# JSJ DECOMPOSITION FOR HANDLEBODY-KNOTS

YI-SHENG WANG

Abstract. The paper applies the JSJ decomposition and Koda-Ozawa's annulus classification to analyze the annulus configuration in a handlebody-knot exterior. We introduce the notion of the annulus diagram, to pack the configuration into a labeled graph, and classify genus two handlebody-knots in terms of their annulus diagrams. Applications to handlebody-knot symmetries are discussed; methods to produce handlebody-knots with various types of annulus diagrams are also presented.

## 1. Introduction

Let $M$ be a compact, connected, orientable, irreducible, $\partial$-irreducible 3-manifold. The JSJ decomposition asserts that, up to isotopy, there is a unique surface $S \subset M$ consisting of essential annuli and tori such that **1.** every component of the exterior $E(S) := M - \mathring{\mathfrak{N}}(S)$ is either I-/Seifert fibered or hyperbolic and **2.** the removal of any component of $S$ causes the first condition to fail, where $\mathring{\mathfrak{N}}(S)$ is an open regular neighborhood of $S \subset M$ [14], [15] (see also [1]). Assign a solid (resp. hollow) node to each fibered (resp. hyperbolic) component of $E(S)$, and to each component $N$ of $\mathring{\mathfrak{N}}(S)$ assign an edge between nodes corresponding to component(s) of $E(S)$ that meets(meet) the frontier of $N$. The resulting graph is called a *characteristic diagram* $\Lambda_M$ of $M$.

The present work concerns the case where $M$ has a *connected* boundary and is *atoroidal*, namely, containing no non-boundary parallel essential tori, and *embeddable* in an oriented 3-sphere $\mathbb{S}^3$. By Fox [6], such $M$ is homeomorphic to a handlebody-knot exterior—the exterior of a tangled handlebody in $\mathbb{S}^3$. Atoroidality and embeddability of $M$ impose strong topological constraints on its JSJ decomposition. If the genus $g(\partial M) = 1$, there is only one way to embed $M$ in $\mathbb{S}^3$ by Gordon-Luecke [10] and its exterior in $\mathbb{S}^3$ is always a solid torus. The characteristic diagram $\Lambda_M$ in this case is either Figs. 1a or 1d. In the former, $M$ is a hyperbolic knot exterior, whereas in the latter $M$ is a torus knot exterior. The main results here are a classification theorem for the characteristic diagram of $M$ with $g(\partial M) = 2$ and its enhancement and application to handlebody-knot theory.

**Classification of characteristic diagrams.** Let $M$ be a compact, $\partial$-irreducible, atoroidal 3-submanifold of $\mathbb{S}^3$ with $\partial M$ connected and $g(\partial M) = 2$.

**Theorem 1.1** (Theorem 2.23)**.** *The characteristic diagram $\Lambda_M$ of $M$ is one of the entries in the table in Fig. 1.*

By Thurston's hyperbolization theorem [22], $M$ is either hyperbolic or cylindrical, namely, $M$ containing an essential annulus; it is the former if and only if $\Lambda_M$ is Fig. 1a. It is an interesting question as to whether all diagrams in Fig. 1 can be realized by such an $M$. To the author's knowledge, there is currently no known example whose characteristic diagram is Figs. 1h, 1k, 1l, 1m or 1n.

Recall that the $W$-system of $M$ introduced by Neumann-Swarup [20] is a maximal set of *canonical* annuli in $M$, where an essential annulus is *canonical* if any other essential

*Date*: October 8, 2023.
2020 *Mathematics Subject Classification.* Primary 57K12, 57K30, 57S05; Secondary, 57M15, 58D19.
*Key words and phrases.* handlebody-knots, characteristic submanifold, essential annulus, knot symmetry.

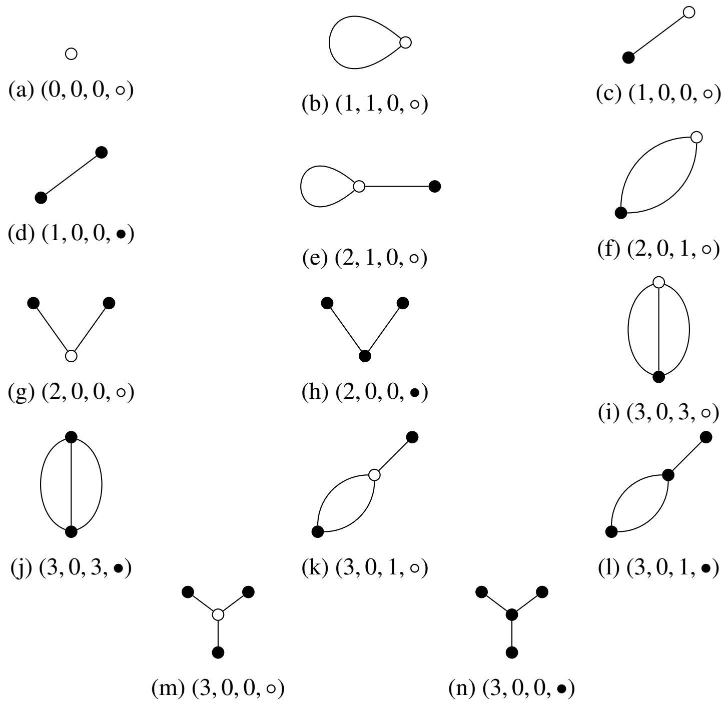



FIGURE 1. Table of characteristic diagrams.

annulus can be isotoped away from it. Theorem 1.1, together with Theorem 3.14 and Proposition 2.21(v), implies the following.

**Corollary 1.2.** *The W-system of M coincides with the JSJ decomposition if $\Lambda_M$ is not one of Figs. 1f, 1k and 1l.*

**Corollary 1.3.** *Up to isotopy, M contains four (resp. five, and infinitely many) essential annuli if $\Lambda_M$ is Figs. 1h or 1f (resp. 1k or 1l, and 1d); otherwise, M contains at most three essential annuli.*

**Applications to handlebody-knot theory.** A genus $g$ handlebody knot $(\mathbb{S}^3, \mathrm{HK})$ is a genus $g$ handlebody HK in $\mathbb{S}^3$. In Sections. 3-4, we apply Theorem 1.1 to study *handlebody-knots of genus* 2, abbreviated to *handlebody-knots* unless otherwise specified. While, up to isotopy, a genus 1 handlebody-knot, equivalent to a classical knot, is determined by its exterior by Gordon-Luecke [10], there are infinitely many *inequivalent*, namely non-isotopic, genus 2 handlebody-knots with homeomorphic exteirors by Motto [19], Lee-Lee [18]. In particular, the characteristic diagram $\Lambda_{E(\mathrm{HK})}$ of the handlebody-knot exterior $E(\mathrm{HK})$ cannot differentiate them, and finer information has to be added.

The present work concerns non-trivial *atoroidal* handlebody-knots $(\mathbb{S}^3, \mathrm{HK})$—that is, $E(\mathrm{HK})$ is atoroidal and not a handlebody; they are of particular interest, being precisely those with a finite symmetry group by Funayoshi-Koda [7], where the (positive) symmetry group $\mathcal{MCG}_{(+)}(\mathbb{S}^3, \mathrm{HK})$ of $(\mathbb{S}^3, \mathrm{HK})$, as defined in Koda [16], is the (positive) mapping class group of the pair $(\mathbb{S}^3, \mathrm{HK})$.

To enhance the characteristic diagram $\Lambda_{E(\mathrm{HK})}$, we recall that Koda-Ozawa [17] and Funayoshi-Koda [7, Lemma 3.2] show that only four types of annuli $A$ can occur as essential annuli in an atoroidal handlebody-knot exterior $E(\mathrm{HK})$. These four types can be described in terms of $\partial A$ in relation to the handlebody HK [17, Proof of Theorem 3.3].

*Type* 2 : Exactly one component $l_A$ of $\partial A$ bounds a disk $\mathcal{D}_A$ in HK; if the disk $\mathcal{D}_A$ is non-separating (resp. separating) in HK, then $A$ is of *type* 2-1 (resp. *type* 2-2). For an example of a type 2-1 annulus, see Fig. 2a.

– The symbol $\mathbf{h}_i$ is reserved for a type 2-$i$ annulus, $i = 1, 2$.

*Type* 3-2 : Components of $\partial A$ are *parallel* in $\partial$HK and bound no disks in HK, and there exists a unique *non-separating* disk $\mathcal{D}_A \subset$ HK disjoint from $\partial A$ [7]. Let $V :=$ HK$-\mathring{\mathfrak{N}}(D)$. Then $A$ is of type 3-2i (resp. type 3-2ii) if $A$ is essential (resp. inessential) in $E(V)$.

– The symbol $\mathbf{k}_*$ is reserved for a type 3-2$*$ annulus.

*Type* 3-3 : Components of $\partial A \subset \partial$HK are *non-parallel* and bound no disks in HK; there exists a unique *separating* essential disk $\mathcal{D}_A$ in HK disjoint from $\partial A$ [24]. The disk $\mathcal{D}_A$ cuts HK into two solid tori, each containing a component of $\partial A$. *The slope pair* of $A$ is the slopes of $\partial A$ with respect to the two solid tori. For instance, the handlebody-knot in Fig. 2b admits a type 3-3 annulus with a slope pair $(1, 1)$.

– The symbol $\mathbf{l}(r_1, r_2)$ denotes a type 3-3 annulus with a slope pair $(r_1, r_2)$; if $(r_1, r_2) = (0, 0)$, we simply write $\mathbf{l}_0$ and say $A$ has a *trivial slope pair*. The slope pair is of either the form $(\frac{p}{q}, \frac{q}{p}), pq \neq 0$ or the form $(\frac{p}{q}, pq), q \neq 0$, where $p, q$ are coprime integers by [24, Lemma 2.12].

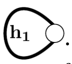
*Type* 4-1 : Components of $\partial A$ are parallel in $\partial$HK and every essential disk in HK meets $\partial A$. Note that the core of the solid torus cut off by $A$ from $E$(HK) is an Eudave-Muñoz knot [4].

– For a type 4-1 annulus the symbol **em** is reserved.

Label each edge of $\Lambda_{E(\mathrm{HK})}$, based on the type of the annulus it represents. Then the resulting edge-labeled diagram, denoted by $\Lambda_{\mathrm{HK}}$, is called the *annulus diagram* of $(\mathbb{S}^3, \mathrm{HK})$. The annulus diagram contains finer information; for instance, $(\mathbb{S}^3, 5_1)$ and $(\mathbb{S}^3, 6_4)$ in the Ishii-Kishimoto-Moriuchi-Suzuki handlebody-kont table [13] have homeomorphic exteriors but different annulus diagrams (Figs. 2a and 2b). By the definition, an essential annulus $A \subset E$(HK) is non-separating if and only if $A$ is of type 2 or of type 3-3.

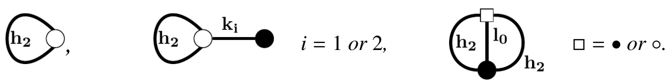
We classify the annulus diagrams of atoroidal handlebody-knots admitting an essential annulus of type 2 or of type 3-3 with specific slope pairs.

**Theorem 1.4** (Theorem 3.18, Proposition 6.10)**.** *Suppose* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal and* $E$(HK) *admits a type* 2 *essential annulus* $A$*.*

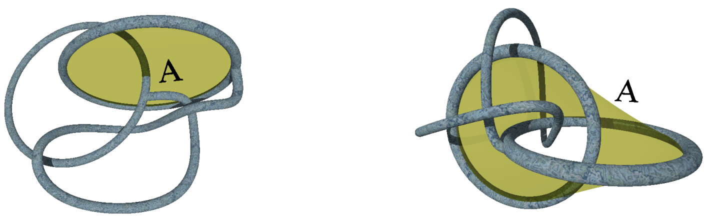
(i) *If* $A$ *is of type* 2-1*, then* $\Lambda_{\mathrm{HK}}$ *is* 

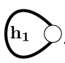
.

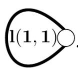
(ii) *If* $A$ *is of type* 2-2*, then* $\Lambda_{\mathrm{HK}}$ *is one of the following:*




(iii) *Every diagram in* (i) *and* (ii) *can be realized by some atoroidal handlebody-knot.*




(a) $\Lambda_{\mathrm{HK}}$ of $(\mathbb{S}^3, 5_1)$: 

. (b) $\Lambda_{\mathrm{HK}}$ of $(\mathbb{S}^3, 6_4)$: 

.

FIGURE 2. Annulus diagrams.

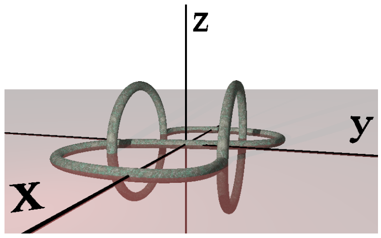
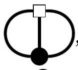
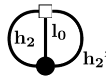


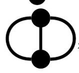

FIGURE 3. Rigid symmetries of $(\mathbb{S}^3, 4_1)$.

In the case the characteristic diagram $\Lambda_{E(\mathrm{HK})}$ is of $\theta$-shape, we show that the annulus diagram $\Lambda_{\mathrm{HK}}$ is determined by $\Lambda_{E(\mathrm{HK})}$, and obtain a characterization of the simplest non-trivial atoroidal handlebody-knot in terms of the characteristic diagram.

**Theorem 1.5** (Theorems 3.14, 3.21)**.** *Suppose* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal.*

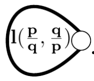
(i) *If* $\Lambda_{E(\mathrm{HK})}$ *is* *, then the annulus diagram* $\Lambda_{\mathrm{HK}}$ *is* 

*, where* $\square = \circ$ *or* $\bullet$.

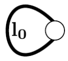
(ii) *If* $\Lambda_{E(\mathrm{HK})}$ *is* *, then* $(\mathbb{S}^3, \mathrm{HK})$ *is equivalent to* $(\mathbb{S}^3, 4_1)$ *in the handlbody-knot table* [13].

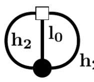
For a type 3-3 annulus $A$, we have the following partial classification.

**Theorem 1.6** (Corollaries 3.10, 3.16, Lemma 3.7)**.** *Suppose* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal, and* $A \subset E(\mathrm{HK})$ *a type* 3-3 *essential annulus.*

(i) *If $A$ has a boundary slope pair of* $(\frac{p}{q}, \frac{q}{p})$*,* $pq \neq 0$*, then* $\Lambda_{\mathrm{HK}}$ *is* 

.

(ii) *If $A$ has a trivial slope pair, then* $\Lambda_{\mathrm{HK}}$ *is* 

*or* 

.

We remark that (i) is Corollary 3.10, and (ii) follows from Theorem 1.4, Lemma 3.7 and Corollary 3.16. Also, Theorem 1.1, Theorem 1.5, and Lemma 3.15 imply that $E(\mathrm{HK})$ can admit at most two type 3-3 essential annuli, up to isotopy, and should this happen, both would have the same boundary slope pair $(\frac{p}{q}, pq)$ with $|p|$ greater than 1.

Applying Theorem 1.4, we compute the symmetry group for atoroidal handlebody-knots whose exteriors contain a type 2 annulus.

**Theorem 1.7** (Theorems 4.9 –4.11)**.** *Suppose* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal and* $A \subset E(\mathrm{HK})$ *a type* 2 *essential annulus.*

(i) *If $A$ is of type* 2-1*, then* $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2$ *and* $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2$.
(ii) *If $A$ is the unique type* 2-2 *annulus in* $E(\mathrm{HK})$*, up to isotopy, then* $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) \simeq \{1\}$ *and* $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2$.
(iii) *If $A$ is the unique type* 2-2 *annulus, but not the unique annulus in* $E(\mathrm{HK})$*, up to isotopy, then* $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) \simeq \{1\} \simeq \mathcal{MCG}(\mathbb{S}^3, \mathrm{HK})$.
(iv) *If $A$ is not the unique type* 2-2 *annulus, up to isotopy, then* $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2$ *and* $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2$.

Note the difference between *"unique annulus"* and *"unique type XXX annulus"*: in the latter, annuli of other types might exist. Theorem 1.7 implies $\mathcal{MCG}(\mathbb{S}^3, 4_1) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathcal{MCG}_+(\mathbb{S}^3, 4_1) \simeq \mathbb{Z}_2$ as the reflection against the xy-plane and rotation around the z-axis by $\pi$ in Fig. 3 represent two non-trivial mapping classes. To our knowledge, $(\mathbb{S}^3, 4_1)$ is the only known example that attains the upper bound in Theorem 1.7 (iv); on the other hand, no handlebody-knot admitting a unique type 2 annulus has been found to have a non-trivial symmetry group so far. We speculate the following sharper statements are both true.

**Problem 1.8.** Under the same assumption as in Theorem 1.7, $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if $(\mathbb{S}^3, \mathrm{HK})$ is equivalent to $(\mathbb{S}^3, 4_1)$.

**Problem 1.9.** Under the same assumption as in Theorem 1.7, suppose $A$ is the unique type 2 annulus in $E(\mathrm{HK})$, up to isotopy. Then $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) \simeq \{1\}$.

The rigid motions shown in Fig. 3 suggest a variant of the Nielsen realization problem.

**Problem 1.10.** Let $(\mathbb{S}^3, \mathrm{HK})$ be a non-trivial atoroidal handlebody-knot. Then there exists a subgroup $G < \mathcal{H}omeo(\mathbb{S}^3, \mathrm{HK})$ such that $\pi_0 : \mathcal{H}omeo(\mathbb{S}^3, \mathrm{HK}) \to \mathcal{MCG}(\mathbb{S}^3, \mathrm{HK})$ restricts to an isomorphism on $G$.

Handlebody-knot symmetry is itself a topic of independent interest. To our knowledge, apart from $(\mathbb{S}^3, 4_1)$, the symmetry group is computed for only five other handlebody-knots in the table [13]:

$$\mathcal{MCG}(\mathbb{S}^3, 5_1) \simeq \mathcal{MCG}(\mathbb{S}^3, 6_1) \simeq \mathcal{MCG}(\mathbb{S}^3, 6_{11}) \simeq \{1\},$$
$$\mathcal{MCG}(\mathbb{S}^3, 5_2) \simeq \mathcal{MCG}_+(\mathbb{S}^3, 5_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathcal{MCG}(\mathbb{S}^3, 6_4) \simeq \mathcal{MCG}_+(\mathbb{S}^3, 6_4) \simeq \mathbb{Z}_2.$$

The first two are computed by Koda [16] using results from Motto [19] and Lee-Lee [18], while the third follows from [23] and Theorem 1.4; the last two are computed in [24]. They all can be realized as subgroups of the homeomorphism groups.

To prove Theorem 1.4(iii), we need to produce *atoroidal* handlebody-knots admitting a type 2 *essential* annulus—a type 2 annulus is not necessarily essential by the definition. Sections 5 and 6 develop essentiality and atoroidality tests and present a systematical approach, via spatial graphs, to generate atoroidal handlebody-knots admitting a type 2 essential annulus.

Our tests make use of an *unknotting operation*: given a type 2 annulus $A \subset E(\mathrm{HK})$, then the union $\mathrm{HK}_A := \mathrm{HK} \cup \mathfrak{N}(A)$ induces a handlebody-knot $(\mathbb{S}^3, \mathrm{HK}_A)$, where $\mathfrak{N}(A) \subset E(\mathrm{HK})$ is a regular neighborhood of $A$. The frontier of $\mathfrak{N}(A) \subset E(\mathrm{HK})$ consists of two annuli in $\partial \mathrm{HK}_A$, whose cores we denote by $l_+, l_- \subset \partial \mathrm{HK}_A$. Recall also that a set of disjoint simple loops $\{l_1, \dots, l_n\}$ in the boundary of a 3-manifold $M$ is *primitive* if there exists a set of disjoint disks $\{D_1, \dots, D_n\}$ in $M$ such that $l_i \cap \partial D_j$ is a point when $i = j$ and empty otherwise. Our essentiality and atoroidality criteria are stated as follows.

**Theorem 1.11** (Propositions 5.8 and 5.9)**.** *Given a handlebody-knot* $(\mathbb{S}^3, \mathrm{HK})$*, and a type* 2 *annulus* $A \subset E(\mathrm{HK})$.

(i) *Suppose A is of type* 2-1*. Then* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal and A is essential if and only if* $(\mathbb{S}^3, \mathrm{HK}_A)$ *is trivial with* $\{l_+, l_-\}$ *not primitive in* $E(\mathrm{HK}_A)$ *or is non-trivial and atoroidal.*

(ii) *Suppose A is of type* 2-2*. Then* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal and A is essential if and only if* $(\mathbb{S}^3, \mathrm{HK}_A)$ *is trivial with* $l_+, l_-$ *not homotopically trivial in* $E(\mathrm{HK}_A)$ *or is non-trivial and atoroidal.*

**Convention.** We work in the piecewise linear category. Given a subpolyhedron $X$ of $M$, we denote by $\overline{X}$, $\mathring{X}$, $\mathfrak{N}(X)$, and $\partial_M X$ the closure, the interior, a regular neighborhood, and the frontier of $X$ in $M$, respectively. The *exterior* $E(X)$ of $X$ in $M$ is defined to be the complement of $\mathring{\mathfrak{N}}(X)$ if $X \subset M$ is of positive codimension, and defined to be the closure of $M - X$ otherwise. Submanifolds of a manifold $M$ are assumed to be proper and in general position except in some obvious cases where submanifolds are in $\partial M$. A surface $S$ other than a disk in a 3-manifold $M$ is *essential* if it is incompressible and $\partial$-incompressible. A disk $D \subset M$ is *essential* if $D$ does not cut a 3-ball off from $M$. When $M$ is a handlebody, an essential disk is also called a *meridian* disk. 3-manifolds here are assumed to be orientable, and given a 3-manifold $M$, $g(\partial M)$ denotes the sum of the genera of its boundary components. Given an oriented loop $l$ in a space $X$, $[l]$ denotes the element represented by $l$ in the first integral homology group $H_1(X)$. We denote by $(\mathbb{S}^3, X)$ an embedding of $X$ in the oriented 3-sphere $\mathbb{S}^3$.

## 2. Characteristic submanifolds

Here we review Johannson's characteristic submanifold theory [15] (see also [2]), and introduce the characteristic diagram and annulus diagram. A completeness criteria needed in Section 3 is also developed.

### 2.1. Characteristic submanifold theory.

**Definition 2.1.** Given a compact $n$-manifold $M$, a *boundary-pattern* $\underline{\underline{m}}$ for $M$ is a finite set of compact, connected $(n-1)$-submanifolds of $\partial M$ such that the intersection of any $i$ of them is either empty or an $(n-i)$-manifold.

We denote by $|\underline{\underline{m}}|$ the union of all elements of $\underline{\underline{m}}$. An *$i$-faced disk* is a disk $D$ whose boundary-pattern $\underline{\underline{d}}$ consists of $i$ elements with $|\underline{\underline{d}}| = \partial D$. When $i \le 3$ (resp. $i = 4$), $(D, \underline{\underline{d}})$ is called a *small-faced disk* (resp. a *square*). The empty boundary-pattern is denoted by $\underline{\underline{\phi}}$, and the *completion* $\overline{\underline{\underline{m}}}$ of a boundary-pattern $\underline{\underline{m}}$ for $M$ is the boundary-pattern given by

$$\overline{\underline{\underline{m}}} := \{\mathtt{G} \in \underline{\underline{m}}\} \cup \{\text{components of } \overline{\partial M - |\underline{\underline{m}}|}\}.$$

Throughout the paper, an annulus (or arc) is assumed to carry the boundary-pattern $\overline{\underline{\underline{\phi}}}$. Given a manifold $(M, \underline{\underline{m}})$ with boundary-pattern and a submanifold $N \subset M$ of positive codimension, if $N \cap \partial M$ meets every intersection of elements of $\overline{\underline{\underline{m}}}$ transversely, then $N$ inherits a natural boundary-pattern given by

$$\underline{\underline{n}} := \{\mathtt{G} \cap \partial N \mid \forall \mathtt{G} \in \underline{\underline{m}}\}. \tag{2.1}$$

Similarly, $\underline{\underline{n}}$ defines a boundary-pattern for a codimension-zero submanifold $N$ of $M$, provided the intersection $\partial_M N \cap \partial M$ meets every intersection of elements in $\overline{\underline{\underline{m}}}$ transversely. The boundary-pattern $\underline{\underline{n}}$ for $N$ is called the *submanifold boundary-pattern*; when $N$ is of codimension zero, we call the completion $\overline{\underline{\underline{n}}}$ the *proper boundary-pattern* for $N$. Throughout the paper, a submanifold $N \subset M$ is assumed to satisfy the transversality condition, and unless otherwise specified, $N$ carries the submanifold boundary-pattern $\underline{\underline{n}}$ except that, when $N$ is regarded as the exterior $E(W)$ of some submanifold $W$ in $M$, the proper boundary-pattern is assumed and denoted by $\tilde{\underline{\underline{m}}}$. We drop $\underline{\underline{n}}$ from the notation when there is no risk of confusion, but specify in the notation the proper boundary-pattern $\overline{\underline{\underline{n}}}$ whenever useful.

**Definition 2.2.** An *arc* $\gamma$ in a surface $(S, \underline{\underline{s}})$ with boundary-pattern is *essential* if no component of the exterior $(E(\gamma), \tilde{\underline{\underline{s}}})$ is a small-faced disk.

A *surface* $S$ in a 3-manifold $(M, \underline{\underline{m}})$ with boundary-pattern is *essential* if no component $X$ of $(E(S), \tilde{\underline{\underline{m}}})$ contains a small-faced disk that meets the frontier $\partial_M X$ in an essential arc in $\partial_M X$. A *codimension-zero submanifold* $N$ in $(M, \underline{\underline{m}})$ is essential if its frontier $\partial_M N$ is *essential* in $(M, \underline{\underline{m}})$.

In the case $\underline{\underline{m}} = \overline{\underline{\underline{\phi}}}$, the definition is equivalent to the one in terms of incompressibility and $\partial$-incompressibility. A 3-manifold $(M, \underline{\underline{m}})$ with boundary-pattern can be *I-fibered* (resp. *Seifert fibered*) if it admits an I-bundle (resp. Seifert bundle) structure $M \xrightarrow{\pi} B$ with $B$ equipped with a boundary-pattern $\underline{\underline{b}}$ such that

$$\underline{\underline{m}} = \{\pi^{-1}(\mathtt{G}) \mid \mathtt{G} \in \underline{\underline{b}}\} \cup \{\text{components of } \overline{\partial M - \pi^{-1}(\partial B)}\}. \tag{2.2}$$

If $(M, \underline{\underline{m}})$ is I-fibered over $(B, \underline{\underline{b}})$, a component of $\overline{\partial M - \pi^{-1}(\partial B)}$ is called a *lid* of $(M, \underline{\underline{m}})$ (with respect to $\pi$), and any other element in $\underline{\underline{m}}$ is called a *side* of $(M, \underline{\underline{m}})$ (with respect to $\pi$). If $(M, \underline{\underline{m}})$ can be I-fibered over an annulus, we call it a *cylindrical shell*. An annulus $A$ in $(M, \underline{\underline{m}})$ is *parallel* to an element $\mathtt{A} \in \underline{\underline{m}}$ (resp. to another annulus $A'$ in $(M, \underline{\underline{m}})$) if a component of $(E(A \cup \mathtt{A}), \tilde{\underline{\underline{m}}})$ (resp. $(E(A \cup A'), \tilde{\underline{\underline{m}}})$) is a cylindrical shell meeting both the regular neighborhoods of $A$ and of $\mathtt{A}$ (resp. of $A'$). The following is a corollary of the vertical-horizontal theorem [15, Proposition 5.6; Corollary 5.7].

**Lemma 2.3.** *Suppose* $(M, \underline{\underline{m}})$ *is I-fibered over* $(B, \underline{\underline{b}})$ *with* $\chi(B) < 0$. *Let A be an essential annulus in* $(M, \underline{\underline{m}})$. *Then the boundary* $\partial A$ *is in the lid(s)* $\mathrm{L} \in \underline{\underline{m}}$, *and there exists an isotopy* $F_t : (A, \partial A) \to (M, \mathrm{L})$ *with* $F_0 = \mathrm{id}$ *and* $F_1(A)$ *the preimage of an essential loop in B.*

**Definition 2.4.** An *$\mathcal{F}$-manifold W* in $(M, \underline{\underline{m}})$ is a codimension-zero essential submanifold of $M$ such that each component of $W$ can be I- or Seifert fibered. An $\mathcal{F}$-manifold $W$ in $M$ is *full* if there exists no component $Y$ of $E(W)$ such that $Y \cup W$ is an $\mathcal{F}$-manifold in $(M, \underline{\underline{m}})$.

**Definition 2.5.** An $\mathcal{F}$-manifold $W$ in $(M, \underline{\underline{m}})$ is *complete* if, for any component $Y$ of $(E(W), \underline{\underline{\tilde{m}}})$ and any essential square, annulus or torus $S$ in $Y$, one of the following holds.

If $S \cap \partial_M Y \neq \emptyset$, then $Y$ can be fibered as a product I-bundle or $S^1$-bundle over $S$. (C1)

If $S \cap \partial_M Y = \emptyset$, then $S$ is parallel to a component of $\partial_M Y$ in $Y$. (C2)

**Definition 2.6.** A *characteristic submanifold W* for $(M, \underline{\underline{m}})$ is a full, complete $\mathcal{F}$-manifold in $(M, \underline{\underline{m}})$.

2.2. **Characteristic submanifiolds of atoroidal manifolds.** Here $M$ is a compact, connected, orientable, irreducible, $\partial$-irreducible 3-manifold containing no essential tori and equipped with the boundary-pattern $\overline{\underline{\underline{\emptyset}}}$. We also assume $\partial M \neq \emptyset$, and allow disconnected $\partial M$; the boundary-pattern $\overline{\underline{\underline{\emptyset}}}$ is dropped from the notation when no confusion may arise. For such an $M$, the existence and uniqueness of characteristic submanifolds are guaranteed.

**Theorem 2.7** ([15, Proposition 9.4; Corollary 10.9])**.** *There exists a characteristic submanifold W for M, and two characteristic submanifolds* $W_1, W_2$ *for M are ambient isotopic.*

Furthermore, characteristic submanifolds have the engulfing property.

**Theorem 2.8** ([15, Proposition 10.8])**.** *Let W be a characteristic submanifold for M. Then, for every $\mathcal{F}$-manifold* $X \subset M$, *there exists an ambient isotopy* $F_t$ *such that* $F_1(X) \subset W$.

The following, a direct consequence of [2, Theorem 2.9.3], gives an alternative description of characteristic submanifolds in terms of simple manifolds.

**Definition 2.9.** A manifold $(X, \underline{\underline{x}})$ with boundary-pattern is *simple* if any component of a characteristic submanifold of $(X, \underline{\underline{\overline{x}}})$ is a regular neighborhood of a square, annulus or torus in $\underline{\underline{x}}$.

**Theorem 2.10.** *Given a full $\mathcal{F}$-manifold* $W \subset M$, *W is a characteristic submanifold for M if and only if, for every component* $Y \subset (E(W), \underline{\underline{\tilde{m}}})$, *Y either is simple or is a cylindrical shell.*

We examine topological properties of submanifolds of $M$ that can be I- or Seifert fibered.

**Lemma 2.11.** *Let X be an essential codimension-zero submanifold of M. Then* $\partial X$ *contains a genus one component if and only if* $(X, \underline{\underline{\overline{x}}})$ *can be Seifert fibered over an n-faced disk with at most one exceptional fiber, and* $\underline{\underline{x}}$ *non-empty and containing disjoint elements; additionally, it has exactly one exceptional fiber when* $n = 2$.

*Proof.* The direction "⇐" is clear. To see the direction "⇒", note first that by the essentiality of $X$ and the boundary-pattern $\overline{\underline{\underline{\emptyset}}}$ on $M$, the intersection $X \cap \partial M$ is non-empty and consists of disjoint annuli $A_1, \dots, A_m$ in $\partial X$. This implies $X$ is a solid torus for no essential torus exists in $M$. Since $M$ is $\partial$-irreducible, $H_1(A_i) \to H_1(X)$ cannot be trivial, and therefore, $(X, \underline{\underline{\overline{x}}})$ can be Seifert fibered over an $n$-faced disk $(D, \underline{\underline{d}})$ with $n = 2m > 0$. In the case $n = 2$, by the essentaility of $\partial_M X$, the Seifert fibering must contain at least one exceptional fiber. □

**Corollary 2.12.** *Let* $X \subset M$ *be an essential codimension-zero submanifold. Then*

(i) $\partial X$ *contains a genus one component if and only if* $\partial X$ *is a torus.*

(ii) $(X, \underline{\underline{x}})$ *can be Seifert fibered if and only if* $(X, \underline{\underline{x}})$ *is a Seifert fibered solid torus.*
(iii) *If* $g(\partial X) = 1$, *then* $(X, \overline{\underline{x}})$ *admits an essential annulus meeting* $\partial_M X$.

*Proof.* (i), (ii) follow directly from Lemma 2.11 For (iii), the frontier of a regular neighborhood of any element in $\underline{\underline{x}}$ is an essential annulus meeting $\partial_M X$. □

**Lemma 2.13.** *Given an essential codimension-zero submanifold* $X \subset M$, *if* $(X, \underline{\underline{x}})$ *is I-fibered over* $(B, \underline{\underline{b}})$, *then* $\underline{\underline{b}} = \underline{\underline{\phi}}$; *that is,* $\underline{\underline{x}}$ *consists of only lids.*

*Proof.* By the definition (2.2), the lid(s) of $(X, \underline{\underline{x}})$ is(are) element(s) in $\underline{\underline{x}}$. On the other hand, since the boundary pattern on $M$ is $\overline{\underline{\underline{\phi}}}$, the submanifold boundary-pattern $\underline{\underline{x}}$ consists of disjoint elements. Thus $\underline{\underline{x}}$ only contains the lid(s). □

**Lemma 2.14.** *Let* $(X, \underline{\underline{x}}) \xrightarrow{\pi} (B, \overline{\underline{\underline{\phi}}})$ *be an I-bundle and* $g(\partial X) > 1$. *Then every essential annulus in* $(X, \underline{\underline{x}})$ *disjoint from the sides of* $(X, \underline{\underline{x}})$ *is parallel to a side* $\mathtt{A} \in \underline{\underline{x}}$ *if and only if* $B$ *is a pair of pants.*

*Proof.* The direction "⇐" follows from Lemma 2.3. We prove the direction "⇒" by contradiction. Observe first that since $g(\partial X) > 1$, the Euler characteristic $\chi(B)$ is less than or equal to $-1$ by the equality $2\chi(B) = 2 - 2g(\partial X)$. In particular, the base $B$ is a closed surface $\hat{B}$ with $k$ open disks removed such that $k$ and the genus $g(\hat{B})$ satisfy $3 - 2g(\hat{B}) \leq k$ when $B$ is orientable and $3 - g(\hat{B}) \leq k$ otherwise. Let $l$ be a non-separating loop in $B$ if $\hat{B}$ is neither a 2-sphere nor a projective plane, or a loop cutting a Möbius band off from $B$ if $\hat{B}$ is a projective space, or a loop cutting a pair of pants off from $B$ if $\hat{B}$ is a 2-sphere. Then if $B$ is not a pair of pants, the preimage of $l$ is an essential annulus in $X$ disjoint from the sides and not parallel to any side of $(X, \underline{\underline{x}})$. □

The following is a corollary of [15, Proposition 4.6].

**Lemma 2.15.** *Let* $S \subset M$ *be a surface consisting of essential annuli, and* $X$ *a component of* $(E(S), \underline{\underline{\tilde{m}}})$. *Then first,* $X$ *contains no essential tori, and secondly, given an annulus* $A \subset X$ *disjoint from* $\partial_M X$, $A$ *is essential in* $X$ *if and only if* $A$ *is essential in* $M$.

**Theorem 2.16** (**Completeness Criterion**). *Let* $W \subset M$ *be a full* $\mathcal{F}$*-manifold. Then* $W$ *is complete if and only if, for every component* $Y$ *of* $(E(W), \underline{\underline{\tilde{m}}})$, *either* $Y$ *is a cylindrical shell or* $g(\partial Y) > 1$, $Y$ *cannot be I-fibered over a pair or pants, and every essential annulus in* $Y$ *disjoint from* $\partial_M Y$ *is parallel to a component of* $\partial_M Y$.

*Proof.* "⇒": Given a component $Y$ of $(E(W), \underline{\underline{\tilde{m}}})$, either $Y$ admits an essential square or annulus that meets $\partial_M Y$ or it does not. By (C1) in Definition 2.5, $Y$ is a cylindrical shell if it is the former. Suppose it is the latter. Then, since $Y$ contains no essential square, it cannot be I-fibered over a pair of pants, and by Corollary 2.12(iii), $g(\partial Y)$ cannot be 1. The rest follows directly from (C2) of Definition 2.5.

"⇐": It is clear that the conditions (C1) and (C2) in Definition (2.5) are satisfied if $Y$ is a cylindrical shell. So, we suppose otherwise; by Theorem 2.10, it suffices to show that $Y$ is simple. Let $W_y$ be the characteristic submanifold of $Y$; note that since $Y$ is a component of $(E(W), \underline{\underline{\tilde{m}}})$, $Y \subset M$ is equipped with the proper boundary-pattern. If $W_y = \emptyset$, then $Y$ is simple by the definition. If $W_y \neq \emptyset$ but $\partial_Y W_y = \emptyset$, then $Y = W_y$. Since $g(\partial Y) > 1$, by Corollary 2.12(i)(ii), it cannot be Seifert fibered, so $Y$ admits an I-bundle structure, contradicting the assumption by Lemma 2.14.

Suppose $\partial_Y W_y \neq \emptyset$, and let $X_y$ be a component of $W_y$, and $A$ be a component of the frontier $\partial_Y X_y$. Then $A$ is disjoint from $\partial_M Y$ since $W_y$ contains a regular neighborhood of $\partial_M Y$ by Theorem 2.8. The component $A \subset \partial_Y X_y$ therefore cannot be a square by the boundary-pattern $\overline{\underline{\underline{\phi}}}$ on $M$; neither can it be a torus because of Lemma 2.15. The component $A$ hence is an annulus. By the assumption, the annulus $A$ is parallel to a component $A'$ of $\partial_M Y$ in $Y$. Let $P \subset Y$ be the cylindrical shell between $A$ and $A'$. Then by the fullness of $W_y$, we have $P \supset X_y$ and $A' \subset X_y$. On the other hand, the essentiality of $X_y$ implies $\partial_P X_y = \emptyset$, so $P = X_y$. In other words, every component of $W_y$ is a regular neighborhood of some component in $\partial_M Y$, so $Y$ is simple. □

2.3. **Characteristic diagram.** Let $M$ be as in the previous subsection.

**Definition 2.17 (Characteristic Surfaces).** A *characteristic surface* $S$ of $M$ is a union of components of $\partial_M W$ such that

- no two components of $S$ are parallel, and
- every component of $\partial_M W$ is parallel to some component of $S$,

where $W \subset M$ is a characteristic submanifold.

The existence of a characteristic surface follows from the existence of a characteristic submanifold $W$ of $M$: for instance, a maximal subset of mutually non-parallel annuli in $\partial_M W$ is a characteristic surface. Characteristic surfaces of $M$ are unique, up to isotopy, by Theorem 2.7.

**Corollary 2.18.** *Given two characteristic surfaces $S_1, S_2$ of $M$, there exists an ambient isotopy $F_t$ satisfying $F_1(S_1) = S_2$.*

Furthermore, by Theorem 2.10, every component of the exterior $E(S) := M - \mathring{\mathfrak{N}}(S)$ is either Seifert/I-fibered or simple.

**Definition 2.19.** Given a characteristic surface $S$ of $M$, the associated *characteristic diagram* $\Lambda_M$ is a graph defined as follows:

- Assign a solid node $\bullet$ to each component of $E(S)$ that can be I-or Seifert fibered.
- Assign a hollow node $\circ$ to each component of $E(S)$ that is simple.
- To each component of $\mathfrak{N}(S)$, assign an edge between node(s) corresponding to component(s) of $E(S)$ meeting the component of $\mathfrak{N}(S)$.

A node in $\Lambda_M$ or the component $X \subset E(S)$ it represents is said to be of *genus* $g$ if $g(\partial X) = g$. In general, $\partial X$ is not connected, but when $M \subset \mathbb{S}^3$, we have the following.

**Lemma 2.20.** *If $M$ is embeddable in $\mathbb{S}^3$ and $\partial M$ is connected, then the boundary $\partial X$ of every component $X \subset E(S)$ is connected.*

*Proof.* By the atoroidality, $S$ consists of only annuli, so every component of $\partial X$ meets $\partial M$. Let $C$ be a component of $\partial X$. Then, by the embeddability of $M$ in $\mathbb{S}^3$, $C$ splits $M$ into two components, one of which, denoted by $M_1$, contains $X$. Connectedness of $\partial M$ implies $\partial M_1 = C$, and therefore $K := C \cap \partial M = \partial M_1 \cap \partial M$.

Suppose $\partial X$ contains another component $C'$. Then $C' \cap C = \emptyset$ implies $C' \cap \partial M \subset \partial M - K$, contradicting $X \subset M_1$ since $\partial M - K \subset M - M_1$. Therefore $\partial X = C$ is connected. □

Two characteristic diagrams are *isomorphic* if there is a graph isomorphism between them sending solid (resp. hollow) nodes to solid (resp. hollow) nodes of the same genus. By Corollary 2.18, the characteristic diagram $\Lambda_M$ of $M$ is determined by $M$, up to isomorphism. We say an annulus $A \subset M$ is *characteristic* if it is isotopic to a component of a characteristic surface $S$ of $M$.

2.4. **Classification and annulus diagram.** Throughout the subsection, $M$ is a compact $\partial$-irreducible, atoroidal 3-submanifold in $\mathbb{S}^3$ with connected $\partial M$ and $g(\partial M) = 2$, and $\Lambda_M$ is its characteristic diagram.

**Proposition 2.21.**

(i) *$\Lambda_M$ has exactly one genus two node, and all the other nodes are of genus one.*
(ii) *Genus one nodes in $\Lambda_M$ are all solid, and each corresponds to a Seifert-fibered solid torus that is not a cylindrical shell.*
(iii) *No loop in $\Lambda_M$ contains a solid node.*
(iv) *All edges in $\Lambda_M$ are adjacent to the genus two node.*

(v) *If the genus two node in $\Lambda_M$ is solid, then it corresponds to an I-bundle over a pair of pants or a Möbius band or Klein bottle with an open disk removed.*
(vi) *If the genus two node in $\Lambda_M$ is solid, then $\Lambda_{E(\mathrm{HK})}$ cannot be a bigon.*
(vii) *Every node in $\Lambda_M$ is at most trivalent.*

*Proof.* Let $W$ be a characteristic submanifold of $M$ and $S$ a corresponding characteristic surface of $M$. Suppose the complement $E(S) := M - \mathring{\mathfrak{N}}(S)$ contains $n$ components $X_1, \dots, X_n$. Then the equality of Euler characteristic

$$-2 = 2 - 2g(\partial M) = \chi(\partial M) = \sum_{i=1}^{n} \chi(\partial X_i) = \sum_{i=1}^{n} (2 - 2g(\partial X_i))$$

implies

$$\sum_{i=1}^{n} (g(\partial X_i) - 1) = 1.$$

In particular, there exists exactly one genus two component in $E(S)$, and other components are of genus one and hence Seifert-fibered by Lemma 2.11 with none of them a cylindrical shell by the definition of $S$. This proves (i) and (ii).

We prove (iii) by contradiction. Suppose there is a loop with a solid node in $\Lambda_M$, and denote by $A$ the annulus corresponding to the loop, and by $X \subset E(S)$ the component corresponding to the solid node. Then the union of $X$ and $\mathfrak{N}(A)$ is either Seifert-fibered or I-fibered, contradicting the fullness of $W$.

To see (iv), it suffices to show there is no edge connecting two genus one solid nodes, given (iii). Suppose such an edge exists, and let $X_1, X_2 \subset E(S)$ be the Seifert components corresponding to the solid nodes. Let $A$ be the annulus corresponding to the edge. Then the union $X_1 \cup \mathfrak{N}(A) \cup X_2$ is Seifert fibered, contradicting the fullness of $W \subset M$.

For (v), we observe first that the component $U$ in $E(S)$ corresponding to a genus two solid node cannot be Seifert fibered by Lemma 2.11, and hence is I-fibered. Since the lid(s) of $U$ has(have) Euler characteristic $-2$, the base is either a pair of pants or a Möbius band, torus, or Klein bottle with one open disk removed. Suppose the base is a torus with one open disk removed. Then $\Lambda_M$ is $\bullet\!\!-\!\!\bullet$ by (iv). Denote by $A$ the annulus corresponding to the edge, and let $V$ be the solid torus corresponding to the genus one node. Choose generators of $H_1(A), H_1(V)$ so the homomorphism $H_1(A) \to H_1(V)$ can be identified with $\mathbb{Z} \xrightarrow{m} \mathbb{Z}, m \geq 0$. Since $A$ is essential, we have $m \neq 0, 1$. The short exact sequence

$$0 \to H_1(A) \xrightarrow{(m,0)} H_1(V) \oplus H_1(U) \to H_1(M) \to 0$$

then implies $H_1(M) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_m$, contradicting $M \subset \mathbb{S}^3$.

We prove (vi) by contradiction. Suppose $\Lambda_M$ is a bigon, and let $U$ (resp. $V$) be the components of $E(S)$ corresponding to the genus two (resp. genus one) node, and $A_1, A_2$ the annuli corresponding to the edges. Then $U$ is an admissible I-bundle over a Möbius band with one open disk removed by (v). Choose generators of $H_1(A_i)$, $H_1(V), H_1(U)$ so that $H_1(A_i) \simeq \mathbb{Z} \xrightarrow{m} \mathbb{Z} \simeq H_1(V)$, $i = 1, 2$, and

$$H_1(A_1) \simeq \mathbb{Z} \xrightarrow{\binom{1}{0}} \mathbb{Z} \oplus \mathbb{Z} \simeq H_1(U) \quad \text{and} \quad H_1(A_2) \simeq \mathbb{Z} \xrightarrow{\binom{\pm 1}{2}} \mathbb{Z} \oplus \mathbb{Z} \simeq H_1(U).$$

Then by the exact sequence

$$0 \to H_1(A_1 \cup A_2) \to H_1(V) \oplus H_1(U) \to H_1(M) \to \tilde{H}_1(A_1 \cup A_2) \to 0,$$

either $(m, 0, 1)$ or $(0, 0, 1)$ in $H_1(V) \oplus H_1(U)$ induces an element of order 2 in $H_1(M)$, contradicting $M \subset \mathbb{S}^3$.

Lastly, in view of (iv), to prove (vii), it suffices to consider the genus two node. The case with a solid genus two node follows from (v), so we assume the genus two node is hollow, and $Y \subset E(S)$ is the corresponding genus two component. Suppose $\partial_M Y$ has more than 3 components. Then there exists an annular component $A$ in $\overline{\partial Y - \partial_M Y}$. Let $A_1, A_2$ be the components of $\partial_M Y$ that meet $\partial A$. Suppose the frontier $A'$ of a regular neighborhood of $A_1 \cup A \cup A_2$ in $Y$ is inessential, then there is an essential square in $Y$, contradicting the completeness of the characteristic submanifold $W \subset M$; on the other hand, since $Y$ is of genus two, the annulus $A'$ is not parallel to any component of $\partial_M Y$; thus by the simpleness of $Y$, neither can $A'$ be essential. □

**Definition 2.22.** We say the characteristic diagram of $M$ is of *type* $(e, l, b, \square)$ if $\Lambda_M$ has $e$ edges, $l$ loops, and $b$ bigons, and $\square = \bullet$ (resp. $\circ$) if the genus two node in $\Lambda_M$ is solid (resp. hollow).

**Theorem 2.23.** *Characteristic diagrams of M are classified, up to isomorphism, by their types* $(e, l, b, \square)$ *into* 13 *classes in the table in Fig. 1.*

*Proof.* Note first characteristic diagrams of the same type are isomorphic. By Proposition 2.21(iv), (vii), we have $1 \leq e \leq 3$, $l = 0$ or 1, and $b = 0, 1, 3$. In addition, $(1, 1, 0, \bullet)$, $(2, 1, 0, \bullet)$ are ruled out by (iii) and $(2, 0, 1, \bullet)$ by (vi) in Proposition 2.21. □

Recall that a handlebody-knot $(\mathbb{S}^3, \mathrm{HK})$ is *irreducible* if $E(\mathrm{HK})$ is $\partial$-irreducible.

**Lemma 2.24.** *Suppose* $(\mathbb{S}^3, \mathrm{HK})$ *is reducible. Then it is trivial if and only if it is atoroidal.*

*Proof.* Observe first that there exists a separating essential disk $D \subset E(\mathrm{HK})$. The disk $D$ splits $E(\mathrm{HK})$ into two knot exteriors $E(K_1)$, $E(K_2)$, for some knots $K_1, K_2$ in $\mathbb{S}^3$. Then $(\mathbb{S}^3, \mathrm{HK})$ is trivial if and only if both $K_1$, $K_2$ are trivial and therefore if and only if $(\mathbb{S}^3, \mathrm{HK})$ is atoroidal. □

**Corollary 2.25.** *Suppose* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal. Then* $(\mathbb{S}^3, \mathrm{HK})$ *is non-trivial if and only if* $(\mathbb{S}^3, \mathrm{HK})$ *is irreducible.*

*Proof.* "$\Leftarrow$" is straightforward, while "$\Rightarrow$" follows from Lemma 2.24. □

**Definition 2.26** (Annulus Diagram)**.** Let $(\mathbb{S}^3, \mathrm{HK})$ be a non-trivial, atoroidal handlebody-knot. Then the annulus diagram $\Lambda_{\mathrm{HK}}$ of $(\mathbb{S}^3, \mathrm{HK})$ is the characteristic diagram $\Lambda_{E(\mathrm{HK})}$ of $E(\mathrm{HK})$ together with a labeling $\mathbf{h}_i, \mathbf{k}_i, \mathbf{l}(r_1, r_2), \mathbf{l}_0$ or **em** for each edge, based on the type of the annulus the edge represents, as defined in Introduction.

## 3. Classification

Throughout the section, $(\mathbb{S}^3, \mathrm{HK})$ is a non-trivial atoroidal handlebody-knot. We examine here combinations of non-separating annuli of various types in $E(\mathrm{HK})$. Let $A \subset E(\mathrm{HK})$ be a non-separating essential annulus, and $\mathrm{HK}_A$ be the union $\mathrm{HK} \cup \mathfrak{N}(A)$. The frontier of $\mathfrak{N}(A)$ in $E(\mathrm{HK})$ are two annuli $A_+, A_-$, whose cores we denote by $l_+, l_-$, respectively. We orient $l_+, l_-$ so as to satisfy $[l_+] = [l_-] \in H_1(\mathfrak{N}(A))$. In the case $A$ is of type 2-2, one of $l_+, l_-$, say $l_-$, is separating in $\partial \mathrm{HK}_A$. We denote the components of $\partial A$ by $l_1, l_2$ if $A$ is of type 3-3, and by $l_A, l$ if $A$ is of type 2 with $l_A$ the one bounding a disk in HK. In addition, by "unique", we understand "unique, up to isotopy", and given a group $G$, we denote by $\langle x_1, \ldots, x_n \rangle$, $x_i \in G$, the subgroup generated by $x_1, \ldots, x_n$.

### 3.1. Annulus configuration.

Recall first a result on type 4-1 annuli [7, Lemma 3.7], [25, Lemma 2.2].

**Lemma 3.1.** *Let* $\hat{A} \subset E(\mathrm{HK})$ *be a type* 4-1 *annulus. Then no non-separating essential annulus in* $E(\mathrm{HK})$ *disjoint from* $\hat{A}$ *exists.*

Given a type 3-3 annulus $A$, we fix an oriented disk $\mathcal{D}_A \subset \mathrm{HK}$ disjoint from $\partial A$. Recall the definition of meridional basis from [24].

**Definition 3.2.** Suppose $A$ is of type 3-3 with a slope pair $(\frac{p}{q}, pq)$. Then a *meridional basis* of $H_1(E(\mathrm{HK}_A))$ is a basis given by the homology classes of the boundary of two oriented, disjoint, non-parallel meridian disks $D_1, D_2 \subset \mathrm{HK}_A$ disjoint from $\mathcal{D}_A$ with $[\partial D_1] - [\partial D_2] = [\partial \mathcal{D}_A] \in H_1(E(\mathrm{HK}_A))$.

**Lemma 3.3.** *Suppose $A$ is of type* 3-3 *with a slope pair $(\frac{p}{q}, pq)$ and $\{b_1, b_2\}$ a meridional basis of $H_1(E(\mathrm{HK}_A))$. If $[l_+] = (p_1, p_2)$ in terms of $\{b_1, b_2\}$, then $[l_-] = (p_1 \mp 1, p_2 \pm 1)$ and $p_1 + p_2 = \pm p$.*

*Proof.* Denote by $V_1, V_2$ the solid tori in $\mathrm{HK} - \mathring{\mathfrak{N}}(\mathcal{D}_A)$, and by $U$ the solid torus $V_1 \cup V_2 \cup \mathfrak{N}(A)$. Then $l_+, l_-$ are two parallel curves in $\partial U$, and they separate the two disk components of the frontier $\partial_{\mathrm{HK}}\mathfrak{N}(\mathcal{D}_A) \subset \partial U$, so $[l_+] - [l_-] = \pm[\partial \mathcal{D}_A] \in H_1(E(\mathrm{HK}_A))$ and therefore the first assertion. Consider the short exact sequence

$$0 \to \langle [\partial \mathcal{D}_A] \rangle \to H_1(E(\mathrm{HK}_A)) \to H_1(E(U)) \simeq \langle b_1 = b_2 \rangle \to 0,$$

and note that the slopes of $l_+, l_- \subset \partial U$ are $\frac{p}{q}$ with respect to $(\mathbb{S}^3, U)$. Hence $p_1 + p_2 = \pm p$. □

**Lemma 3.4.** *Suppose $A$ is of type* 3-3 *with a boundary slope pair $(r_1, r_2)$.*

*If $(r_1, r_2) = (\frac{p}{q}, \frac{q}{p})$, $pq \neq 0$, then $\{[l_+], [l_-]\}$ is a basis of $H_1(E(\mathrm{HK}_A))$.*

*If $(r_1, r_2) = (\frac{p}{q}, pq)$, $pq \neq 0$, then $\langle [l_+], [l_-] \rangle$ is a subgroup of $H_1(E(\mathrm{HK}_A))$ with index $|p|$.*

*If $(r_1, r_2) = (0, 0)$, then $\langle [l_+], [l_-] \rangle$ is a rank one subgroup of $H_1(E(\mathrm{HK}_A))$.*

*Proof.* Denote by $V_1, V_2$ the solid tori in $\mathrm{HK} - \mathring{\mathfrak{N}}(\mathcal{D}_A)$, and by $U$ the union $V_1 \cup V_2 \cup \mathfrak{N}(A)$.

Suppose $(r_1, r_2) = (\frac{p}{q}, \frac{q}{p})$, $|p|, |q| > 1$. Then $U$ is a Seifert fibered space with two exceptional fibers, and therefore the exterior $W := E(U)$ of $U$ in $\mathbb{S}^3$ is a solid torus, whose core is a $(p, q)$-torus knot in $\mathbb{S}^3$. Since $l_+, l_-$ are parallel to the core of $W$ in $W$ by [21], $[l_+] = [l_-]$ generates $H_1(W)$. On the other hand, we have $E(\mathrm{HK}_A) = W - \mathfrak{N}(\mathcal{D}_A)$; that is, $E(\mathrm{HK}_A)$ is obtained by removing a regular neighborhood of an arc in $W$ dual to $\mathcal{D}_A$, so $H_1(W, E(\mathrm{HK}_A)) = 0$. This together with $H_2(W) = 0$ implies the short exact sequence

$$0 \to H_2(W, E(\mathrm{HK}_A)) \to H_1(E(\mathrm{HK}_A)) \to H_1(W) \to 0$$

given by the inclusion $E(\mathrm{HK}_A) \hookrightarrow W$. Because of the facts that $\langle [\mathcal{D}_A] \rangle = H_2(W, E(\mathrm{HK}_A))$, and $\pm[\partial \mathcal{D}_A] = [l_+] - [l_-]$, and $[l_+] = [l_-]$ generates $H_1(W)$, we have $\{[l_+], [l_-]\}$ is a basis of $H_1(E(\mathrm{HK}_A))$.

Suppose $(r_1, r_2) = (\frac{p}{q}, pq)$, $q \neq 0$. Then by Lemma 3.3, $[l_+] = (p_1, p_2)$ and $[l_-] = (p_1 \mp 1, p_2 \pm 1)$ with $p_1 + p_2 = p$ in terms of a meridional basis of $H_1(E(\mathrm{HK}_A))$, and hence the determinant

$$\begin{vmatrix} p_1 & p_2 \\ p_1 \mp 1 & p_2 \pm 1 \end{vmatrix} = \pm(p_1 + p_2) = \pm p.$$

When $p \neq 0$, $\langle [l_+], [l_-] \rangle < H_1(E(\mathrm{HK}_A))$ is a subgroup of rank two with index $|p|$. When $p = 0$, since $[l_+] - [l_-] = \mp(1, -1)$, at least one of $[l_+], [l_-] \in H_1(E(\mathrm{HK}_A))$ is non-trivial, so $\langle [l_+], [l_-] \rangle$ is a subgroup isomorphic to $\mathbb{Z}$. □

**Corollary 3.5.** *Suppose $A$ is of type* 3-3 *with a non-trivial slope pair, and $A'$ is a non-separating annulus disjoint from $A$. Then $\partial A, \partial A'$ are parallel in $\partial \mathrm{HK}$. In particular, $A'$ is of type* 3-3 *with the same slope pair.*

*Proof.* Choose a regular neighborhood $\mathfrak{N}(A)$ with $\mathfrak{N}(A) \cap A' = \emptyset$. Let $P$ be the planar surface $\partial E(\mathrm{HK}_A) - \mathring{A}_+ \cup \mathring{A}_-$. Denote by $l_{1\pm}, l_{2\pm}$ the components of $\partial A_\pm$ and by $l'_1, l'_2$ the components of $\partial A'$. Since $l'_1, l'_2 \subset P$, one of $l'_1, l'_2$ is parallel to one of $l_{1\pm}, l_{2\pm}$; it may be assumed that $l'_1$ is parallel to $l_{1+}$. By Lemma 3.4, $[l_+] \neq \pm[l_-]$ and none of $[l_+], [l_-]$ is trivial in $H_1(E(\mathrm{HK}_A))$. These, together with $[l'_1] = [l'_2] \in H_1(E(\mathrm{HK}_A))$, imply that $l'_2$ is parallel to either $l_{2+}$ or $l_{1+}$. The latter is impossible since $l'_1, l'_2$ are not parallel in $\partial \mathrm{HK}$ and hence not parallel in $P$. Therefore $\partial A'$ is parallel to $\partial A_+$ and hence to $\partial A$. □

There is an analog of Lemma 3.4 for type 2 annuli.

**Lemma 3.6.** *If $A$ is of type* 2-1*, then $\{[l_+], [l_-]\}$ is a basis of $H_1(E(\mathrm{HK}_A))$. If $A$ is of type* 2-2*, then $[l_-]$ is trivial and the quotient $H_1(E(\mathrm{HK}_A))/\langle [l_+] \rangle \simeq \mathbb{Z}$.*

*Proof.* It follows from the fact that $l_+, l_-$ bound non-parallel, non-separating meridian disks in $\mathrm{HK}_A$ if $A$ is of type 2-1, and $l_-$ (resp. $l_+$) bounds a separating (resp. non-separating) disk in $\mathrm{HK}_A$ if $A$ is of type 2-2. □

**Lemma 3.7.** *Suppose $A$ is of type* 3-3 *with a trivial slope pair, and $A'$ is a type* 3-3 *annulus disjoint from $A$. Then $A, A'$ are parallel in $E(\mathrm{HK})$.*

*Proof.* Suppose $\partial A$ and $\partial A'$ are parallel in $\partial\mathrm{HK}$. Let $B_1, B_2 \subset \partial\mathrm{HK}$ be the annuli cut off by $\partial A, \partial A'$. Then $A \cup A' \cup B_1 \cup B_2$ bounds a solid torus $V$ in $E(\mathrm{HK})$ by the atoroidality of $(\mathbb{S}^3, \mathrm{HK})$. Since $A$ has a trivial slope pair, the linking number $\ell k(l_1, l_2)$ is 0 and hence the core of $A$ is a preferred longitude with respect to $(\mathbb{S}^3, V)$; this implies $H_1(A) \to H_1(V)$ is an isomorphism, so $A, A'$ are parallel through $V$.

Suppose $\partial A$ and $\partial A'$ are not parallel. Let $l'_1, l'_2$ be the components of $\partial A'$. Then since $\partial\mathrm{HK} - \partial A$ is a four-times punctured sphere, it may be assumed that $l_1, l'_1$ are parallel in $\partial\mathrm{HK}$, and $l_2, l'_2$ are not. Let $B_1 \subset \partial\mathrm{HK}$ be the annulus cut off by $l_1.l'_1$. Then $B_1 \cup A \cup A'$ induces an annulus $A'' \subset E(\mathrm{HK})$ disjoint from $A \cup A'$ with $\partial A''$ parallel to $l_2, l'_2$. Let $B_2, B_3 \subset \partial\mathrm{HK}$ be the annuli cut off by $\partial A''$ and $l_2 \cup l'_2$. Then the torus $B_1 \cup B_2 \cup B_3 \cup A \cup A' \cup A''$ bounds a solid torus $W$ in $E(\mathrm{HK})$ since $(\mathbb{S}^3, \mathrm{HK})$ is atoroidal.

Let $P_1, P_2 \subset \partial\mathrm{HK}$ be the pairs of pants cut off by $B_1 \cup B_2 \cup B_3$. Then $P_1, P_2$ can be regarded as a planar surface in $E(W)$. By [17, Lemma 3.5], $P_1, P_2$ are inessential in $E(W)$.

**Case** 1**: $P_1$ is compressible.** Let $D$ be a compressing disk of $P_1$ that minimizes

$$\#\{D \cap P_2 \mid D \text{ a compressing disk of } P_1\}.$$

**Subcase** 1.1**: $D \cap P_2 = \emptyset$.** The disk $D$ is either in HK or in $E(\mathrm{HK})$. Since $\partial D$ is essential in $P_1$, $\partial D$ is essential in $\partial\mathrm{HK}$, so $D$ is a compressing disk of $\partial\mathrm{HK}$ in $\mathbb{S}^3$. On the other hand, $\partial A \cup \partial A' \cup \partial A''$ contains three mutually non-parallel simple loops in $\partial\mathrm{HK}$ that bound no disks in HK, so every meridian disk in HK meets $\partial A \cup \partial A' \cup \partial A''$, and hence $D \subset E(\mathrm{HK})$, but this contradicts the fact that $(\mathbb{S}^3, \mathrm{HK})$ is irreducible.

**Subcase** 1.2**: $D \cap P_2 \neq \emptyset$.** Note first that $D \cap P_2$ only contains circles. Let $D' \subset D$ be the disk cut off by a circle in $D \cap P_2$ innermost in $D$. By the minimality $\partial D'$ is essential in $P_2$; hence $D'$ is a compressing disk of $\partial\mathrm{HK}$ in $\mathbb{S}^3$, a contradiction as in **Subcase** 1.1.

The same argument applies to the case where $P_2$ is compressible.

**Case** 2**: $P_1, P_2$ are incompressible.** First observe that, since none of the components of $\partial A \cup \partial A' \cup \partial A''$ is separating in $\partial\mathrm{HK}$, $P_1$ (resp. $P_2$) meets $B_i$ for each $i$. Let $D$ be a $\partial$-compressing disk of $P_1$ that minimizes

$$\#\{D \cap P_2 \mid D \text{ a } \partial\text{-compressing disk of } P_1\}.$$

Then by the minimality and incompressibility of $P_2$, $D \cap P_2$ is either empty or some arcs.

**Subcase** 2.1**: $D \cap P_2 = \emptyset$.** Denote by $\gamma$ the arc $D \cap E(W)$, and note that $\gamma \subset B_u := B_1 \cup B_2 \cup B_3$ if $D \subset \mathrm{HK}$; otherwise $\gamma \subset A_u := A \cup A' \cup A''$. In addition, $\gamma$ is inessential in either case: in the former, it follows from the fact that none of $B_i$, $i = 1, 2, 3$, has two boundary components lying in $P_1$, whereas in the latter, it results from the $\partial$-incompressibility of $A, A', A''$.

Let $D'$ be the disk cut off by $\gamma$ from $B_u$ (resp. $A_u$). Then $D \cup D'$ induces a disk $D''$ disjoint from $B_u$ (resp. $A_u$). Since $D$ is a $\partial$-compressing disk of $P_1$ in $E(W)$, $\partial D''$ is essential in $P_1$, contradicting the incompressibility of $P_1$.

**Subcase** 2.2**: $D \cap P_2 \neq \emptyset$.** Let $D' \subset D$ be a disk cut off by an arc in $D \cap P_2$ outermost in $D$. Denote by $\gamma$ the arc $D' \cap \partial W$; as with **Subcase** 2.1, $\gamma$ is either in $A_u$ or in $B_u$, and inessential whichever way. Let $D''$ be the disk cut off by $\gamma$ from $A_u$ or $B_u$. Then $D' \cup D''$ induces a disk $D'''$ disjoint from $P_1$ with $\partial D'' \subset P_2$. By the minimality of $\#D \cap P_2$, $\partial D'''$ is essential in $P_2$, contradicting the incompressibility of $P_2$. □

**Lemma 3.8.** *If $\{[l_+], [l_-]\}$ is a basis of $H_1(E(\mathrm{HK}_A))$, then $A$ is the unique annulus in $E(\mathrm{HK})$.*

*Proof.* By Theorem 2.8, it suffices to show that $\mathfrak{N}(A) \subset E(\mathrm{HK})$ is a characteristic submanifold of $(E(\mathrm{HK}), \overline{\underline{\phi}})$. To see this, we employ Theorem 2.16. Since $\mathfrak{N}(A)$ is a full $\mathcal{F}$-manifold of $(E(\mathrm{HK}), \overline{\underline{\phi}})$, it amounts to showing that every essential annulus $A'$ in $E(\mathrm{HK}_A)$ disjoint from $A_+, A_-$ is parallel to $A_+, A_-$, where $E(\mathrm{HK}_A) \subset (E(\mathrm{HK}), \overline{\underline{\phi}})$ is endowed with the proper boundary pattern. Denote by $l'$ a core of $A'$.

**Case 1: $A'$ is non-separating in $E(\mathrm{HK})$.** Since $\{[l_+], [l_-]\}$ is a basis of $H_1(E(\mathrm{HK}_A))$, the argument for Corollary 3.5 applies and thus $\partial A'$ is parallel to $\partial A_+$ or $\partial A_-$ in $\partial E(\mathrm{HK}_A)$; it may be assumed that it is the former, and denote by $B_1, B_2$ the annuli cut off by $\partial A_+, \partial A'$ from $\partial E(\mathrm{HK}_A)$. Since every compressing disk of a torus in $E(\mathrm{HK}_A)$ can be isotoped away from $A$ by the essentiality of $A$, we have $E(\mathrm{HK}_A)$ is atoroidal. Particularly, $A_+ \cup A' \cup B_1 \cup B_2$ bounds a solid torus $W$ in $E(\mathrm{HK}_A)$. Let $X$ be the closure of the complement $E(\mathrm{HK}_A) - W$ and $l_w$ a core of $W$, and orient $l', l_w$ so that $[l'] = [l_+]$ and $[l'] = k[l_w]$, $k > 0$, in $H_1(W)$. Consider the short exact sequence

$$0 \to H_1(A') \xrightarrow{(\iota_1,\iota_2)} H_1(W) \oplus H_1(X) \xrightarrow{\iota_3-\iota_4} H_1(E(\mathrm{HK}_A)) \to 0,$$

where $\iota_i$, $i = 1, 2, 3, 4$, are induced by the inclusions. Note that $\iota_4$ sends $[l']$ to $[l_+]$ and $[l_-]$ to itself, and $\iota_1$ sends $[l']$ to $k[l_w]$. Since $\{[l_+], [l_-]\}$ is a basis of $H_1(E(\mathrm{HK}_A))$, the image of $[l_w]$ under $\iota_3$ is $m[l_+] + n[l_-]$, for some $m, n \in \mathbb{Z}$. Then the identity $\iota_3 \circ \iota_1 = \iota_4 \circ \iota_2$ gives us $km[l_+] + kn[l_-] = [l_+]$, and therefore $n = 0, k = m = 1$. This implies $H_1(A') \xrightarrow{\iota_1} H_1(W)$ is an isomorphism, and hence $A'$ is parallel to $A_+$ through $W$ in $E(\mathrm{HK}_A)$.

**Case 2: $A'$ is separating in $E(\mathrm{HK})$.** Since the components of $\partial A'$ are parallel and do not separate the components of $\partial A$ in $\partial\mathrm{HK}$, the components of $\partial A'$ are also parallel in $\partial E(\mathrm{HK}_A)$. Let $B \subset \partial E(\mathrm{HK}_A)$ be the annulus cut off by $\partial A'$. Then $B \cup A'$ bounds a solid torus $W$ in $E(\mathrm{HK}_A)$. Set $X := \overline{E(\mathrm{HK}_A) - W}$, and consider the short exact sequence

$$0 \to H_1(A') \xrightarrow{(\iota_1,\iota_2)} H_1(W) \oplus H_1(X) \xrightarrow{\iota_3-\iota_4} H_1(E(\mathrm{HK}_A)) \to 0,$$

where $\iota_i$, $i = 1, 2, 3, 4$, are induced by the inclusions. Let $l_w$ be a core of $W$. Then one can orient $l', l_w$ so that $[l'] = k[l_w]$ with $k > 1$ by the essentiality of $A'$. Since $[l_+], [l_-] \in H_1(X)$ and $H_2(E(\mathrm{HK}_A), X) = 0$, we have the homomorphism $\iota_4 : H_1(X) \to H_1(E(\mathrm{HK}_A))$ is an isomorphism and $\langle [l_+], [l_-] \rangle = H_1(X)$. Let the image of $[l_w]$ under $\iota_3$ be $m[l_+] + n[l_-]$, and the image of $[l']$ under $\iota_2$ be $m'[l_+] + n'[l_-]$, for some $m, n, m', n' \in \mathbb{Z}$. Then $x = ([l_w], m[l_+] + n[l_-]) \in H_1(W) \oplus H_1(X)$ is in the kernel of $\iota_3 - \iota_4$, and therefore, there exists $c \in \mathbb{Z}$ such that the image of $c[l']$ under $(\iota_1, \iota_2)$ is $x$; in other words, we have the equality

$$(kc[l_w], m'c[l_+] + n'c[l_-]) = ([l_w], m[l_+] + n[l_-]) \in H_1(W) \oplus H_1(X),$$

but this implies $k = c = 1, m = m', n = n'$, contradicting $k > 1$. □

**Lemma 3.9.** *The pair $\{[l_+], [l_-]\}$ forms a basis of $H_1(E(\mathrm{HK}_A))$ if and only if $A$ is of type* 2-1 *or of type* 3-3 *with the slope pair $(\frac{p}{q}, \frac{q}{p})$, $pq \neq 0$.*

*Proof.* "⇐" follows from Lemmas 3.6 and 3.4. "⇒" also results from the same lemmas as $\{[l_+], [l_-]\}$ can form a basis of $H_1(E(\mathrm{HK}_A))$ only if $A$ is of type 2 or of type 3-3. □

Lemmas 3.8 and 3.9 give us the following uniqueness result.

**Corollary 3.10.** *If $A$ is of type* 2-1 *or of type* 3-3 *with the slope pair $(\frac{p}{q}, \frac{q}{p})$, $pq \neq 0$, then $A$ is the unique annulus in $E(\mathrm{HK})$.*

**Lemma 3.11.** *Let $A, A'$ be two disjoint type* 2-2 *annuli in $E(\mathrm{HK})$. If $\partial A, \partial A'$ are parallel in $\partial\mathrm{HK}$, then $A, A'$ are parallel in $E(\mathrm{HK})$.*

*Proof.* Let $B_1, B_2 \subset \partial\mathrm{HK}$ be the annuli cut off by $\partial A, \partial A'$. Then $B_1 \cup B_2 \cup A \cup A'$ bounds a solid torus $W$ in $E(\mathrm{HK})$ by the atoroidality of $(\mathbb{S}^3, \mathrm{HK})$. Observe that $l_A$ is a longitude of $(\mathbb{S}^3, W)$ since it bounds a disk in HK. This implies $H_1(A) \to H_1(W)$ is an isomorphism, and hence $A, A'$ are parallel through $W$ in $E(\mathrm{HK})$. □

**Corollary 3.12.** *Let $A, A', A''$ be three disjoint type* 2-2 *annuli in* $E(\mathrm{HK})$. *Then at least two of them are parallel in* $E(\mathrm{HK})$.

*Proof.* Let $l' \subset \partial A', l'' \subset \partial A''$ be the components that do not bound a disk in HK, and $l_{A'} \subset \partial A', l_{A''} \subset \partial A''$ the other components. Then $l_A, l_{A'}, l_{A''}$ are parallel in $\partial$HK by the definition of a type 2-2 annulus.

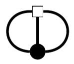
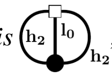
Suppose $A, A'$ are not parallel in $E(\mathrm{HK})$. Then $l, l'$ are longitudes of the solid tori $V, V'$ in $\mathrm{HK} - \mathring{U}$, where $U \subset \mathrm{HK}$ is the 3-ball cut off by the disks bounded by $l_A, l_{A'}$. In particular, $l''$ is parallel to either $l$ or $l'$, so by Lemma 3.11, $A''$ is parallel to $A$ or $A'$. □

**Lemma 3.13.** *Suppose $A$ is of type* 2-2. *Then there exists another type* 2-2 *annulus $A'$ disjoint from and non-parallel to $A$ if and only if there exists a type* 3-3 *annulus $A''$ with a trivial slope pair disjoint from $A$.*

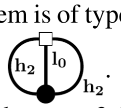
*Proof.* "⇒": Let $l_{A'} \subset \partial A'$ be the component that bounds a disk in HK and $l' \subset \partial A'$ another component. Then $l_A, l_{A'}$ are parallel and bound an annulus $B$ in $\partial$HK, and $l, l'$ are non-parallel in $\partial$HK by Lemma 3.11. The union $A \cup A' \cup B$ induces a type 3-3 annulus, which has a trivial slope pair since $\ell k(l, l') = \ell k(l_A, l_{A'}) = 0$.

"⇐": Let $l''_1, l''_2$ be components of $\partial A''$. Then one of them, say $l''_1$, is parallel to $l$ in $\partial$HK. Let $B \subset \partial\mathrm{HK}$ be the annulus cut off by $l, l''_1$. Then the union $A \cup B \cup A''$ induces a type 2-2 annulus disjoint from and non-parallel to $A$ with boundary components parallel to $l_A, l''_2$. □

3.2. **Classification theorems.** Let $\Lambda_{E(\mathrm{HK})}$ be the characteristic diagram of $E(\mathrm{HK})$, and $\Lambda_{\mathrm{HK}}$ the annulus diagram of $(\mathbb{S}^3, \mathrm{HK})$.

**Theorem 3.14** (**$\theta$-shape characteristic diagram).** *If $\Lambda_{E(\mathrm{HK})}$ is* *, then $\Lambda_{\mathrm{HK}}$ is* 

*, □ = • or ∘, and the Seifert fibered solid torus has no exceptional fiber.*

*Proof.* Let $A, A', A''$ be the non-separating annuli corresponding to the edges of $\Lambda_{E(\mathrm{HK})}$. None of them is of type 2-1 by Corollary 3.10 or of type 3-3 with a non-trivial slope pair by Corollaries 3.10 and 3.5 since no two of them separate $E(\mathrm{HK})$. Therefore, $A, A', A''$ are of type 2-2 or of type 3-3 with a trivial slope. By Lemma 3.7, at most one of them is of type 3-3, whereas by Corollary 3.12, at most two of them are of type 2-2, so $\Lambda_{\mathrm{HK}}$ is 

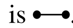
.

Let $W$ be the component corresponding to the genus one node, and $A$ the type 3-3 annulus. If a core of $A$ is a $(p, q)$-curve with respect to $(\mathbb{S}^3, W)$, then the linking number of the components of $\partial A$ in $\mathbb{S}^3$ is $\pm pq$. Since $A$ has a trivial slope pair, $pq = 0$, and by the essentiality of $A$, $q \neq 0$ and therefore $(p, q) = (0, \pm 1)$. Thus $W$ has no exceptional fiber. □

**Lemma 3.15.** *The exterior $E(\mathrm{HK})$ contains a non-characteristic, non-separating annulus $A$ if and only if $\Lambda_{E(\mathrm{HK})}$ is •—•. In addition, $A$ is of type* 3-3 *with a boundary slope pair $(\frac{p}{q}, pq)$, $pq \neq 0$, and is the unique non-separating annulus in* $E(\mathrm{HK})$.

*Proof.* "⇐": Let $X$ be the component corresponding to the genus two node. By Proposition 2.21, $X$ is I-fibered over a Klein bottle $B$ with an open disk removed. Any non-separating simple loop $l$ in $B$ induces an essential annulus $A$ in $X$ and hence in $E(\mathrm{HK})$ by Lemma 2.15. Since $l$ cannot be isotoped away from essential separating loops that are not parallel to $\partial B$ in $B$ by [8, Theorem 3.3], $A$ is not characteristic.

"⇒": By Theorem 2.8 and Lemma 2.15, we may assume $A$ is an essential annulus in a component $X$ of a characteristic submanifold of $E(\mathrm{HK})$ with $A$ non-parallel to any component of $\partial_{E(\mathrm{HK})} X$. By Proposition 2.21, $X$ is either an I-bundle with $\chi(\partial X) < 0$ or a Seifert fibered solid torus. The latter is impossible because $\#\partial_{E(\mathrm{HK})} X \leq 3$ by Theorem 2.23 and $X$ has no exceptional fiber by Theorem 3.14 when $\#\partial_{E(\mathrm{HK})} X = 3$.

Therefore, $X$ is an I-bundle over a Möbius band or Klein bottle with an open disk removed; in particular, $\Lambda_{E(\mathrm{HK})}$ is  or . The former is ruled out by Proposition 2.21(vi), so $X$ is an I-bundle over a Klein bottle with an opened disk removed $B$, and $\Lambda_{E(\mathrm{HK})}$ is .

By [8, Theorem 3.3], every two non-separating simple loops in a Klein bottle with an opened disk removed are isotopic, so $A$ is the unique non-separating annulus in $E(\mathrm{HK})$. Now, to determine the type of $A$, first note that the annulus $A' := \partial_{E(\mathrm{HK})}X \subset E(\mathrm{HK})$ is an annulus non-isotopic to $A$, so $A$ is not of type 2-1 or of type 3-3 with a slope pair $(\frac{p}{q}, \frac{q}{p})$, $pq \neq 0$, by Corollary 3.10. Denote by $X'$ the solid torus $\overline{E(\mathrm{HK}) - X}$ and observe that, by the essentiality of $A' = X \cap X' \subset E(\mathrm{HK})$, the homomorphism

$$H_1(A') \simeq \mathbb{Z} \xrightarrow{k} \mathbb{Z} \simeq H_1(X')$$

induced by the inclusion neither is trivial nor is an isomorphism, namely $k \neq 0, \pm 1$. On the other hand, the decomposition $E(\mathrm{HK}_A) = (X - \mathring{\mathfrak{N}}(A)) \cup X'$ gives us the isomorphism:

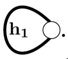
$$H_1(E(\mathrm{HK}_A)) \simeq \langle v_+, v_-, u \rangle / (v_+ + v_- = \pm k u), \tag{3.1}$$

where $u$ is a generator of $H_1(X')$, $v_\pm = [l_\pm]$, and $l_\pm$ are the cores of the frontier $\partial_{E(\mathrm{HK})}\mathfrak{N}(A)$. If $A$ is of type 2-2, then $v_-$ is trivial in $H_1(E(\mathrm{HK}_A))$ by Lemma 3.6, so $H_1(E(\mathrm{HK}_A)) \simeq \mathbb{Z}$, a contradiction. If $A$ is of type 3-3 with a trivial slope pair, then at least one of $v_+, v_-$ is not a generator by Lemma 3.3, contradicting (3.1), as both $\{v_+, u\}$ and $\{v_-, u\}$ form a basis of $H_1(E(\mathrm{HK}_A))$. Therefore $A$ is of type 3-3 with a slope pair $(\frac{p}{q}, pq)$, $pq \neq 0$. □

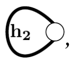
**Corollary 3.16.** *If $A$ is of type 2 or of type 3-3 with a trivial slope pair or a slope pair $(\frac{p}{q}, \frac{q}{p})$, $pq \neq 0$, then $A$ is characteristic.*

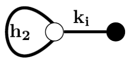
**Corollary 3.17.** *Up to isotopy, non-separating annuli in $E(\mathrm{HK})$ are mutually disjoint.*

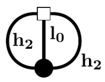
**Theorem 3.18 (Classification Theorem).**

(i) *If $A$ is of type 2-1, then $\Lambda_{\mathrm{HK}}$ is* 
.

(ii) *If $A$ is of type 2-2, then $\Lambda_{\mathrm{HK}}$ is one of the following:*

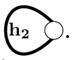


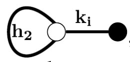
, 

$i = 1$ *or* 2, 

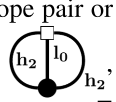
□ = • *or* ∘.

*Proof.* (i) follows from Corollary 3.10. To see (ii), let $S$ be a characteristic surface of $E(\mathrm{HK})$. By Theorem 2.23, $S$ consists of at most three annuli, one of which is $A$ by Corollary 3.16.

**Case 1: $\#S = 1$.** This implies $\Lambda_{\mathrm{HK}}$ is 
.

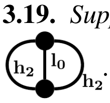
**Case 2: $\#S = 2$.** Let $A' \in S$ be the other annulus. Then by Corollaries 3.10 and 3.5, it is not of type 2-1 or of type 3-3 with a non-trivial slope pair. By Lemma 3.13 and Corollary 3.16, it is not of type 2-2 or of type 3-3 with a trivial slope pair since $\#S = 2$. Therefore $A'$ is separating, and by Lemma 3.1, it is not of type 4-1, so $\Lambda_{\mathrm{HK}}$ is 

, $i = 1$ or 2.

**Case 3: $\#S = 3$.** Let $A', A''$ be the other two annuli. Then at least one of them, say $A'$, is non-separating by Theorem 2.23. On the other hand, $A'$ cannot be of type 2-1 or of type 3-3 with a non-trivial slope by Corollaries 3.10 and 3.5, so $A'$ is of type 2-2 or of type 3-3 with a trivial slope pair; this implies that $A''$ is of type 3-3 with a trivial slope pair or of type 2-2, respectively, by Lemma 3.13 and Corollary 3.16. Therefore $\Lambda_{\mathrm{HK}}$ is 

, □ = • or ∘. □

We now give a characterization of $(\mathbb{S}^3, 4_1)$ in terms of characteristic diagrams.

**Lemma 3.19.** *Suppose the annulus diagrams of the handlebody-knots $(\mathbb{S}^3, \mathrm{HK}), (\mathbb{S}^3, \widetilde{\mathrm{HK}})$ are both* 

*. Then $(\mathbb{S}^3, \mathrm{HK})$ and $(\mathbb{S}^3, \widetilde{\mathrm{HK}})$ are equivalent.*

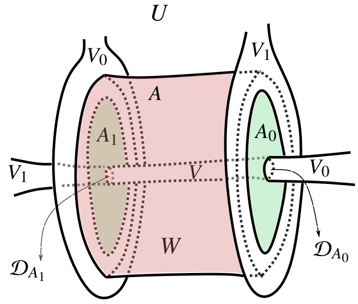



FIGURE 4. Decompose $E(\mathrm{HK})$ and HK.

*Proof.* Let $A$ (resp. $\tilde{A}$) and $A_0, A_1$ (resp. $\tilde{A}_0, \tilde{A}_1$) be the type 3-3 annulus and the two type 2-2 annuli in $E(\mathrm{HK})$ (resp. $E(\widetilde{\mathrm{HK}})$), respectively, and denote by $l_{A_0}, l_{A_1}$ (resp. $\tilde{l}_{A_0}, \tilde{l}_{A_1}$ ) the boundary components of $A_0, A_1$ (resp. $\tilde{A}_0, \tilde{A}_1$) that bound disks $\mathcal{D}_{A_0}, \mathcal{D}_{A_1}$ (resp. $\mathcal{D}_{\tilde{A}_0}, \mathcal{D}_{\tilde{A}_1}$) in HK (resp. $\widetilde{\mathrm{HK}}$), respectively. Also, let $U \subset E(\mathrm{HK}), \tilde{U} \subset E(\widetilde{\mathrm{HK}})$ be the I-bundles and $W, \tilde{W}$ their exteriors in $E(\mathrm{HK}), E(\widetilde{\mathrm{HK}})$, respectively. Note that $W$ (resp. $\tilde{W}$) is a Seifert fibered solid torus whose frontier in $E(\mathrm{HK})$ (resp. $E(\widetilde{\mathrm{HK}})$) is the union $A \cup A_0 \cup A_1$ (resp. $\tilde{A} \cup \tilde{A}_0 \cup \tilde{A}_1$), and $l_{A_0}, l_{A_1}$ (resp. $\tilde{l}_{A_0}, \tilde{l}_{A_1}$) lie in different lids of $U$ (resp. $\tilde{U}$); see Fig. 4.

To show $(\mathbb{S}^3, \mathrm{HK}), (\mathbb{S}^3, \widetilde{\mathrm{HK}})$ are equivalent, we first construct a homeomorphism

$$f_0 : (U, A, A_0, A_1, l_{A_0}, l_{A_1}) \to (\tilde{U}, \tilde{A}, \tilde{A}_0, \tilde{A}_1, l_{\tilde{A}_0}, l_{\tilde{A}_1}).$$

To do this, we identify $U, \tilde{U}$ with $P \times I, \tilde{P} \times I$, respectively, where $P, \tilde{P}$ are pairs of pants. Let $C, C_0, C_1$ (resp. $\tilde{C}, \tilde{C}_0, \tilde{C}_1$) be the components of $\partial P$ (resp. $\partial \tilde{P}$), and identify $(C_0 \times I, C_0 \times 0)$ and $(C_1 \times I, C_1 \times 1)$ with $(A_0, l_{A_0})$ and $(A_1, l_{A_1})$ (resp. $(\tilde{C}_0 \times I, \tilde{C}_0 \times 0)$ and $(\tilde{C}_1 \times I, \tilde{C}_1 \times 1)$ with $(\tilde{A}_0, \tilde{l}_{A_0})$ and $(\tilde{A}_1, \tilde{l}_{A_1})$), respectively.

It is not difficult to see there exist homeomorphisms $g_i : P \times i \to \tilde{P} \times i$ that map $(C \times i, C_0 \times i, C_1 \times i)$ to $(\tilde{C} \times i, \tilde{C}_0 \times i, \tilde{C}_1 \times i)$, $i = 0, 1$. On the other hand, since the mapping class group of a three-times punctured sphere is given by the permutation group on the punctures, $g_0, g_1$ can be extended to $f_0$.

Now, let $V, V_0, V_1 \subset \mathrm{HK}$ (resp. $\tilde{V}, \tilde{V}_0, \tilde{V}_1 \subset \widetilde{\mathrm{HK}}$) be the 3-ball and two solid tori cut off by $\mathcal{D}_{A_0}, \mathcal{D}_{A_1}$ (resp. $\mathcal{D}_{\tilde{A}_0}, \mathcal{D}_{\tilde{A}_1}$) such that $\mathcal{D}_{A_i}, P \times i \subset \partial V_i$ (resp. $\mathcal{D}_{\tilde{A}_i}, \tilde{P} \times i \subset \partial \tilde{V}_i$), $i = 0, 1$. Then the exterior $E(V \cup W)$ (resp. $E(\tilde{V} \cup \tilde{W})$) of $V \cup W$ (resp. $\tilde{V} \cup \tilde{W}$) in $\mathbb{S}^3$ is $U \cup V_0 \cup V_1$ (resp. $\tilde{U} \cup \tilde{V}_0 \cup \tilde{V}_1$); see Fig. 4, and $f_0$ can be extended to a homeomorphism

$$f_1 : (E(V \cup W), U, V_0, V_1) \to (E(\tilde{V} \cup \tilde{W}), \tilde{U}, \tilde{V}_0, \tilde{V}_1)$$

as follows. Extend first the restriction $f_0|_{P \times i}$ to a homeomorphism

$$\bar{f}_0 : \partial(V_0 \cup V_1) \to \partial(\tilde{V}_0 \cup \tilde{V}_1)$$

that sends a meridian of $V_i$ to a meridian of $\tilde{V}_i$, $i = 0, 1$; this can be done because $\partial V_i - \mathring{P} \times i$ consists of an annulus and the disk $\mathcal{D}_{A_i}$. Then extend $\bar{f}_0$ to a homeomorphism from $V_0 \cup V_1$ to $\tilde{V}_0 \cup \tilde{V}_1$, which, together with $f_0$, induces $f_1$.

Observe that $E(V \cup W)$ (resp. $E(\tilde{V} \cup \tilde{W})$) meets $W$ (resp. $\tilde{W}$) at an annulus $A^\flat$ (resp. $\tilde{A}^\flat$) Thus we can extend the restriction $f_1|_{A^\flat}$ to a homeomorphism

$$\bar{f}_1 : (W, A^\flat) \to (\tilde{W}, \tilde{A}^\flat).$$

Gluing $\bar{f}_1$ and $f_1$ together yields a homeomorphism

$$f_2 : (E(V), U, V_1, V_2, W) \to (E(\tilde{V}), \tilde{U}, \tilde{V}_1, \tilde{V}_2, \tilde{W}).$$

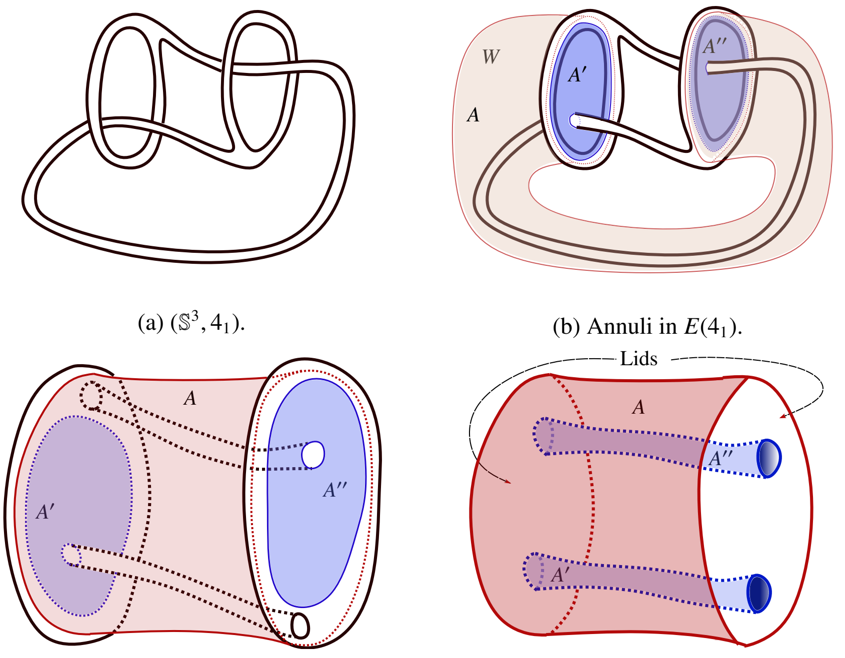
Since $V \subset \mathrm{HK}$, $\tilde{V} \subset \widetilde{\mathrm{HK}}$ are 3-balls, by the Alexander trick, $f_2|_{\partial V}$ can be extended to a homeomorphism

$$\bar{f}_2 : (V, \partial V) \to (\tilde{V}, \partial\tilde{V}).$$

Gluing $\bar{f}_2$ and $f_2$ together yields a homeomorphism

$$(\mathbb{S}^3, U, W, V_1, V_2, V) \to (\mathbb{S}^3, \tilde{U}, \tilde{W}, \tilde{V}_1, \tilde{V}_2, \tilde{V}),$$

and hence an equivalence between $(\mathbb{S}^3, \mathrm{HK})$ and $(\mathbb{S}^3, \widetilde{\mathrm{HK}})$. □



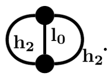

(a) $(\mathbb{S}^3, 4_1)$.

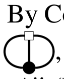
(b) Annuli in $E(4_1)$.

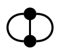
(c) $E(4_1) - \mathring{W}$.

(d) I-bundle over a twice-punctured disk.

FIGURE 5. Annulus diagram of $(\mathbb{S}^3, 4_1)$.

**Lemma 3.20.** *The annulus diagram of* $(\mathbb{S}^3, 4_1)$ *is* .

*Proof.* Recall that $(\mathbb{S}^3, 4_1)$ is equivalent to the handlebody-knot in Fig. 5a, and its exterior admits three annuli $A, A', A''$ as depicted in Fig. 5b, where $A$ is of type 3-3, and $A', A''$ are of type 2-2. By Corollary 3.16, they are characteristic and hence the characteristic diagram of $E(4_1)$ is , $\square = \circ$ or $\bullet$. Let $W \subset E(\mathrm{HK})$ be the Seifert fibered solid torus cut off by $A \cup A' \cup A''$ (Fig. 5b). Then as shown in Figs. 5c and 5d, the exterior of $W \subset E(\mathrm{HK})$ together with $A \cup A' \cup A''$ is an I-bundle over a pair of pants, and hence the assertion. □

**Theorem 3.21.** *The characteristic diagram* $\Lambda_{E(\mathrm{HK})}$ *is*  *if and only if* $(\mathbb{S}^3, \mathrm{HK})$ *is equivalent to* $(\mathbb{S}^3, 4_1)$.

*Proof.* This follows from Theorem 3.14 and Lemmas 3.19 and 3.20. □

## 4. HANDLEBODY-KNOT SYMMETRIES

In this section, we compute the symmetry groups of handlebody-knots whose exteriors contain a type 2 annulus, based on the classification in Theorem 3.18.

4.1. **Mapping class group.** We recall some properties of mapping class groups. Given subpolyhedra $X_1, \ldots, X_n$ of an oriented manifold $M$, the space of self-homeomorphisms of $M$ preserving $X_i$, $i = 1, \ldots, n$, setwise (resp. pointwise) is denoted by

$$\mathcal{H}omeo(M, X_1, \ldots, X_n) \qquad (\text{resp. } \mathcal{H}omeo(M, \operatorname{rel} X_1, \ldots, X_n)\ ),$$

and the mapping class group of $(M, X_1, \ldots, X_n)$ is defined as

$$\mathcal{MCG}(M, X_1, \ldots, X_n) := \pi_0(\mathcal{H}omeo(M, X_1, \ldots, X_n))$$
$$\Big(\text{resp. } \mathcal{MCG}(M, \operatorname{rel} X_1, \ldots, X_n) := \pi_0(\mathcal{H}omeo(M, \operatorname{rel} X_1, \ldots, X_n))\Big).$$

The "+" subscript is added when only orientation-preserving homeomorphisms are used:

$$\mathcal{H}omeo_+(M, X_1, \ldots, X_n) \quad (\text{resp. } \mathcal{H}omeo_+(M, \operatorname{rel} X_1, \ldots, X_n)),$$
$$\mathcal{MCG}_+(M, X_1, \ldots, X_n) \quad (\text{resp. } \mathcal{MCG}_+(M, \operatorname{rel} X_1, \ldots, X_n)).$$

Given $f \in \mathcal{H}omeo(M, X_1, \cdots, X_n)$, $[f]$ denotes the mapping class it represents. If $M = \mathbb{S}^3$, then we call the mapping class group the *symmetry group* of $(M, X_1, \ldots, X_n)$, and every 3-submanifold of $\mathbb{S}^3$ carries the induced orientation.

**Lemma 4.1** (Cutting Homomorphism, [5, Proposition 3.20])**.** *Let $\Sigma$ be an oriented closed surface and $\alpha_1, \ldots, \alpha_n$ mutually disjoint and non-homotopic simple loops in $\Sigma$. Then there is a well-defined homomorphism*

$$\text{cut} : \mathcal{MCG}_+(\Sigma, [\alpha_1], \ldots, [\alpha_n]) \to \mathcal{MCG}_+(\Sigma - \mathfrak{N}(\alpha_1 \cup \cdots \cup \alpha_n))$$

*whose kernel is generated by the Dehn twists about $\alpha_1, \ldots, \alpha_n$, where the group*

$$\mathcal{MCG}_+(\Sigma, [\alpha_1], \ldots, [\alpha_n])$$

*is the subgroup of $\mathcal{MCG}_+(\Sigma)$ given by homeomorphisms that preserve the isotopy classes of $\alpha_1, \ldots, \alpha_n$, respectively.*

Then next two lemmas are proved in [3] and [7] (see also [23, Remark 2.1]).

**Lemma 4.2** ([3, Lemma 2.3])**.** *If $(\mathbb{S}^3, \mathrm{HK})$ is atoroidal, then*

$$\mathcal{MCG}_+(E(\mathrm{HK}), \operatorname{rel} \partial E(\mathrm{HK})) \simeq \{1\}.$$

**Lemma 4.3** ([7])**.** *The symmetry group $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK})$ is finite if and only if $(\mathbb{S}^3, \mathrm{HK})$ is non-trivial and atoroidal.*

**Lemma 4.4.** *Let $(W, \underline{\underline{w}})$ be an oriented solid torus with boundary pattern, where $\underline{\underline{w}} = \{\mathtt{G}_1, \mathtt{G}_2, \ldots, \mathtt{G}_n\}$, and $\mathtt{G}_i$, $i = 1, \ldots, n$, are all annuli, and $|\underline{\underline{w}}| = \partial W$.*

*Suppose $f \in \mathcal{H}omeo_+(W, \mathtt{G}_1, \ldots, \mathtt{G}_n)$ does not swap the components of $\partial \mathtt{G}_1$—which holds automatically when $n > 2$. Then $f$ is isotopic to* id *in $\mathcal{H}omeo_+(W, \mathtt{G}_1, \ldots, \mathtt{G}_n)$.*

*Proof.* Without loss of generality, it may be assumed that $\mathtt{G}_i \cap \mathtt{G}_j \neq \emptyset$ if and only if $|i-j| \leq 1$. Denote by $\mathtt{U}_k$ the union $\mathtt{G}_1 \cup \cdots \cup \mathtt{G}_k$ and set $\mathtt{U}_0 = \emptyset$. Observe that, if $f|_{\mathtt{U}_{k-1}} = \mathrm{id}$, $1 \leq k \leq n$, then $f$ can be isotoped in

$$\mathcal{H}omeo_+(W, \mathtt{G}_k, \ldots, \mathtt{G}_n, \operatorname{rel} \mathtt{U}_{k-1}), \tag{4.1}$$

so that $f|_{\mathtt{U}_k} = \mathrm{id}$. To see this, we first isotope $f|_{\mathtt{U}_k}$ to id in $\mathcal{H}omeo(\mathtt{U}_k, \operatorname{rel} \mathtt{U}_{k-1})$ as follows: In the case $k = 1$, it results from the assumption that $f$ does not swap components of $\partial \mathtt{G}_1$, whereas if $1 < k < n$, it follows from the fact that $\mathcal{MCG}(\mathtt{U}_k, \operatorname{rel} \mathtt{U}_{k-1}) = \{1\}$. If $k = n$, then it is a consequence of $f$ sending meridian disks of $W$ to themselves. Via a regular neighborhood of $\mathtt{U}_k$ in $W$, the isotopy of $f|_{\mathtt{U}_k}$ can be extended to an isotopy in (4.1) that isotopes $f$ so that $f|_{\mathtt{U}_k} = \mathrm{id}$. Hence by induction, we may assume $f \in \mathcal{H}omeo(W, \operatorname{rel} \partial W)$, and the assertion follows since $\mathcal{MCG}(W, \operatorname{rel} W) \simeq \{1\}$. □

**Lemma 4.5.** *Let $W$ be a solid torus in $\mathbb{S}^3$ and $A \subset \partial W$ an annulus with $H_1(A) \to H_1(W)$ non-trivial and not an isomorphism. Then $\mathcal{MCG}(\mathbb{S}^3, W, A) \simeq \mathcal{MCG}_+(\mathbb{S}^3, W, A)$.*

*Proof.* Orient the cores $c_A, c_W$ of $A, W$, respectively, so that the induced homomorphism $H_1(A) \to H_1(W)$ sends $[c_A]$ to $q[c_W]$, $q \geq 0$. Since $H_1(A) \to H_1(W)$ is non-trivial, and not an isomorphism, we have $q \neq 0, 1$. Since $q \neq 1$, the linking number $\ell k(c_A, c_W)$ is non-zero. On the other hand, $q \neq 0$ implies any self-homeomorphism $f$ of $(\mathbb{S}^3, W, A)$ either preserves or reverses the orientations of both $c_A, c_W$, and hence $\ell k(c_A, c_W) = \ell k(f(c_A), f(c_W))$, and $f$ is therefore orientation-preserving, given $\ell k(c_A, c_W) \neq 0$. □

**Lemma 4.6.** *Let $W$ be an oriented solid torus, and $A_1, A_2 \subset \partial W$ two disjoint annuli with $H_1(A_i) \to H_1(W)$, $i = 1, 2$, isomorphisms. Then $\mathcal{MCG}_+(W, A_1 \cup A_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathcal{MCG}(W, A_1 \cup A_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

*Proof.* Identify $W$ with $\mathrm{Q} \times S^1 \subset \mathbb{R}^2 \times \mathbb{C}$, where $S^1$ is the unit circle $\{z = e^{i\theta}\}$ and Q is the square given by

$$\{(x, y) \mid -1 \leq x, y \leq 1\}.$$

Identify $A_1, A_2$ with the annuli given by $y = \pm 1$, and their cores $c_1, c_2$ the loops given by $x = 0$, and denote by $B_1, B_2$ the annuli in the closure of $\partial W - A$.

Consider $\mathtt{r}_i \in \mathcal{H}omeo_+(W, A_1 \cup A_2)$, $i = 1, 2$, defined by the assignments:

$$\begin{aligned} \mathrm{Q} \times S^1 &\to \mathrm{Q} \times S^1 \\ (x, y, z) &\mapsto (-x, -y, z), \\ (x, y, z) &\mapsto (-x, y, \bar{z}) \end{aligned}$$

respectively. Note that $\mathtt{r}_1, \mathtt{r}_2$ both are of order 2 and commute with each other. In addition, $\mathtt{r}_1$ swaps $A_1, A_2$ and also $B_1, B_2$, whereas $\mathtt{r}_2$ swaps $A_1, A_2$ but preserves $B_1, B_2$, so their composition $\mathtt{r}_1 \circ \mathtt{r}_2$ swaps $B_1, B_2$ but preserves $A_1, A_2$. This implies they represent distinct mapping classes. Since every $f \in \mathcal{H}omeo(W, A_1 \cup A_2)$ either swaps $A_1, A_2$ (resp. $B_1, B_2$) or preserves them, by Lemma 4.4, $\{[\mathtt{r}_1], [\mathtt{r}_2]\}$ generates $\mathcal{MCG}_+(W, A_1 \cup A_2)$.

To see $\mathcal{MCG}(W, A_1 \cup A_2) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, consider $\mathtt{m} \in \mathcal{H}omeo(W, A_1 \cup A_2)$ defined by the assignment

$$\begin{aligned} \mathrm{Q} \times S^1 &\to \mathrm{Q} \times S^1 \\ (x, y, z) &\mapsto (-x, y, z), \end{aligned}$$

which is orientation-reversing, commutes with $\mathtt{r}_i, i = 1, 2$, and together with $\mathtt{r}_i, i = 1, 2$, generates $\mathcal{MCG}(W, A_1 \cup A_2)$. □

**Lemma 4.7.** *Let $W$ be an oriented solid torus and $A_1, A_2, A_3 \subset \partial W$ three disjoint annuli with $H_1(A_i) \to H_1(W)$, $i = 1, 2, 3$, isomorphisms. Then $\mathcal{MCG}_+(W, A_1, A_2 \cup A_3) \simeq \mathbb{Z}_2$ and $\mathcal{MCG}(W, A_1, A_2 \cup A_3) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.*

*Proof.* Identify $W$ with $\mathcal{H} \times S^1 \subset \mathbb{C} \times \mathbb{C}$, where $S^1 \subset \mathbb{C}$ is the unit circle, and $\mathcal{H} \subset \mathbb{C}$ the regular hexagon with center at the origin and vertices $v_k = e^{\frac{2\pi k}{6}}$, $k = 1, \ldots, 6$. Identify $A_k$ with the product of $S^1$ and the edge $e_k$ connecting $v_{2k-1}, v_{2k}$, $k = 1, 2, 3$. Denote by $\mathtt{r} \in \mathcal{H}omeo_+(W, A_1, A_2 \cup A_3)$ the homeomorphism given by

$$\begin{aligned} \mathcal{H} \times S^1 &\to \mathcal{H} \times S^1 \\ (z_1, z_2) &\mapsto (-\bar{z}_1, \bar{z}_2); \end{aligned}$$

$\mathtt{r}$ swaps $A_2, A_3$ and hence represents a non-trivial mapping class in $\mathcal{MCG}_+(W, A_1, A_2 \cup A_3)$. Since every $f \in \mathcal{H}omeo_+(W, A_1, A_2 \cup A_3)$ either swaps $A_2, A_3$ or preserves them, by Lemma 4.4, either $[f] = [\mathtt{r}]$ or $[f]$ is trivial, so $\mathcal{MCG}_+(W, A_1, A_2 \cup A_3) \simeq \mathbb{Z}_2$. On the other hand, there is an orientation-reversing homeomorphism $\mathtt{m} \in \mathcal{H}omeo(W, A_1, A_2 \cup A_3)$ defined by

$$\begin{aligned} \mathcal{H} \times S^1 &\to \mathcal{H} \times S^1 \\ (z_1, z_2) &\mapsto (z_1, \bar{z}_2), \end{aligned}$$

which is of order 2 and commutes with $\mathtt{r}$, and $\{[\mathtt{r}], [\mathtt{m}]\}$ generates $\mathcal{MCG}(W, A_1, A_2 \cup A_3)$. □

The next lemma follows from [11, Section 2] (see also [12, Theorem 1]).

**Lemma 4.8.** *Given a handlebody-knot* $(\mathbb{S}^3, \mathrm{HK})$ *and an essential surface* $S$ *in* $E(\mathrm{HK})$*, the natural homomorphisms*

$$\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}, S) \to \mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}),$$
$$\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(S)) \to \mathcal{MCG}(\mathbb{S}^3, \mathrm{HK})$$

*are injective.*

4.2. **Symmetry groups of handlebody-knots.** Here $(\mathbb{S}^3, \mathrm{HK})$ is an atoroidal handlebody-knot, and $A \subset E(\mathrm{HK})$ a type 2 essential annulus. The symbols $l, l_A, \mathrm{HK}_A, A_+, A_-$ are as in Section 3. In addition, we identify the intersection $\mathfrak{N}(A)\cap\partial\mathrm{HK}$ with $\mathfrak{N}(l\cup l_A) = \mathfrak{N}(l)\cup\mathfrak{N}(l_A)$.

**Theorem 4.9.** *If* $A$ *is of type* 2-1*, then* $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2$ *and* $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2\times\mathbb{Z}_2$.

*Proof.* Note first that the injection $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A)) \to \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK})$ in Lemma 4.8 is an isomorphism since $A$ is unique by Theorem 3.18, and composing its inverse with the homomorphism $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A)) \xrightarrow{\Phi} \mathcal{MCG}_+(\mathfrak{N}(A), A_+ \cup A_-)$ given by restriction to $\mathfrak{N}(A)$ yields the homomorphism

$$\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) \simeq \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A)) \to \mathcal{MCG}_+(\mathfrak{N}(A), A_+ \cup A_-).$$

Since no self-homeomorphism of $(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A))$ can swap $\mathfrak{N}(l), \mathfrak{N}(l_A)$, by Lemma 4.6, it suffices to show the injectivity of $\Phi$ as it implies the injectivity of

$$\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A)) \to \mathcal{MCG}(\mathfrak{N}(A), A_+ \cup A_-).$$

To see $\Phi$ is injective, let $[f] \in \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A))$ with $\Phi([f]) = 1$. This implies $f|_{\partial\mathrm{HK}-\mathfrak{N}(l\cup l_A)}$ does not permute punctures of the four-times punctured sphere $\partial\mathrm{HK}-\mathfrak{N}(l\cup l_A)$, and thus $[f|_{\partial\mathrm{HK}-\mathfrak{N}(l\cup l_A)}] = 1 \in \mathcal{MCG}_+(\partial\mathrm{HK}-\mathfrak{N}(l\cup l_A))$ since $[f|_{\partial\mathrm{HK}-\mathfrak{N}(l\cup l_A)}]$ is of finite order by Lemma 4.3. Again by Lemma 4.3, $[f|_{\partial\mathrm{HK}}]$ is of finite order in $\mathcal{MCG}_+(\partial\mathrm{HK}, [l], [l_A])$; hence by Lemma 4.1, it is the identity. Because $f|_{\partial\mathrm{HK}}$ is isotopic to id, $f$ can be isotoped in $\mathcal{H}omeo(\mathbb{S}^3, \mathrm{HK})$ so that $f|_{\partial\mathrm{HK}} = \mathrm{id}$. Applying Lemma 4.2, one can further isotope $f$ to id in $\mathcal{H}omeo(\mathbb{S}^3, \mathrm{rel}\,\partial\mathrm{HK})$. □

**Theorem 4.10.** *If* $A \subset E(\mathrm{HK})$ *is the unique type* 2-2 *annulus, then* $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) \simeq \{1\}$ *and* $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2$*. If in addition* $E(\mathrm{HK})$ *admits an annulus* $A'$ *of another type, then* $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) \simeq \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) \simeq \{1\}$.

*Proof.* As in the previous case, the uniqueness of $A$ gives us the homomorphism

$$\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) \simeq \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A)) \xrightarrow{\Phi} \mathcal{MCG}_+(\mathfrak{N}(A), A_+ \cup A_-).$$

The first assertion follows once we show the injectivity of $\Phi$ because, given any $f \in \mathcal{H}omeo_+(\mathbb{S}^3, \mathrm{HK}, \mathfrak{N}(A))$, it can neither swap $A_+, A_-$ nor swap $\mathfrak{N}(l), \mathfrak{N}(l_A)$ by the definition of a type 2-2 annulus. The second assertion can be derived from the first as follows: by Theorem 3.18, the annulus $A'$ is the unique type 3-2 annulus in $E(\mathrm{HK})$. Let $W \subset E(\mathrm{HK})$ be the solid torus cut off by $A'$. Then by the essentiality of $A'$, $H_1(A') \to H_1(W)$ is non-trivial and not an isomorphism. On the other hand, by Lemma 4.8, there is a homomorphism

$$\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) \simeq \mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}, A') = \mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}, W) \to \mathcal{MCG}(\mathbb{S}^3, W, A').$$

Now, if $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK})$ is non-trivial, then by the first assertion, the mapping class group $\mathcal{MCG}(\mathbb{S}^3, W, A')$ contains a mapping class represented by an orientation-reversing homeomorphism, contradicting Lemma 4.5.

We now prove the injectivity of $\Phi$. Let $[f] \in \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK})$ with $\Phi([f]) = 1 \in \mathcal{MCG}_+(\mathfrak{N}(A), A_+ \cup A_-)$. We can isotope $g := f|_{\partial\mathrm{HK}}$ in $\mathcal{H}omeo_+(\partial\mathrm{HK}, \mathfrak{N}(l \cup l_A))$ so that $g|_{\mathfrak{N}(l\cup l_A)} = \mathrm{id}$. Let $D$ be the meridian disk disjoint from $l_A$ and dual to $l$. Then one can further isotope $g$ in $\mathcal{H}omeo_+(\partial\mathrm{HK}, \mathrm{rel}\,\mathfrak{N}(l \cup l_A))$ so that $g|_{\mathfrak{N}(\partial D)} = \mathrm{id}$. In other words,

$f|_{\partial \mathrm{HK}}$ represents a mapping class in $\mathcal{MCG}_+(\partial \mathrm{HK}, \mathrm{rel}\, \mathfrak{N}(\partial D \cup l))$. Now, the homomorphism induced by the inclusion

$$\mathcal{MCG}_+(\partial \mathrm{HK}, \mathrm{rel}\, \mathfrak{N}(\partial D \cup l)) \to \mathcal{MCG}_+(\partial \mathrm{HK})$$

is injective by [5, Theorem 3.18], and by Lemma 4.3, $[f|_{\partial \mathrm{HK}}] \in \mathcal{MCG}_+(\partial \mathrm{HK})$ is of finite order, so $[f|_{\partial \mathrm{HK}}] \in \mathcal{MCG}_+(\partial \mathrm{HK}, \mathrm{rel}\, \mathfrak{N}(\partial D \cup l))$ is also of finite order. The group $\mathcal{MCG}_+(\partial \mathrm{HK}, \mathrm{rel}\, \mathfrak{N}(\partial D \cup l))$ is, however, torsion free, and hence $f|_{\partial \mathrm{HK}}$ is isotopic to id in $\mathcal{H}omeo_+(\partial \mathrm{HK})$. We may thence isotope $f$ in $\mathcal{H}omeo_+(\mathbb{S}^3, \mathrm{HK})$ so that $f|_{\partial \mathrm{HK}} = \mathrm{id}$. By Lemma 4.2, $f$ can be further isotoped to id in $\mathcal{H}omeo_+(\mathbb{S}^3, \mathrm{rel}\, \partial \mathrm{HK})$. □

**Theorem 4.11.** *If $A \subset E(\mathrm{HK})$ is of type 2-2 but not the unique type 2-2 annulus, then $\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2$ and $\mathcal{MCG}(\mathbb{S}^3, \mathrm{HK}) < \mathbb{Z}_2 \times \mathbb{Z}_2$.*

*Proof.* By Theorem 3.18, $E(\mathrm{HK})$ admits a unique type 3-3 annulus $A_0$, and exactly two non-isotopic type 2-2 annuli $A_1, A_2$; the three annuli together cut off a solid torus $W \subset E(\mathrm{HK})$, and form a characteristic surface of $E(\mathrm{HK})$, and induce, via Lemma 4.8, the homomorphism

$$\begin{aligned}\mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}) &\simeq \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, A_0, A_1 \cup A_2)\\ &= \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, W) \xrightarrow{\Phi} \mathcal{MCG}_+(W, A_0, A_1 \cup A_2).\end{aligned}$$

It suffices to prove that $\Phi$ is injective, in view of Lemma 4.7.

Let $[f] \in \mathcal{MCG}_+(\mathbb{S}^3, \mathrm{HK}, W)$ with $\Phi([f]) = 1 \in \mathcal{MCG}_+(W, A_0, A_1 \cup A_2)$. Note that $\partial \mathrm{HK} \cap W$ consists of three annuli $B_0, B_1, B_2$; denote by $c_0, c_1, c_2$ their cores, respectively. Since $\Phi([f]) = 1$, $f|_{\partial \mathrm{HK} - (B_0 \cup B_1 \cup B_2)}$ does not permute punctures of $\partial \mathrm{HK} - (B_0 \cup B_1 \cup B_2)$, which is two copies of the three-times punctured sphere, and therefore $[f|_{\partial \mathrm{HK} - (B_0 \cup B_1 \cup B_2)}] = 1 \in \mathcal{MCG}_+(\partial \mathrm{HK} - (B_0 \cup B_1 \cup B_2))$. On the other hand by Lemma 4.3, $[f|_{\partial \mathrm{HK}}]$ is of finite order in $\mathcal{MCG}_+(\partial \mathrm{HK}, [c_0], [c_1], [c_2])$, and hence trivial therein by Lemma 4.1; in particular, $f|_{\partial \mathrm{HK}}$ is isotopic to id in $\mathcal{H}omeo_+(\partial \mathrm{HK})$. We then isotope $f$ in $\mathcal{H}omeo_+(\mathbb{S}^3, \mathrm{HK})$ so that $f|_{\partial \mathrm{HK}} = \mathrm{id}$; by Lemma 4.2, we can further isotope $f$ to id in $\mathcal{H}omeo_+(\mathbb{S}^3, \mathrm{rel}\, \partial \mathrm{HK})$. □

## 5. Irreducibility and atoroidality

Let $(\mathbb{S}^3, \mathrm{HK})$ be a handlebody-knot, not necessarily atoroidal, and $A \subset E(\mathrm{HK})$ a type 2 annulus, not necessarily essential. The symbols $l_A, l \subset \partial A$, $\mathrm{HK}_A$, and $A_+, A_-, l_+, l_- \subset \partial \mathrm{HK}_A$ are as in Section 3.

### 5.1. Essentiality, irreducibility and triviality.

**Lemma 5.1.** *Let $A$ be of type 2-1 and consider the following statements:*

*(i)* $(\mathbb{S}^3, \mathrm{HK})$ *is trivial.*
*(ii)* *$A$ is inessential.*
*(iii)* $(\mathbb{S}^3, \mathrm{HK}_A)$ *is reducible and there exists a disk $D$ meeting $l_+ \cup l_-$ at one point.*
*(iv)* $(\mathbb{S}^3, \mathrm{HK})$ *is reducible.*

*Then (i) ⇒ (ii) ⇔ (iii) ⇒ (iv).*

*Proof.* Note first that by the definition $A$ is incompressible.

(i) ⇒(ii): Let $D \subset E(\mathrm{HK})$ be a compressing disk of $\partial E(\mathrm{HK})$. Minimize $\# D \cap A$ in the isotopy class of $A$. If $D \cap A = \emptyset$, then, by the incompressibility of $A$ and the fact that $D$ does not separate $l, l_A$ in $E(\mathrm{HK})$, the union $\partial D \cup \partial A$ cuts $\partial E(\mathrm{HK})$ into two pairs of pants $P, P'$, and each is bounded by $l, l_A$, and $\partial D$. The union $P \cup A \cup D$ thus is a torus, and bounds a solid torus $W \subset E(\mathrm{HK})$ by the triviality of $(\mathbb{S}^3, \mathrm{HK})$. Since the core of $A$ is a longitude of $(\mathbb{S}^3, W)$, every meridian disk of $W$ disjoint from $D$ is a $\partial$-compressing disk of $A$. If $D \cap A \neq \emptyset$, then, since $A$ is incompressible, any outermost disk in $D$ cut off by $D \cap A$ is a $\partial$-compressing disk of $A$ by the minimality.

(ii)⇒(iii) & (ii) ⇒(iv): Since $A$ is incompressible, it is $\partial$-compressible. Let $D$ be a $\partial$-compressing disk of $A$. Then $D$ induces a disk in $E(\mathrm{HK}_A)$ meeting $l_+ \cup l_-$ at one point, and hence $(\mathbb{S}^3, \mathrm{HK}_A)$ is reducible. On the other hand, the frontier of a regular neighborhood of $A \cup D \subset E(\mathrm{HK})$ is a $\partial$-compressing disk of $\partial E(\mathrm{HK})$, so $(\mathbb{S}^3, \mathrm{HK})$ is reducible.

(iii)⇒(ii): The disk $D$ induces a $\partial$-compressing disk of $A$. □

**Lemma 5.2.** *Let $A$ be of type* 2-1*. Then $(\mathbb{S}^3, \mathrm{HK})$ is trivial if and only if $(\mathbb{S}^3, \mathrm{HK}_A)$ is trivial and $\{l_+, l_-\}$ is primitive.*

*Proof.* "⇒": By (i)⇒(iii) in Lemma 5.1, there exists a disk $D$ meeting $l_+ \cup l_-$ at one point, say $D \cap l_+ \neq \emptyset$. Then the frontier of a regular neighborhood $\mathfrak{N}(A_+ \cup D)$ of $A_+ \cup D \subset E(\mathrm{HK}_A) - l_-$ is an essential separating disk $D' \subset E(\mathrm{HK}_A)$, which splits $E(\mathrm{HK}_A)$ into two parts: a solid torus where $l_+$ lies and $D$ is a meridian disk and the exterior $E(K)$ of a knot $(\mathbb{S}^3, K)$ where $l_- \subset \partial E(K)$ is a meridian of $(\mathbb{S}^3, K)$. If $(\mathbb{S}^3, \mathrm{HK}_A)$ is non-trivial, then $(\mathbb{S}^3, K)$ is non-trivial and $\partial E(K)$ induces an incompressible torus $T$ in $E(\mathrm{HK}_A)$, which is also incompressible in $E(\mathrm{HK})$, for given any compressing disk $D$ of $T$, one can always isotope $A$ away from $D$ by the incompressibility of $A$; this contradicts $(\mathbb{S}^3, \mathrm{HK})$ is trivial. So $(\mathbb{S}^3, K)$ is trivial, and $E(K)$ is a solid torus with $l_-$ primitive in $E(K)$, and hence the assertion.

"⇐": By [26] (see also [9]), there exists a basis $\{x_+, x_-\}$ of $\pi_1(E(\mathrm{HK}_A))$ with $x_\pm$ in the conjugate classes determined by $l_\pm$, respectively. Since $\pi_1(E(\mathrm{HK}))$ is the HNN extension of $\pi_1(E(\mathrm{HK}_A))$ with respect to $\pi_1(A)$, $\pi_1(E(\mathrm{HK}))$ is free, so $(\mathbb{S}^3, \mathrm{HK})$ is trivial. □

**Lemma 5.3.** *If $A$ is of type* 2-2*, then the following are equivalent:*

*(i)* $(\mathbb{S}^3, \mathrm{HK})$ *is reducible.*
*(ii) $A$ is inessential.*
*(iii)* $(\mathbb{S}^3, \mathrm{HK}_A)$ *is reducible and $l_-$ is homotopically trivial in $E(\mathrm{HK}_A)$.*

*Proof.* (i)⇒(ii): Let $D$ be an essential disk in $E(\mathrm{HK})$. Minimize $\#D \cap A$ in the isotopy class of $A$. Suppose $D \cap A = \emptyset$. Then $\partial D$ lies in the once-punctured torus $T$ in $\partial \mathrm{HK}_A - l_+ \cup l_-$. If $\partial D$ is separating, then it may be assumed that $\partial D$ is parallel to $l_-$, and so $A$ is compressible. If $\partial D$ is non-separating, then there is a loop $l$ in $T$ meeting $\partial D$ once. The frontier of a regular neighborhood of $D \cup l$ in $E(\mathrm{HK}_A) - l_-$ is an essential separating disk disjoint from $A$, and therefore, as in the previous case, the annulus $A$ is compressible. If $D \cap A$ contains a circle, then any innermost disk in $D$ cut off by $D \cap A$ is a compressing disk of $A$. If $D \cap A$ contains only arcs, then an outermost disk $D'$ in $D$ cut off by $D \cap A$ either is a $\partial$-compressing disk of $A$ or induces an essential disk $D''$ disjoint from $A$ in $E(\mathrm{HK})$; either way implies $A$ is inessential.

(ii) ⇒ (iii) & (ii) ⇒ (i): Consider first the case $A$ is compressible. Then any compressing disk $D$ induces a disk $D' \subset E(\mathrm{HK}_A)$ with $\partial D' = l_-$ and a disk $D'' \subset E(\mathrm{HK})$ with $\partial D'' = l_A$, and therefore (iii) and (i). Now if $A$ is $\partial$-compressible, and $D$ is a $\partial$-compressing disk of $A$, then $D$ induces a disk $D' \subset E(\mathrm{HK}_A)$ with $D' \cap l_+$ a point and $D' \cap A_- = \emptyset$; the frontier of a regular neighborhood $\mathfrak{N}(A_+ \cup D')$ in $E(\mathrm{HK}_A) - A_-$ is a separating disk $D''$ with $\partial D''$ parallel to $l_-$; this implies $A$ is compressible, that is, the previous case.

(iii) ⇒ (i) & (iii) ⇒ (ii) follow from Dehn's lemma. □

**Lemma 5.4.** *If $(\mathbb{S}^3, \mathrm{HK}_A)$ is trivial and $l_-$ is homotopically trivial, then $(\mathbb{S}^3, \mathrm{HK})$ is trivial.*

*Proof.* Denote by $D \subset E(\mathrm{HK}_A)$ a disk bounded by $l_-$. Then $D$ splits $E(\mathrm{HK}_A)$ into two solid tori, in one of which $l_+$ is primitive. Therefore $\pi_1(E(\mathrm{HK}_A))$ has a basis $\{x, y\}$ with $x$ in the conjugacy class determined by $l_+$. The assertion then follows from the fact that $\pi_1(E(\mathrm{HK}))$ is the HNN extension of $\pi_1(E(\mathrm{HK}_A))$ with respect to $\pi_1(A)$. □

The converse of Lemma 5.4 is not true in general. As a corollary of Corollary 2.25 and the assertions (ii)⇒(iv) in Lemma 5.1 and (ii)⇒(i) in Lemma 5.3, we have the following.

**Corollary 5.5.** *If $(\mathbb{S}^3, \mathrm{HK})$ is non-trivial and atoroidal, then $A$ is essential.*

5.2. **Non-triviality and atoroidality.** We present here criteria for $(\mathbb{S}^3, \mathrm{HK})$ to be non-trivial and atoroidal in terms of $(\mathbb{S}^3, \mathrm{HK}_A)$ and $l_+, l_-$. Recall first two results on atoroidality:

**Corollary 5.6.** *If* $(\mathbb{S}^3, \mathrm{HK})$ *is non-trivial and atoroidal, then* $(\mathbb{S}^3, \mathrm{HK}_A)$ *is atoroidal.*

*Proof.* This follows from [24, Lemma 4.1], but can also be deduced from Corollary 5.5 since $A$ is essential by Corollary 5.5, if there exists an incompressible torus $T \subset E(\mathrm{HK}_A)$, then any compressing disk of $T$ can be isotoped away from $A$, contradicting the atoroidality of $(\mathbb{S}^3, \mathrm{HK})$. □

**Corollary 5.7.** [24, Lemma 4.9] *Suppose* $(\mathbb{S}^3, \mathrm{HK}_A)$ *is atoroidal, and* $l_- \subset E(\mathrm{HK}_A)$ *is not homotopically trivial if* $A$ *is of type* 2-2. *Then* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal.*

**Proposition 5.8.** *Suppose* $A$ *is of type* 2-1. *Then* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal and* $A$ *is essential if and only if* $(\mathbb{S}^3, \mathrm{HK}_A)$ *either is trivial with* $\{l_+, l_-\}$ *not primitive in* $E(\mathrm{HK}_A)$ *or is non-trivial and atoroidal.*

*Proof.* "⇒": By Corollary 5.6, $(\mathbb{S}^3, \mathrm{HK}_A)$ is atoroidal. If $(\mathbb{S}^3, \mathrm{HK}_A)$ is trivial, then $\{l_+, l_-\}$ cannot be primitive in $E(\mathrm{HK}_A)$ by Lemma 5.2 since $A$ is essential and hence $(\mathbb{S}^3, \mathrm{HK})$ is non-trivial by (i)⇒(ii) in Lemma 5.1.

"⇐": The handlebody-knot $(\mathbb{S}^3, \mathrm{HK})$ is atoroidal by Corollary 5.7, and is non-trivial by Lemma 5.2, so $A$ is essential by Corollary 5.5. □

**Proposition 5.9.** *Suppose* $A$ *is of type* 2-2. *Then* $(\mathbb{S}^3, \mathrm{HK})$ *is atoroidal and* $A$ *is essential if and only if* $(\mathbb{S}^3, \mathrm{HK}_A)$ *either is trivial with* $l_- \subset E(\mathrm{HK}_A)$ *not homotopically trivial or is non-trivial and atoroidal.*

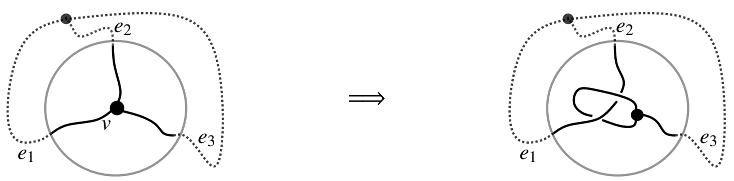
*Proof.* "⇒": By Corollary 5.6, $(\mathbb{S}^3, \mathrm{HK}_A)$ is atoroidal. If $(\mathbb{S}^3, \mathrm{HK}_A)$ is trivial, then by (iii)⇒(ii) in Lemma 5.3, $l_-$ cannot be homotopically trivial since $A$ is essential.

"⇐": By Lemma 5.7, $(\mathbb{S}^3, \mathrm{HK})$ is atoroidal. If $A$ is inessential, then by (ii)⇒(iii) in Lemma 5.3, the handlebody-knot $(\mathbb{S}^3, \mathrm{HK}_A)$ is reducible with $l_- \subset E(\mathrm{HK}_A)$ homotopically trivial, contradicting the assumption and Lemma 2.24. □

## 6. Examples

Here we construct atoroidal handlebody-knots that admit a type 2 essential annulus, and show that annulus diagrams in Theorem 3.18 can all be realized by such handlebody-knots.

6.1. **Looping trivalent spatial graphs.** Let $(\mathbb{S}^3, \Gamma)$ be a spatial graph with $\Gamma$ either a $\theta$-graph or a handcuff graph. Then we can produce a new spatial graph $(\mathbb{S}^3, \Gamma^\circ)$ by replacing a small neighborhood of a trivalent node[1] in $\Gamma$ with a loop as shown in Fig. 6.



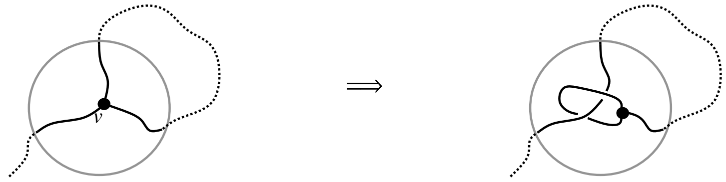

(a) Neighborhood of a trivalent node $v \in \Gamma$. (b) Replacing $v$ with a loop.

Figure 6. Looping of a spatial $\theta$-graph.

Label the trivalent node with $v$ and its three adjacent edges $e_1, e_2, e_3$ as in Fig. 6a. Then the new spatial graph $(\mathbb{S}^3, \Gamma^\circ)$ in Fig. 6b is said to be obtained by *looping* $e_1e_2$ *at* $v$, and $(\mathbb{S}^3, \Gamma^\circ)$ is called a *looping* of $(\mathbb{S}^3, \Gamma)$, provided the resulting spatial graph is connected (see

[1] A neighborhood $\mathfrak{N}(v) \in \Gamma$ of the trivalent node $v$ is a regular neighborhood of $v \subset \mathbb{S}^3$ such that $(\mathfrak{N}(v), \mathfrak{N}(v) \cap \Gamma)$ is homeomorphic to a unit 3-ball with three non-negative axes.

Fig. 7); there are six possible loopings for a spatial $\theta$-graph, and four for a spatial handcuff graph.




(a) Neighborhood of a trivalent node $v \in \Gamma$. (b) Replacing $v$ with a loop.

FIGURE 7. Looping of a spatial handcuff graph.

A double looping $(\mathbb{S}^3, \Gamma^{\circledcirc})$ of $(\mathbb{S}^3, \Gamma)$ is the spatial graph obtained by looping at both trivalent nodes of $\Gamma$. Taking a regular neighborhood of a looping $\Gamma^\circ$ (resp. double looping $\Gamma^{\circledcirc}$) in $\mathbb{S}^3$ gives us a handlebody-knot, denoted by $(\mathbb{S}^3, \mathrm{HK}^\circ_\Gamma)$ (resp. $(\mathbb{S}^3, \mathrm{HK}^{\circledcirc}_\Gamma)$), whose exterior contains a canonical type 2 annulus $A^\circ_\Gamma$ induced by the created loop in $(\mathbb{S}^3, \Gamma^\circ)$.

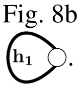
A spatial graph $(\mathbb{S}^3, \Gamma)$ is said to be *nontrivially atoroidal* if the induced handlebody-knot $(\mathbb{S}^3, \mathfrak{N}(\Gamma))$ is non-trivial and atoroidal.

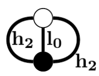
**Lemma 6.1.** *If* $(\mathbb{S}^3, \Gamma)$ *is nontrivially atoroidal, then* $(\mathbb{S}^3, \mathrm{HK}^\circ_\Gamma)$ *induced by a looping of* $(\mathbb{S}^3, \Gamma)$ *is atoroidal, and* $A^\circ_\Gamma \subset E(\mathrm{HK}^\circ_\Gamma)$ *is essential. Furthermore* $A^\circ_\Gamma$ *is of type* 2-1 *and is the unique annulus if* $\Gamma$ *is a* $\theta$*-graph, and is of type* 2-2 *if* $\Gamma$ *is a handcuff graph.*

*Proof.* The disk bounded by a component of $\partial A^\circ_\Gamma$ in $\mathrm{HK}^\circ_\Gamma$ is dual to the two edges being looped, so $A^\circ_\Gamma$ is of type 2-1 if $\Gamma$ is a $\theta$-graph and is of type 2-2 otherwise. The essentiality of $A^\circ_\Gamma$ and atoroidality of $(\mathbb{S}^3, \mathrm{HK}^\circ_\Gamma)$ follow from Propositions 5.8 and 5.9. □

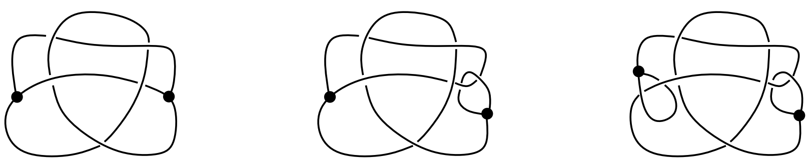
**Corollary 6.2.** *If* $(\mathbb{S}^3, \Gamma)$ *is nontrivially atoroidal, then any handlebody-knot* $(\mathbb{S}^3, \mathrm{HK}^{\circledcirc}_\Gamma)$ *obtained by a double looping of* $(\mathbb{S}^3, \Gamma)$ *is atoroidal, and its exterior contains two non-isotopic type* 2-2 *essential annuli.*

*Proof.* The two canonical annuli are of type 2-2 since any looping $(\mathbb{S}^3, \Gamma^\circ)$ is a spatial handcuff graph. The rest follows from Lemma 6.1. □

As an application of Lemma 6.1 and Corollary 6.2, we consider the spine $(\mathbb{S}^3, \Gamma)$ of $(\mathbb{S}^3, 5_2)$ in [13] as shown in Fig. 8a. Then Fig. 8b is a looping of $(\mathbb{S}^3, \Gamma)$, whose associated handlebody-knot has the annulus diagram 

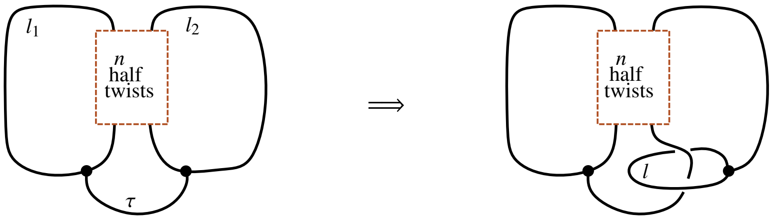
. On the other hand, the double looping of $(\mathbb{S}^3, \Gamma)$ in Fig. 8c induces a handlebody-knot whose annulus diagram is 

.



(a) Spine of $(\mathbb{S}^3, 5_2)$. (b) Looping. (c) Double looping.

FIGURE 8. Handlebody-knots with a type 2 annulus.




(a) $(n,2)$-torus link $(\mathbb{S}^3, L_n)$ with a tunnel $\tau$. (b) (Tunnel) looping of $(\mathbb{S}^3, L_n \cup \tau)$.

FIGURE 9. Construction of Koda's handlebody-knot family.

6.2. **Unknotting annuli of type** 2. As opposed to Lemma 6.1 and Corollary 6.2, here we present a looping operation that yields atoroidal handlebody-knots that admit an essential *unknotting* type 2 annulus.

Let $(\mathbb{S}^3, \Gamma)$ be a spatial $\theta$-graph that is a union of a non-trivial knot $(\mathbb{S}^3, K)$ and a tunnel $\tau$ of $(\mathbb{S}^3, K)$. Let $\kappa_1, \kappa_2$ be the arcs of $K$ cut off by $\tau$. Then a *tunnel* looping of $(\mathbb{S}^3, K \cup \tau)$ is a looping obtained by looping $\kappa_i\tau$ at a trivalent node of $\Gamma = K \cup \tau$, $i = 1$ or $2$.

**Lemma 6.3.** *The handlebody-knot* $(\mathbb{S}^3, \mathrm{HK}^\circ_\Gamma)$ *induced by a tunnel looping of* $(\mathbb{S}^3, \Gamma)$ *is atoroidal, and* $A^\circ_\Gamma$ *is an unknotting essential type* 2-1 *annulus.*

*Proof.* It follows from the "only if " part of Proposition 5.8 since $(\mathbb{S}^3, K)$ is non-trivial. □

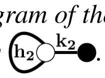
Now, let $(\mathbb{S}^3, \Gamma)$ be the union of a non-split link $(\mathbb{S}^3, L)$ and a tunnel $\tau$ of $(\mathbb{S}^3, L)$.

**Lemma 6.4.** *The handlebody-knot* $(\mathbb{S}^3, \mathrm{HK}^\circ_\Gamma)$ *induced by a looping of* $(\mathbb{S}^3, \Gamma)$ *is atoroidal, and* $A^\circ_\Gamma$ *is an unknotting essential type* 2-2 *annulus.*

*Proof.* Use $(\mathbb{S}^3, L)$ being non-split and apply the "only if" part of Proposition 5.9. □

To show that all annulus diagrams in Theorem 3.18 can be realized by some atoroidal handlebody-knots, we consider the union of an $(n,2)$-torus link $(\mathbb{S}^3, L_n = l_1 \cup l_2)$, $n \in \mathbb{Z}$, with a tunnel $\tau$ as depicted in Fig. 9a. Denote by $(\mathbb{S}^3, \mathrm{HK}_n)$ the handlebody-knot induced by the looping of $(\mathbb{S}^3, L_n \cup \tau)$ in Fig. 9b. Note that $(\mathbb{S}^3, \mathrm{HK}_2)$ is equivalent to $(\mathbb{S}^3, 4_1)$, while $\{(\mathbb{S}^3, \mathrm{HK}_n)\}_{n>2}$ is Koda's handlebody-knot family in [16, Example 3]; Lemmas 6.3 and 6.4 give an alternative way to see they are irreducible, in view of Corollary 2.25.

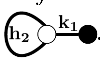
Observe that if $n > 2$ and is even, the handlebody-knot exterior $E(\mathrm{HK}_n)$ contains a type 3-2 annulus $A$ given as follows: let $A_c$ be a cabling annulus in $E(L_n) := \mathbb{S}^3 - \mathring{\mathfrak{N}}(L_n)$ with $\tau \cap E(L_n) \subset A_c$. Let $\mathfrak{N}(l_i)$ be the component of $\mathfrak{N}(L_n)$ containing $l_i$, $i = 1, 2$, and perform the looping construction entirely in $\mathring{\mathfrak{N}}(l_2)$. Then the frontier of $\mathfrak{N}(l_2) \cup \mathfrak{N}(A_c)$ in $E(l_1) := \mathbb{S}^3 - \mathring{\mathfrak{N}}(l_1)$ is an essential annulus $A \subset E(\mathrm{HK}_n)$ of type 3-2ii as $A$ is $\partial$-compressible in $E(l_1)$.

**Corollary 6.5.** *Suppose $n > 2$ and is even. Then the annulus diagram of the handlebody-knot* $(\mathbb{S}^3, \mathrm{HK}_n)$ *obtained by the looping of* $(\mathbb{S}^3, L_n \cup \tau)$ *in Fig. 9b is* 

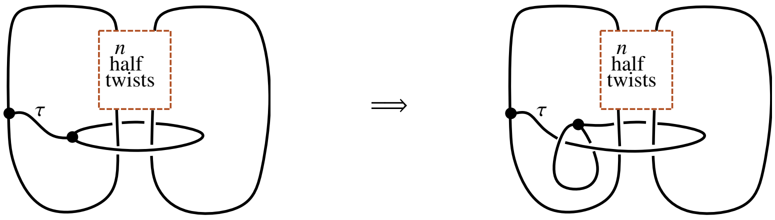
.

**Remark 6.6.** Let $l_+, l_-$ be the cores of the two annuli in the frontier of a regular neighborhood of the type 2-2 annulus in $E(\mathrm{HK}_n)$. Then one of $l_+, l_-$ is primitive in $E(L_n \cup \tau)$. Thus the union $l_+ \cup l_-$ in Lemma 5.1 (iii) cannot be replaced with a single $l_+$ or $l_-$.

Next, we consider the union of the 2-component link $(\mathbb{S}^3, L'_n)$ with $n$ odd and the tunnel $\tau$ in Fig. 10a. Then the looping of $(\mathbb{S}^3, L'_n \cup \tau)$ in Fig. 10b induces a handlebody-knot $(\mathbb{S}^3, \mathrm{HK}'_n)$ whose exterior contains a type 3-2i annulus given by the cabling annulus of the $(n,2)$-torus knot component of $(\mathbb{S}^3, L'_n)$, so we have the following.

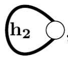
**Corollary 6.7.** *The annulus diagram of the handlebody-knot* $(\mathbb{S}^3, \mathrm{HK}'_n)$ *obtained by the looping of* $(\mathbb{S}^3, L'_n \cup \tau)$ *in Fig. 10b is* 

.




(a) $(\mathbb{S}^3, L'_n)$, $n$ odd, with the tunnel $\tau$. (b) Looping of $(\mathbb{S}^3, L'_n \cup \tau)$.

FIGURE 10. Handlebody-knot exteriors that contain a type 3-2i annulus.

Lastly, to produce handlebody-knots with the annulus diagram 

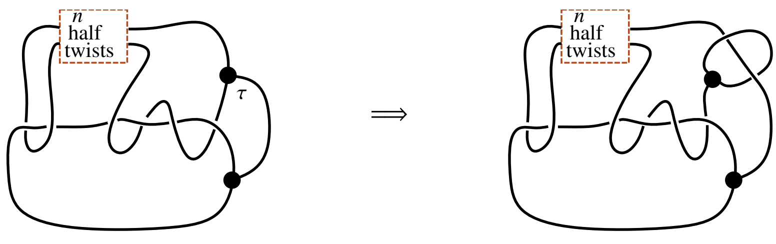
, we observe that, given a handlebody-knot $(\mathbb{S}^3, \mathrm{HK})$ with a type 2-2 annulus $A \subset E(\mathrm{HK})$, the loops $l_+, l_-$ bound two disks $D_+, D_-$ in $\mathrm{HK}_A$, respectively, and $D_+, D_-$ determine a spine $\Gamma_A$ of $\mathrm{HK}_A$; denote by $l_1, l_2$ the constituent loops in $\Gamma_A$ with $l_2$ disjoint from $D_+$ in $\mathrm{HK}_A$, and orient $l_1, l_2$. Then we have the following criterion for the non-uniqueness of $A \subset E(\mathrm{HK})$.

**Lemma 6.8.** (1) *Suppose* $E(\mathrm{HK})$ *contains a type* 3-2 *annulus* $A'$. *Then* $\ell k(l_1, l_2) \neq \pm 1$.
(2) *Suppose* $E(\mathrm{HK})$ *contains a type* 2-2 *annulus* $A'$ *not isotopic to A, and* $(\mathbb{S}^3, l_1)$ *is a trivial knot. Then* $(\mathbb{S}^3, l_1 \cup l_2)$ *is either a trivial link or a Hopf link.*

*Proof.* (1): **Case** 1**:** $A'$ **is of type** 3-2**i.** Let $W \subset E(\mathrm{HK})$ be the solid torus cut off by $A'$, and $l_w$ an oriented core of $W$. Note that the core of $A'$ is a $(p, q)$-curve on $\partial W$ with $|q| > 1$ since $A' \subset E(\mathrm{HK})$ is essential. If the linking number $\ell k(l_1, l_w)$ is $m$, then the linking number $\ell k(l_1, l_2)$ is $\pm qm \neq \pm 1$.

**Case** 2**:** $A'$ **is of type** 3-2**ii.** Let $D \subset \mathrm{HK}_A$ be a non-separating disk dual to $l_1$, and denote by $V$ the solid torus $\mathrm{HK}_A - \mathring{\mathfrak{N}}(D)$. The annulus $A'$ cuts $E(V)$ into two solid tori, one of which, denoted by $W$, contains $D$. Note that the core of the annulus $W \cap V$ has a slope of $\frac{p}{q}$, $|p| > 1$, with respect to $(\mathbb{S}^3, l_2)$. Let $D_w$ be an oriented meridian disk of $W$. If the linking number $\ell k(l_1, \partial D_w) = n$, then the linking number $\ell k(l_1, l_2) = \pm np \neq \pm 1$.

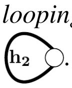
(2): Observe first that $(\mathbb{S}^3, l_2)$ is trivial by the existence of $A'$. Therefore, $(\mathbb{S}^3, l_1 \cup l_2)$ is trivial if it is split. Suppose it is non-split. Then there exists an essential disk $D \subset E(l_2)$ meeting $l_1$ at exactly one point. Denote by $W$ the 3-ball $\overline{E(l_2) - \mathfrak{N}(D)}$. Then since $(\mathbb{S}^3, l_1)$ is trivial, the ball-arc pair $(W, l_1 \cap W)$ is trivial, so $(\mathbb{S}^3, l_1 \cup l_2)$ is a Hopf link. □




(a) $(\mathbb{S}^3, L''_n)$, $n$ even, with the tunnel $\tau$. (b) Looping of $(\mathbb{S}^3, L''_n \cup \tau)$.

FIGURE 11. Handlebody-knots with a unique type 2 annulus.

Consider now the handcuff graph given by the union of the 2-component link $(\mathbb{S}^3, L''_n)$ with $n$ even and the tunnel $\tau$ in Fig. 11a.

**Corollary 6.9.** *The handlebody-knot induced by the looping of* $(\mathbb{S}^3, L''_n \cup \tau)$ *in Fig. 11b with even* $n \neq 0$ *is atoroidal with the annulus diagram* 

.

*Proof.* It follows from Lemmas 6.4 and 6.8 since the linking number of $(\mathbb{S}^3, L''_n)$ is $\pm 1$, and it is not a Hopf link, for every even $n \neq 0$. □

Handlebody-knots induced by Figs. 8b, 8c, 9b, 10b, and 11b imply the following.

**Proposition 6.10.** *Annulus diagrams in Theorem 3.18 can all be realized.*

## Acknowledgment

The author thanks Makoto Sakuma and Yuya Koda for the helpful and constructive discussions. The work was supported by National Sun Yat-sen University and Academia Sinica, and MoST (grant no. 110-2115-M-001-004-MY3), Taiwan.

## References

[1] F. Bonahon: *Geometric structure on* 3*-manifolds*, In: Handbook of Geometric Topology, R.J. Daverman and R.B. Sher (eds.), Elsevier (2001), 93–164.
[2] R. D. Canary, D. McCullough: *Homotopy Equivalences of 3-manifold and Deformation Theory of Kleinian Groups*, Mem. Amer. Math. Soc. **172** (2004).
[3] S. Cho, Y. Koda: *Topological symmetry groups and mapping class groups for spatial graphs*, Michigan Math. J. **62** (2013), 131–142.
[4] M. Eudave-Muñoz: *Non-hyperbolic manifolds obtained by Dehn surgery on hyperbolic knots*, in: W. Kazez (Ed.), Proceedings of the 1993 Internalational Georgia Topology Conference, AMS/IP Stud. in Adv. Math., Vol. 2, AMS, Providence, RI, 1997, pp. 35–61.
[5] B. Farb, D. Margalit: *A Primer on Mapping Class Groups*, Princeton University Press, (2011).
[6] R. H. Fox: *On the imbedding of polyhedra in 3-space*, Ann. of Math. **2**(49) (1948), 462–470.
[7] K. Funayoshi, Y. Koda: *Extending automorphisms of the genus-*2 *surface over the* 3*-sphere*, Q. J. Math. **71** (2020), 175–196.
[8] D. Gomez: *The fundamental group of the punctured Klein bottle and the simple loop conjecture*, Graduate J. Math. **2** (2017), 59–65.
[9] C. Gordon: *On primitive sets of loops in the boundary of a handlebody*, Topology Appl. **27** (1987), 285–299.
[10] C. Gordon, J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. 2 **2** (1989), 371–415.
[11] A. Hatcher: *Homeomorphisms of sufficiently large $P^2$-irreducible 3-manifolds*, Topology **15** (1976) 343–347.
[12] A. Hatcher: *Spaces of incompressible surfaces*, arXiv:math/9906074 [math.GT].
[13] A. Ishii, K. Kishimoto, H. Moriuchi, M. Suzuki: *A table of genus two handlebody-knots up to six crossings*, J. Knot Theory Ramifications **21**(4), (2012) 1250035.
[14] W. Jaco, P. B. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs Amer. Math. Soc. **220**, American Mathematical Society, Providence, 1979.
[15] K. Johannson: *Homotopy Equivalences of 3-Manifolds with Boundaries*, Lecture Notes in Math. **761**, Springer, Berlin, Heidelberg, 1979.
[16] Y. Koda: *Automorphisms of the 3-sphere that preserve spatial graphs and handlebody-knots*, Math. Proc. Cambridge Philos. Soc, **159** (2015), 1–22.
[17] Y. Koda, M. Ozawa, with an appendix by C. Gordon: *Essential surfaces of non-negative Euler characteristic in genus two handlebody exteriors*, Trans. Amer. Math. Soc. **367** (2015), no. 4, 2875–2904.
[18] J. H. Lee, S. Lee: *Inequivalent handlebody-knots with homeomorphic complements*, Algebr. Geom. Topol. **12** (2012), 1059–1079.
[19] M. Motto: *Inequivalent genus two handlebodies in $S^3$ with homeomorphic complements*, Topology Appl. **36**, (1990), 283–290.
[20] W. D. Neumann, G. A. Swarup: *Canonical decompositions of* 3*.manifolds*, Geom. Topol. **1** (1997), 21–40.
[21] H. Seifert: *Topologie dreidimensionaler gefaserter Räume*, Acta Math. **60** (1933), 147–288.
[22] W. P. Thurston: *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.
[23] Y.-S. Wang: *Unknotting annuli and handlebody-knot symmetry*, Topology Appl. **305** (2021), 107884.
[24] Y.-S. Wang: *Rigidity and symmetry of cylindrical handlebody-knots*, Osaka J. Math. **60** (2023), 267-304.
[25] Y.-S. Wang: *Annulus configuration in handlebody-knot exteriors*, arXiv:2301.06379 [math.GT].
[26] H. Zieschang: *On simple systems of paths on complete pretzels*, Amer. Math. Soc. Transl., **92** (1970), 127–137

National Sun Yat-sen University, Kaohsiung 804, Taiwan
*Email address*: yisheng@mail.nsysu.edu.tw