See Remark 10 in Section 8.

4. Reduction to the abstract theory

For the moment, we fix t and treat the bi-Lipschitz homeomorphism $T = T_t : \Omega \to \Omega_t = T_t\Omega$. Let

$$\tilde{\Omega} = \Omega_t, \quad f(y) = g(x), \; y = Tx,$$

and confirm the chain rule for this transformation of variables, that is,

$$\nabla g = \nabla f \ DT, \quad dy = (\det DT)dx$$

for $\nabla g = (\frac{\partial g}{\partial x_1}, \cdots, \frac{\partial g}{\partial x_n})$, $\nabla f = (\frac{\partial f}{\partial y_1}, \cdots, \frac{\partial f}{\partial y_n})$, and

$$DT = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} \cdots \frac{\partial y_1}{\partial x_n} \\ \cdot \qquad \cdot \\ \frac{\partial y_n}{\partial x_1} \cdots \frac{\partial y_n}{\partial x_n} \end{pmatrix}.$$

Putting $\tilde{\gamma}_i = \gamma_{it} = T\gamma_i$ for $i = 0, 1$, we take the eigenvalue problem

$$-\Delta u = \lambda u \text{ in } \tilde{\Omega}, \quad u = 0 \text{ on } \tilde{\gamma}_0, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \tilde{\gamma}_1, \tag{32}$$

that is, (9) for $T = T_t$. Let

$$\tilde{V} = \{v \in H^1(\tilde{\Omega}) \mid v|_{\tilde{\gamma}_0} = 0\},$$

and introduce the weak form of (32),

$$u \in \tilde{V}, \ \tilde{A}(u, v) = \lambda \tilde{B}(u, v), \quad \forall v \in \tilde{V}, \tag{33}$$

where

$$\tilde{A}(u, v) = \int_{\tilde{\Omega}} \nabla_y u \cdot \nabla_y v \ dy, \quad \tilde{B}(u, v) = \int_{\tilde{\Omega}} uv \ dy. \tag{34}$$

Given $\phi \in V$, put

$$\psi(y) = \phi(x), \quad y = Tx.$$

Then it holds that

$$\phi \in V \ \Leftrightarrow \ \psi \in \tilde{V}$$

for $V \subset H^1(\Omega)$ defined by (4). Writing

$$U(x) = u(y), \ V(x) = v(y), \quad y = Tx, \tag{35}$$

we obtain

$$\begin{aligned} \nabla_y u \cdot \nabla_y v &= [\nabla_x U (DT)^{-1}] \cdot [\nabla_x V (DT)^{-1}] \\ &= (\nabla_x U)(DT)^{-1}(DT)^{-1T}(\nabla_x V)^T \\ &= (\nabla_x U) Q (\nabla_x V)^T = Q[\nabla_x U, \nabla_x V] \end{aligned}$$

for $Q = (DT)^{-1}(DT)^{-1T}$, where F^T denotes the transpose of the matrix F. Then, (33) means

$$\int_\Omega Q[\nabla_x U, \nabla_x V] a \ dx = \lambda \int_\Omega UVa \ dx$$

for $a = \det DT$. The condition of normalization

$$\int_{\tilde{\Omega}} u^2 \; dy = 1$$

is also transformed into

$$\int_{\Omega} U^2 a \; dx = 1.$$

Under the family $\{T_t\}$ of homeomorphisms, therefore, the weak form (10) of (9), is equivalent to (12) for $V \subset H^1(\Omega)$ defined by (4) and B_t, A_t defined by (13)-(14). Here we confirm the following lemma.

Lemma 7. *The j-th eigenvalue of (9) is equal to that of (12).*

Proof. For the moment, let $\tilde{\lambda}_j(t)$ be the j-th eigenvalues of (9) and let $\lambda_j(t)$ be that of (12) for (13) and (14). By the mini-max principle (6), it holds that

$$\tilde{\lambda}_j(t) = \min_{\tilde{L}_j} \max_{v \in \tilde{L}_j \setminus \{0\}} \tilde{R}_t[v] = \max_{\tilde{W}_j} \min_{v \in \tilde{W}_j \setminus \{0\}} \tilde{R}_t[v], \tag{36}$$

where $V_t \subset H^1(\Omega_t)$ is defined by (11),

$$\tilde{R}_t[v] = \frac{\tilde{A}_t(v,v)}{\tilde{B}_t(v,v)}, \quad \tilde{A}_t(u,v) = \int_{\Omega_t} \nabla u \cdot \nabla v \; dx, \quad \tilde{B}_t(u,v) = \int_{\Omega_t} uv \; dx,$$

and $\{\tilde{L}_j\}$ and $\{\tilde{W}_j\}$ be the families of all subspaces of V_t with dimension and codimension j and $j-1$, respectively.

It holds also that

$$\lambda_j(t) = \min_{L_j} \max_{v \in L_j \setminus \{0\}} R_t[v] = \max_{W_j} \min_{v \in W_j \setminus \{0\}} R_t[v], \tag{37}$$

for

$$R_t[v] = \frac{A_t(v,v)}{B_t(v,v)}, \tag{38}$$

and $\{L_j\}$ and $\{W_j\}$ denote the families of all subspaces of V with dimension and codimension j and $j-1$, respectively.

If the set L is a j-dimensional subspace of V there is $\phi_\ell \in L$, $1 \le \ell \le j$, such that

$$\int_{\Omega} \phi_\ell \phi_{\ell'} dx = \delta_{\ell\ell'}$$

and

$$\phi = \sum_{\ell=1}^{j} c_\ell \phi_\ell, \quad c_\ell = \int_{\Omega} \phi \phi_\ell \; dx$$

for any $\phi \in L$, which implies

$$\psi = \sum_{\ell=1}^{j} c_\ell \psi_\ell$$

for $\psi = \phi \circ T_t^{-1}$ and $\psi_\ell = \phi \circ T_t^{-1}$. Hence we obtain dim $\tilde{L}_t \leq$ dim L for

$$\tilde{L}_t = \{\phi \circ T_t^{-1} \mid \phi \in L\}.$$

The reverse inequality follows similarly, and hence it holds that

$$\dim \tilde{L}_t = \dim L = j.$$

Since $T_t : \Omega \to \Omega_t$ is a bi-Lipschitz homeomorphism, furthermore, $L \subset V$ if and only if $\tilde{L}_t \subset V_t$. We thus obtain

$$\tilde{\lambda}_j(t) = \lambda_j(t) \tag{39}$$

by (36)-(37). □

We are ready to develop an abstract theory, writing L^2 norm in $X = L^2(\Omega)$ as $|\cdot|_X$. With V in (4), we recall that $\|\cdot\|_V$ denotes the norm in V and that the inclusion $V \hookrightarrow X$ is compact. It holds also that

$$|v|_X \leq K\|v\|_V, \quad v \in V \tag{40}$$

for $K = 1$.

Henceforth, C denotes the generic positive constant. The above $A_t : V \times V \to \mathbb{R}$ and $B_t : X \times X \to \mathbb{R}$ for $t \in I$ are symmetric bilinear forms, satisfying

$$|A_t(u,v)| \leq C\|u\|_V\|v\|_V, \ A_t(u,u) \geq \delta\|u\|_V^2, \quad u,v \in V \tag{41}$$

and

$$|B_t(u,v)| \leq C|u|_X|v|_X, \ B_t(u,u) \geq \delta|u|_X^2, \quad u,v \in X \tag{42}$$

for some $\delta > 0$. Then the eigenvalues of (12) are denoted by

$$0 < \lambda_1(t) \leq \lambda_2(t) \leq \cdots \to +\infty.$$

The weak and the strong convergences of $\{u_j\} \subset Y$ to $u \in Y$ for $Y = X$ or $Y = V$ are, furthermore, indicated by

$$\mathrm{w} - \lim_{j\to\infty} u_j = u \text{ in } Y$$

and

$$\mathrm{s} - \lim_{j\to\infty} u_j = u \text{ in } Y,$$

respectively.

5. Continuity of eigenvalues and eigenspaces

Let $t \in I$ be fixed. We begin with the following theorem valid under the abstract setting in the previous section.

Theorem 8. *The conditions*

$$\begin{aligned} &\lim_{h\to 0} \sup_{\|u\|_V, \|v\|_V \leq 1} |A_{t+h}(u,v) - A_t(u,v)| = 0 \\ &\lim_{h\to 0} \sup_{|u|_X, |v|_X \leq 1} |B_{t+h}(u,v) - B_t(u,v)| = 0, \end{aligned} \tag{43}$$

imply

$$\lim_{h\to 0} \lambda_j(t+h) = \lambda_j(t) \tag{44}$$

for any $j = 1, 2, \cdots$.

Proof. We note that the j-th eigenvalue of (9) is given by the mini-max principle as in (37), for the Rayleigh quotient $R_t[v]$ defined by (38).

Given t, let

$$\begin{aligned} \alpha(h) &= \sup_{\|u\|_V, \|v\|_V \leq 1} |(A_{t+h} - A_t)(u,v)| \\ \beta(h) &= \sup_{|u|_X, |v|_X \leq 1} |(B_{t+h} - B_t)(u,v)|. \end{aligned} \tag{45}$$

We obtain

$$(A_{t+h} - A_t)(v,v) \geq -\alpha(h)\|v\|_V^2 \geq -\frac{\alpha(h)}{\delta} A_t(v,v)$$

and

$$(A_{t+h} - A_t)(v,v) \leq \alpha(h)\|v\|_V^2 \leq \frac{\alpha(h)}{\delta} A_t(v,v)$$

by (41) and (43), which implies

$$(1 - \delta^{-1}\alpha(h))A_t(v,v) \leq A_{t+h}(v,v) \leq (1 + \delta^{-1}\alpha(h))A_t(v,v).$$

Similarly, there arises that

$$(1 - \delta^{-1}\beta(h))B_t(v,v) \leq B_{t+h}(v,v) \leq (1 + \delta^{-1}\beta(h))B_t(v,v)$$

for any $v \in V$.

Then it follows that

$$(1 - o(1))R_t[v] \leq R_{t+h}[v] \leq (1 + o(1))R_t[v]$$

uniformly in $v \in V \setminus \{0\}$, and hence

$$(1 - o(1))\lambda_j(t) \leq \lambda_j(t+h) \leq (1 + o(1))\lambda_j(t)$$

by (43). Thus we obtain (44). □

Let $u_j(t) \in V$ be the eigenfunction of (12) corresponding to the eigenvalue $\lambda_j(t)$:

$$B_t(u_j(t), u_{j'}(t)) = \delta_{jj'}, \quad A_t(u_j(t), v) = \lambda_j(t) B_t(u_j(t), v), \ \forall v \in V. \tag{46}$$

Fix t, assume (19), and define λ by (21). Although this multiplicity m is not stable under the perturbation of t, we obtain the following theorem concerning the continuity of eigenspaces with respect to t.

Let

$$Y_t^\lambda = \langle u_j(t) \mid k \leq j \leq k+m-1 \rangle \tag{47}$$

be the subspace of X generated by the above eigenfunctions $u_j(t)$ for $k \leq j \leq k+m-1$.

LEMMA 9. *Under the above situation, any $h_\ell \to 0$ admits a subsequence, denoted by the same symbol, such that the limits*

$$s - \lim_{\ell \to \infty} u_j(t+h_\ell) = \phi_j \in Y_t^\lambda \ \text{in } V, \quad k \leq j \leq k+m-1 \tag{48}$$

exists. In particular it holds that

$$B_t(\phi_j, \phi_{j'}) = \delta_{jj'}, \quad k \leq j, j' \leq k+m-1. \tag{49}$$

PROOF. We have

$$\lim_{h \to 0} \lambda_j(t+h) = \lambda_j(t), \quad k \leq j \leq k+m-1 \tag{50}$$

by Theorem 8. It holds also that

$$\|u_j(t')\|_V \leq C, \quad k \leq j \leq k+m-1, \ |t'| < \varepsilon_0 \tag{51}$$

by (40)-(42) and (46):

$$\delta \|u_j(t')\|_V^2 \leq A_{t'}(u_j(t'), u_j(t')) = \lambda_j(t') B_t(u_j(t'), u_j(t')) = \lambda_j(t').$$

Given $h_\ell \to 0$, therefore, we have a subsequence, denoted by the same symbol, which admits the weak limits

$$\mathrm{w} - \lim_{\ell \to \infty} u_j(t+h_\ell) = \phi_j \ \text{in } V, \quad k \leq j \leq k+m-1, \tag{52}$$

for some $\phi_j \in V$. From (46) it follows that

$$A_t(\phi_j, v) = \lambda_j(t) B_t(\phi_j, v), \quad \forall v \in V$$

and hence

$$\phi_j \in Y_t^\lambda, \quad k \leq j \leq k+m-1.$$

Since $V \hookrightarrow X$ is compact, the weak convergence (52) implies the strong convergence

$$\text{s} - \lim_{\ell\to\infty} u_j(t+h_\ell) = \phi_j \text{ in } X, \tag{53}$$

and hence (49). Now we improve this weak convergence (52) to the strong convergence (48) in V, using (43).

For this purpose, we put

$$v = u_j(t+h_\ell) - \phi_j$$

and recall (21) and (50). Then there arises that

$$\begin{aligned}
\delta\|v\|_V^2 &\le A_{t+h_\ell}(u_j(t+h_\ell) - \phi_j, v) \\
&= A_{t+h_\ell}(u_j(t+h_\ell), v) - (A_{t+h_\ell} - A_t)(\phi_j, v) - A_t(\phi_j, v) \\
&= \lambda_j(t+h_\ell)B_{t+h_\ell}(u_j(t+h_\ell), v) - (A_{t+h_\ell} - A_t)(\phi_j, v) - \lambda_j(t)B_t(\phi_j, v) \\
&= (\lambda_j(t+h_\ell) - \lambda_j(t))B_{t+h_\ell}(u_j(t+h_\ell), v) \\
&\quad + \lambda_j(t)(B_{t+h_\ell} - B_t)(u_j(t+h_\ell), v) + \lambda_j(t)B_t(u_j(t+h_\ell) - \phi_j, v) \\
&\quad - (A_{t+h_\ell} - A_t)(\phi_j, v) \\
&\le C|\lambda_j(t+h_\ell) - \lambda_j(t)| \cdot |u_j(t+h_\ell)|_X \cdot K\|v\|_V \\
&\quad + \lambda_j(t) \cdot \beta(h_\ell) \cdot |u_j(t+h_\ell)|_X \cdot K\|v\|_V \\
&\quad + \lambda_j(t) \cdot C|u_j(t+h_\ell) - \phi_j|_X \cdot K\|v\|_V + \alpha(h_\ell)\|\phi_j\|_V \cdot \|v\|_V,
\end{aligned}$$

and hence

$$\begin{aligned}
\|v\|_V &\le C\{|\lambda_j(t+h_\ell) - \lambda_j(t)| + \beta(h_\ell) + |u_j(t+h_\ell) - \phi_j|_X + \alpha(h_\ell)\} \\
&= o(1)
\end{aligned}$$

by (43) and (53). □

Remark 5. *If $m = 1$, the eigenfunction $u_j(t)$ in (58) is uniquely determined by (46) up to the multiplication of ± 1, which implies $\phi_j = \pm u_j(t)$. In the other case of $m \ge 2$, the eigenfunction which attains (39) does not satisfy this property. Hence we have*

$$\phi_j = \sum_{i=1}^m q_j^i u_i(t)$$

for $Q = (q_j^i)$ satisfying $Q^T = Q^{-1}$ in Theorem 9. In other words, the eigenfunction $u_j(t)$ corresponding to $\lambda_j(t)$ has more varieties than ± 1 multiplication, although the eigenspace Y_t^λ is determined. By this ambiguity the limit ϕ_j in (48) depends on the sequence $h_\ell \to 0$, which makes the argument below to be complicated.

6. First derivatives

If $\{T_t\}$ is 1-differentiable in the setting of Section 1, we can put

$$\dot{A}_t(u,v) = \int_\Omega \dot{Q}_t[\nabla u, \nabla v]a_t + Q_t[\nabla u, \nabla v]\dot{a}_t \ dx, \quad u,v \in V$$

and

$$\dot{B}_t(u,v) = \int_\Omega uv\dot{a}_t \ dx, \quad u,v \in X.$$

These $\dot{A}_t : V \times V \to \mathbb{R}$ and $\dot{B}_t : X \times X \to \mathbb{R}$ in (13) are bilinear forms satisfying

$$\begin{aligned} &|\dot{A}_t(u,v)| \leq C\|u\|_V\|v\|_V, \quad u,v \in V \\ &|\dot{B}_t(u,v)| \leq C|u|_X|v|_X, \quad u,v \in X \end{aligned} \tag{54}$$

and

$$\begin{aligned} &\lim_{h\to 0}\frac{1}{h}\sup_{\|u\|_V,\|v\|_V\leq 1}\left|\left(A_{t+h}-A_t-h\dot{A}_t\right)(u,v)\right| = 0 \\ &\lim_{h\to 0}\frac{1}{h}\sup_{|u|_X,|v|_X\leq 1}\left|\left(B_{t+h}-B_t-h\dot{B}_t\right)(u,v)\right| = 0. \end{aligned} \tag{55}$$

Hence Theorem 1 in Section 4 is reduced to the following abstract theorem. In this theorem, the assumption made by Theorem 8 is valid, and therefore, there arises that (44).

Theorem 10. *Let X, V be Hilbert spaces over $\mathbb{R}$, with compact embedding $V \hookrightarrow X$. Let $A_t : V \times V \to \mathbb{R}$ and $B_t : X \times X \to \mathbb{R}$ be symmetric bilinear forms satisfying (41)-(42) for any $t \in I$. Given t, assume the existence of the bilinear forms $\dot{A}_t : V \times V \to \mathbb{R}$ and $\dot{B}_t : X \times X \to \mathbb{R}$ such that (54)-(55). Assume, finally, (19) with $\ell = k$ and $k = m$ for $\lambda_j(t)$, $k \leq k+m-1$, defined by (37)-(38). Then the conclusion of Theorem 1 holds.*

Before proceeding to the rigorous proof, we develop a formal argument, writing (12) as

$$u_t \in V, \ B_t(u_t,u_t) = 1, \quad A_t(u_t,v) = \lambda_t B_t(u_t,v), \ v \in V. \tag{56}$$

First, taking a formal differentiation in t in this equality, we obtain

$$\dot{A}_t(u_t,v) + A_t(\dot{u}_t,v) = \dot{\lambda}_t B_t(u_t,v) + \lambda_t\dot{B}_t(u_t,v) + \lambda_t B_t(\dot{u}_t,v), \quad \forall v \in V. \tag{57}$$

Putting $v = u_t$, we obtain

$$\dot{A}_t(u_t,u_t) + A_t(\dot{u}_t,u_t) = \dot{\lambda}_t + \lambda_t\dot{B}_t(u_t,u_t) + \lambda_t B_t(\dot{u}_t,u_t) \tag{58}$$

by

$$B_t(u_t,u_t) = 1. \tag{59}$$

To eliminate $\dot{u}_t$ in (58), second, we use (59) to deduce

$$\dot{B}_t(u_t, u_t) + 2B_t(\dot{u}_t, u_t) = 0. \tag{60}$$

From

$$\lambda_t = \lambda_t B_t(u_t, u_t) = A_t(u_t, u_t)$$

it is derived also that

$$\dot{\lambda}_t = \dot{A}_t(u_t, u_t) + 2A_t(\dot{u}_t, u_t) \tag{61}$$

and then (58) is replaced by

$$\dot{A}_t(u_t, u_t) + \frac{1}{2}\dot{\lambda}_t - \frac{1}{2}\dot{A}_t(u_t, u_t) = \dot{\lambda}_t + \lambda_t \dot{B}_t(u_t, u_t) - \frac{1}{2}\lambda_t \dot{B}_t(u_t, u_t),$$

or,

$$\dot{\lambda}_t = \dot{A}_t(u_t, u_t) - \lambda_t \dot{B}_t(u_t, u_t). \tag{62}$$

As is noticed in Remark 5 in Section 5, if the eigenspace

$$Y_t^\lambda = \langle u_j(t) \mid k \leq j \leq k + m - 1 \rangle$$

corresponding to the eigenvalue $\lambda = \lambda_j(t)$, $k \leq j \leq k+m-1$, to (12), is one-dimensional as in $m = 1$, the eigenfunction u_t in (56) is unique up to the multiplication of ± 1, and this ambiguity is canceled in (62). This property is valid even if $m \geq 2$ as in Remark 1.

Turning to the rigorous proof, we use the following lemma, recalling $u_j(t) \in V$, Y_t^λ, and ϕ_j in (46), (47), and (48), respectively.

LEMMA 11. *Under the assumption of Theorem 10, any $h_\ell \to 0$ admits a subsequence, denoted by the same symbol, such that*

$$\lim_{\ell \to \infty} \frac{1}{h_\ell}\{\lambda_j(t + h_\ell) - \lambda_j(t)\} = \dot{A}_t(\phi_j, \phi_j) - \lambda_j(t)\dot{B}_t(\phi_j, \phi_j) \tag{63}$$

for $k \leq j \leq k + m - 1$. It holds also that

$$\dot{A}_t(\phi_j, \phi_{j'}) - \lambda_j(t)\dot{B}_t(\phi_j, \phi_{j'}) = 0, \quad k \leq j \neq j' \leq k + m - 1. \tag{64}$$

PROOF. Let $k \leq j, j' \leq k + m - 1$. Since

$$A_{t+h}(u_j(t+h) - \phi_j, u_{j'}(t+h) - \phi_{j'}) = A_{t+h}(u_j(t+h), u_{j'}(t+h)) \\ -A_{t+h}(u_j(t+h), \phi_{j'}) - A_{t+h}(\phi_j, u_{j'}(t+h)) + A_{t+h}(\phi_j, \phi_{j'})$$

and

$$A_t(u_j(t+h) - \phi_j, u_{j'}(t+h) - \phi_{j'}) = A_t(u_j(t+h), u_{j'}(t+h)) \\ -A_t(u_j(t+h), \phi_{j'}) - A_t(\phi_j, u_{j'}(t+h)) + A_t(\phi_j, \phi_{j'}),$$

it holds that

$$\begin{aligned} & h(A_{t+h}-A_t)(\frac{u_j(t+h)-\phi_j}{h},\frac{u_{j'}(t+h)-\phi_{j'}}{h}) \\ & \quad = \frac{1}{h}(A_{t+h}-A_t)(u_j(t+h)-\phi_j,u_{j'}(t+h)-\phi_{j'}) \\ & \quad = \frac{1}{h}(A_{t+h}-A_t)(u_j(t+h),u_{j'}(t+h)) + \frac{1}{h}(A_{t+h}-A_t)(\phi_j,\phi_{j'}) \\ & \qquad -\frac{1}{h}(A_{t+h}-A_t)(u_j(t+h),\phi_{j'}) - \frac{1}{h}(A_{t+h}-A_t)(\phi_j,u_{j'}(t+h)). \end{aligned} \tag{65}$$

In (65), first, we obtain

$$\frac{1}{h}(A_{t+h}-A_t)(u_j(t+h),u_{j'}(t+h)) = \dot{A}_t(u_j(t+h),u_{j'}(t+h)) + o(1)$$

as $h \to 0$ by (51) and (55). Then (48) implies

$$\begin{aligned} & \lim_{\ell\to\infty}\frac{1}{h_\ell}(A_{t+h_\ell}-A_t)(u_j(t+h_\ell),u_{j'}(t+h_\ell)) \\ & \quad = \lim_{\ell\to\infty}\dot{A}_t(u_j(t+h_\ell,u_{j'}(t+h_\ell)) = \dot{A}_t(\phi_j,\phi_{j'}) \end{aligned} \tag{66}$$

and also

$$\lim_{\ell\to\infty}\frac{1}{h_\ell}(A_{t+h_\ell}-A_t)(\phi_j,\phi_{j'}) = \dot{A}_t(\phi_j,\phi_{j'}). \tag{67}$$

Second, it holds that

$$\begin{aligned} A_{t+h}(u_j(t+h),\phi_{j'}) &= \lambda_j(t+h)B_{t+h}(u_j(t+h),\phi_{j'}) \\ A_t(u_j(t+h),\phi_{j'}) &= \lambda_j(t)B_t(u_j(t+h),\phi_{j'}) \end{aligned}$$

by $\phi_{j'} \in Y_t^\lambda$, which implies

$$\begin{aligned} & \frac{1}{h_\ell}(A_{t+h_\ell}-A_t)(u_j(t+h_\ell),\phi_{j'}) \\ & \quad = \frac{1}{h_\ell}(\lambda_j(t+h_\ell)B_{t+h_\ell}(u_j(t+h_\ell),\phi_{j'}) - \lambda_j(t)B_t(u_j(t+h_\ell),\phi_{j'})) \\ & \quad = \frac{1}{h_\ell}(\lambda_j(t+h_\ell)-\lambda_j(t))B_{t+h_\ell}(u_j(t+h_\ell),\phi_{j'}) \\ & \qquad +\frac{1}{h_\ell}\lambda_j(t)(B_{t+h_\ell}-B_t)(u_j(t+h_\ell),\phi_{j'}) \\ & \quad = \frac{1}{h_\ell}(\lambda_j(t+h_\ell)-\lambda_j(t))(\delta_{jj'}+o(1)) + \lambda_j(t)\dot{B}_t(\phi_j,\phi_{j'}) + o(1) \end{aligned} \tag{68}$$

by

$$\begin{aligned} B_{t+h_\ell}(u_j(t+h_\ell),\phi_{j'}) &= B_t(u_j(t+h_\ell),\phi_{j'}) + o(1) \\ &= B_t(\phi_j,\phi_{j'}) + o(1) = \delta_{jj'} + o(1) \end{aligned}$$

and (55). Similarly, it follows that

$$\begin{aligned} &\frac{1}{h_\ell}(A_{t+h_\ell}-A_t)(\phi_j,u_{j'}(t+h_\ell)) \\ &= \frac{1}{h_\ell}(\lambda_j(t+h_\ell)-\lambda_j(t))(\delta_{jj'}+o(1))+\lambda_j(t)\dot{B}_t(\phi_j,\phi_{j'})+o(1). \end{aligned} \tag{69}$$

Finally, we obtain

$$\begin{aligned} &\left| h_\ell(A_{t+h_\ell}-A_t)(\frac{u_j(t+h_\ell)-\phi_j}{h_\ell},\frac{u_{j'}(t+h_\ell-\phi_{j'})}{h_\ell})\right| \\ &\le Ch_\ell^2\|\frac{u_j(t+h_\ell)-\phi_j}{h_\ell}\|_V\cdot\|\frac{u_{j'}(t+h_\ell)-\phi_{j'}}{h_\ell}\|_V \\ &= C\|u_j(t+h_\ell)-\phi_j\|_V\cdot\|u_{j'}(t+h_\ell)-\phi_{j'}\|_V=o(1) \end{aligned} \tag{70}$$

by (48) and (55). Then equalities (63)-(64) follow from (66)-(70) as

$$\begin{aligned} 0 = 2\dot{A}_t(\phi_j,\phi_{j'}) &- \frac{2}{h_\ell}(\lambda_j(t+h_\ell)-\lambda_j(t))(\delta_{jj'}+o(1)) \\ &-2\lambda_j(t)\dot{B}_t(\phi_j,\phi_{j'})+o(1), \quad \ell\to\infty. \end{aligned}$$

. □

Below we confirm that the process of taking subsequence in the previous lemma is not necessary, if $h_\ell \to 0$ is unilateral as in $h_\ell \to +0$ or $h_\ell \to -0$. Theorem 1 is thus reduced to the following theorem.

THEOREM 12. *Under the assumption of Theorem 10, the unilateral limits*

$$\dot{\lambda}_j^{\pm}(t)=\lim_{h\to\pm 0}\frac{1}{h}\{\lambda_j(t+h)-\lambda_j(t)\} \tag{71}$$

exist, and it holds that

$$\dot{\lambda}_j^{+}(t)=\mu_{j-k+1},\ \dot{\lambda}_j^{-}(t)=\mu_{k+m-j},\quad k\le j\le k+m-1. \tag{72}$$

Here, μ_q*,* $1\le q\le m$*, is the* q*-th eigenvalue of*

$$u\in Y_t^\lambda,\quad E_t^\lambda(u,v)=\mu B_t(u,v),\ \forall v\in Y_t^\lambda, \tag{73}$$

where Y_t^λ *is the* m*-dimensional eigenspace of (12) corresponding to the eigenvalue* λ *of (21) defined by (47), and*

$$E_t^\lambda=\dot{A}_t-\lambda\dot{B}_t. \tag{74}$$

In particular, it holds that

$$\dot{\lambda}_j^{+}(t)=\dot{\lambda}_{2k+m-1-j}^{-}(t),\quad k\le j\le k+m-1.$$

PROOF. Since Y_t^λ is m-dimensional, the eigenvalue problem (73) admits m-eigenvalues denoted by

$$\mu_1 \leq \cdots \leq \mu_m.$$

By Lemma 11, on the other hand, any $h_\ell \to 0$ takes a subsequence, denoted by the same symbol, satisfying (63)-(64) for some

$$\phi_j \in Y_t^\lambda, \quad k \leq j \leq k+m-1,$$

with (49).

This lemma ensures also the existence of

$$\tilde{\mu}_j = \lim_{\ell\to\infty} \frac{1}{h_\ell}(\lambda_j(t+h_\ell) - \lambda_j(t)), \tag{75}$$

and the equalities

$$E_t^\lambda(\phi_j, \phi_{j'}) = \delta_{jj'}\tilde{\mu}_j, \quad k \leq j, j' \leq k+m-1. \tag{76}$$

We thus obtain

$$\phi_j \in Y_t^\lambda, \quad B_t(\phi_j, \phi_j) = 1, \quad E_t^\lambda(\phi_j, v) = \tilde{\mu}_j B_t(\phi_j, v), \ \forall v \in Y_t^\lambda,$$

and therefore, $\mu = \tilde{\mu}_j$ is an eigenvalue of (73).

If $h_\ell \to +0$, there arises that

$$\tilde{\mu}_k \leq \cdots \leq \tilde{\mu}_{k+m-1}$$

by

$$\lambda_k(t+h) \leq \cdots \leq \lambda_{k+m-1}(t+h),$$

and hence

$$\tilde{\mu}_j = \mu_{j-k+1}, \quad k \leq j \leq k+m-1.$$

Then we obtain the result because the value $\tilde{\mu}_j$ in (75) is independent of the sequence $h_\ell \to +0$.

In the other case of $h_\ell \to -0$, we obtain

$$\tilde{\mu}_j = \mu_{k+m-j}, \quad k \leq j \leq k+m-1,$$

and the result follows similarly. □

Theorem 2 is reduced to the following abstract theorem.

THEOREM 13. *Let the assumption of Theorem 10 hold in I. Fix $t \in I$, and assume*

$$\lim_{h\to 0} \sup_{\|u\|_V, \|v\|_V \leq 1} \left| \dot{A}_{t+h}(u,v) - \dot{A}_t(u,v) \right| = 0$$

$$\lim_{h\to 0} \sup_{|u|_X, |v|_X \leq 1} \left| \dot{B}_{t+h}(u,v) - \dot{B}_t(u,v) \right| = 0. \tag{77}$$

Then, it follows that

$$\lim_{h\to\pm 0}\dot{\lambda}_j^{\pm}(t+h)=\dot{\lambda}_j^{\pm}(t).$$

Proof. Assume (19)-(21) and take $k\le j\le k+m-1$. Since the assumption of Theorem 10 holds in I, any $t'\in I$ admits $u_j(t')\in V$ such that

$$B_{t'}(u_j(t'),u_j(t'))=1,\ A_{t'}(u_j(t'),u_j(t'))=\lambda_j(t')B_{t'}(u_j(t'),u_j(t')) \tag{78}$$

and

$$\dot{\lambda}_j^{+}(t')=\dot{A}_{t'}(u_j(t'),u_j(t'))-\lambda_j(t')\dot{B}_{t'}(u_j(t'),u_j(t')). \tag{79}$$

by Lemma 11 and Theorem 12.

Given t in this theorem, and take $h_\ell\to+0$ and $u_j(t')$ in (78) for $t'=t+h_\ell$. Hence there is a subsequence denoted by the same symbol such that (48) with $\phi_j\in V$. Then it holds that

$$\dot{\lambda}_j^{+}(t)=\dot{A}_t(\phi_j,\phi_j)-\lambda_j(t)\dot{B}_t(\phi_j,\phi_j).$$

We thus obtain

$$\begin{aligned}\lim_{\ell\to\infty}\dot{\lambda}_j^{+}(t+h_\ell)&=\lim_{\ell\to\infty}\{\dot{A}_{t+h_\ell}(u_j(t+h_\ell),u_j(t+h_\ell))\\&\quad-\lambda_j(t+h_\ell)\dot{B}_{t+h_\ell}(u_j(t+h_\ell),u_j(t+h_\ell))\}\\&=\dot{A}_t(\phi_j,\phi_j)-\lambda_j(t)\dot{B}_t(\phi_j,\phi_j)=\dot{\lambda}_j^{+}(t)\end{aligned}$$

by (79), and hence

$$\lim_{h\to+0}\dot{\lambda}_j^{+}(t+h)=\dot{\lambda}_j^{+}(t)$$

because $h_\ell\to+0$ is arbitrary.

The proof of

$$\lim_{h\to-0}\dot{\lambda}_j^{-}(t+h)=\dot{\lambda}_j^{-}(t)$$

is similar. □

Remark 6. *The limits (75) exist for any j under the conditions (41), (42), (54), and (55). If these conditions are satisfied for any $t\in(-\varepsilon_0,\varepsilon_0)$, the limits (71) are unilaterally locally uniform in $t\in I$. In fact, if not, there are, for example, $t_k\downarrow t_0\in(-\varepsilon_0,\varepsilon_0)$, $\delta>0$, and $h_\ell\to 0$, such that*

$$\left|\frac{1}{h_\ell}(\lambda_j(t_k+h_\ell)-\lambda_j(t_k))-\dot{\lambda}_j^{+}(t_k)\right|\ge\delta.$$

Then we obtain

$$\left|\frac{1}{h_\ell}(\lambda_j(t_0+h_\ell)-\lambda_j(t_0))-\dot{\lambda}_j^{+}(t_0)\right|\ge\delta,$$

a contradiction with $\ell \to \infty$.

7. Rearrangement of eigenvalues

For simplicity we introduce the following notations to prove Theorem 3. Recall $I = (-\varepsilon_0, \varepsilon_0)$, and let $f_j \in C^0(I)$, $1 \leq j \leq m$, satisfy

$$f_1(t) \leq \cdots \leq f_m(t), \quad t \in I. \tag{80}$$

Assume the existence of the unilateral limits

$$\dot{f}_j^{\pm}(t) = \lim_{h \to \pm 0} \frac{1}{h}(f_j(t+h) - f_j(t)) \tag{81}$$

and

$$\lim_{h \to \pm 0} \dot{f}_j(t+h) = \dot{f}_j^{\pm}(t) \tag{82}$$

for any j and t. Assume, finally,

$$\dot{f}_j^{+}(t) = \dot{f}_{2k+n-j-1}^{-}(t), \quad k \leq j \leq k+n-1, \tag{83}$$

provided that

$$f_{k-1}(t) < f_k(t) = \cdots = f_{k+n-1}(t) < f_{k+n}(t), \tag{84}$$

where $1 \leq n \leq m$, $1 \leq k \leq m-n+1$, and $t \in I$. In (84) we understand

$$f_0(t) = -\infty, \quad f_{m+1}(t) = +\infty.$$

We call

$$K = K_{k,n}(t) = \{k, \cdots, k+n-1\}$$

the *n-cluster* at t with entry k if (84) arises, and also

$$p(K) = \max\{j \mid k \leq j \leq k + [\frac{n}{2}] - 1, \ \dot{f}_j^{+}(t) < \dot{f}_{2k+n-1-j}^{+}(t)\} - k + 1$$

its *p-value*. Here, we understand $p(K) = 0$ if

$$\dot{f}_k^{+}(t) = \dot{f}_{k+n-1}^{+}(t),$$

noting

$$k \leq i \leq j \leq k+n-1 \quad \Rightarrow \quad \dot{f}_i^{+}(t) \leq \dot{f}_j^{+}(t).$$

We construct a rearrangement of C^0-curves

$$C_j = \{f_j(t) \mid t \in I\}, \quad 1 \leq j \leq m,$$

denoted by $\tilde{C}_j$, $1 \leq j \leq m$, so that are C^1 in $t \in I$. This rearrangement is done only on

$$I_1 = \{t \in I \mid \text{there exists a cluster } K \text{ at } t \text{ such that } p(K) \geq 1.\}.$$

To introduce this rearrangement, we note the following facts in advance. First, given $2 \leq n \leq m$, let

$$I_1^n = \{t \in I \mid \text{there exists an } n\text{-cluster } K \text{ at } t \text{ such that } p(K) \geq 1\}.$$

If $t \in I_1^n$ and $K = K_{k,n}(t)$ satisfies $p(K) \geq 1$, it holds that

$$f_{k+n-1}(t') > f_k(t'), \quad 0 < |t' - t| \ll 1. \tag{85}$$

Hence this t is an isolated point of I_1^n. In particular, each I_1^n, $2 \leq n \leq m$, is at most countable, and hence so is I_1 by

$$I_1 = \bigcup_{n=2}^{m} I_1^n.$$

Second, given $t \in I_1$ and $1 \leq j \leq m$, if $j \in K = K_{k,n}(t)$ holds for some K in $p(K) \geq 1$, this K is unique.

Definition 3. *The curves $\tilde{C}_j$, $1 \leq j \leq m$, are called the transversal rearrangement of C_j, $1 \leq j \leq m$, if the following operations are done.*

1. *If $t \in I_1$, $1 \leq j \leq m$, and $j \in K$ hold for $K = K_{k,n}(t)$ with $p(K) \geq 1$, the curve C_j for $k \leq j \leq k+p-1$ and $k-n-p \leq j \leq k-n-1$ on the right, is connected to $C_{2k+n-j-1}$ on the left at t, where $p = p(K)$.*
2. *No rearrangements to C_j, $1 \leq j \leq m$, are done otherwise.*

The curves $\tilde{C}_j$, $1 \leq j \leq m$, are uniquely constructed from C_j, $1 \leq j \leq m$, by this transversal rearrangement. From the results in the previous section, Theorem 3 is a consequence of the following theorem.

Theorem 14. *Under the above situation, the C^0 curves $\tilde{C}_j$, $1 \leq j \leq m$, made by the transversal rearrangement of C_j, $1 \leq j \leq m$, are C^1.*

Proof. This theorem is obvious if $m = 1$. Now we show it by an induction on m, assuming the assertion up to $m-1$.

Take $t_0 \in I \setminus \overline{I_1}$, and make the transversal rearrangement of C_j, $1 \leq j \leq q$, toward left and right diections. Let t_ℓ, $\ell = 1, 2, \cdots$, be the successive points of I_1 in the left direction:

$$t_\ell \in I_1,\ t_{\ell-1} > t_\ell,\ (t_\ell, t_{\ell-1}) \cap I_1 = \varnothing, \quad \ell = 1, 2, \cdots.$$

If $\{t_\ell\}$ is finite, these C_j's are successfully rearranged to C^1 curves on $(-\varepsilon_0, t_0)$. If not, there is

$$t_* = \lim_{\ell \to \infty} t_\ell \in [-\varepsilon_0, t_0].$$

Then the case $t_* = -\varepsilon_0$ ensures the same conclusion.

Letting $t_* > -\varepsilon_0$, we show that $\{\tilde{C}_j \mid 1 \leq j \leq m\}$ are C^1 curves on $(t_* - \delta, t_* + \delta)$ for $0 < \delta \ll 1$. Once this fact is proven, we can repeat this process up to $t = -\varepsilon_0$ by a covering argument. Turning to the right direction, we conclude that $\tilde{C}_j$, $1 \leq j \leq m$, are C^1 curves on $I = (-\varepsilon_0, \varepsilon_0)$.

To this end we distinguish the cases $t_* \in I_1$ and $t_* \notin I_1$.

If $t_* \in I_1$, first, the above assertion follows from the assumption of induction. In fact, each $K = K_{k,n}(t_*)$ with $p(K) \geq 1$ admits (85), while

$$\tilde{f}_j(t) = \begin{cases} f_j(t), & 0 < t - t_* \ll 1 \\ f_{2k+n-j-1}(t), & 0 < t_* - t \ll 1 \end{cases}, \quad j = k, \ j = k+n-1$$

are C^1 around $t = t_*$. Then we apply the assumption of induction to C_j, $k+1 \leq j \leq k+n-2$, to get n-C^1 curves in $(t_* - \delta, t_* + \delta)$ made by C_j, $k \leq j \leq k+n-1$. Operating this process to any $K = K_{k,n}(t_*)$ with $p(K) \geq 1$ at $t = t_*$, we get C^1 curves $\tilde{C}_j$, $1 \leq j \leq m$, in $(t_* - \delta, t_* + \delta)$ by this transversal rearrangement of C_j, $1 \leq j \leq m$, at $t = t_*$.

If $t_* \notin I_1$, second, we take the cluster decomposition of $\{1, \cdots, m\}$ at $t = t_*$, that is,

$$1 = k_1 < k_1 + n_1 = k_2 < \cdots < k_{s-1} + n_{s-1} = k_s < k_s + n_s = m$$

satisfying

$$\bigcup_{r=1}^{s} K_{k_r,n_r}(t_*) = \{1, \cdots, m\}.$$

Since

$$p(K_{k_r,n_r}(t_*)) = 0, \quad 1 \leq r \leq s$$

holds by the assumption, there are a_r, $1 \leq r \leq s$, such that

$$\dot{f}_j^+(t_*) = a_r, \quad \forall j \in K_{k_r,n_r}.$$

We obtain, on the other hand,

$$f_{k_{r-1}}(t) < f_{k_r}(t), \ |t - t_*| \ll 1, \ 1 \leq r \leq s+1$$

under the agreement

$$f_{k_0}(t) = -\infty, \quad f_{k_{s+1}}(t) = +\infty,$$

and hence $\tilde{C}_j$, $1 \leq j \leq m$, are C^1 on $[t_*, t_* + \delta)$ for $0 < \delta \ll 1$. If t_* is not a right accumulating point of I_1, therefore, these $\tilde{C}_j$, $1 \leq j \leq m$, made by the transversal rearrangement of C_j, $1 \leq j \leq m$, are C^1 on $(t_* - \delta, t_* + \delta)$.

In the other case that t_* is a right accumulating point of I_1, these $\tilde{C}_j$, $1 \leq j \leq m$, are C^1 on $(t_* - \delta, t_*]$, similarly. Then it holds that

$$\dot{f}_j^-(t_*) = a_r, \quad \forall j \in K_{k_r,n_r} = K_{k_r,n_r}(t_*)$$

by $p(K_{k_r,n_r})=0$, and hence $\tilde{C}_j$, $1\leq j\leq m$, are C^1 on $(t_*-\delta,t_*+\delta)$. □

8. Second derivatives

If $T_t:\Omega\to\Omega_t$ is 2-differentiable, we have the other bilinear forms $\ddot{A}_t:V\times V\to\mathbb{R}$ and $\ddot{B}_t:X\times X\to\mathbb{R}$ satisfying

$$\begin{aligned} &|\ddot{A}_t(u,v)|\leq C\|u\|_V\|v\|_v, \quad u,v\in V\\ &|\ddot{B}_t(u,v)|\leq C|u|_X|v|_X, \quad u,v\in X \end{aligned} \tag{86}$$

uniformly in t and

$$\begin{aligned} &\lim_{h\to 0}\frac{1}{h^2}\sup_{\|u\|_V,\|v\|_V\leq 1}\left|\left(A_{t+h}-A_t-h\dot{A}_t-\frac{h^2}{2}\ddot{A}_t\right)(u,v)\right|=0\\ &\lim_{h\to 0}\frac{1}{h^2}\sup_{|u|_X,|v|_X\leq 1}\left|\left(B_{t+h}-B_t-h\dot{B}_t-\frac{h^2}{2}\ddot{B}_t\right)(u,v)\right|=0 \end{aligned} \tag{87}$$

for each t. Hence Theorem 4 is reduced to the following abstract theorem.

Theorem 15. *In Theorem 10, assume, futhermore, (86)-(87). Then the conclusion of Theorem 4 holds.*

For the moment we develop a formal argument as in the first derivative. Assuming (62), first, we deduce

$$\begin{aligned} \ddot{\lambda}_t&=\ddot{A}_t(u_t,u_t)+2\dot{A}_t(\dot{u}_t,u_t)-\dot{\lambda}_t\dot{B}_t(u_t,u_t)-\lambda_t\ddot{B}_t(u_t,u_t)-2\lambda_t\dot{B}_t(\dot{u}_t,u_t)\\ &=2(\dot{A}_t-\lambda_t\dot{B}_t)(\dot{u}_t,u_t)+D_t(u_t,u_t) \end{aligned} \tag{88}$$

for

$$D_t(u,v)=\ddot{A}_t(u,v)-\dot{\lambda}_t\dot{B}_t(u,v)-\lambda_t\ddot{B}_t(u,v), \quad u,v\in V. \tag{89}$$

Putting $v=\dot{u}_t$ in (57), second, we reach

$$(\dot{A}_t-\lambda_t\dot{B}_t)(u_t,\dot{u}_t)=-(A_t-\lambda_tB_t)(\dot{u}_t,\dot{u}_t)+\dot{\lambda}_tB_t(u_t,\dot{u}_t). \tag{90}$$

Then, (60), (88), and (90) imply

$$\begin{aligned} \ddot{\lambda}_t&=-2(A_t-\lambda_tB_t)(z_*,z_*)+2\dot{\lambda}_tB_t(u_t,z_*)+D_t(u_t,u_t)\\ &=-2(A_t-\lambda_tB_t)(z_*,z_*)-\dot{\lambda}_t\dot{B}_t(u_t,u_t)+D_t(u_t,u_t)\\ &=-2(A_t-\lambda_tB_t)(z_*,z_*)+\ddot{A}_t(u_t,u_t)-2\dot{\lambda}_t\dot{B}_t(u_t,u_t)-\lambda_t\ddot{B}_t(u_t,u_t) \end{aligned} \tag{91}$$

for $z_*=\dot{u}_t$.

We observe that $\dot{u}_t\in V$ is not uniquely determined by (57), which is derive formally also. It has, more precisely, the ambiguity of addition of an element in Y_t^λ. This ambiguity, however, cancels in (91) by equality (92) below.

We now develop a rigorous argument valid even to (19). Define λ by (21) and recall

$Y_t^\lambda = \langle u_j(t) \mid k \leq j \leq k+m-1 \rangle$ for $u_j(t)$ satisfying (46). Let, also,

$$C_{t'}^j = A_{t'} - \lambda_j(t')B_{t'}, \quad k \leq j \leq k+m-1,\ t' \in I.$$

Then we obtain

$$C_t^j = A_t - \lambda B_t \equiv C_t$$

and

$$C_t(u,v) = 0, \quad \forall (u,v) \in Y_t^\lambda \times V. \tag{92}$$

By Lemma 11, given $h_\ell \to 0$, which may not be of definite sign, we have a subsequence, denoted by the same symbol, satisfying (48),

$$\mathrm{s} - \lim_{\ell\to\infty} u_j(t+h_\ell) = \phi_j \in Y_t^\lambda \text{ in } V, \quad k \leq j \leq k+m-1. \tag{93}$$

There exists

$$\dot\lambda_j^* = \lim_{\ell\to\infty} \frac{1}{h_\ell}(\lambda_j(t+h_\ell) - \lambda_j(t)), \quad k \leq j \leq k+m-1 \tag{94}$$

passing to a subsequence, with the equality

$$\dot\lambda_j^* \delta_{jj'} = \dot A_t(\phi_j, \phi_{j'}) - \lambda \dot B_t(\phi_j, \phi_{j'}), \quad k \leq j, j' \leq k+m-1.$$

Remark 7. *If we take the subsequence of $h_\ell \to 0$ to be unilateral, the limit $\dot\lambda_j^*$ in (94) exists and is either $\dot\lambda_j^+(t)$ or $\dot\lambda_j^-(t)$.*

Let

$$\dot C_t^{*j} = \dot A_t - \dot\lambda_j^* B_t - \lambda \dot B_t.$$

It holds that

$$\lim_{\ell\to\infty} \sup_{\|u\|_V \leq 1, \|v\|_V \leq 1} \left| \frac{1}{h_\ell}(C_{t+h_\ell}^j - C_t)(u,v) - \dot C_t^{*j}(u,v) \right| = 0, \quad k \leq j \leq k+m-1$$

by (55), and also

$$\dot C_t^{*j}(u,v) = 0, \quad \forall u, v \in Y_t^\lambda,\ k \leq j \leq k+m-1 \tag{95}$$

by Lemma 11. Let

$$z_\ell^j = \frac{1}{h_\ell}(u_j(t+h_\ell) - \phi_j).$$

Lemma 16. *It holds that*

$$\lim_{\ell\to\infty} C_t(z_\ell^j, v) = -\dot C_t^{*j}(\phi_j, v), \quad \forall v \in V. \tag{96}$$

Proof. Given $v \in V$, we obtain

$$C_t(z_\ell^j, v) = \frac{1}{h_\ell} C_t(u_j(t+h_\ell) - \phi_j, v) = \frac{1}{h_\ell} C_t(u_j(t+h_\ell), v)$$
$$= -\frac{1}{h_\ell}(C_{t+h_\ell}^j - C_t)(u_j(t+h_\ell), v) = -\dot{C}_t^{*j}(u_j(t+h_\ell), v) + o(1)$$

by (51) and (92). Then (96) follows from (48). □

Recall that

$$R : X \to Y_t^\lambda = \langle u_j(t) \mid k \leq j \leq k+m-1 \rangle$$

is the orthogonal projection with respect to $B_t(\cdot, \cdot)$, and $P = I - R$. There is a unique $z_*^j \in PV$ satisfying

$$C_t(z_*^j, v) = -\dot{C}_t^{*j}(\phi_j, v), \quad \forall v \in PV. \tag{97}$$

Remark 8. *Equality (97) ensures*

$$\gamma_{\dot{\lambda}_j^*}(\phi_j) = z_*^j, \quad k \leq j \leq k+m-1 \tag{98}$$

under the notation of Definition 2.

Lemma 17. *It holds that*

$$w - \lim_{\ell \to \infty} P z_\ell^j = z_*^j \quad in\ V, \quad k \leq j \leq k+m-1. \tag{99}$$

Proof. Lemma 16 ensures that $\{P z_\ell^j\}$ converges weakly in PV, and hence is bounded there:

$$\|P z_\ell^j\|_V \leq C.$$

Then, passing to a subsequence denoted by the same symbol, there is $\tilde{z}_j \in PV$ such that

$$\mathrm{w} - \lim_{\ell \to \infty} P z_\ell^j = \tilde{z}_j,$$

which satisfies

$$C_t(\tilde{z}_j, v) = -\dot{C}_t^{*j}(\phi_j, v), \quad \forall v \in V \tag{100}$$

by Lemma 16. Since such $\tilde{z}_j \in PV$ is unique, we obtain the result with $\tilde{z}_j = z_*^j$. □

Remark 9. *Since (100) holds with $\tilde{z}_j = z_*^j$, this $z_j^* \in PV$ defined by (97) satisfies*

$$C_t(z_*^j, v) = -\dot{C}_t^{*j}(\phi_j, v), \quad \forall v \in V.$$

Remark 10. *Generally, the inequality*

$$\|z_\ell^j\|_V \leq C$$

is not expected to hold, which causes the other difficulty in later arguments. In fact, if $m = 1$ and $\phi_j = u_j(t)$, for example, this property means

$$|B_t(u_j(t), w^j_{h_\ell})| \le C, \quad w^j_h = \frac{1}{h}(u_j(t+h) - u_j(t)).$$

In the formal argument, we have actually (60), which, however, does not assure the actual convergence

$$\lim_{h\to 0} B_t(u_j(t), w^j_h) = -\frac{1}{2}\dot{B}_t(u_j(t), u_j(t)). \tag{101}$$

In fact, the equality

$$1 = B_{t+h}(u_j(t+h), u_j(t+h)) = B_t(u_j(t), u_j(t))$$

just implies

$$\begin{aligned}
0 &= \frac{1}{h}\{B_{t+h}(u_j(t+h), u_j(t+h)) - B_t(u_j(t), u_j(t))\} \\
&= \frac{1}{h}(B_{t+h} - B_t)(u_j(t+h), u_j(t+h)) \\
&\quad + \frac{1}{h}\{B_t(u_j(t+h), u_j(t+h)) - B_t(u_j(t), u_j(t))\} \\
&= \dot{B}_t(u_j(t), u_j(t)) + o(1) + \frac{1}{h}B_t(u_j(t+h) + u_j(t), u_j(t+h) - u_j(t)) \\
&= \dot{B}_t(u_j(t), u_j(t)) + 2B_t(\frac{u_j(t+h) + u_j(t)}{2}, w^j_h) + o(1)
\end{aligned}$$

and hence

$$\lim_{h\to 0} B_t(\frac{u_j(t+h) + u_j(t)}{2}, w^j_h) = -\frac{1}{2}\dot{B}_t(u_j(t), u_j(t)) \tag{102}$$

differently from (101). Here, the condition $\|w^j_h\|_X = O(1)$ is necessary to conclude

$$B_t(u_j(t+h), w^j_h) = B_t(u_j(t), w^j_h) + o(1)$$

in the left-hand side of (102) from $\phi_j = u_j(t)$ in (48). Our purpose, however, was to assure $\|w^j_h\|_V = O(1)$, which is reduced to $\|w^j_h\|_X = O(1)$ by $\|P_h w^j_h\|_V = O(1)$. This is a circular reasoning.

Lemma 18. *It holds that*

$$s - \lim_{\ell\to\infty} P z^j_\ell = z^j_* \quad \text{in } V, \qquad k \le j \le k+m-1. \tag{103}$$

Proof. Since $V \hookrightarrow X$ is compact, we have

$$\mathrm{s} - \lim_{\ell\to\infty} P z^j_\ell = z^j_* \quad \text{in } X \tag{104}$$

in the previous lemma. Then we obtain

$$\begin{aligned}\delta\|Pz_\ell^j - z_*^j\|_V^2 &\le A_t(Pz_\ell^j - z_*^j, Pz_\ell^j - z_*^j) = C_t(Pz_\ell^j - z_*^j, Pz_\ell^j - z_*^j) + o(1)\\ &= C_t(Pz_\ell^j, Pz_\ell^j - z_*^j) + o(1) = C_t(z_\ell^j, Pz_\ell^j - z_*^j) + o(1)\end{aligned}$$

by (99) and (104).

Since $\phi_j \in Y_t^\lambda$ it holds that

$$\begin{aligned}C_t(z_\ell^j, Pz_\ell^j - z_*^j) &= \frac{1}{h_\ell} C_t(u_j(t+h_\ell), Pz_\ell^j - z_*^j)\\ &= \frac{1}{h_\ell}(C_t - C_{t+h_\ell}^j)(u_j(t+h_\ell), Pz_\ell^j - z_*^j)\\ &= -\dot{C}_t^{*j}(u_j(t+h_\ell), Pz_\ell^j - z_*^j) + o(1)\\ &= -\dot{C}_t^{*j}(\phi_j, Pz_\ell^j - z_*^j) + o(1) = o(1)\end{aligned}$$

by (48) and (99), because

$$v \in V \ \mapsto \ \dot{C}_t^{*j}(\phi_j, v) \in \mathbb{R}$$

is a bounded linear mapping. Then the result follows as

$$\lim_{\ell\to\infty} \|Pz_\ell^j - z_*^j\|_V = 0.$$

□

REMARK 11. *The limit z_*^j in (103) depends on the sequence $h_\ell \to 0$ because it is prescribed by (93) and (98). The limit $\ddot{\lambda}_j^*$ in the following lemma depends also $h_\ell \to 0$, but this ambiguity is cancelled unilaterally. Hence these limits are uniquely determined as either $h \downarrow 0$ or $h \uparrow 0$, because they are characterized by an eigenvalue problem on a finite dimsnsional space, similarly to the first derivative of $\lambda_j(t)$. See Theorem 22 below.*

LEMMA 19. *There exists*

$$\ddot{\lambda}_j^* \equiv \lim_{\ell\to\infty} \frac{2}{h_\ell^2}(\lambda_j(t+h_\ell) - \lambda_j(t) - h_\ell \dot{\lambda}_j^*), \quad k \le j \le k+1-m \tag{105}$$

with

$$\ddot{\lambda}_j^* = (\ddot{A}_t - \lambda \ddot{B}_t - 2\dot{\lambda}_j^* \dot{B}_t)(\phi_j, \phi_j) - 2C_t(z_*^j, z_*^j).$$

PROOF. By (95) and Lemma 18, we have

$$\begin{aligned}C_t(z_\ell^j, z_\ell^j) &= C_t(Pz_\ell^j, Pz_\ell^j) = C_t(z_*^j, z_*^j) + o(1)\\ &= C_t(Pz_\ell^j, z_*^j) + o(1) = C_t(z_\ell^j, z_*^j) + o(1).\end{aligned} \tag{106}$$

It holds that

$$C_t(z_\ell^j, z_*^j) = \frac{1}{h_\ell} C_t(u_j(t+h_\ell) - \phi_j, z_*^j) = \frac{1}{h_\ell} C_t(u_j(t+h_\ell), z_*^j)$$

$$= \frac{1}{h_\ell}(C_t - C^j_{t+h_\ell})(u_j(t+h_\ell), z^j_*) = -\dot{C}^{*j}_t(u_j(t+h_\ell), z^j_*) + o(1)$$
$$= -\dot{C}^{*j}_t(\phi_j, z^j_*) + o(1)$$

by (92) and $\phi_j \in Y^\lambda_t$, which implies

$$C_t(z^j_*, z^j_*) = -\dot{C}^{*j}_t(\phi_j, z^j_*) + o(1) \tag{107}$$

by (106). It holds also that

$$\begin{aligned}\dot{C}^{*j}_t(\phi_j, z^j_*) &= \frac{1}{h_\ell}\dot{C}^{*j}_t(\phi_j, P(u_j(t+h_\ell) - \phi_j)) + o(1) \\ &= \frac{1}{h_\ell}\dot{C}^{*j}_t(\phi_j, u_j(t+h_\ell) - \phi_j) + o(1) \\ &= \frac{1}{h_\ell}\dot{C}^{*j}_t(\phi_j, u_j(t+h_\ell)) + o(1) \qquad (108)\end{aligned}$$

by (95) and $\phi_j \in Y^\lambda_t$.

Here, we use the asymptotics

$$C^j_{t+h_\ell}(\phi_j, u_j(t+h_\ell)) = C^j_t(\phi_j, u_j(t+h_\ell)) + h_\ell \cdot \dot{C}^{*j}_t(\phi_j, u_j(t+h_\ell))$$
$$+\frac{1}{2}h^2_\ell \cdot \ddot{C}^{*j}_{t,\ell}(\phi_j, u_j(t+h_\ell)) + o(h^2_\ell)$$

for

$$\ddot{C}^{*j}_{t,\ell} = \ddot{A}_t - \lambda\ddot{B}_t - 2\dot{\lambda}^*_j\dot{B}_t - \frac{2}{h^2_\ell}(\lambda_j(t+h_\ell) - \lambda_j(t) - h\dot{\lambda}^*_j)B_t, \tag{109}$$

derived from (55) and (87). Since

$$C^j_{t+h_\ell}(\phi_j, u_j(t+h_\ell)) = C^j_t(\phi_j, u_j(t+h_\ell)) = 0$$

holds by (92), we obtain

$$\begin{aligned}C^j_t(z^j_*, z^j_*) &= -\dot{C}^{*j}_t(\phi_j, z^j_*) + o(1) = -\frac{1}{h_\ell}\dot{C}^{*j}_t(\phi_j, u_j(t+h_\ell)) + o(1) \\ &= \frac{1}{2}\ddot{C}^{*j}_{t,\ell}(\phi_j, u_j(t+h_\ell)) + o(1) = \frac{1}{2}\ddot{C}^{*j}_{t,\ell}(\phi_j, \phi_j) + o(1) \\ &= \frac{1}{2}(\ddot{A}_t - \lambda\ddot{B}_t - 2\dot{\lambda}^*_j\dot{B}_t)(\phi_j, \phi_j) - \frac{1}{h^2_\ell}(\lambda_j(t+h_\ell) - \lambda_j(t) - h_\ell\dot{\lambda}^*_j) + o(1)\end{aligned}$$

by (107)-(108) and $B_t(\phi_j, \phi_j) = 1$. Then it follows that

$$\lim_{\ell\to\infty}\frac{2}{h^2_\ell}(\lambda_j(t+h_\ell) - \lambda_j(t) - h_\ell\dot{\lambda}^*_j) = (\ddot{A}_t - \lambda\ddot{B}_t - 2\dot{\lambda}^*_j\dot{B}_t)(\phi_j, \phi_j) - 2C_t(z^j_*, z^j_*),$$

and the proof is complete. □

Lemma 20. *If $\dot{\lambda}_* \equiv \dot{\lambda}^*_j = \dot{\lambda}^*_{j'}$ arises for some $k \le j \ne j' \le k+m-1$, then it holds* *that*

$$(\ddot{A}_t - \lambda \ddot{B}_t - 2\dot{\lambda}_* \dot{B}_t)(\phi_j, \phi_{j'}) = 2C_t(z_*^j, z_*^{j'}).$$

PROOF. As in the previous lemma we obtain

$$\begin{aligned}
C_t(z_*^j, z_*^{j'}) &= C_t(Pz_\ell^j, z_*^{j'}) + o(1) = C_t(z_\ell^j, z_*^{j'}) + o(1) \\
&= \frac{1}{h_\ell} C_t(u_j(t+h_\ell) - \phi_j, z_*^{j'}) + o(1) = \frac{1}{h_\ell} C_t(u_j(t+h_\ell), z_*^{j'}) + o(1) \\
&= \frac{1}{h_\ell}(C_t - C_{t+h_\ell})(u_j(t+h), z_*^{j'}) + o(1) \\
&= -\dot{C}_t^{*j}(u_j(t+h_\ell), z_*^{j'}) + o(1) = -\dot{C}_{t*}^{j}(\phi_j, z_*^{j'}) + o(1) \\
&= -\frac{1}{h_\ell}\dot{C}_t^{*j}(\phi_j, P(u_{j'}(t+h_\ell) - \phi_{j'})) + o(1) \\
&= -\frac{1}{h_\ell}\dot{C}_t^{*j}(\phi_j, u_{j'}(t+h_\ell) - \phi_{j'}) + o(1) \\
&= -\frac{1}{h_\ell}\dot{C}_t^{*j}(\phi_j, u_{j'}(t+h_\ell)) + o(1)
\end{aligned}$$

by $\phi_j, \phi_{j'} \in Y_t^\lambda$. Then it holds that

$$\begin{aligned}
&C_{t+h_\ell}^j(\phi_j, u_{j'}(t+h_\ell)) = C_t(\phi_j, u_{j'}(t+h_\ell)) + h_\ell \cdot \dot{C}_t^{*j}(\phi_j, u_{j'}(t+h_\ell)) \\
&\quad + \frac{1}{2} h_\ell^2 \cdot \ddot{C}_{t,\ell}^{*j}(\phi_j, u_{j'}(t+h_\ell)) + o(h_\ell^2)
\end{aligned}$$

with

$$C_t(\phi_j, u_{j'}(t+h_\ell)) = 0$$

and

$$\begin{aligned}
C_{t+h_\ell}^j(\phi_j, u_{j'}(t+h_\ell)) &= (C_{t+h_\ell}^j - C_{t+h_\ell}^{j'})(\phi_j, u_{j'}(t+h_\ell)) \\
&= (\lambda_{j'}(t+h_\ell) - \lambda_j(t+h_\ell)) B_{t+h_\ell}(\phi_j, u_{j'}(t+h_\ell))
\end{aligned}$$

by (92), to conclude

$$\begin{aligned}
C_t(z_*^j, z_*^{j'}) &= \frac{1}{h_\ell^2}(\lambda_j(t+h_\ell) - \lambda_{j'}(t+h_\ell)) B_{t+h_\ell}(\phi_j, u_{j'}(t+h_\ell)) \\
&\quad + \frac{1}{2}\ddot{C}_{t,\ell}^{*j}(\phi_j, u_{j'}(t+h_\ell)) + o(1). \qquad (110)
\end{aligned}$$

Then we use

$$\begin{aligned}
\lambda_j(t+h_\ell) &= \lambda + h_\ell \dot{\lambda}_j^* + \frac{h_\ell^2}{2}\ddot{\lambda}_j^* + o(h_\ell^2) \\
\lambda_{j'}(t+h_\ell) &= \lambda + h_\ell \dot{\lambda}_{j'}^* + \frac{h_\ell^2}{2}\ddot{\lambda}_{j'}^* + o(h_\ell^2)
\end{aligned}$$

with $\dot{\lambda}_j^* = \dot{\lambda}_{j'}^*$, to deduce

$$\lim_{\ell\to\infty} \frac{1}{h_\ell^2}(\lambda_j(t+h_\ell) - \lambda_{j'}(t+h_\ell))B_{t+h_\ell}(\phi_j, u_{j'}(t+h_\ell))$$
$$= \frac{1}{2}(\ddot{\lambda}_j^* + \ddot{\lambda}_{j'}^*)B_t(\phi_j, \phi_{j'}) = 0.$$

Then the result follows from (109), (110), and the previous lemma. □

Recall $F_t^{\lambda,\lambda'}$ in Definition 2.

LEMMA 21. *Define $\tilde{\mu}_k \leq \cdots \leq \tilde{\mu}_{k+m-1}$ by*

$$\{\tilde{\mu}_j \mid k \leq j \leq k+m-1\} = \{\dot{\lambda}_j^* \mid k \leq j \leq k+m-1\},$$

and assume $k \leq \ell < r \leq k+m$ be such that

$$\tilde{\mu}_{\ell-1} < \tilde{\mu} \equiv \tilde{\mu}_\ell = \cdots = \tilde{\mu}_{r-1} < \tilde{\mu}_r.$$

under the agreement of

$$\tilde{\mu}_{k-1} = -\infty, \quad \tilde{\mu}_{k+m} = +\infty.$$

Then, $\sigma = \ddot{\lambda}_j^*$, $\ell \leq j \leq r-1$, is an eigenvalue of

$$u \in Y_{\lambda,t}^{\ell,r}, \quad F_t^{\lambda,\tilde{\mu}}(u,v) = \sigma B_t(u,v), \ \forall v \in Y_{\lambda,t}^{\ell,r},$$

for

$$Y_{\lambda,t}^{\ell,r} = \langle u_j(t) \mid \ell \leq j \leq r-1\rangle \subset Y_t^\lambda.$$

PROOF. Lemmas 19 and 20 imply

$$F_t^{\lambda,\tilde{\mu}}(\phi_j, \phi_{j'}) = \delta_{jj'}\ddot{\lambda}_j^*, \quad \ell \leq j, j' \leq r-1,$$

and hence the result follows from

$$Y_{\lambda,t}^{\ell,r} = \langle \phi_j \mid \ell \leq j \leq r-1\rangle, \quad B_t(\phi_j, \phi_{j'}) = \delta_{jj'}.$$

□

Theorem 15 is now reduced to the following theorem.

THEOREM 22. *Fix $t \in I$, and assume (19) for $\ell = k$ and $n = m$. Put (21) and let $k \leq \ell < r \leq k+m$ be such that*

$$\dot{\lambda}_{\ell-1}^+(t) < \lambda' \equiv \dot{\lambda}_\ell^+(t) = \cdots = \dot{\lambda}_{r-1}^+(t) < \dot{\lambda}_r^+(t). \tag{111}$$

Then there exists

$$\lambda_j''(t) = \lim_{h\to+0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda - h\lambda'), \quad \ell \leq j \leq r-1. \tag{112}$$

It holds, furthermore, that

$$\lambda_j''(t) = \sigma_{j-\ell+1}, \quad \ell \leq j \leq r-1, \tag{113}$$

where σ_q, $1 \leq q \leq r-\ell$, *denotes the q-th eigenvalue of*

$$u \in Y_{\lambda,t}^{\ell,r}, \qquad F_t^{\lambda,\lambda'}(u,v) = \sigma B_t(u,v), \ \forall v \in Y_{\lambda,t}^{\ell,r}.$$

If

$$\dot{\lambda}_{\ell-1}^-(t) > \lambda' \equiv \dot{\lambda}_\ell^-(t) = \cdots = \dot{\lambda}_{r-1}^-(t) > \dot{\lambda}_r^-(t),$$

there arises that

$$\lambda_j''(t) = \lim_{h\to -0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda - h\lambda'), \quad \ell \leq j \leq r-1$$

with (113).

PROOF. In the previous lemma, we obtain

$$\ddot{\lambda}_\ell^* \leq \cdots \leq \ddot{\lambda}_{r-1}^*.$$

Hence the result follows similarly to Theorem 12. □

REMARK 12. *By the above theorem and Remark 7, the limits (20),*

$$\ddot{\lambda}_j^\pm(t) = \lim_{h\to\pm 0} \frac{2}{h^2}(\lambda_j(t+h) - \lambda_j(t) - h\dot{\lambda}_j^\pm(t))$$

exist for any j, provided that the conditions (41), (42), (54), (55), (86), and (87) hold. If these conditions hold for any t locally uniformly in I, these limits are locally uniform in $t \in I$.

Theorems 5 is reduced to the following abstract theorem. The proof is similar to that of Theorem 13.

THEOREM 23. *Let the assumption of Theorem 15 hold for any t. Fix* $t \in I$, *and assume, furthermore,*

$$\lim_{h\to 0} \sup_{\|u\|_V,\|v\|_V \leq 1} \left| \ddot{A}_{t+h}(u,v) - \ddot{A}_t(u,v) \right| = 0$$
$$\lim_{h\to 0} \sup_{|u|_X,|v|_X \leq 1} \left| \ddot{B}_{t+h}(u,v) - \ddot{B}_t(u,v) \right| = 0. \tag{114}$$

Then it holds that

$$\lim_{h\to\pm 0} \ddot{\lambda}_j^\pm(t+h) = \ddot{\lambda}_j^\pm(t).$$

Finally, Theorem 6 is reduced to the following abstract theorem.

THEOREM 24. *If (114) is valid to any t in the previous theorem,* C^1 *curves* $\tilde{C}_j$,

$1 \leq j \leq m$, *in Theorem 3 are* C^2.

PROOF. Define $\tilde{\lambda}_j(t)$ by

$$\tilde{C}_j = \{\tilde{\lambda}_j(t) \mid t \in I\}, \ 1 \leq j \leq m.$$

From the proof of Theorem 14, Theorems 22 and 23 guarantee the existence of

$$\tilde{\lambda}''_j(t) = \lim_{h \to 0} \frac{1}{h^2}(\tilde{\lambda}_j(t+h) - \tilde{\lambda}_j(t) - h\tilde{\lambda}'_j(t)) \tag{115}$$

together with its continuity in t,

$$\lim_{h \to \pm 0} \tilde{\lambda}''_j(t+h) = \tilde{\lambda}''_j(t)$$

for any t and j. This convergence (115), furthermore, is locally uniform in $t \in I$ by Remark 12.

Then it follows that

$$\tilde{\lambda}_j(t+h) = \tilde{\lambda}_j(t) + h\tilde{\lambda}'_j(t) + \frac{h^2}{2}\tilde{\lambda}''_j(t) + o(h^2)$$
$$\tilde{\lambda}_j(t) = \tilde{\lambda}_j(t+h) - h\tilde{\lambda}'_j(t+h) + \frac{h^2}{2}\tilde{\lambda}''_j(t+h) + o(h^2),$$

as $h \to 0$, which implies

$$\lim_{h \to 0} \frac{1}{h}(\tilde{\lambda}'_j(t+h) - \tilde{\lambda}'_j(t)) = \lim_{h \to 0} \frac{1}{2}(\tilde{\lambda}''_j(t+h) + \tilde{\lambda}''_j(t)) = \tilde{\lambda}''_j(t)$$

for any $t \in I$. Hence these $\tilde{C}_j$'s are C^2. □

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