ALMOST COHERENCE OF HIGHER DIRECT IMAGES

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Abstract. For a flat proper morphism of finite presentation between schemes with almost coherent structural sheaves (in the sense of Faltings), we prove that the higher direct images of quasi-coherent and almost coherent modules are quasi-coherent and almost coherent. Our proof uses Noetherian approximation, inspired by Kiehl's proof of the pseudo-coherence of higher direct images. Our result allows us to extend Abbes-Gros' proof of Faltings' main p-adic comparison theorem in the relative case for projective log-smooth morphisms of schemes to proper ones, and thus also their construction of the relative Hodge-Tate spectral sequence.

Contents

1. Introduction

1.1. One of the first important results in algebraic geometry is the fact that the coherence for modules is preserved by higher direct images by a proper morphism. The Noetherian case is due to Grothendieck [EGA III_1, 3.2.1], and the general case is due to Kiehl [Kie72, 2.9']. The goal of this article is to extend the following corollary to almost algebra, motivated by applications in p-adic Hodge theory.

Theorem 1.2 (Kiehl [Kie72, 2.9'], see [Abb10, 1.4.8])**.** *Let $f : X \to S$ be a morphism of schemes satisfying the following conditions:*

(1) *f is proper and of finite presentation, and*
(2) *$\mathcal{O}_S$ is universally coherent.*

Then, for any coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $q \in \mathbb{N}$, $\mathrm{R}^q f_* \mathcal{M}$ is a coherent $\mathcal{O}_S$-module.

We say that $\mathcal{O}_S$ is *universally coherent* if there is a covering $\{S_i = \mathrm{Spec}(A_i)\}_{i\in I}$ of S by affine open subschemes such that the polynomial algebra $A_i[T_1, \ldots, T_n]$ is a coherent ring for any $i \in I$ and $n \in \mathbb{N}$. Indeed, such a condition on $\mathcal{O}_S$ implies that the coherent $\mathcal{O}_X$-module $\mathcal{M}$ is actually *pseudo-coherent* relative to S, which roughly means that if we embed X locally as a closed subscheme of $\mathbb{A}^n_{S_i}$, then $\mathcal{M}$ admits a resolution by finite free modules over $\mathbb{A}^n_{S_i}$. Theorem 1.2 is a direct corollary of Kiehl's result [Kie72, 2.9'], saying that the derived pushforward $\mathrm{R}f_*$ sends a relative pseudo-coherent complex to a pseudo-coherent complex.

1.3. Almost algebra was introduced by Faltings [Fal88, Fal02] for the purpose of developing p-adic Hodge theory. The setting is a pair $(R, \mathfrak{m})$ consisting of a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$, and the rough idea is to replace the category of R-modules by its quotient by $\mathfrak{m}$-torsion modules. An "almost"

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analogue of Theorem 1.2 is necessary for Faltings' approach to p-adic Hodge theory. Indeed, under the same assumptions of 1.2, Abbes-Gros [AG20, 2.8.14] proved that $\mathrm{R}^q f_*$ sends a quasi-coherent and almost coherent $\mathcal{O}_X$-module to a quasi-coherent and almost coherent $\mathcal{O}_S$-module, by reducing directly to 1.2. This result plays a crucial role in the proof of Faltings' main p-adic comparison theorem in the absolute case (see [AG20, 4.8.13]), and thus of the Hodge-Tate decomposition (see [AG20, 6.4.14]). Later, Zavyalov [Zav21, 5.1.6] extended the same almost coherence result to formal schemes.

However, the almost coherence result [AG20, 2.8.14] is not enough for Faltings' main p-adic comparison theorem in the relative case (thus neither for the relative Hodge-Tate spectral sequence), since we inevitably encounter the situation where $\mathcal{O}_S$ is *universally almost coherent* but not universally coherent. Thus, under the assumptions that

(1) f is projective, flat and of finite presentation, and that
(2) $\mathcal{O}_S$ is universally almost coherent,

Abbes-Gros proved an almost coherence result [AG20, 2.8.18] by adapting the arguments of [SGA 6, III.2.2], where the projectivity condition on f plays a crucial role. This is the reason why Faltings' main p-adic comparison theorem in the relative case (and thus the relative Hodge-Tate spectral sequence) was only proved for projective log-smooth morphisms in [AG20, 5.7.4 (and 6.7.5)].

1.4. In this article, we generalize the almost coherence result [AG20, 2.8.18] to proper morphisms, which allows us to extend Abbes-Gros' proof of Faltings' main p-adic comparison theorem in the relative case to proper log-smooth morphisms, and thus also their construction of the relative Hodge-Tate spectral sequence (see Section 8).

Let R be a ring with an ideal $\mathfrak{m}$ such that for any integer $l \geq 1$, the l-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$. The pair $(R, \mathfrak{m})$ will be our basic setup for almost algebra (see Section 6). The main theorem of this article is the following

Theorem 1.5 (see 7.1). *Let $f : X \to S$ be a morphism of R-schemes satisfying the following conditions:*

(1) *f is proper, flat and of finite presentation, and*
(2) *$\mathcal{O}_X$ and $\mathcal{O}_S$ are almost coherent.*

Then, for any quasi-coherent and almost coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $q \in \mathbb{N}$, $\mathrm{R}^q f_*\mathcal{M}$ is a quasi-coherent and almost coherent $\mathcal{O}_S$-module.

Our proof is close to Kiehl's proof of [Kie72, 2.9], see Section 7. We roughly explain our ideas in the following:

(1) We may assume without loss of generality that S is affine. As $\mathcal{M}$ is not of finite presentation over X in general, we couldn't descend it by Noetherian approximation. But $\mathcal{M}$ is almost coherent, for any $\pi \in \mathfrak{m}$ we can "π-resolve" $\mathcal{M}$ over a truncated Čech hypercovering $X_\bullet = (X_n)_{[n]\in\Delta_{\leq k}}$ (by affine open subschemes) of X by finite free modules $\mathcal{F}_\bullet^\bullet$ as in [Kie72, 2.2], where each $\mathcal{F}_n^\bullet$ is a "resolution" of $\mathcal{M}|_{X_n}$ modulo π-torsion, see Section 6. By Noetherian approximation, we obtain a proper flat morphism $f_\lambda : X_\lambda \to S_\lambda$ of Noetherian schemes together with a complex of finite free modules $\mathcal{F}_{\lambda,\bullet}^\bullet$ over a truncated Čech hypercovering of X_λ descending f and $\mathcal{F}_\bullet^\bullet$.
(2) As in [Kie72, 1.4], the descent data of $\mathcal{M}$ over $X_\bullet$ are encoded as null homotopies of the multiplication by a certain power of π on the cone of $\alpha^*\mathcal{F}_m^\bullet \to \mathcal{F}_n^\bullet$ (where $\alpha : [m] \to [n]$ is a morphism in the truncated simplicial category $\Delta_{\leq k}$), see Section 3. We can descend the latter by Noetherian approximation, from which we produce some coherent modules over X_λ, see Section 5.
(3) Applying the classical coherence result for $f_\lambda : X_\lambda \to S_\lambda$, we see that the Čech complex of $\mathcal{F}_{\lambda,\bullet}^\bullet$ is "pseudo-coherent" modulo certain power of π, see Section 4. The same thing holds for the Čech complex of $\mathcal{F}_\bullet^\bullet$ by base change (due to the flatness of f_λ). Since this Čech complex computes $\mathrm{R}\Gamma(X, \mathcal{M})$ up to certain degree and modulo certain power of π, the conclusion follows by varying π in $\mathfrak{m}$.

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2. Notation and Conventions

2.1. All rings considered in this article are unitary and commutative.

2.2. Let $\mathscr{A}$ be an abelian category. When we consider "a complex in $\mathscr{A}$", we always refer to a cochain complex in $\mathscr{A}$, and we denote it by $M^\bullet$ with differential maps $\mathrm{d}^n : M^n \to M^{n+1}$ ($n \in \mathbb{Z}$). For any $a \in \mathbb{Z}$, we denote by $\tau^{\geq a} M^\bullet$ (resp. $\sigma^{\geq a} M^\bullet$) the canonical (resp. stupid) truncation of $M^\bullet$, see [Sta22, 0118].

2.3. Let Δ be the category formed by finite ordered sets $[n] = \{0, 1, \ldots, n\}$ ($n \in \mathbb{N}$) with non-decreasing maps ([Sta22, 0164]). For $k \in \mathbb{N} \cup \{\infty\}$, we denote by $\Delta_{\leq k}$ the full subcategory of Δ formed by objects $[0], [1], \ldots, [k]$. For a category C, a contravariant functor from $\Delta_{\leq k}$ to C sending $[n]$ to X_n is called a *k-truncated simplicial object* of C, denoted by $X_\bullet$. Let $\mathcal{P}$ be a property for objects of C. We say that $X_\bullet$ has property $\mathcal{P}$ if each X_n has property $\mathcal{P}$.

3. Isomorphisms up to Bounded Torsion

In this section, we fix a ring R and an element π of R. Consider an abelian category $\mathscr{A}$ with a ring homomorphism $R \to \mathrm{End}(\mathrm{id}_{\mathscr{A}})$, where $\mathrm{End}(\mathrm{id}_{\mathscr{A}})$ is the ring of endomorphisms of the identity functor. Thus, π defines a functorial endomorphism on any object M of $\mathscr{A}$. We denote by $\mathbf{K}(\mathscr{A})$ the homotopy category of complexes in $\mathscr{A}$.

Definition 3.1. (1) We say that an object M in $\mathscr{A}$ is *π-null* if it is annihilated by π. We say that a morphism $f : M \to N$ in $\mathscr{A}$ is a *π-isomorphism* if its kernel and cokernel are π-null.

(2) We say that a complex $M^\bullet$ in $\mathscr{A}$ is *π-exact* if the cohomology group $H^n(M^\bullet)$ is π-null for any $n \in \mathbb{Z}$. We say that a morphism of complexes $f : M^\bullet \to N^\bullet$ in $\mathscr{A}$ is a *π-quasi-isomorphism* if it induces a π-isomorphism on the cohomology groups $H^n(f) : H^n(M^\bullet) \to H^n(N^\bullet)$ for any $n \in \mathbb{Z}$.

Lemma 3.2 ([AG20, 2.6.3])**.** *Let $f : M \to N$ be a morphism in $\mathscr{A}$.*

(1) *If there exists a morphism $g : N \to M$ in $\mathscr{A}$ such that $g \circ f = \pi \cdot \mathrm{id}_M$ and $f \circ g = \pi \cdot \mathrm{id}_N$, then f is a π-isomorphism.*

(2) *If f is a π-isomorphism, then $\pi \cdot \mathrm{id}_N$ and $\pi \cdot \mathrm{id}_M$ uniquely factor through $N \to \mathrm{Im}(f)$ and $\mathrm{Im}(f) \to M$ respectively, whose composition $g : N \to M$ satisfies that $g \circ f = \pi^2 \cdot \mathrm{id}_M$ and $f \circ g = \pi^2 \cdot \mathrm{id}_N$. In particular, the morphism g is functorial in f.*

Lemma 3.3. *Let $f : M^\bullet \to N^\bullet$ be a morphism of complexes in $\mathscr{A}$. Assume the following conditions:*

(1) *for any $i > 0$, $M^i = N^i = 0$;*

(2) *there is $n \in \mathbb{N}$ such that for any $-n \leq i \leq 0$, M^i is projective and $\pi \cdot H^i(N^\bullet) = 0$.*

Then, there exists a morphism $s^i : M^i \to N^{i-1}$ for any $-n \leq i \leq 0$, such that

$$\pi^{1-i} \cdot f^i = \pi \cdot s^{i+1} \circ \mathrm{d}^i + \mathrm{d}^{i-1} \circ s^i \tag{3.3.1}$$

as morphisms from M^i to N^i, where we put $s^1 = 0$. In particular, the morphism of canonically truncated complexes

$$\pi^{n+1} \cdot f : \tau^{\geq -n} M^\bullet \longrightarrow \tau^{\geq -n} N^\bullet \tag{3.3.2}$$

is homotopic to zero.

Proof. We construct s^i by induction. Setting $0 = s^1 = s^2 = \cdots$, we may assume that we have already constructed the homomorphisms for any degree strictly bigger than i with identities (3.3.1). As $\pi^{-i} \cdot f^{i+1} = \pi \cdot s^{i+2} \circ \mathrm{d}^{i+1} + \mathrm{d}^i \circ s^{i+1}$, we see that

$$\mathrm{d}^i \circ (\pi^{-i} \cdot f^i - s^{i+1} \circ \mathrm{d}^i) = \pi^{-i} \cdot f^{i+1} \circ \mathrm{d}^i - (\pi^{-i} \cdot f^{i+1} - \pi \cdot s^{i+2} \circ \mathrm{d}^{i+1}) \circ \mathrm{d}^i = 0. \tag{3.3.3}$$

Thus, the map $\pi^{-i} \cdot f^i - s^{i+1} \circ \mathrm{d}^i : M^i \to N^i$ factors through $\mathrm{Ker}(\mathrm{d}^i : N^i \to N^{i+1})$. The assumption $\pi \cdot H^i(N^\bullet) = 0$ implies that the map $\pi^{1-i} \cdot f^i - \pi \cdot s^{i+1} \circ \mathrm{d}^i : M^i \to N^i$ factors through $\mathrm{Im}(\mathrm{d}^{i-1} : N^{i-1} \to N^i)$. As M^i is projective, there exists a morphism $s^i : M^i \to N^{i-1}$ such that $\pi^{1-i} \cdot f^i - \pi \cdot s^{i+1} \circ \mathrm{d}^i = \mathrm{d}^{i-1} \circ s^i$, which completes the induction. In particular, for any $i \geq -n$, we have

$$\pi^{n+1} \cdot f^i = (\pi^{n+i+1} \cdot s^{i+1}) \circ \mathrm{d}^i + \mathrm{d}^{i-1} \circ (\pi^{n+i} \cdot s^i). \tag{3.3.4}$$

Recall that $\tau^{\geq -n} M^\bullet = (0 \to M^{-n}/\mathrm{Im}(\mathrm{d}^{-n-1}) \to M^{1-n} \to \cdots \to M^0 \to 0)$. Thus, we see that $\pi^{n+1} \cdot f : \tau^{\geq -n} M^\bullet \to \tau^{\geq -n} N^\bullet$ is homotopic to zero. $\square$

Proposition 3.4. *Let $P^\bullet$ be a complex of projective objects in $\mathscr{A}$, and let $M^\bullet$ be a π-exact complex in $\mathscr{A}$. Assume that there are integers $a \le b$ such that P^i and M^i vanish for any $i \notin [a,b]$. Then, the R-module $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet)$ is π^{b-a+1}-null.*

Proof. It follows directly from 3.3. □

Corollary 3.5. *Let $P^\bullet$ be a complex of projective objects in $\mathscr{A}$, and let $f : M^\bullet \to N^\bullet$ be a π-quasi-isomorphism of complexes in $\mathscr{A}$. Assume that there are integers $a \le b$ such that P^i, M^i and N^i vanish for any $i \notin [a,b]$. Then, the map $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet)$ induced by f is a $\pi^{2(b-a+3)}$-isomorphism of R-modules.*

Proof. There is an exact sequence of R-modules

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet[-1]) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet) \tag{3.5.1}$$

where $C^\bullet$ is the cone of f ([Sta22, 0149]). As $C^\bullet$ and $C^\bullet[-1]$ are π^2-exact and vanish outside $[a-1, b+1]$, the outer two R-modules are $\pi^{2(b-a+3)}$-null by 3.4, whence we draw the conclusion. □

Lemma 3.6. *Let $g : P^\bullet \to N^\bullet$ and $f : M^\bullet \to N^\bullet$ be morphisms of complexes in $\mathscr{A}$. Assume that there are integers $a \le b$ such that*

1. *$M^i = N^i = 0$ for any $i > b$, and that*
2. *P^i is projective for any $i \in [a,b]$ and zero for any $i \notin [a,b]$, and that*
3. *the map $H^i(f) : H^i(M^\bullet) \to H^i(N^\bullet)$ is a π-isomorphism for $i > a$ and π-surjective for $i = a$.*

Then, $\pi^{2(b-a+1)} \cdot g$ lies in the image of the map $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet)$ induced by f.

Proof. Let $C^\bullet$ be the cone of f, and let $\iota : N^\bullet \to C^\bullet$ be the canonical morphism. Applying the homological functor $\mathrm{Hom}_{\mathbf{D}(\mathscr{A})}(P^\bullet, -)$ to the distinguished triangle $\tau^{\le a-1}C^\bullet \to C^\bullet \to \tau^{\ge a}C^\bullet \to (\tau^{\le a-1}C^\bullet)[1]$ in the derived category $\mathbf{D}(\mathscr{A})$, we obtain an exact sequence of R-modules ([Sta22, 0149, 064B])

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, \tau^{\le a-1}C^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, \tau^{\ge a}C^\bullet). \tag{3.6.1}$$

The first term is zero by assumption (2), and $\tau^{\ge a}C^\bullet$ is π^2-exact by assumption (3). As $\tau^{\ge a}C^\bullet$ vanishes outside $[a,b]$ by assumption (1), the third term is $\pi^{2(b-a+1)}$-null by 3.4. We see that $\pi^{2(b-a+1)} \cdot (\iota \circ g)$ is zero in $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet)$. Therefore, the conclusion follows from the exact sequence ([Sta22, 0149])

$$\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, M^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, N^\bullet) \to \mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P^\bullet, C^\bullet). \tag{3.6.2}$$

□

4. Pseudo-coherence up to Bounded Torsion

In this section, we fix integers $a \le b$, a ring R and an element π of R. We remark that the universal bound l that shall appear in each statement of this section depends only on the difference $b - a$ but not on R or π.

Definition 4.1. Let $M^\bullet$ be a complex of R-modules.

1. A π-$[a,b]$-*pseudo resolution* of $M^\bullet$ is a morphism $f : P^\bullet \to M^\bullet$ of complexes of R-modules, where $P^\bullet$ is a complex of finite free R-modules such that $P^i = 0$ for any $i \notin [a,b]$, and where the map of cohomology groups $H^i(f) : H^i(P^\bullet) \to H^i(M^\bullet)$ is a π-isomorphism for $i > a$ and π-surjective for $i = a$.
2. We say that $M^\bullet$ is π-$[a,b]$-*pseudo-coherent* if $M^i = 0$ for any $i > b$ and if it admits a π-$[a,b]$-pseudo resolution. We say that an R-module M is π-$[a,b]$-*pseudo-coherent* if the complex $M[0]$ is π-$[a,b]$-pseudo-coherent.

We follow the presentation of [Sta22, 064N] to establish some basic properties of this notion. The author does not know whether this notion is Zariski local on R or not (cf. [Sta22, 066D]). This ad hoc notion only serves for the proof of our main theorem.

Lemma 4.2. *For any integers $a' \ge a$ and $b' \ge b$ with $a' \le b'$, a π-$[a,b]$-pseudo-coherent complex of R-modules is also π-$[a',b']$-pseudo-coherent.*

Proof. We only need to treat the case $a = a'$ and the case $b = b'$ separately. If $a = a'$, then it is clear that $M^i = 0$ for any $i > b'$ and a π-$[a,b]$-pseudo resolution of $M^\bullet$ is also a π-$[a,b']$-pseudo resolution. If $b = b'$, then a π-$[a,b]$-pseudo resolution $P^\bullet \to M^\bullet$ induces a π-$[a',b]$-pseudo resolution $\sigma^{\ge a'}P^\bullet \to M^\bullet$. □

Lemma 4.3. *Let $M^\bullet$ and $N^\bullet$ be complexes of R-modules vanishing in degrees $> b$, and let $\alpha : M^\bullet \to N^\bullet$ be a morphism inducing a π-isomorphism on cohomology groups $H^i(\alpha) : H^i(M^\bullet) \to H^i(N^\bullet)$ for any $i \geq a$.*

(1) *If $M^\bullet$ is π-$[a,b]$-pseudo-coherent, then $N^\bullet$ is π^2-$[a,b]$-pseudo-coherent.*
(2) *If $N^\bullet$ is π-$[a,b]$-pseudo-coherent, then $M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*

Proof. (1) We take a π-$[a,b]$-pseudo resolution $f : P^\bullet \to M^\bullet$. In particular, $H^i(f)$ is a π-isomorphism for any $i > a$ and π-surjective for $i = a$. Hence, $H^i(\alpha \circ f) = H^i(\alpha) \circ H^i(f)$ is π^2-isomorphism for any $i > a$ and π^2-surjective for $i = a$, which shows that $\alpha \circ f : P^\bullet \to N^\bullet$ is a π^2-$[a,b]$-pseudo resolution.

(2) Let $g : P^\bullet \to N^\bullet$ be a π-$[a,b]$-pseudo resolution. We obtain from 3.6 a morphism $f : P^\bullet \to M^\bullet$ lifting $\pi^l \cdot g$ up to homotopy for $l = 2(b-a+1)$. Thus, for any $i \in \mathbb{Z}$, we have

$$H^i(\pi^l \cdot g) = H^i(\alpha) \circ H^i(f). \tag{4.3.1}$$

Notice that $H^i(\pi^l \cdot g)$ is a π^{l+1}-isomorphism for $i > a$ and π^{l+1}-surjective for $i = a$, and that $H^i(\alpha)$ is a π-isomorphism for $i \geq a$. We see that $H^i(f)$ is a π^{l+2}-isomorphism for $i > a$ and π^{l+2}-surjective for $i = a$. Thus, $f : P^\bullet \to M^\bullet$ is a π^{l+2}-$[a,b]$-pseudo resolution. □

Proposition 4.4. *Let $M^\bullet$ and $N^\bullet$ be two complexes of R-modules vanishing in degree $> b$. Assume that they are isomorphic in the derived category $\mathbf{D}(R)$. Then, if $M^\bullet$ is π-$[a,b]$-pseudo-coherent, then $N^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*

Proof. Let $P^\bullet \to N^\bullet$ be a bounded above projective resolution with the same top degree. The assumption implies that there is a quasi-isomorphism of complexes $P^\bullet \to M^\bullet$ ([Sta22, 064B]). The conclusion follows from applying 4.3 to $P^\bullet \to M^\bullet$ and $P^\bullet \to N^\bullet$. □

Lemma 4.5. *Let $\alpha : M_1^\bullet \to M_2^\bullet$ be a morphism of π-$[a,b]$-pseudo-coherent complexes of R-modules. Given π-$[a,b]$-pseudo resolutions $f_i : P_i^\bullet \to M_i^\bullet$ $(i = 1,2)$, there exists a morphism of complexes $\alpha' : P_1^\bullet \to P_2^\bullet$ such that $(\pi^l \cdot \alpha) \circ f_1$ is homotopic to $f_2 \circ \alpha'$ for an integer $l \geq 0$ depending only on $b-a$.*

Proof. It follows directly from 3.6. □

Lemma 4.6. *Let $M_1^\bullet \xrightarrow{\alpha} M_2^\bullet \xrightarrow{\beta} M_3^\bullet \xrightarrow{\gamma} M_1^\bullet[1]$ be a distinguished triangle in the homotopy category $\mathbf{K}(R)$. Assume that $M_1^\bullet$ is π-$[a+1,b+1]$-pseudo-coherent, $M_2^\bullet$ is π-$[a,b]$-pseudo-coherent, and $M_3^i = 0$ for any $i > b$. Then, $M_3^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*

Proof. We take a π-$[a+1,b+1]$-pseudo resolution $f_1 : P_1^\bullet \to M_1^\bullet$ and a π-$[a,b]$-pseudo resolution $f_2 : P_2^\bullet \to M_2^\bullet$. By 4.5, there exists a morphism $\alpha' : P_1^\bullet \to P_2^\bullet$ lifting $\pi^l \cdot \alpha$ in $\mathbf{K}(R)$ for an integer $l \geq 0$ depending only on $b-a$. If we denote its cone by $P_3^\bullet$, then we have a morphism of distinguished triangles in $\mathbf{K}(R)$.

$$\begin{array}{ccccccc} P_1^\bullet & \xrightarrow{\alpha'} & P_2^\bullet & \xrightarrow{\beta'} & P_3^\bullet & \xrightarrow{\gamma'} & P_1^\bullet[1] \\ \downarrow{\scriptstyle \pi^l\cdot f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \pi^l\cdot f_1[1]} \\ M_1^\bullet & \xrightarrow{\alpha} & M_2^\bullet & \xrightarrow{\beta} & M_3^\bullet & \xrightarrow{\gamma} & M_1^\bullet[1] \end{array} \tag{4.6.1}$$

Let $C_1^\bullet$, $C_2^\bullet$, $C_3^\bullet$ be the cones of $\pi^l \cdot f_1$, f_2, f_3 respectively. We obtain a distinguished triangle $C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to C_1^\bullet[1]$ in $\mathbf{K}(R)$ ([Sta22, 05R0]). By assumption, $\tau^{\geq(a+1)} C_1^\bullet$ is $\pi^{2(l+1)}$-exact and $\tau^{\geq a} C_2^\bullet$ is π^2-exact. Thus, we see that $\tau^{\geq a} C_3^\bullet$ is $\pi^{2(l+2)}$-exact. As $P_3^\bullet$ vanishes outside $[a,b]$, $P_3^\bullet \to M_3^\bullet$ is a $\pi^{2(l+2)}$-$[a,b]$-pseudo resolution. □

Proposition 4.7. *Let $0 \longrightarrow M_1^\bullet \xrightarrow{\alpha} M_2^\bullet \xrightarrow{\beta} M_3^\bullet \longrightarrow 0$ be an exact sequence of complexes of R-modules.*

(1) *Assume that $M_1^\bullet$ is π-$[a+1,b+1]$-pseudo-coherent and $M_2^\bullet$ is π-$[a,b]$-pseudo-coherent. Then, $M_3^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*
(2) *Assume that $M_1^\bullet$ and $M_3^\bullet$ are π-$[a,b]$-pseudo-coherent. Then, $M_2^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*
(3) *Assume that $M_2^\bullet$ is π-$[a-1,b-1]$-pseudo-coherent and $M_3^\bullet$ is π-$[a-2,b-1]$-pseudo-coherent. Then, $M_1^\bullet$ is π^l-$[a-1,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*

Proof. Let $C^\bullet$ be the cone of $\alpha : M_1^\bullet \to M_2^\bullet$. Then, the natural morphism $C^\bullet \to M_3^\bullet$ is a quasi-isomorphism.

(1) In this case, $M_3^i = 0$ and $C^i = 0$ for any $i > b$, and actually $C^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by applying 4.6 to the distinguished triangle $M_1^\bullet \to M_2^\bullet \to C^\bullet \to M_1^\bullet[1]$ in $\mathbf{K}(R)$. Thus, $M_3^\bullet$ is π^{2l}-$[a,b]$-pseudo-coherent by 4.3.(1).

(2) In this case, $M_2^i = 0$ and $C^i = 0$ for any $i > b$, and $C^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by 4.3.(2). Thus, $M_2^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by applying 4.6 to the distinguished triangle $C^\bullet[-1] \to M_1^\bullet \to M_2^\bullet \to C^\bullet$ in $\mathbf{K}(R)$.

(3) In this case, $M_1^i = 0$ and $C^i = 0$ for any $i > b-1$, and $C^\bullet$ is π^l-$[a-2,b-1]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by 4.3.(2). Thus, $M_1^\bullet$ is π^l-$[a-1,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by applying 4.6 to the distinguished triangle $M_2^\bullet[-1] \to C^\bullet[-1] \to M_1^\bullet \to M_2^\bullet$ in $\mathbf{K}(R)$. ☐

Corollary 4.8. *Let $M^\bullet$ be a complex of R-modules vanishing in degrees $> b$. Assume that the cohomology group $H^i(M^\bullet)$ is π-$[a-i,b-i]$-pseudo-coherent for any $i \in [a,b]$. Then, $M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$.*

Proof. We proceed by induction on $b-a$. If $a = b$, then $\tau^{\geq a} M^\bullet = H^a(M^\bullet)[-a]$ is π-$[a,b]$-pseudo-coherent by assumption. Thus, $M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by 4.3.(2). In general, consider the exact sequence of complexes of R-modules

$$0 \longrightarrow \tau^{\leq (b-1)} M^\bullet \longrightarrow M^\bullet \longrightarrow (M^{b-1}/\mathrm{Ker}(\mathrm{d}^{b-1})[1-b] \to M^b[-b]) \longrightarrow 0. \tag{4.8.1}$$

As the natural morphism of complexes $N^\bullet = (M^{b-1}/\mathrm{Ker}(\mathrm{d}^{b-1})[1-b] \to M^b[-b]) \to H^b(M^\bullet)[-b]$ is a quasi-isomorphism, $N^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a$ by 4.3.(2). Notice that $\tau^{\leq (b-1)} M^\bullet$ is π^l-$[a,b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b-a-1$ by induction. The conclusion follows from 4.7.(2). ☐

Lemma 4.9. *Let $M^\bullet$ be a π-exact complex of R-modules vanishing outside $[a,b]$, and let $N^\bullet$ be a complex of R-modules. Then, $M^\bullet \otimes_R^{\mathrm{L}} N^\bullet$ is π^l-exact (i.e. any complex representing the derived tensor product $M^\bullet \otimes_R^{\mathrm{L}} N^\bullet$ is π^l-exact) for an integer $l \geq 0$ depending only on $b-a$.*

Proof. We proceed by induction on $b-a$. If $a = b$, then the multiplication by π on $M^\bullet = H^a(M^\bullet)[-a]$ factors through zero. Hence, $M^\bullet \otimes_R^{\mathrm{L}} N^\bullet$ is π-exact. In general, consider the distinguished triangle in the derived category $\mathbf{D}(R)$,

$$(\tau^{\leq (b-1)} M^\bullet) \otimes_R^{\mathrm{L}} N^\bullet \longrightarrow M^\bullet \otimes_R^{\mathrm{L}} N^\bullet \longrightarrow H^b(M^\bullet)[-b] \otimes_R^{\mathrm{L}} N^\bullet \longrightarrow (\tau^{\leq (b-1)} M^\bullet) \otimes_R^{\mathrm{L}} N^\bullet[1]. \tag{4.9.1}$$

Notice that $(\tau^{\leq (b-1)} M^\bullet) \otimes_R^{\mathrm{L}} N^\bullet$ is π^l-exact for an integer $l \geq 0$ depending only on $b-a$ by induction. By the long exact sequence of cohomology groups, we see that $M^\bullet \otimes_R^{\mathrm{L}} N^\bullet$ is π^l-exact for an integer $l \geq 0$ depending only on $b-a$. ☐

Proposition 4.10. *Let $M^\bullet$ be a π-$[a,b]$-pseudo-coherent complex of R-modules, and let S be an R-algebra. Then, $\tau^{\geq a}(S \otimes_R^{\mathrm{L}} M^\bullet)$ is represented by a π^l-$[a,b]$-pseudo-coherent complex of S-modules for an integer $l \geq 0$ depending only on $b-a$.*

Proof. We take a bounded above flat resolution $F^\bullet \to M^\bullet$ with the same top degree. By 4.3.(2), $\sigma^{\geq a-1} F^\bullet$ is a π^l-$[a,b]$-pseudo-coherent complex of flat R-modules for an integer $l \geq 0$ depending only on $b-a$. Let $P^\bullet \to \sigma^{\geq a-1} F^\bullet$ be a π^l-$[a,b]$-pseudo resolution, and let $C^\bullet$ be its cone. Consider the distinguished triangle in $\mathbf{K}(S)$,

$$S \otimes_R P^\bullet \longrightarrow S \otimes_R \sigma^{\geq a-1} F^\bullet \longrightarrow S \otimes_R C^\bullet \longrightarrow S \otimes_R P^\bullet[1]. \tag{4.10.1}$$

Notice that $\tau^{\geq a} C^\bullet$ is a π^{2l}-exact complex vanishing outside $[a,b]$ and that $S \otimes_R C^\bullet \cong S \otimes_R^{\mathrm{L}} C^\bullet$ in $\mathbf{D}(S)$ by construction. After enlarging l by 4.9, we may assume that $\tau^{\geq a}(S \otimes_R C^\bullet) = \tau^{\geq a}(S \otimes_R^{\mathrm{L}} \tau^{\geq a} C^\bullet)$ is π^l-exact. By the long exact sequence associated to (4.10.1), we see that $S \otimes_R P^\bullet \to S \otimes_R \sigma^{\geq a-1} F^\bullet$ is a π^{2l}-$[a,b]$-pseudo resolution of complexes of S-modules, and thus so is the composition

$$S \otimes_R P^\bullet \longrightarrow S \otimes_R \sigma^{\geq a-1} F^\bullet \longrightarrow \tau^{\geq a}(S \otimes_R \sigma^{\geq a-1} F^\bullet) = \tau^{\geq a}(S \otimes_R F^\bullet), \tag{4.10.2}$$

where the target is a complex of S-modules representing $\tau^{\geq a}(S \otimes_R^{\mathrm{L}} M^\bullet)$ and vanishing in degrees $> b$. ☐

Definition 4.11. Let M be an R-module. We say that M is *of π-finite type* if there exists $n \in \mathbb{N}$ and a π-surjective R-linear homomorphism $R^{\oplus n} \to M$.

This definition is a special case of 6.1 below.

Lemma 4.12. *Assume that R is Noetherian. Let M be an R-module.*

(1) *If M is of π-finite type, then it is π-$[a, b]$-pseudo-coherent for any integers $a \leq 0 \leq b$. Conversely, if M is π-$[a, b]$-pseudo-coherent for some integers $a \leq 0 \leq b$, then M is of π-finite type.*
(2) *If M is of π-finite type, then so are its subquotients. Conversely, if M admits a finite filtration of length l (*[Sta22, 0121]*) whose graded pieces are of π-finite type, then M is of π^l-finite type.*

Proof. (1) If M is of π-finite type, then there is a finitely generated R-submodule N of M such that $\pi M \subseteq N$. Since R is Noetherian, N is pseudo-coherent ([Sta22, 066E]). Hence, M is π-$[a, b]$-pseudo-coherent for any integers $a \leq 0 \leq b$. Conversely, if M is π-$[a, b]$-pseudo-coherent, we take a π-$[a, b]$-pseudo resolution $P^\bullet \to M[0]$. As a subquotient of a finitely generated R-module, $H^0(P^\bullet)$ is also finitely generated as R is Noetherian. Hence, $M = H^0(M[0])$ is of π-finite type.

(2) Let N be a finitely generated R-submodule of M such that $\pi M \subseteq N$. Let $M_0 \subseteq M_1$ be two R-submodules of M. Notice that $N \cap M_1$ is a finitely generated R-module as R is Noetherian. The conclusion follows from the π-surjectivity of $N \cap M_1 \to M_1/M_0$. Conversely, assume that there is a finite filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$ such that M_{i+1}/M_i is of π-finite type. Then, we see that M is of π^l-finite type by inductively using [AG20, 2.7.14.(ii)]. □

Proposition 4.13. *Assume that R is Noetherian. Let $M^\bullet$ be a complex of R-modules.*

(1) *If $H^i(M^\bullet)$ is of π-finite type for any $i \geq a$ and if $M^i = 0$ for any $i > b$, then $M^\bullet$ is π^l-$[a, b]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$.*
(2) *If $M^\bullet$ is π-$[a, b]$-pseudo-coherent, then $H^i(M^\bullet)$ is of π-finite type for any $i \geq a$.*

Proof. (1) follows from 4.12.(1) and 4.8; and (2) follows from the same argument of 4.12.(1). □

5. Glueing Sheaves up to Bounded Torsion

In this section, we fix a ring R and an element π of R.

5.1. Let E/C be a fibred site, and let $\mathcal{O} = (\mathcal{O}_\alpha)_{\alpha \in \mathrm{Ob}(C)}$ be a sheaf of R-algebras over the total site E ([SGA 4_{II}, VI.7.4.1]). We say that an $\mathcal{O}$-module $\mathcal{F} = (\mathcal{F}_\alpha)_{\alpha \in \mathrm{Ob}(C)}$ on E is *π-Cartesian*, if for every morphism $f : \beta \to \alpha$ in C, the induced map $f^*\mathcal{F}_\alpha \to \mathcal{F}_\beta$ is a π-isomorphism of $\mathcal{O}_\beta$-modules.

5.2. Let E be a category. Recall that a semi-representable object of E is a family $\{U_i\}_{i \in I}$ of objects of E. A morphism $\{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ of semi-representable objects of E is given by a map $\alpha : I \to J$ and for every $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ ([Sta22, 01G0]). Assume that E is a site ([SGA 4_{I}, II.1.1.5]) where fibred products are representable. For a semi-representable object $K = \{U_i\}_{i \in I}$ of objects of E, let $E_{/K} = \coprod_{i \in I} E_{/U_i}$ be the disjoint union of the localizations of E at U_i ([Sta22, 09WK]). We note that for any morphism $K' \to K$ of semi-representable objects of E, the canonical morphism of sites $E_{/K'} \to E_{/K}$ induced by the cocontinuous forgetful functor $E_{/K'} \to E_{/K}$ is also induced by the continuous base change functor $E_{/K} \to E_{/K'}$ ([Sta22, 0D85, 0D87]).

Let $r \in \mathbb{N} \cup \{\infty\}$. For an r-truncated simplicial semi-representable object $K_\bullet = (K_n)_{[n] \in \mathrm{Ob}(\Delta_{\leq r})}$ of E (where each K_n is a semi-representable object of E), we denote by $E_{/K_\bullet}$ the fibred site over the r-truncated simplicial category $\Delta_{\leq r}$ whose fibre over $[n]$ is $E_{/K_n}$ ([Sta22, 0D8A]). We denote by $\nu : E_{/K_\bullet} \to E$ the augmentation, and by $\nu_n : E_{/K_n} \to E$ the corresponding morphism of sites for any $n \in \mathbb{N}_{\leq r}$ ([Sta22, 0D8B]).

Lemma 5.3. *Let E be a site where fibred products are representable, let*

(5.3.1)
$$\begin{array}{ccc} Y' & \xrightarrow{g'} & X' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \end{array}$$

be a commutative diagram in E, and let $\mathcal{F}$ a sheaf on $E_{/X'}$. Then, the following diagram is commutative

$$
\begin{array}{ccc}
f'_*f'^*g^*f_*\mathcal{F} & \xrightarrow{\sim} & f'_*g'^*f^*f_*\mathcal{F}\\
{\scriptstyle \alpha_{f'}|_{g^*f_*\mathcal{F}}}\uparrow & & \downarrow{\scriptstyle f'_*g'^*\beta_f|_{\mathcal{F}}}\\
g^*f_*\mathcal{F} & & f'_*g'^*\mathcal{F}\\
{\scriptstyle g^*f_*\alpha_{g'}|_{\mathcal{F}}}\downarrow & & \uparrow{\scriptstyle \beta_g|_{f'_*g'^*\mathcal{F}}}\\
g^*f_*g'_*g'^*\mathcal{F} & \xrightarrow{\sim} & g^*g_*f'_*g'^*\mathcal{F}
\end{array}
\tag{5.3.2}
$$

where $\alpha_{f'}$ (resp. $\alpha_{g'}$) is the adjunction morphism $\mathrm{id} \to f'_*f'^*$ (resp. $\mathrm{id} \to g'_*g'^*$), β_f (resp. β_g) is the adjunction morphism $f^*f_* \to \mathrm{id}$ (resp. $g^*g_* \to \mathrm{id}$), and the horizontal isomorphisms are induced by the canonical isomorphisms $f'^*g^* \xrightarrow{\sim} (g\circ f')^* = (f\circ g')^* \xleftarrow{\sim} g'^*f^*$ and $f_*g'_* \xrightarrow{\sim} (f\circ g')_* = (g\circ f')_* \xleftarrow{\sim} g_*f'_*$. We call the morphism

$$
g^*f_*\mathcal{F} \longrightarrow f'_*g'^*\mathcal{F} \tag{5.3.3}
$$

defined by the composition in either upper or lower way of (5.3.2) *the* base change morphism*.*

Moreover, if the diagram (5.3.1) *is Cartesian, then the base change morphism $g^*f_*\mathcal{F} \to f'_*g'^*\mathcal{F}$ is an isomorphism.*

Proof. As $(E_{/U})_{U\in\mathrm{Ob}(E)}$ forms a fibred site over E ([SGA 4_{II}, VI.7.3.2]), the commutativity of (5.3.2) is a special case of [SGA 4_{III}, XVII.2.1.3] applied to the asssociated fibred topos. One can also check it by unwinding the definitions. Indeed, for any object V of $E_{/Y}$, we have

$$
g^*f_*\mathcal{F}(V/Y) = f_*\mathcal{F}(V/X) = \mathcal{F}(V\times_X X'/X'), \tag{5.3.4}
$$

$$
f'_*g'^*\mathcal{F}(V/Y) = g'^*\mathcal{F}(V\times_Y Y'/Y') = \mathcal{F}(V\times_Y Y'/X'). \tag{5.3.5}
$$

One can check by definitions that the composition $g^*f_*\mathcal{F}(V/Y) \to f'_*g'^*\mathcal{F}(V/Y)$ in either upper or lower way of (5.3.2) coincides with the restriction map of $\mathcal{F}$ along the canonical morphism $V\times_Y Y' \to V\times_X X'$ over X'. Moreover, if the diagram (5.3.1) is Cartesian, then $V\times_Y Y' \to V\times_X X'$ is an isomorphism so that $g^*f_*\mathcal{F}(V/Y) \to f'_*g'^*\mathcal{F}(V/Y)$ is an isomorphism. □

Proposition 5.4. *Let E be a site where fibred products are representable, let $\mathcal{O}$ be a sheaf of R-algebras on E, and let $\{U_i \to X\}_{i\in I}$ be a covering in E. Consider the 2-truncated Čech hypercovering (*[Sta22, 01G6]*)*

$$
K_\bullet = (\{U_i\times_X U_j\times_X U_k\}_{i,j,k\in I} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \{U_i\times_X U_j\}_{i,j\in I} \rightrightarrows \{U_i\}_{i\in I}), \tag{5.4.1}
$$

regarded as a 2-truncated simplicial semi-representable object of $E_{/X}$. Let $\mathcal{F}_\bullet = (\mathcal{F}_n)_{[n]\in\Delta_{\leq 2}}$ be a π-Cartesian $\mathcal{O}_{/K_\bullet}$-module over the 2-truncated simplicial ringed site $E_{/K_\bullet}$, and we put $\mathcal{F} = \nu_*\mathcal{F}_\bullet$ where $\nu : E_{/K_\bullet} \to E_{/X}$ is the augmentation. Then, the canonical map $\nu_0^*\mathcal{F} \to \mathcal{F}_0$ is a π^8-isomorphism.

Proof. For any $i, j, k \in I$, we denote by $f_i : U_i \to X$, $f_{ij} : U_i\times_X U_j \to X$, $f_{ijk} : U_i\times_X U_j\times_X U_k \to X$ the canonical morphisms, and denote by $\mathcal{G}_i$, $\mathcal{G}_{ij}$, $\mathcal{G}_{ijk}$ the restrictions of $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$ to U_i, $U_i\times_X U_j$, $U_i\times_X U_j\times_X U_k$ respectively. By definition ([Sta22, 09WM]), we have

$$
\mathcal{F} = \mathrm{Eq}(\prod_{j\in I} f_{j*}\mathcal{G}_j \rightrightarrows \prod_{j,k\in I} f_{jk*}\mathcal{G}_{jk}). \tag{5.4.2}
$$

We need to show that the canonical map (note that the restriction functor f_i^* of sheaves commute with any limits as it admits a left adjoint $f_{i!}$),

$$
f_i^*\mathcal{F} = \mathrm{Eq}(\prod_{j\in I} f_i^*f_{j*}\mathcal{G}_j \rightrightarrows \prod_{j,k\in I} f_i^*f_{jk*}\mathcal{G}_{jk}) \longrightarrow \mathcal{G}_i \tag{5.4.3}
$$

given by composing the projection on the i-th component with the adjunction morphism $f_i^*f_{i*}\mathcal{G}_i \to \mathcal{G}_i$, is a π^8-isomorphism for any $i \in I$. Fixing $i \in I$, for any $j, k \in I$, we name some natural arrows as indicated in the following commutative diagram

$$
\begin{array}{ccc}
U_i \times_X U_j \times_X U_k & \xrightarrow{h_{jk}} & U_j \times_X U_k \\
\downarrow{\scriptstyle \beta_k} & & \downarrow{\scriptstyle \alpha_k} \\
U_i \times_X U_j & \xrightarrow{h_j} & U_j \\
\downarrow{\scriptstyle g_j} & & \downarrow{\scriptstyle f_j} \\
U_i & \xrightarrow{f_i} & X
\end{array}
\qquad (g_{jk} : U_i \times_X U_j \times_X U_k \to U_i,\ f_{jk} : U_j \times_X U_k \to X) \tag{5.4.4}
$$

Thus, we have a canonical commutative diagram of sheaves on $E_{/U_i}$,

$$
\begin{array}{ccccccc}
g_{j*}g_j^*\mathcal{G}_i & \longrightarrow & g_{j*}\mathcal{G}_{ij} & \longleftarrow & g_{j*}h_j^*\mathcal{G}_j & \xleftarrow{\sim} & f_i^*f_{j*}\mathcal{G}_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g_{jk*}g_{jk}^*\mathcal{G}_i & \longrightarrow & g_{jk*}\mathcal{G}_{ijk} & \longleftarrow & g_{jk*}h_{jk}^*\mathcal{G}_{jk} & \xleftarrow{\sim} & f_i^*f_{jk*}\mathcal{G}_{jk} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
g_{k*}g_k^*\mathcal{G}_i & \longrightarrow & g_{k*}\mathcal{G}_{ik} & \longleftarrow & g_{k*}h_k^*\mathcal{G}_k & \xleftarrow{\sim} & f_i^*f_{k*}\mathcal{G}_k
\end{array} \tag{5.4.5}
$$

obtained by the following steps:

(1) The structure of $\mathcal{F}_\bullet$ gives a canonical commutative diagram

$$
\begin{array}{ccccccccc}
\beta_k^*g_j^*\mathcal{G}_i & \longrightarrow & \beta_k^*\mathcal{G}_{ij} & \longleftarrow & \beta_k^*h_j^*\mathcal{G}_j & \longleftarrow & \beta_k^*h_j^*f_j^*f_{j*}\mathcal{G}_j & = & h_{jk}^*\alpha_k^*f_j^*f_{j*}\mathcal{G}_j \\
\| & & \downarrow & & \downarrow & & & & \downarrow \\
g_{jk}^*\mathcal{G}_i & \longrightarrow & \mathcal{G}_{ijk} & \longleftarrow & h_{jk}^*\mathcal{G}_{jk} & \longleftarrow & h_{jk}^*f_{jk}^*f_{jk*}\mathcal{G}_{jk} & = & h_{jk}^*\alpha_k^*f_j^*f_{j*}\alpha_{k*}\mathcal{G}_{jk}
\end{array} \tag{5.4.6}
$$

where the two horizontal arrows on the right square are induced by the adjunction morphisms $f_j^*f_{j*} \to \mathrm{id}$ and $f_{jk}^*f_{jk*} \to \mathrm{id}$. The right square is commutative since the adjunction morphism $f_{jk}^*f_{jk*} \to \mathrm{id}$ is the composition of $f_j^*f_{j*} \to \mathrm{id}$ with $\alpha_k^*\alpha_{k*} \to \mathrm{id}$.

(2) Applying β_{k*} to (5.4.6) and composing with the adjunction morphism $\mathrm{id} \to \beta_{k*}\beta_k^*$, we obtain a canonical commutative diagram

$$
\begin{array}{ccccccc}
g_j^*\mathcal{G}_i & \longrightarrow & \mathcal{G}_{ij} & \longleftarrow & h_j^*\mathcal{G}_j & \longleftarrow & h_j^*f_j^*f_{j*}\mathcal{G}_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\beta_{k*}g_{jk}^*\mathcal{G}_i & \longrightarrow & \beta_{k*}\mathcal{G}_{ijk} & \longleftarrow & \beta_{k*}h_{jk}^*\mathcal{G}_{jk} & \longleftarrow & h_j^*f_j^*f_{jk*}\mathcal{G}_{jk}
\end{array} \tag{5.4.7}
$$

where we used the fact that the vertical arrow on the right of (5.4.6) is the the image of the vertical arrow on the right of (5.4.7) under β_k^*.

(3) Applying g_{j*} to (5.4.7) and composing with the adjunction morphisms $\mathrm{id} \to g_{j*}g_j^*$ on the right, we obtain a canonical commutative diagram

$$
\begin{array}{ccccccc}
g_{j*}g_j^*\mathcal{G}_i & \longrightarrow & g_{j*}\mathcal{G}_{ij} & \longleftarrow & g_{j*}h_j^*\mathcal{G}_j & \longleftarrow & f_i^*f_{j*}\mathcal{G}_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g_{jk*}g_{jk}^*\mathcal{G}_i & \longrightarrow & g_{jk*}\mathcal{G}_{ijk} & \longleftarrow & g_{jk*}h_{jk}^*\mathcal{G}_{jk} & \longleftarrow & f_i^*f_{jk*}\mathcal{G}_{jk}
\end{array} \tag{5.4.8}
$$

(4) Going through the construction and using the fact that the adjunction morphism $\mathrm{id} \to g_{jk*}g_{jk}^*$ is the composition of $\mathrm{id} \to \beta_{k*}\beta_k^*$ with $\mathrm{id} \to g_{j*}g_j^*$, we see that the two horizontal arrows in the right square of (5.4.8) are the base change isomorphisms defined in 5.3. Hence, we obtain the first row of (5.4.5). Replacing the second row of (5.4.4) by $h_k : U_i \times_X U_k \to U_k$, we obtain the second row of (5.4.5) in the same way.

Since

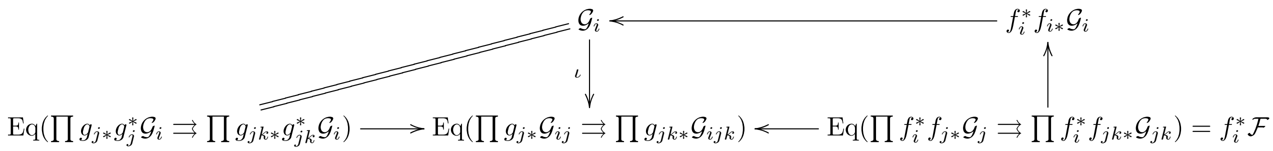
$$
\{g_{jk} : U_i \times_X U_j \times_X U_k \to U_i\}_{j,k \in I} \rightrightarrows \{g_j : U_i \times_X U_j \to U_i\}_{j \in I} \tag{5.4.9}
$$

is a 1-truncated Čech hypercovering of U_i, the equalizer corresponding to the first column in (5.4.5) is equal to

(5.4.10)
$$\mathcal{G}_i = \mathrm{Eq}(\prod_{j\in I} g_{j*}g_j^*\mathcal{G}_i \rightrightarrows \prod_{j,k\in I} g_{jk*}g_{jk}^*\mathcal{G}_i)$$

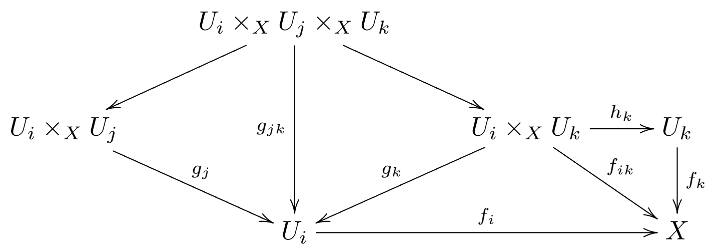
by the sheaf property of $\mathcal{G}_i$ on $E_{/U_i}$. Since $\mathcal{F}_\bullet$ is π-Cartesian, the horizontal arrows in (5.4.5) are π^2-isomorphisms by 3.2 (see [He22, 7.3]). Therefore, the morphisms between the equalizers corresponding to each column in (5.4.5) (see the second row of (5.4.11) in the following) are π^4-isomorphisms. In order to show that the canonical map $f_i^*\mathcal{F} \to \mathcal{G}_i$ (5.4.3) is a π^8-isomorphism, it remains to prove the square in the following natural diagram is commutative,

(5.4.11)



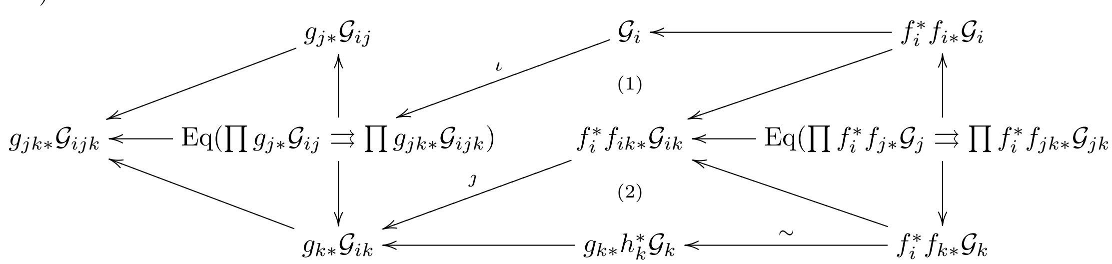
where ι is the natural map making the left triangle commutative, and in each equalizer, j goes through I for the first product, and j,k go through I for the second product. Consider the commutative diagram for any $j,k\in I$,

(5.4.12)



from which we obtain the following natural commutative diagram

(5.4.13)



Indeed, the natural map $\jmath : f_i^*f_{ik*}\mathcal{G}_{ik} = f_i^*f_{i*}g_{k*}\mathcal{G}_{ik} \to g_{k*}\mathcal{G}_{ik}$ is defined by applying the adjunction morphism $f_i^*f_{i*} \to \mathrm{id}$ to $g_{k*}\mathcal{G}_{ik}$, and other natural arrows have appeared in the diagrams (5.4.5) and (5.4.11). Thus, the commutativity of (1) follows from applying the adjunction morphism $f_i^*f_{i*} \to \mathrm{id}$ to the canonical map $\mathcal{G}_i \to g_{k*}\mathcal{G}_{ik}$, and the commutativity of (2) follows from the following natural commutative diagram

(5.4.14)
$$\begin{array}{ccccccc}
g_{k*}\mathcal{G}_{ik} & \xleftarrow{\jmath} & f_i^*f_{i*}g_{k*}\mathcal{G}_{ik} & = & f_i^*f_{k*}h_{k*}\mathcal{G}_{ik} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
g_{k*}h_k^*\mathcal{G}_k & \longleftarrow & f_i^*f_{i*}g_{k*}h_k^*\mathcal{G}_k & = & f_i^*f_{k*}h_{k*}h_k^*\mathcal{G}_k & \longleftarrow & f_i^*f_{k*}\mathcal{G}_k
\end{array}$$

where the vertical arrows are induced by the canonical map $h_k^*\mathcal{G}_k \to \mathcal{G}_{ik}$, and the composition of the second row is the base change isomorphism $f_i^*f_{k*}\mathcal{G}_k \xrightarrow{\sim} g_{k*}h_k^*\mathcal{G}_k$ by 5.3 (we indeed used both two constructions of the base change isomorphism, see the construction of (5.4.5)). In particular, we see that the natural diagram extracted from (5.4.13),

$$
\begin{array}{ccc}
\mathcal{G}_i & \longleftarrow & f_i^* f_{i*}\mathcal{G}_i \\
\downarrow{\scriptstyle\iota} & & \uparrow \\
\mathrm{Eq}(\prod g_{j*}\mathcal{G}_{ij} \rightrightarrows \prod g_{jk*}\mathcal{G}_{ijk}) & & \mathrm{Eq}(\prod f_i^* f_{j*}\mathcal{G}_j \rightrightarrows \prod f_i^* f_{jk*}\mathcal{G}_{jk}) \\
\downarrow & & \downarrow \\
g_{k*}\mathcal{G}_{ik} & \longleftarrow & f_i^* f_{k*}\mathcal{G}_k
\end{array}
\tag{5.4.15}
$$

is commutative for any $k \in I$. This shows that the diagram (5.4.11) is commutative, which completes the proof. □

Remark 5.5. We expect a generalization to any 2-truncated hypercovering $K_\bullet$ of X as in [Sta22, 0D8E].

Example 5.6. Let X be a quasi-compact and separated scheme, and let $K_0 = \{U_i \to X\}_{0\leq i\leq k}$ be a finite open covering of X consisting of affine open subschemes. For any $n \in \mathbb{N}$, we define a semi-representable object of the Zariski site X_{Zar} of X,

$$K_n = \{U_{i_0} \cap \cdots \cap U_{i_n} \to X\}_{0\leq i_0,\cdots,i_n\leq k}. \tag{5.6.1}$$

These K_n naturally form a simplicial semi-representable object of X_{Zar}, $K_\bullet = (K_n)_{[n]\in \mathrm{Ob}(\Delta)}$, called the Čech hypercovering associated to K_0 of X. We put

$$X_n = \coprod_{0\leq i_0,\dots,i_n\leq k} U_{i_0} \cap \cdots \cap U_{i_n} \tag{5.6.2}$$

which is a finite disjoint union of affine open subschemes of X, and denote by $\nu_n : X_n \to X$ the canonical morphism. It is clear that the site X_{Zar/K_n} is naturally equivalent to the Zariski site $X_{n,\mathrm{Zar}}$. We also obtain a simplicial affine scheme $X_\bullet = (X_n)_{[n]\in\mathrm{Ob}(\Delta)}$, and an augmentation $\nu : X_\bullet \to X$ (where we omit the subscript "Zar").

For any $\mathcal{O}_{X_\bullet}$-module $\mathcal{F}_\bullet$, we consider the ordered Čech complex $\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet)$, whose degree-$n$ term is the R-module ([Sta22, 01FG])

$$\check{C}^n_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet) = \prod_{0\leq i_0<\cdots<i_n\leq k} \mathcal{F}_n(U_{i_0} \cap \cdots \cap U_{i_n}). \tag{5.6.3}$$

In general, for any complex of $\mathcal{O}_{X_\bullet}$-modules $\mathcal{F}^\bullet_\bullet$, we consider the total complex of the ordered Čech complexes $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$, whose degree-$n$ term is the R-module (see [Sta22, 01FP])

$$\mathrm{Tot}^n(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)) = \bigoplus_{p+q=n} \prod_{0\leq i_0<\cdots<i_p\leq k} \mathcal{F}^q_p(U_{i_0} \cap \cdots \cap U_{i_p}). \tag{5.6.4}$$

Indeed, it depends only on the k-truncation $(\mathcal{F}^\bullet_n)_{[n]\in\mathrm{Ob}(\Delta_{\leq k})}$. If $\mathcal{F}^\bullet_\bullet$ is the pullback of a complex of quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}^\bullet$ (i.e. $\mathcal{F}^\bullet_\bullet = \nu^*\mathcal{F}^\bullet$), then there is an isomorphism in the derived category $\mathbf{D}(X)$ (see [Sta22, 01FK, 01FM, 0FLH]),

$$\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)) \xrightarrow{\sim} \mathrm{R}\Gamma(X, \mathcal{F}^\bullet). \tag{5.6.5}$$

Lemma 5.7. *Under the assumptions in* 5.6 *and with the same notation, for any quasi-coherent $\mathcal{O}_{X_\bullet}$-module $\mathcal{F}_\bullet$, $\mathcal{F} = \nu_*\mathcal{F}_\bullet$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. By definition, $\mathcal{F} = \mathrm{Eq}(\nu_{0*}\mathcal{F}_0 \rightrightarrows \nu_{1*}\mathcal{F}_1)$. As ν_i is affine ($i = 0, 1$), $\nu_{i*}\mathcal{F}_i$ is a quasi-coherent $\mathcal{O}_X$-module. Hence, the equalizer $\mathcal{F}$ is also a quasi-coherent $\mathcal{O}_X$-module. □

Proposition 5.8. *Under the assumptions in* 5.6 *and with the same notation, let $r \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, $\mathrm{sk}_r(X_\bullet) = (X_n)_{[n]\in\mathrm{Ob}(\Delta_{\leq r})}$ the r-truncation of $X_\bullet$, $\mathcal{F}_\bullet$ a quasi-coherent $\mathcal{O}_{\mathrm{sk}_r(X_\bullet)}$-module of finite type. Assume that $\mathcal{F}_\bullet$ is π-Cartesian (see* 5.1*). Then, there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite type and a π^{25}-isomorphism $\nu^*\mathcal{F} \to \mathcal{F}_\bullet$.*

Proof. If we also denote by $\nu : \mathrm{sk}_r(X_\bullet) \to X$ the augmentation, then $\mathcal{F}' = \nu_*\mathcal{F}_\bullet$ is a quasi-coherent $\mathcal{O}_X$-module by 5.7 and the canonical morphism $\nu_0^*\mathcal{F}' \to \mathcal{F}_0$ is a π^8-isomorphism by 5.4. We claim that the canonical morphism $\nu^*\mathcal{F}' \to \mathcal{F}_\bullet$ is a π^9-isomorphism. Indeed, for any integer $0 \leq n \leq r$ the morphism $\nu_n : X_n \to X$ is the composition of $\nu_0 : X_0 \to X$ with a projection $f : X_n \to X_0$ associated to a morphism $[0] \to [n]$ in Δ. Since $f^*\mathcal{F}_0 \to \mathcal{F}_n$ is a π-isomorphism by assumption, we see that $\nu_n^*\mathcal{F}' \to \mathcal{F}_n$ is a π^9-isomorphism.

For any $0 \leq i \leq k$, we denote by $\mathcal{G}_i$ the restriction of $\mathcal{F}_0$ to the component U_i, which is a quasi-coherent $\mathcal{O}_{X_i}$-module of finite type by assumption. By 3.2.(1), there exists a π^{16}-isomorphism $\mathcal{G}_i \to \mathcal{F}'|_{U_i}$. Notice that $\mathcal{F}'$ is the filtered union of its quasi-coherent $\mathcal{O}_X$-submodules of finite type by [EGA $\mathrm{I}_{\mathrm{new}}$, 6.9.9]. Thus, there is a sufficiently large quasi-coherent $\mathcal{O}_X$-submodule $\mathcal{F}$ of finite type such that the π^{16}-surjection $\mathcal{G}_i \to \mathcal{F}'|_{U_i}$ factors through $\mathcal{F}|_{U_i}$ for any $0 \leq i \leq k$. Thus, the inclusion $\mathcal{F} \subseteq \mathcal{F}'$ is π^{16}-surjective, which implies that the induced morphism $\nu^*\mathcal{F} \to \mathcal{F}_\bullet$ is a π^{25}-isomorphism. □

Lemma 5.9. *Under the assumptions in* 5.6 *and with the same notation, let $\mathcal{F}_\bullet^\bullet \to \mathcal{G}_\bullet^\bullet$ be a π-isomorphism of complexes of quasi-coherent $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$-modules. Then, the map*

$$\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{G}_\bullet^\bullet)) \tag{5.9.1}$$

is a π-isomorphism. In particular, it is a π^2-quasi-isomorphism.

Proof. By taking sections on an affine scheme, $\mathcal{F}_p^q(U_{i_0} \cap \cdots \cap U_{i_p}) \to \mathcal{G}_p^q(U_{i_0} \cap \cdots \cap U_{i_p})$ is still a π-isomorphism. This shows that (5.9.1) is a π-isomorphism. The second assertion follows from 3.2. □

Lemma 5.10. *Under the assumptions in* 5.6 *and with the same notation, let a be an integer, $\mathcal{F}_\bullet^\bullet \to \mathcal{G}_\bullet^\bullet$ a morphism of complexes of quasi-coherent $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$-modules. Assume that for any $0 \leq n \leq k$, the map $H^i(\mathcal{F}_n^\bullet) \to H^i(\mathcal{G}_n^\bullet)$ is a π-isomorphism for any $i > a$ and π-surjective for $i = a$. Then, the map*

$$H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))) \longrightarrow H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{G}_\bullet^\bullet))) \tag{5.10.1}$$

is a $\pi^{4(k+1)}$-isomorphism for any $i > a + k$ and $\pi^{2(k+1)}$-surjective for $i = a + k$.

Proof. As the functor $\check{C}^p_{\mathrm{ord}}(X_\bullet, -)$ (5.6.3) on quasi-coherent $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$-modules is exact, we see that $H^q(\check{C}^p_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)) = \check{C}^p_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet))$, where $H^q(\mathcal{F}_\bullet^\bullet) = (H^q(\mathcal{F}_n^\bullet))_{[n] \in \mathrm{Ob}(\Delta_{\leq k})}$ is a quasi-coherent $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$-module. Thus, there is a spectral sequence

$$E_2^{pq} = H^p(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet))) \Rightarrow H^{p+q}(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))), \tag{5.10.2}$$

which is convergent, since $\check{C}^p_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^q) = 0$ unless $0 \leq p \leq k$ ([Sta22, 0132]).

Let $\mathcal{K}_\bullet^\bullet$ be the cone of $\mathcal{F}_\bullet^\bullet \to \mathcal{G}_\bullet^\bullet$. The assumption implies that $H^i(\mathcal{K}_n^\bullet)$ is π^2-null for any $0 \leq n \leq k$ and $i \geq a$. The convergent spectral sequence (5.10.2) for $\mathcal{K}_\bullet^\bullet$ implies that for any $i \in \mathbb{Z}$, there is a finite filtration of length $\leq (k+1)$ on $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{K}_\bullet^\bullet)))$ whose graded pieces are subquotients of $E_2^{p,i-p}$ where $0 \leq p \leq k$. Since $E_2^{p,i-p}$ is π^2-null for any $i \geq a+k$, we see that $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{K}_\bullet^\bullet)))$ is $\pi^{2(k+1)}$-null for such i. The conclusion follows from the long exact sequence of cohomology groups associated to the distinguished triangle in $\mathbf{K}(R)$,

$$\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{G}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{K}_\bullet^\bullet)) \longrightarrow \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))[1]. \tag{5.10.3}$$

□

Proposition 5.11. *Under the assumptions in* 5.6 *and with the same notation, let $a \leq b$ be two integers, $\mathcal{F}_\bullet^\bullet$ a complex of quasi-coherent $\mathcal{O}_{\mathrm{sk}_k(X_\bullet)}$-modules vanishing in degrees $> b$. Assume that*

1. *R is Noetherian, and that*
2. *the R-module $H^p(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet)))$ is of π-finite type for any $0 \leq p \leq k$ and $q \geq a$ (see 4.11).*

Then, the complex of R-modules $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))$ is π^l-$[a+k, b+k]$-pseudo-coherent for an integer $l \geq 0$ depending only on $b - a$ and k.

Proof. Consider the convergent spectral sequence (5.10.2). Notice that for any $0 \leq p \leq k$ and $q \geq a$, a subquotient of $E_2^{pq} = H^p(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, H^q(\mathcal{F}_\bullet^\bullet)))$ is of π-finite type by 4.12.(2). Since there is a finite filtration of length $\leq (k+1)$ on $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)))$ whose graded pieces are subquotients of $E_2^{p,i-p}$ where $0 \leq p \leq k$, we see that $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet)))$ is of π^{k+1}-finite type for any $i \geq a + k$ by 4.12.(2). As $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}_\bullet^\bullet))$ vanishes in degrees $> b + k$ by definition, the conclusion follows from 4.13.(1). □

6. Almost Coherent Modules

In this section, we fix a ring R with an ideal $\mathfrak{m}$ such that for any integer $l \geq 1$, the l-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$ (in particular, $\mathfrak{m} = \mathfrak{m}^2$, see [GR03, 2.1.6.(B)]). Let E be a site with a final object $*$, and let $\mathcal{O}$ be a sheaf of R-algebras on E.

Definition 6.1 ([AG20, 2.7.3])**.** Let M be an $\mathcal{O}$-module on E.

(1) We say that M is *almost zero* if it is π-null for any $\pi \in \mathfrak{m}$. We say that a morphism $f : M \to N$ of $\mathcal{O}$-modules is an *almost isomorphism* if its kernel and cokernel are almost zero.
(2) We say that M is *of π-finite type* for some element $\pi \in R$ if there exists a covering $\{U_i \to *\}_{i\in I}$ in E such that for any $i \in I$ there exist finitely many sections $s_1, \dots, s_n \in M(U_i)$ such that the induced morphism of $\mathcal{O}|_{U_i}$-modules $\mathcal{O}^{\oplus n}|_{U_i} \to M|_{U_i}$ has π-null cokernel. We say that M is *of almost finite type* if it is of π-finite type for any $\pi \in \mathfrak{m}$.
(3) We say that M is *almost coherent* if M is of almost finite type, and if for any object U of E and any finitely many sections $s_1, \dots, s_n \in M(U)$, the kernel of the induced morphism of $\mathcal{O}|_U$-modules $\mathcal{O}^{\oplus n}|_U \to M|_U$ is an $\mathcal{O}|_U$-module of almost finite type.

We refer to Abbes-Gros [AG20, 2.7, 2.8] for a more detailed study of almost coherent modules. They work in a slightly restricted basic setup for almost algebra [AG20, 2.6.1], but most of their arguments still work in our setup $(R, \mathfrak{m})$ by adding the following lemmas.

Lemma 6.2. *Let M be an $\mathcal{O}$-module on E, and let $\pi_1, \pi_2 \in R$. If M is of π_i-finite type for $i = 1, 2$, then it is of $(x\pi_1 + y\pi_2)$-finite type for any $x, y \in R$. In particular, if there exists an integer $l \geq 1$ such that M is of π^l-finite type for any $\pi \in \mathfrak{m}$, then M is of almost finite type.*

Proof. The problem is local on E. We may assume that there exist morphisms of $\mathcal{O}$-modules f_i : $\mathcal{O}^{\oplus n_i} \to M$ $(i = 1, 2)$ with π_i-null cokernels. Thus, the cokernel of $f_1 \oplus f_2 : \mathcal{O}^{\oplus n_1} \oplus \mathcal{O}^{\oplus n_2} \to M$ is killed by $x\pi_1 + y\pi_2$. The "in particular" part follows from the assumption that the ideal $\mathfrak{m}$ is generated by the subset $\{\pi^l \mid \pi \in \mathfrak{m}\}$. □

Lemma 6.3. *Let M be an $\mathcal{O}$-module.*

(1) *Assume that there exists an integer $l \geq 1$ such that for any $\pi \in \mathfrak{m}$, there exists an almost coherent $\mathcal{O}$-module M_π and a π^l-isomorphism $M \to M_\pi$. Then, M is almost coherent.*
(2) *Assume that there exists an integer $l \geq 1$ such that for any $\pi \in \mathfrak{m}$, there exists an almost coherent $\mathcal{O}$-module M_π and a π^l-isomorphism $M_\pi \to M$. Then, M is almost coherent.*

Proof. (1) The π^l-isomorphism $M \to M_\pi$ induces a π^{2l}-isomorphism $M_\pi \to M$ by 3.2. Such an argument shows that (1) implies (2). We also see that M is of π^{3l}-finite type. Hence, M is of almost finite type by 6.2. For any object U of E, and any morphism of $\mathcal{O}|_U$-modules $f : \mathcal{O}^{\oplus n}|_U \to M|_U$, consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \xrightarrow{f} & M \\ & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & \mathrm{Ker}(f_\pi) & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \xrightarrow{f_\pi} & M_\pi \end{array} \tag{6.3.1}$$

It is clear that $\mathrm{Ker}(f) \to \mathrm{Ker}(f_\pi)$ is a π^l-isomorphism. Since M_π is almost coherent by assumption, $\mathrm{Ker}(f_\pi)$ is of almost finite type. Hence, $\mathrm{Ker}(f)$ is of π^{3l}-finite type by the argument in the beginning. Thus, $\mathrm{Ker}(f)$ is of almost finite type by 6.2. This verifies the almost coherence of M. □

We collect some basic properties about almost coherence that will be used in the rest of this article. Their proofs are essentially given in [AG20], and we only give a brief sketch here.

Proposition 6.4 ([AG20, 2.7.16])**.** *Let $0 \longrightarrow M_1 \xrightarrow{u} M_2 \xrightarrow{v} M_3 \longrightarrow 0$ be an almost exact sequence of $\mathcal{O}$-modules on E. If two of M_1, M_2, M_3 are almost coherent, then so is the third.*

Proof. Since almost isomorphisms preserve almost coherence by 6.3, we may assume that $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact.

Assume that M_2 and M_3 are almost coherent. Then, M_1 is of almost finite type by [AG20, 2.7.14.(iii)] and 6.2. Hence, M_1 is almost coherent as a submodule of an almost coherent $\mathcal{O}$-module M_2 by definition.

Assume that M_1 and M_2 are almost coherent. Then, M_3 is of almost finite type as a quotient of M_2. Let U be an object of E. We need to show that any homomorphism $f_3 : \mathcal{O}^{\oplus n}|_U \to M_3|_U$ has kernel of π^2-finite type for any $\pi \in \mathfrak{m}$ by 6.2. The problem is local on E. Thus, we may take a π-surjection $f_1 : \mathcal{O}^{\oplus m}|_U \to M_1|_U$ and a lifting $f_3' : \mathcal{O}^{\oplus n}|_U \to M_2|_U$ of f_3. We put $f_2 = (f_1, f_3') : \mathcal{O}^{\oplus m+n}|_U \to M_2|_U$, and obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}^{\oplus m}|_U & \longrightarrow & \mathcal{O}^{\oplus m+n}|_U & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & M_1|_U & \longrightarrow & M_2|_U & \longrightarrow & M_3|_U & \longrightarrow & 0 \end{array} \tag{6.4.1}$$

The snake lemma shows that $\mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3)$ is π-surjective. Since $\mathrm{Ker}(f_2)$ is of almost finite type as M_2 is almost coherent, we see that $\mathrm{Ker}(f_3)$ is of π^2-finite type.

Assume that M_1 and M_3 are almost coherent. Then, M_2 is of almost finite type by [AG20, 2.7.14.(ii)] and 6.2. Let U be an object of E. We need to show that any homomorphism $f_2 : \mathcal{O}^{\oplus n}|_U \to M_2|_U$ has kernel of π^2-finite type for any $\pi \in \mathfrak{m}$ by 6.2. The problem is local on E. Thus, we may take a π-surjection $\mathcal{O}^{\oplus m}|_U \to \mathrm{Ker}(v \circ f_2)$ as $\mathrm{Ker}(v \circ f_2)$ is of almost finite type (M_3 is almost coherent). Thus, we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathcal{O}^{\oplus m}|_U & \longrightarrow & \mathcal{O}^{\oplus n}|_U & \longrightarrow & M_3|_U & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \| & & \\ 0 & \longrightarrow & M_1|_U & \xrightarrow{u} & M_2|_U & \xrightarrow{v} & M_3|_U & \longrightarrow & 0 \end{array} \tag{6.4.2}$$

By snake lemma, we see that $\mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2)$ is π-surjective. Since $\mathrm{Ker}(f_1)$ is of almost finite type as M_1 is almost coherent, we see that $\mathrm{Ker}(f_2)$ is of π^2-finite type. ☐

Corollary 6.5 ([AG20, 2.7.17]). *For any morphism $f : M \to N$ of almost coherent $\mathcal{O}$-modules,* $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ *and* $\mathrm{Coker}(f)$ *are almost coherent.*

Corollary 6.6. *Assume that $\mathcal{O}$ is almost coherent as an $\mathcal{O}$-module. Then, any cohomology group of a complex of finite free $\mathcal{O}$-modules is almost coherent.*

Proof. A finite free $\mathcal{O}$-module is almost coherent by 6.4. Thus, a cohomology group of a complex of finite free $\mathcal{O}$-modules is almost coherent by 6.5. ☐

Proposition 6.7 ([AG20, 2.8.7]). *Let $X = \mathrm{Spec}(A)$ be an affine scheme over R, let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module, and let $\pi \in \mathfrak{m}$. Then, the $\mathcal{O}_X$-module $\mathcal{F}$ is of π-finite type (resp. of almost finite type, almost coherent) on the Zariski site of X if and only if the A-module $\mathcal{F}(X)$ is of π-finite type (resp. of almost finite type, almost coherent) on the trivial site of a single point.*

Proof. It is clear that the statement for "of π-finite type" implies that for "of almost finite type" and thus implies that for "almost coherent". It remains to show that for any $\pi \in \mathfrak{m}$ and any finitely many elements $f_1, \dots, f_n \in A$ generating A as an ideal, an A-module M is of π-finite type if and only if the A_{f_i}-module M_{f_i} is of π-finite type for any $1 \le i \le n$. The necessity is obvious. For the sufficiency, we write $M = \bigcup_{\lambda \in \Lambda} M_\lambda$ as a filtered union of its A-submodules of finite type. There exists $\lambda_0 \in \Lambda$ large enough such that $M_{\lambda_0, f_i} \to M_{f_i}$ is a π-isomorphism for any $1 \le i \le n$. Hence, $M_{\lambda_0} \to M$ is a π-isomorphism, which completes the proof. ☐

Lemma 6.8 (cf. [Kie72, 2.2]). *Let $k \in \mathbb{N}$, let $X_\bullet = (X_n)_{[n] \in \mathrm{Ob}(\Delta_{\le k})}$ be a k-truncated simplicial affine scheme over R, and let $M_\bullet$ be a quasi-coherent $\mathcal{O}_{X_\bullet}$-module. Assume that the $\mathcal{O}_{X_\bullet}$-modules $\mathcal{O}_{X_\bullet}$ and $M_\bullet$ are almost coherent. Then, for any $\pi \in \mathfrak{m}$, there exists a π-exact sequence of quasi-coherent $\mathcal{O}_{X_\bullet}$-modules*

$$\cdots \longrightarrow F_\bullet^{-1} \longrightarrow F_\bullet^0 \longrightarrow M_\bullet, \tag{6.8.1}$$

such that F_n^i is a finite free $\mathcal{O}_{X_n}$-module for any $i \le 0$ and $0 \le n \le k$.

Proof. Firstly, we construct $F_\bullet^0$. For each $0 \le n \le k$, we take a finite free $\mathcal{O}_{X_n}$-module N_n and a π-surjection $h_n : N_n \to M_n$ (as M_n is of π-finite type, see 6.7). We put

$$F_n^0 = \bigoplus_{0 \le m \le k} \bigoplus_{\alpha \in \mathrm{Hom}_\Delta([m],[n])} \alpha^* N_m. \tag{6.8.2}$$

It forms naturally a finite free $\mathcal{O}_{X_\bullet}$-module $F^0_\bullet$. There is a natural morphism $F^0_n \to M_n$ defined on the (m,α)-component by the composition

$$\alpha^*N_m \xrightarrow{\alpha^*(h_m)} \alpha^*M_m \longrightarrow M_n. \tag{6.8.3}$$

It induces a π-surjective homomorphism of $\mathcal{O}_{X_\bullet}$-modules $F^0_\bullet \to M_\bullet$ (cf. the proof of [Kie72, 2.2]). Notice that its kernel $M^{-1}_\bullet$ is also a quasi-coherent $\mathcal{O}_{X_\bullet}$-module which is almost coherent by 6.5. Thus, we can apply the previous procedure to $M^{-1}_\bullet$ and we construct $F^i_\bullet$ inductively for any $i \leq 0$. □

7. Proof of the Main Theorem

This section is devoted to proving the following theorem.

Theorem 7.1. *Let R be a ring with an ideal $\mathfrak{m}$ such that for any integer $l \geq 1$, the l-th powers of elements of $\mathfrak{m}$ generate $\mathfrak{m}$. Consider a flat proper morphism of finite presentation $f : X \to S$ between R-schemes. Assume that $\mathcal{O}_X$ and $\mathcal{O}_S$ are almost coherent as modules over themselves. Then, for any quasi-coherent and almost coherent $\mathcal{O}_X$-module $\mathcal{M}$ and any $q \in \mathbb{N}$, $\mathrm{R}^q f_*\mathcal{M}$ is a quasi-coherent and almost coherent $\mathcal{O}_S$-module.*

Proof. The problem is local on S. Thus, we may assume that $S = \mathrm{Spec}(A)$ is affine. Since f is quasi-compact and quasi-separated, $\mathrm{R}^q f_*\mathcal{M}$ is a quasi-coherent $\mathcal{O}_S$-module for any $q \in \mathbb{N}$. Thus, it remains to prove that $H^q(X,\mathcal{M})$ is an almost coherent A-module by 6.7. We write X as a finite union of affine open subschemes $X = \bigcup_{0\leq i\leq k} U_i$ for some $k \in \mathbb{N}_{\geq 2}$, and consider the k-truncated Čech hypercovering $X_\bullet = (X_n)_{[n]\in\mathrm{Ob}(\Delta_{\leq k})}$ (see 5.6), where for any $0 \leq n \leq k$,

$$X_n = \coprod_{0\leq i_0,\dots,i_n\leq k} U_{i_0} \cap \cdots \cap U_{i_n}, \tag{7.1.1}$$

which is a finite disjoint union of affine open subschemes of X as X is separated. Let $\nu : X_\bullet \to X$ denote the augmentation.

We fix an element $\pi \in \mathfrak{m}$ and a negative integer $a < -(k+2)$ in the following, and take a sequence of quasi-coherent $\mathcal{O}_{X_\bullet}$-modules by 6.8,

$$0 \longrightarrow \mathcal{F}^a_\bullet \longrightarrow \cdots \longrightarrow \mathcal{F}^{-1}_\bullet \longrightarrow \mathcal{F}^0_\bullet \longrightarrow \mathcal{M}_\bullet = \nu^*\mathcal{M}, \tag{7.1.2}$$

such that $\mathcal{F}^\bullet_n \to \mathcal{M}_n[0]$ is a π-$[a,0]$-pseudo resolution (see 4.1) for any $0 \leq n \leq k$. In other words, for any $a \leq i \leq 0$ and $0 \leq n \leq k$,

(1) $\mathcal{F}^i_n$ is a finite free $\mathcal{O}_{X_n}$-module, and
(2) $H^i(\mathcal{F}^\bullet_n)$ is π-null for any $a < i < 0$, and $H^0(\mathcal{F}^\bullet_n) \to \mathcal{M}_n$ is a π-isomorphism.

For any morphism $\alpha : [m] \to [n]$ in $\Delta_{\leq k}$ (regarded also as a morphism $X_n \to X_m$), we denote by $C^\bullet_\alpha$ the cone of the induced map $\alpha^*\mathcal{F}^\bullet_m \to \mathcal{F}^\bullet_n$ of complexes of finite free $\mathcal{O}_{X_n}$-modules.

Lemma 7.2. *For any morphism $\alpha : [m] \to [n]$ in $\Delta_{\leq k}$, there exists a homomorphism of finite free $\mathcal{O}_{X_n}$-modules $s^i_\alpha : C^i_\alpha \to C^{i-1}_\alpha$ for any $i \geq a+1$ such that*

$$\pi^{-4a} \cdot \mathrm{id}_{C^i_\alpha} = s^{i+1}_\alpha \circ \mathrm{d}^i_\alpha + \mathrm{d}^{i+1}_\alpha \circ s^i_\alpha \tag{7.2.1}$$

for any $i \geq a+1$, where $\mathrm{d}^i_\alpha : C^i_\alpha \to C^{i+1}_\alpha$ is the differential map.

Proof. We firstly note that $C^i_\alpha = \mathcal{F}^i_n \oplus \alpha^*\mathcal{F}^{i+1}_m$ is a finite free $\mathcal{O}_{X_n}$-module. In particular, $C^\bullet_\alpha$ vanishes outside $[a-1,0]$. By definition, $H^i(\mathcal{F}^\bullet_n) \to H^i(\mathcal{M}_n[0])$ is a π-isomorphism for any $i > a$. Notice that the induced map $\alpha^*H^i(\mathcal{M}_m[0]) \to H^i(\mathcal{M}_n[0])$ is an isomorphism since $\mathcal{M}_\bullet = \nu^*\mathcal{M}$. Thus, the induced map $\alpha^*H^i(\mathcal{F}^\bullet_m) \to H^i(\mathcal{F}^\bullet_n)$ is a π^2-isomorphism of $\mathcal{O}_{X_n}$-modules for any $i > a$, which implies that $H^i(C^\bullet_\alpha)$ is π^4-null for any $i > a$. Thus, the conclusion follows directly from 3.3 (see (3.3.4)). □

Now we write A as a filtered union of finitely generated $\mathbb{Z}$-subalgebras $A = \mathrm{colim}_{\lambda\in\Lambda} A_\lambda$. By [EGA IV$_3$, 8.5.2, 8.8.2, 8.10.5, 11.2.6], there exists an index $\lambda_0 \in \Lambda$ such that $\pi \in A_{\lambda_0}$,

(1) a flat proper morphism of finite presentation $f_{\lambda_0} : X_{\lambda_0} \to S_{\lambda_0} = \mathrm{Spec}(A_{\lambda_0})$ whose base change along $S \to S_{\lambda_0}$ is f,
(2) affine open subschemes $U_{\lambda_0,i}$ $(0 \leq i \leq k)$ of X_{λ_0} whose base change along $S \to S_{\lambda_0}$ is U_i,

(3) a complex of finite free $\mathcal{O}_{X_{\lambda_0,\bullet}}$-modules $0 \to \mathcal{F}^a_{\lambda_0,\bullet} \to \cdots \to \mathcal{F}^{-1}_{\lambda_0,\bullet} \to \mathcal{F}^0_{\lambda_0,\bullet} \to 0$ whose pullback along $S \to S_{\lambda_0}$ is $\mathcal{F}^\bullet_\bullet$ (where $X_{\lambda_0,\bullet}$ is the k-truncated Čech hypercovering associated to $X_{\lambda_0,0} = \coprod_{0\leq i\leq k} U_{\lambda_0,i} \to X_{\lambda_0}$),

(4) a homomorphism of finite free $\mathcal{O}_{X_{\lambda_0,n}}$-modules $s^i_{\lambda_0,\alpha} : C^i_{\lambda_0,\alpha} \to C^{i-1}_{\lambda_0,\alpha}$ for any $i \geq a+1$ and any morphism $\alpha : [m] \to [n]$ in $\Delta_{\leq k}$, such that

$$\pi^{-4a} \cdot \mathrm{id}_{C^i_{\lambda_0,\alpha}} = s^{i+1}_{\lambda_0,\alpha} \circ \mathrm{d}^i_{\lambda_0,\alpha} + \mathrm{d}^{i+1}_{\lambda_0,\alpha} \circ s^i_{\lambda_0,\alpha} \tag{7.2.2}$$

for any $i \geq a+1$, where $C^\bullet_{\lambda_0,\alpha}$ is the cone of $\alpha^*\mathcal{F}^\bullet_{\lambda_0,m} \to \mathcal{F}^\bullet_{\lambda_0,n}$, and $\mathrm{d}^i_{\lambda_0,\alpha} : C^i_{\lambda_0,\alpha} \to C^{i+1}_{\lambda_0,\alpha}$ is the differential map.

We note that X_{λ_0} and S_{λ_0} are Noetherian schemes.

Lemma 7.3. *For any morphism $\alpha : [m] \to [n]$ in $\Delta_{\leq k}$, the induced map of coherent $\mathcal{O}_{X_{\lambda_0,n}}$-modules $\alpha^* H^i(\mathcal{F}^\bullet_{\lambda_0,m}) \to H^i(\mathcal{F}^\bullet_{\lambda_0,n})$ is a π^{-4a}-isomorphism for any $i \geq a+2$.*

Proof. It follows directly from the relation (7.2.2). □

Lemma 7.4. *For any $i \geq a+2$, there exists a coherent $\mathcal{O}_{X_{\lambda_0}}$-module $\mathcal{G}^i_{\lambda_0}$ and a π^{-100a}-isomorphism $\nu^*_{\lambda_0}\mathcal{G}^i_{\lambda_0} \to H^i(\mathcal{F}^\bullet_{\lambda_0,\bullet})$, where $\nu_{\lambda_0} : X_{\lambda_0,\bullet} \to X_{\lambda_0}$ is the augmentation.*

Proof. It follows by applying directly 5.8 to the coherent $\mathcal{O}_{X_{\lambda_0,\bullet}}$-module $H^i(\mathcal{F}^\bullet_{\lambda_0,\bullet})$, whose condition is satisfied by 7.3. □

Lemma 7.5. *The A_{λ_0}-module $H^i(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, H^j(\mathcal{F}^\bullet_{\lambda_0,\bullet})))$ is of π^{-200a}-finite type for any $0 \leq i \leq k$ and $j \geq a+2$.*

Proof. Notice that by (5.6.5) and 7.4, we have

$$H^i(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \nu^*_{\lambda_0}\mathcal{G}^j_{\lambda_0})) = H^i(X_{\lambda_0}, \mathcal{G}^j_{\lambda_0}), \tag{7.5.1}$$

which is an A_{λ_0}-module of finite type, since X_{λ_0} is proper over the Noetherian scheme $\mathrm{Spec}(A_{\lambda_0})$. Thus, $H^i(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, H^j(\mathcal{F}^\bullet_{\lambda_0,\bullet})))$ is of π^{-200a}-finite type by 5.9 and 7.4. □

Lemma 7.6. *The complex of A_{λ_0}-modules $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet}))$ is π^l-$[a+k+2,k]$-pseudo-coherent for an integer $l \geq 0$ depending only on a and k.*

Proof. It follows directly from 5.11 whose conditions are satisfied by 7.5. □

Lemma 7.7. *The complex of A-modules $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$ is π^l-$[a+k+2,k]$-pseudo-coherent for an integer $l \geq 0$ depending only on a and k.*

Proof. By 4.10 and 7.6, $\tau^{\geq(a+k+2)}(A \otimes^{\mathrm{L}}_{A_{\lambda_0}} \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet})))$ is represented by a π^l-$[a+k+2,k]$-pseudo-coherent complex of A-modules for an integer $l \geq 0$ depending only on a and k. Since $f_{\lambda_0} : X_{\lambda_0} \to S_{\lambda_0}$ is flat, $A \otimes^{\mathrm{L}}_{A_{\lambda_0}} \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet}))$ is also represented by $A \otimes_{A_{\lambda_0}} \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_{\lambda_0,\bullet}, \mathcal{F}^\bullet_{\lambda_0,\bullet})) = \mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$. Hence, $\tau^{\geq(a+k+2)}(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$ is π^l-$[a+k+2,k]$-pseudo-coherent for an integer $l \geq 0$ depending only on a and k by 4.4. The conclusion follows from applying 4.3.(2) to $\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet)) \to \tau^{\geq(a+k+2)}(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))$. □

Lemma 7.8. *The complex of A-modules $\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet)$ is π^l-$[a+k+2,k]$-pseudo-coherent for an integer $l \geq 0$ depending only on a and k.*

Proof. Since $\mathcal{F}^\bullet_\bullet \to \mathcal{M}_\bullet[0]$ is a π-$[a,0]$-pseudo resolution, the map $H^i(\mathrm{Tot}(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{F}^\bullet_\bullet))) \to H^i(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet))$ is a $\pi^{4(k+1)}$-isomorphism for any $i \geq a+k+1$ by 5.10. The conclusion follows from 4.3.(1) and 7.7. □

Recall that $\mathrm{R}\Gamma(X, \mathcal{M})$ is represented by the ordered Čech complex $\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet)$ by (5.6.5). As we have taken $a < -(k+2)$ in the beginning, we see that for any $q \in \mathbb{N}$, $H^q(X, \mathcal{M}) = H^q(\check{C}^\bullet_{\mathrm{ord}}(X_\bullet, \mathcal{M}_\bullet))$ is π^l-isomorphic to the q-th cohomology H'^q of a complex of finite free A-modules for an integer $l \geq 0$ depending only on a and k by 7.8 and 4.1 (taking $a = -k-3$ at first is actually enough for this argument). Since A is an almost coherent A-module by applying 6.7 to $\mathcal{O}_S$, we see that H'^q is an almost coherent A-module by applying 6.6 to the trivial site of a single point with structural sheaf given by A. Since l is independent of the choice of $\pi \in \mathfrak{m}$, the A-module $H^q(X, \mathcal{M})$ is almost coherent by 6.3, which completes the proof of our main theorem 7.1. □

8. Remark on Abbes-Gros' Construction of the Relative Hodge-Tate Spectral Sequence

8.1. Let K be a complete discrete valuation field of characteristic 0 with algebraically closed residue field of characteristic $p > 0$. Let $(f, g) : (X'^{\triangleright} \to X') \to (X^{\circ} \to X)$ be a morphism of open immersions of quasi-compact and quasi-separated schemes over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$. Consider the following conditions:

(1) The associated log schemes $(X', \mathscr{M}_{X'})$ and $(X, \mathscr{M}_X)$ endowed with the compactifying log structures are adequate in the sense of [AGT16, III.4.7] (which holds for instance if the open immersions $X'^{\triangleright} \to X'$ and $X^{\circ} \to X$ are semi-stable over $\mathrm{Spec}(K) \to \mathrm{Spec}(\mathcal{O}_K)$, see [He21, 10.11]).
(2) The morphism of log schemes $(X', \mathscr{M}_{X'}) \to (X, \mathscr{M}_X)$ is smooth and saturated.
(3) The morphism of schemes $g : X' \to X$ is projective.

Under these assumptions, Abbes-Gros proved Faltings' main p-adic comparison theorem in the relative case for the morphism (f, g) [AG20, 5.7.4], and constructed a relative Hodge-Tate spectral sequence [AG20, 6.7.5] (for an explicit local version, see [AG20, 6.9.6] and [He21, 1.4]). We explain that their proof and construction are still valid if we replace the assumption (3) by the following assumption

(3)' The morphism of schemes $g : X' \to X$ is proper.

8.2. The assumption on the projectivity of g has been only used in the proof of [AG20, 5.3.31]. There, they encountered a Cartesian diagram of schemes

(8.2.1)
$$\begin{array}{ccc} \overline{X}'^{(\infty)} & \longrightarrow & X' \\ {\scriptstyle g^{(\infty)}}\downarrow & & \downarrow{\scriptstyle g} \\ \overline{X}^{(\infty)} & \longrightarrow & X \end{array}$$

where $\overline{X}^{(\infty)}$ is an $\mathcal{O}_{\overline{K}}$-scheme such that $\mathcal{O}_{\overline{X}^{(\infty)}}$ and $\mathcal{O}_{\overline{X}'^{(\infty)}}$ are almost coherent as modules over themselves ([AG20, 5.3.5.(ii)]), and a quasi-coherent and almost coherent $\mathcal{O}_{\overline{X}'^{(\infty)}}$-module $\mathscr{G}$. For proving the almost coherence of $\mathrm{R}^i g^{(\infty)}_* \mathscr{G}$, they applied [AG20, 2.8.18] where the assumption on the projectivity of g has been used.

Now we replace the assumption (3) by the assumption (3)', by replacing [AG20, 2.8.18] by our main theorem 7.1. Indeed, the morphism g is flat by the assumption (2) (see [Kat89, 4.5]), proper by the assumption (3)', of finite presentation by the assumptions (3)' and (1) (as X is locally Noetherian). Hence, so is the morphism $g^{(\infty)}$ by base change. Therefore, we deduce the almost coherence of $\mathrm{R}^i g^{(\infty)}_* \mathscr{G}$ from our main theorem 7.1.

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