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Essential holonomy of Cantor actions

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Abstract. A minimal group action has essential holonomy if the set of points with non-trivial holonomy has positive measure. If such an action is topologically free, then having essential holonomy is equivalent to the action not being essentially free, which means that the set of points with non-trivial stabilizer has positive measure. In this paper, we investigate the relation between the property of having essential holonomy and structure of the acting group for minimal equicontinuous actions on Cantor sets. We show that if such a group action is locally quasi-analytic and has essential holonomy, then every commutator subgroup in the group lower central series has elements with positive measure set of points with non-trivial holonomy. In particular, we prove that a minimal equicontinuous Cantor action by a nilpotent group has no essential holonomy. We also show that the property of having essential holonomy is preserved under return equivalence and continuous orbit equivalence of minimal equicontinuous Cantor actions. Finally, we give examples to show that the assumption on the action that it is locally quasi-analytic is necessary.

1. Introduction

We say that $(\mathfrak{X}, \Gamma, \Phi)$ is a *Cantor action* if Γ is a countable group, $\mathfrak{X}$ is a Cantor space, and $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is an action by homeomorphisms. In this paper we consider Cantor actions that are minimal and equicontinuous. Under these assumptions, $(\mathfrak{X}, \Gamma, \Phi)$ has a unique ergodic invariant probability measure μ. We recall further basic definitions and constructions for Cantor actions in Section 2, as used in the formulation of our results below.

The dynamical properties of Cantor actions, even under assumptions of minimality and equicontinuity, can have surprisingly subtle aspects, especially for the case where Γ is non-abelian, as revealed by the many examples in the literature. One approach to classifying minimal equicontinuous Cantor actions is via their dynamical properties. The main result of this work makes a new contribution to this classification scheme, as it relates the algebraic properties of Γ and the dynamics of the action, through the study of the property that an action has *essential holonomy*; see Definition 1.3 below.

First recall some standard notions concerning the fixed point sets for a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$. We use the notation $g \cdot x = \Phi(g)(x)$, for $g \in \Gamma$ and $x \in \mathfrak{X}$. The set $\mathfrak{X}_g = \{x \in \mathfrak{X} \mid g \cdot x = x\}$ consists of fixed points for g, and the *stabilizer* of a point $x \in \mathfrak{X}$ is the subgroup $\Gamma_x = \{g \in \Gamma \mid g \cdot x = x\}$. Let

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$$\mathfrak{X}_\Gamma = \bigcup_{e \neq g \in \Gamma} \mathfrak{X}_g = \{x \in \mathfrak{X} \mid \Gamma_x \neq \{e\}\} \tag{1}$$

be the set of all points fixed by some non-identity element $g \in \Gamma$.

DEFINITION 1.1. *A minimal Cantor action* $(\mathfrak{X}, \Gamma, \Phi)$ *with invariant probability measure* μ *is:*

1. free *if* $\mathfrak{X}_\Gamma$ *is empty.*
2. topologically free *if* $\mathfrak{X}_\Gamma$ *is a meager set in* $\mathfrak{X}$*.*
3. essentially free *if* $\mu(\mathfrak{X}_\Gamma) = 0$*.*

Recall that a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is *effective* if the only element of Γ that acts as the identity on $\mathfrak{X}$ is the identity of Γ. It is elementary to show that if Γ is abelian, then every effective minimal Cantor action of Γ must be free. The topologically free Cantor actions have an important role in the study of the C*-algebras associated with the actions, as studied for example in [**4, 8, 24, 27, 30**].

Kambites, Silva and Steinberg showed in [**25**, Theorem 4.3] that the action of a group generated by finite automata on a rooted tree is topologically free if and only if it is essentially free. Joseph proved in [**22**, Corollary 2.4] that if Γ has countably many subgroups, then a minimal Cantor action of Γ is topologically free if and only if it is essentially free.

Bergeron and Gaboriau [**7**] showed that if Γ is a non-amenable group which is a free product of two residually finite groups, then Γ admits a minimal equicontinuous Cantor action which is topologically free and not essentially free. Abért and Elek [**1**] proved a similar result for finitely generated non-abelian free groups Γ. Joseph [**22**] proved that any non-amenable surface group admits a continuum of pairwise non-conjugate and measurably non-isomorphic minimal equicontinuous Cantor actions which are topologically free and not essentially free.

Examples of effective minimal equicontinuous Cantor actions which are not topologically free include some actions of branch and weakly branch groups on the boundaries of rooted trees [**15, 29, 31**], some actions of nilpotent groups [**21**], and actions of topological full groups [**10, 15**].

The work by Gröger and Lukina [**16**] introduced a refinement of the notion of essentially free actions, called the *essential holonomy* property. The idea is to consider the dynamics of the action in small neighborhoods of fixed points. In place of the set of points with trivial (resp. non-trivial) stabilizers, one considers the set of points with trivial (resp. non-trivial) holonomy. The analog of an essentially free Cantor action is an action which has *no essential holonomy.*

DEFINITION 1.2. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a Cantor action. Say that* $x \in \mathfrak{X}$ *is a* point of non-trivial holonomy for $g \in \Gamma$ *if* $\Phi(g)(x) = x$*, and for each open set* $U \subset \mathfrak{X}$ *with* $x \in U$*, there exists* $y \in U$ *such that* $\Phi(g)(y) \neq y$*.*

We say that a fixed point $x \in \mathfrak{X}$ is a *point of trivial holonomy* for $g \in \Gamma$, if x is fixed by $\Phi(g)$, and x has an open neighborhood $U_{x,g}$ where every point is fixed by $\Phi(g)$. We say that $x \in \mathfrak{X}$ is a *point of trivial holonomy*, if x is a point of trivial holonomy for all $g \in \Gamma$ with $g \cdot x = x$. We say that $x \in \mathfrak{X}$ is a *point of non-trivial holonomy*, if x is a point of non-trivial holonomy for some $g \in \Gamma$ with $g \cdot x = x$.

Let $\mathfrak{X}_g^{hol}$ denote the (possibly empty) subset of points of non-trivial holonomy in $\mathfrak{X}_g$, and set

$$\mathfrak{X}_\Gamma^{hol} = \bigcup_{g \in \Gamma} \mathfrak{X}_g^{hol} \subset \bigcup_{g \in \Gamma} \mathfrak{X}_g \subset \mathfrak{X}_\Gamma \ . \tag{2}$$

The set $\mathfrak{X}_\Gamma^{hol}$ is invariant under the action of Γ, thus if $(\mathfrak{X}, \Gamma, \Phi)$ admits an ergodic invariant probability measure, then $\mathfrak{X}_\Gamma^{hol}$ has either μ-measure 0 or 1. The following concept was formulated in [**16**]:

DEFINITION 1.3. *A measure-preserving Cantor action* $(\mathfrak{X}, \Gamma, \Phi)$ *with ergodic invariant probability measure* μ *has* essential holonomy *if the set* $\mathfrak{X}_\Gamma^{hol}$ *of points with non-trivial holonomy has positive* μ*-measure. Otherwise, it has* no essential holonomy*.*

We note that topologically the set $\mathfrak{X}_\Gamma^{hol}$ is always ‘small’, namely, it is a meager set. Recall that a set is meager if it is contained in a countable union of nowhere dense sets. However, $\mathfrak{X}_\Gamma^{hol}$ need not be ‘small’ with respect to the measure μ, and this paper investigates minimal equicontinuous Cantor actions where the set $\mathfrak{X}_\Gamma^{hol}$ is measurably ‘large’, namely, has full μ-measure.

Vorobets [**32**] showed that the standard action of the Grigorchuk group on the boundary of a binary rooted tree has only a countable set of points with non-trivial holonomy, which implies that it has no essential holonomy. Gröger and Lukina proved in [**16**] that the action of a group generated by finite automata on a rooted tree has no essential holonomy; their proof does not require the Cantor action to be topologically free. They also gave a criterion for when a group action on a rooted tree has no essential holonomy for its boundary Cantor action.

The decomposition (2) of $\mathfrak{X}_\Gamma^{hol}$ as a countable union of fixed point sets implies that an action has essential holonomy if and only if, for at least one $g \in \Gamma$, the set $\mathfrak{X}_g^{hol}$ has positive μ-measure. In particular, this implies that whether or not a Cantor action has essential holonomy is a local property. This remark is the basis for the following invariance result, with details and proof given in Section 3.

PROPOSITION 1.4.

1. *Suppose that minimal equicontinuous Cantor actions* $(\mathfrak{X}_i, \Gamma_i, \Phi_i)$*, for* $i = 1, 2$*, are continuous orbit equivalent. Then either both have essential holonomy, or both have no essential holonomy.*
2. *Suppose that minimal equicontinuous Cantor actions* $(\mathfrak{X}_i, \Gamma_i, \Phi_i)$*, for* $i = 1, 2$*, are return equivalent. Then either both have essential holonomy, or both have no essential holonomy.*

For a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, the assumption that μ is a probability measure, and that the action is minimal, implies that μ is a continuous measure; that is, the measure of any point $x \in \mathfrak{X}$ is zero. Thus, if a Cantor action has essential holonomy, then the intersection of $\mathfrak{X}_g^{hol}$ with the support of μ must be an uncountable set. This remark is the basis for the result by Joseph in [**22**, Corollary 2.4] that if Γ has only countably many subgroups, then a topologically free minimal Cantor action by Γ is essentially free, and hence has no essential holonomy. The argument in [**22**] holds for minimal actions by homeomorphisms on any compact Hausdorff space with an invariant finite ergodic measure.

Our main result below relates the dynamical properties of a minimal equicontinuous Cantor action of a finitely generated group Γ, with the lower central series $\Gamma = \gamma_1(\Gamma) \supset \gamma_2(\Gamma) \supset \cdots \supset \gamma_n(\Gamma) \supset \cdots$. Intuitively, a *locally quasi-analytic* minimal Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is an action such that every homeomorphism $\Phi(g)$ has a unique extension from small open sets to open sets of uniform diameter over $\mathfrak{X}$, see Section 2.5 for a precise definition and discussion of this property. In particular, the locally quasi-analytic property is a localized form of the topologically free property.

THEOREM 1.5. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a locally quasi-analytic minimal equicontinuous Cantor action with invariant ergodic probability measure* μ. *If the action has essential holonomy, then for every* $n \geq 1$ *there exists* $\phi_n \in \gamma_n(\Gamma)$ *such that the action of* ϕ_n *has a positive measure set of points with non-trivial holonomy.*

The proof of Theorem 1.5 is given in Section 4. We observe the following corollary for $n = 2$:

COROLLARY 1.6. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a locally quasi-analytic minimal equicontinuous Cantor action with ergodic invariant probability measure* μ. *Let* $[\Gamma, \Gamma] \subset \Gamma$ *be the commutator subgroup of* Γ. *If the action has essential holonomy, then there exists* $\phi \in [\Gamma, \Gamma]$ *such that the action of* ϕ *has a positive measure set of points with non-trivial holonomy.*

It is tempting to apply Corollary 1.6 inductively, so that the conclusion of Theorem 1.5 applies to the derived series of Γ. This argument does not go through, though, because while the action of the commutator subgroup $[\Gamma, \Gamma]$ on $\mathfrak{X}$ is again equicontinuous and locally quasi-analytic, it need not be minimal, and the minimality assumption on the action is critical for the proof of Theorem 1.5.

The family of examples constructed in the proof of Theorem 5.1 show that the assumption that a minimal equicontinuous Cantor action is locally quasi-analytic is essential for the conclusions of Theorem 1.5 and Corollary 1.6. The actions of groups Γ and the commutator subgroups $[\Gamma, \Gamma]$ in Theorem 5.1 are minimal and not locally quasi-analytic, and the actions of the group Γ have essential holonomy, while the actions of $[\Gamma, \Gamma]$ have no essential holonomy.

If Γ is a Noetherian group, then every Cantor action by Γ must be locally quasi-analytic by [**19**, Theorem 1.6]. For a nilpotent group Γ, the lower central series terminates, and Γ is Noetherian, so as a consequence we obtain the following.

COROLLARY 1.7. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a minimal equicontinuous Cantor action. If* Γ *is a finitely generated nilpotent group, then* $(\mathfrak{X}, \Gamma, \Phi)$ *has no essential holonomy.*

In the case when $(\mathfrak{X}, \Gamma, \Phi)$ is topologically free, Corollary 1.7 recovers the result of Joseph [**22**], who proved that a minimal Cantor action of a finitely generated nilpotent group (and more generally, of a group with countable number of subgroups acting on a compact Hausdorff topological space) is topologically free if and only if it is essentially free. The paper [**22**] uses the method of invariant random subgroups, which is very different from our proof. While the result in [**22**] applies to topologically free minimal actions which are not necessarily equicontinuous, the result of Corollary 1.7 applies to minimal actions that are equicontinuous but need not be topologically free.

Up until recently, it was an open problem to show that if Γ is a finitely generated amenable group, then every minimal equicontinuous Cantor action of Γ has no essential holonomy. In the paper [**23**], Joseph constructs a family of wreath products of two finitely generated amenable groups which are amenable groups, and which admit topologically free and not essentially free minimal equicontinuous actions. Actions in Theorem 5.1 in our paper are not locally quasi-analytic (and therefore not topologically free) actions of infinitely generated amenable groups which have essential holonomy. It would be interesting to obtain a criterion for when an amenable group admits an action with essential holonomy.

Theorem 1.5 suggests that the property of having essential holonomy is an interesting invariant of Cantor actions, intrinsically related to the structure of the acting group, to be further explored.

2. Cantor actions

We recall some basic properties of Cantor actions. More details can be found in [**5, 9, 10, 19, 20, 26**].

2.1. Basic notions

Let $(\mathfrak{X}, \Gamma, \Phi)$ denote a topological action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$. We write $g \cdot x$ for $\Phi(g)(x)$ when appropriate. The orbit of $x \in \mathfrak{X}$ is the subset $\mathcal{O}(x) = \{g \cdot x \mid g \in \Gamma\}$. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x)$ is dense in $\mathfrak{X}$. The action is said to be *effective*, or *faithful*, if the homomorphism $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ is injective.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in \Gamma$ we have that $d_{\mathfrak{X}}(x, y) < \delta$ implies $d_{\mathfrak{X}}(g \cdot x, g \cdot y) < \varepsilon$. The property of being equicontinuous is independent of the choice of the metric on $\mathfrak{X}$ which is compatible with the topology of $\mathfrak{X}$.

For the rest of the section, we assume that a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is minimal and equicontinuous. We say that $U \subset \mathfrak{X}$ is *adapted* to the action if U is a non-empty clopen subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ then $\Phi(g)(U) = U$. Recall a basic property of minimal equicontinuous Cantor actions (see [**19**, Section 3]).

PROPOSITION 2.1. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action, and let $d_{\mathfrak{X}}$ be an invariant metric on $\mathfrak{X}$ compatible with the topology on $\mathfrak{X}$. Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, there exists an adapted clopen set $U \subset \mathfrak{X}$ with $x \in U$ and* $\mathrm{diam}(U) < \varepsilon$.

COROLLARY 2.2. *For a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, the adapted clopen sets form a subbasis for the topology of $\mathfrak{X}$.*

For an adapted set U, the set of "return times" to U,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\} \tag{3}$$

is a subgroup of Γ, called the *stabilizer* of U. Indeed, for $g, g' \in \Gamma$ with $g \cdot U \cap g' \cdot U \neq \emptyset$ we have $g^{-1} g' \cdot U = U$, hence $g^{-1} g' \in \Gamma_U$. Moreover, the translates $\{g \cdot U \mid g \in \Gamma\}$ form a finite clopen partition of $\mathfrak{X}$, and are in 1-1 correspondence with the quotient space $X_U = \Gamma/\Gamma_U$. Then Γ acts by permutations of the finite set X_U and so the stabilizer group $\Gamma_U \subset \Gamma$ has finite index. Note that this implies that if $V \subset U$ is a proper inclusion of adapted sets, then the inclusion $\Gamma_V \subset \Gamma_U$ is also proper.

Let U be an adapted set for the action $(\mathfrak{X}, \Gamma, \Phi)$, then the action of Γ restricts on U to an action of Γ_U, so we have a homomorphism $\Phi_U \colon \Gamma_U \to \mathrm{Homeo}(U)$. Let $\mathcal{H}_U$ denote the image of this action. Note that the map $\Phi_U \colon \Gamma_U \to \mathcal{H}_U$ is injective if the action is topologically free.

2.2. Group chains

Given a basepoint x, by iterating the process in Proposition 2.1 one can always construct the following:

DEFINITION 2.3. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$, with $U_0 = \mathfrak{X}$, is said to be an* adapted neighborhood basis *at $x \in \mathfrak{X}$ for the action Φ if each U_ℓ is adapted to the action Φ, $U_{\ell+1} \subset U_\ell$ for all $\ell \geq 0$, and $\cap\, U_\ell = \{x\}$.*

Let $\Gamma_\ell = \Gamma_{U_\ell}$ denote the stabilizer group of U_ℓ given by (3). Then we obtain a descending chain of finite index subgroups $\mathcal{G}^x_{\mathcal{U}} : \Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$. Note that each Γ_ℓ has finite index in Γ, and is not assumed to be a normal subgroup. Also note that while the intersection of the chain $\mathcal{U}$ is a single point $\{x\}$, the intersection of the stabilizer groups in $\mathcal{G}^x_{\mathcal{U}}$ need not be the trivial group.

Next, set $X_\ell = \Gamma/\Gamma_\ell$ and note that Γ acts transitively on the left on X_ℓ. The inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a natural Γ-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$X \;=\; \varprojlim \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} = \{(x_\ell) = (x_0, x_1, \ldots) \mid p_{\ell+1}(x_{\ell+1}) = x_\ell\} \tag{4}$$

which is a Cantor space with the Tychonoff topology. Thus elements of X are infinite sequences with entries in X_ℓ, $\ell \geq 0$. The actions of Γ on the factors X_ℓ induce a minimal equicontinuous action, denoted by $\Phi_x \colon \Gamma \times X \to X$, which reads

$$(g, (x_\ell)) \mapsto g \cdot (x_\ell) = (g \cdot x_\ell) = (g \cdot x_0, g \cdot x_1, \ldots). \tag{5}$$

For each $\ell \geq 0$, we have the "partition coding map" $\Theta_\ell \colon \mathfrak{X} \to X_\ell$ which is Γ-equivariant. The maps $\{\Theta_\ell\}$ are compatible with the map on quotients in (4), and so define a limit map $\Theta_x \colon \mathfrak{X} \to X$. The fact that the diameters of the clopen sets $\{U_\ell\}$ tend to zero, implies that Θ_x is a homeomorphism. This is proved in detail in [**12**, Appendix A]. Moreover, $\Theta_x(x) = e_\infty = (e\Gamma_\ell) \in X$, the basepoint of the inverse limit (4), where $e\Gamma_\ell = \Gamma_\ell$ is the coset of the identity $e \in \Gamma$. Let X have an ultrametric metric such that Γ acts on X by isometries, for instance, let

$$d_X\left((x_\ell),(y_\ell)\right) = 2^{-m}, \qquad \text{where } m = \max\{\ell \mid x_\ell = y_\ell,\, \ell \geq 0\}. \tag{6}$$

Then let $d_{\mathfrak{X}}$ be the ultrametric metric on $\mathfrak{X}$ induced from d_X by the homeomorphism Θ_x. The minimal equicontinuous action (X, Γ, Φ_x) is called the *odometer model* centered at x for $(\mathfrak{X}, \Gamma, \Phi)$.

The group chain $\mathcal{G}^x_{\mathcal{U}}$ depends on x and $\mathcal{U}$, and one can introduce an equivalence relation which, for a given group Γ, identifies the class of group chains with topologically conjugate associated odometer models. We refer the interested reader to [**12**].

2.3. Unique ergodic invariant measure

Given a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, choose an adapted neighborhood basis $\mathcal{U}$ and consider the corresponding group chain $\mathcal{G}^x_{\mathcal{U}}$ and the odometer model (4). The group Γ acts transitively on the coset space $X_\ell = \Gamma/\Gamma_\ell$ and we define a Γ-invariant probability measure μ_ℓ on X_ℓ by giving equal weight to each point (coset) in X_ℓ. Thus one has

$$\mu_\ell(h\Gamma_\ell) = \frac{1}{|\Gamma : \Gamma_\ell|}, \text{ for all } h\Gamma_\ell \in X_\ell \text{ and all } \ell \geq 0, \tag{7}$$

where $|\Gamma : \Gamma_\ell|$ denotes the index of Γ_ℓ in Γ. The unique Γ-invariant measure on the inverse limit X is defined as the limit of the pull-backs of these measures under the projection maps $X \to X_\ell$. Then the invariant measure μ on $\mathfrak{X}$ is the pull-back via the homeomorphism $\Theta_x \colon \mathfrak{X} \to X$.

Alternately, consider the closure $E = \overline{\Phi(\Gamma)} \subset \text{Homeo}(\mathfrak{X})$ in the uniform topology. It is a profinite compact group, called the *Ellis* or *enveloping group* [**5, 13**]. (If the action $(\mathfrak{X}, \Gamma, \Phi)$ is not assumed to be equicontinuous, then $\overline{\Phi(\Gamma)}$ is only a semi-group.) The group E acts on $\mathfrak{X}$ with isotropy group $E_x = \{g \in E \mid g \cdot x = x\}$ for $x \in \mathfrak{X}$. Then E_x is a closed subgroup of E, and we have $\mathfrak{X} = E/E_x$. The profinite group E has a unique Haar measure $\widehat{\mu}$, which is invariant with respect to the action of E on itself. The measure $\widehat{\mu}$ on E pushes down to the measure μ on $\mathfrak{X}$.

2.4. Lebesgue density theorem

Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action with probability measure μ and ultrametric $d_{\mathfrak{X}}$ induced from (6). Let $B(x, \epsilon) = \{y \in \mathfrak{X} \mid d_{\mathfrak{X}}(x, y) < \epsilon\}$ denote the open ball with center x of radius $\varepsilon > 0$. The proof of the *Lebesgue Density Theorem* in the formulation below can be found, for instance, in [**28**, Proposition 2.10].

THEOREM 2.4. *Let $\mathfrak{X}$ be a Polish space, and suppose $\mathfrak{X}$ has an ultrametric $d_{\mathfrak{X}}$ compatible with its topology. Let μ be a probability measure on $\mathfrak{X}$, and let A be a Borel set of positive measure. Then for μ-almost every $x \in A$, the* Lebesgue density *of x in A, given by*

$$\lim_{\epsilon \to 0} \frac{\mu(A \cap B(x, \epsilon))}{\mu(B(x, \epsilon))} \tag{8}$$

exists and is equal to 1.

We give an important consequence of the Lebesgue Density Theorem.

LEMMA 2.5. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a minimal equicontinuous Cantor action, with invariant probability measure* μ*. Assume there exists an element* $g \in \Gamma$ *for which* $\mathfrak{X}_g^{hol}$ *has positive* μ*-measure. Then there exists* $x \in \mathfrak{X}_g^{hol}$ *such that, for all* $0 < \varepsilon < 1$*, there exists an adapted set* U_ε *with* $x \in U_\varepsilon$ *and* $\mu(U_\varepsilon \cap \mathfrak{X}_g^{hol}) \geq (1-\varepsilon) \cdot \mu(U_\varepsilon)$*.*

PROOF. Since $\mathfrak{X}_g^{hol}$ has positive μ-measure, by the Lebesgue Density Theorem 2.4 there exists a point $x \in \mathfrak{X}_g^{hol}$ of full Lebesgue density. For this point, choose an adapted neighborhood basis $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ at x. By the convergence of the limit in Theorem 2.4 there exists ℓ_ε so that $\mu(U_\ell \cap \mathfrak{X}_g^{hol}) \geq (1-\varepsilon) \cdot \mu(U_\ell)$ for $\ell \geq \ell_\varepsilon$. Then set $U_\varepsilon = U_{\ell_\varepsilon}$. □

2.5. Locally quasi-analytic

The quasi-analytic property for Cantor actions was introduced by Álvarez López and Candel in [**3**, Definition 9.4] as a generalization of the notion of a *quasi-analytic action* studied by Haefliger for actions of pseudogroups of real-analytic diffeomorphisms in [**17**]. The authors introduced a local form of the quasi-analytic property in [**18, 19**]:

DEFINITION 2.6. [**19**, Definition 2.1] *A topological action* $(\mathfrak{X}, \Gamma, \Phi)$ *on a metric Cantor space* $\mathfrak{X}$*, is* locally quasi-analytic *if there exists* $\varepsilon > 0$ *such that for any non-empty open set* $U \subset \mathfrak{X}$ *with* $\mathrm{diam}(U) < \varepsilon$*, and for any non-empty open subset* $V \subset U$*, and elements* $g_1, g_2 \in \Gamma$

$$\textit{if the restrictions } \Phi(g_1)|V = \Phi(g_2)|V, \quad \textit{then } \Phi(g_1)|U = \Phi(g_2)|U. \tag{9}$$

The action is said to be quasi-analytic *if* (9) *holds for* $U = \mathfrak{X}$*.*

In other words, $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic if for every $g \in \Gamma$, the homeomorphism $\Phi(g)$ has unique extensions on the sets of diameter $\varepsilon > 0$ in $\mathfrak{X}$, with ε uniform over $\mathfrak{X}$. We note that an effective minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is topologically free if and only if it is quasi-analytic [**19**, Proposition 2.2].

Recall that a group Γ is *Noetherian* [**6**] if every increasing chain of subgroups has a maximal element. Equivalently, a group is Noetherian if every subgroup of Γ is finitely generated.

THEOREM 2.7. [**19**, Theorem 1.6] *Let* Γ *be a Noetherian group. Then a minimal equicontinuous Cantor action* $(\mathfrak{X}, \Gamma, \Phi)$ *is locally quasi-analytic.*

A finitely generated nilpotent group is Noetherian, so as a corollary we obtain that all minimal equicontinuous Cantor actions by finitely generated nilpotent groups are locally quasi-analytic. Examples of locally quasi-analytic actions which are not quasi-analytic are easy to construct; see for instance [**19**, Example A.4].

3. Invariance

We recall notions of equivalence for Cantor actions, considered as topological dynamical systems. For each notion considered, we show that the measurable dynamical systems property that a minimal equicontinuous Cantor action has essential holonomy is preserved for equivalent actions. The work [**14**] gives a comparison of the various notions of equivalence for the case of Cantor actions by $\Gamma = \mathbb{Z}^n$.

First, we recall the most basic equivalence of actions.

DEFINITION 3.1. *Cantor actions* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *are said to be* isomorphic*, or* conjugate*, if there is a homeomorphism* $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ *and a group isomorphism* $\Theta\colon \Gamma_1 \to \Gamma_2$ *so that*

$$\Phi_1(\gamma) = h^{-1} \circ \Phi_2(\Theta(\gamma)) \circ h \text{ for all } \gamma \in \Gamma_1 \ . \tag{10}$$

The statement below holds for measure-preserving actions which are not necessarily equicontinuous.

PROPOSITION 3.2. *Let* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *be isomorphic minimal Cantor actions, with respective unique ergodic invariant probability measures* μ_1 *and* μ_2*. Then* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *has essential holonomy if and only if* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *has essential holonomy.*

PROOF. Let a homeomorphism $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ and an isomorphism $\Theta\colon \Gamma_1 \to \Gamma_2$ be as in Definition 3.1. Then $h^*(\mu_2)$ is an invariant probability measure for $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$, hence by uniqueness $\mu_1 = h^*(\mu_2)$. Moreover, for $g \in \Gamma_1$ the action $\Phi_1(g)$ has essential holonomy if and only if $\Phi_2(\Theta(g))$ has essential holonomy, so $h(\mathfrak{X}^{hol}_{\Gamma_1}) = \mathfrak{X}^{hol}_{\Gamma_2}$, and the claim follows. □

The notion of *return equivalence* is the analog for Cantor actions of Morita equivalence for C*-algebras. This equivalence is weaker than the notion of isomorphism, and is natural when considering the Cantor actions defined by the holonomy actions for matchbox manifolds, as in [**18, 19**]. For convenience in the definition below we restrict to minimal equicontinuous Cantor actions, but the notion of return equivalence can be defined for more general minimal actions. In a more general case there are no adapted neighborhoods, and the closure of the action in $\text{Homeo}(U_i)$, $i = 1, 2$, need not be a group, and so more care is required in the definition.

DEFINITION 3.3. *Minimal equicontinuous Cantor actions* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *are said to be* return equivalent *if there exist non-empty clopen subsets* $U_i \subset \mathfrak{X}_i$*, for* $i = 1, 2$*, such that* U_i *is adapted to the action* Φ_i*, and there is a homeomorphism* $h\colon U_1 \to U_2$ *whose induced homomorphism* $h_*\colon \text{Homeo}(U_1) \to \text{Homeo}(U_2)$ *restricts to an isomorphism* $\Theta\colon \mathcal{H}_{U_1} \to \mathcal{H}_{U_2}$*.*

Note that when $U_i = \mathfrak{X}_i$ and both actions are effective, then this definition reduces to the usual notion of isomorphism of the actions, with induced group isomorphism $\Theta\colon \Gamma_1 \cong \mathcal{H}_{\mathfrak{X}_1} \to \mathcal{H}_{\mathfrak{X}_2} \cong \Gamma_2$.

PROPOSITION 3.4. *Let* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *be return equivalent minimal equicontinuous Cantor actions, with respective unique ergodic invariant probability measures* μ_1 *and* μ_2*. Then* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *has essential holonomy if and only if* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *has essential holonomy.*

PROOF. Assume that $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ has essential holonomy, and let $U_1 \subset \mathfrak{X}_1$ and $U_2 \subset \mathfrak{X}_2$ be clopen sets such that the restricted actions are isomorphic by $\Theta\colon \mathcal{H}_{U_1} \to \mathcal{H}_{U_2}$. Let $g \in \Gamma_1$ be such that $\mathfrak{X}_g^{hol}$ has positive μ_1-measure, and thus there exists $x \in \mathfrak{X}_1$ such that x is fixed by $\Phi_1(g)$ and $\mathfrak{X}_{1,g}^{hol}$ has Lebesgue density 1 at x. The action Φ_1 is minimal on $\mathfrak{X}_1$ so there exists $k \in \Gamma_1$ such that $k \cdot x \in U_1$. Then $g' = kgk^{-1} \in \Gamma_1$ has a fixed point $x' = kx$ which is a point of Lebesgue density 1 in the set $\mathfrak{X}_{1,g'}^{hol}$. As $x' \in U_1 \cap \Phi_1(g')(U_1)$ and U_1 is adapted, we have $U_1 = \Phi_1(g')(U_1)$ and so $g' \in \Gamma_{1,U}$.

The action of Γ_{1,U_1} on U_1 is minimal, so the renormalized measure $\mu_1' = \mu(U_1)^{-1}(\mu_1|U_1)$ is the unique invariant probability measure for the restricted action of Γ_{1,U_1} on U_1.

The set $\mathfrak{X}_{1,g'}^{hol} \cap U_1$ has Lebesgue density 1 at x', hence its image $h(x') \in h(\mathfrak{X}_{1,g'}^{hol} \cap U_1) \subset U_2$ is also a point of Lebesgue density 1 for the action of $\Theta(g')$ on U_2, with corresponding renormalized measure μ_2' on U_2. Thus, $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ has essential holonomy. The converse follows similarly. □

The notion of *continuous orbit equivalence* for Cantor actions was introduced in [**8**]. It is the analogue for topological dynamics of measurable orbit equivalence for measurable actions, as first introduced by Dye [**11**]. Continuous orbit equivalence plays a fundamental role in the classification of group actions on Cantor sets (see for example [**30**]).

DEFINITION 3.5. *Minimal Cantor actions* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *are said to be* continuously orbit equivalent *if there exists a homeomorphism* $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ *and continuous functions*

1. $\alpha\colon G_1 \times \mathfrak{X}_1 \to G_2$, $h(\Phi_1(g_1)(x_1)) = \Phi_2(\alpha(g_1, x_1))(h(x_1))$ *for all* $g_1 \in G_1$ *and* $x_1 \in \mathfrak{X}_1$*;*
2. $\beta\colon G_2 \times \mathfrak{X}_2 \to G_1$, $h^{-1}(\Phi_2(g_2)(x_2)) = \Phi_1(\beta(g_2, x_2))(h^{-1}(x_2))$ *for all* $g_2 \in G_2$ *and* $x_2 \in \mathfrak{X}_2$.

The homeomorphism h is called a *continuous orbit equivalence* between the two actions. Note that the functions α and β are not assumed to satisfy the cocycle property.

We have the following result of Cortez and Medynets:

THEOREM 3.6. [**10**] *Let* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *be topologically free minimal equicontinuous Cantor actions. If the actions* Φ_1 *and* Φ_2 *are continuously orbit equivalent, then they are return equivalent.*

This result was generalized by the authors:

THEOREM 3.7. [**19**] *Let* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *be locally quasi-analytic minimal equicontinuous Cantor actions. If the actions* Φ_1 *and* Φ_2 *are continuously orbit equivalent, then they are return equivalent.*

In fact, equicontinuity of minimal Cantor actions is preserved under continuous orbit equivalence, as shown in the authors' work [**21**]:

PROPOSITION 3.8. [**21**, Proposition 3.1] *Suppose that minimal Cantor actions* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *are continuously orbit equivalent. If both* Γ_1 *and* Γ_2 *are finitely generated groups, and* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *is equicontinuous, then* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *is equicontinuous.*

For the general case (not necessarily locally quasi-analytic) of minimal Cantor actions with unique invariant probability measures, we have the following:

PROPOSITION 3.9. *Let* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *be continuously orbit equivalent minimal Cantor actions, with respective unique ergodic invariant probability measures* μ_1 *and* μ_2*. Then* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *has essential holonomy if and only if* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *has essential holonomy.*

PROOF. Let $h\colon \mathfrak{X}_1 \to \mathfrak{X}_2$ be the homeomorphism given by Definition 3.5. Then the pull-back $\widetilde{\mu}_2 = h^*(\mu_2)$ is a probability measure on $\mathfrak{X}_1$. Condition (1) of Definition 3.5 implies that $\widetilde{\mu}_2$ is invariant under the action $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$, hence by uniqueness of measure $\mu_1 = \widetilde{\mu}_2$.

Assume that $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ has essential holonomy, and let $g \in \Gamma_1$ be such that $\mathfrak{X}_g^{hol}$ has positive μ_1-measure, so there exists $x \in \mathfrak{X}_1$ such that $\Phi_1(g)(x) = x$ and $\mathfrak{X}_g^{hol}$ has μ_1-Lebesgue density 1 at x.

Let $\alpha\colon G_1 \times \mathfrak{X}_1 \to G_2$ be the map given by condition (1) so that $h(\Phi_1(g_1)(x_1)) = \Phi_2(\alpha(g_1, x_1))(h(x_1))$ for all $g_1 \in G_1$ and $x_1 \in \mathfrak{X}_1$. Then for $g_1 = g$ there exists a clopen set $U_1 \subset \mathfrak{X}_1$ so that $x \in U_1$ and $\alpha(g, y) \in \Gamma_2$ is continuous for $y \in U_1$. Since Γ_2 is a discrete space, we can assume that U_1 is sufficiently small so that $g_2 = \alpha(g, y) \in \Gamma_2$ is constant for $y \in U$.

We then have $h(\Phi_1(g)(y)) = \Phi_2(g_2)(h(y))$ for all $y \in U_1$. Set $U_2 = h(U_1)$ then this states that $h_{U_1}\colon U_1 \to U_2$ conjugates the action of $\Phi_1(g_1)$ on U_1 with the action of $\Phi_2(g_2)$ on U_2. Thus, $\Phi_2(g_2)$ has non-trivial holonomy at the fixed point $z = h(x)$ and this is a point of Lebesgue density 1 for the action of $\Phi_2(g_2)$. In particular, the action $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ has essential holonomy.

The converse implication is proved similarly, using condition (2) of Definition 3.5. □

4. Dynamics and the lower central series

Theorem 1.5 relates the non-trivial essential holonomy property for a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, with the lower central series of Γ. In this section, we show that if a minimal equicontinuous Cantor action of Γ is locally quasi-analytic and has essential holonomy, then every commutator subgroup in the lower central series of Γ has elements with positive measure sets of points with non-trivial holonomy.

First, recall the construction of the lower central series for Γ.

Set $\gamma_1(\Gamma) = \Gamma$, and for $i \geq 1$, let $\gamma_{i+1}(\Gamma) = [\Gamma, \gamma_i(\Gamma)]$ be the commutator subgroup, which is a normal subgroup of Γ. Then for $a \in \gamma_i(\Gamma)$ and $b \in \gamma_j(\Gamma)$ the commutator $[a, b] \in \gamma_{i+j}(\Gamma)$. Moreover, these subgroups form a descending chain

$$\Gamma = \gamma_1(\Gamma) \supset \gamma_2(\Gamma) \supset \cdots \supset \gamma_n(\Gamma) \supset \cdots \ . \tag{11}$$

Note that each quotient group $\gamma_i(\Gamma)/\gamma_{i+1}(\Gamma)$ is abelian.

The group Γ is nilpotent of length n_Γ if there exists an $n_\Gamma > 0$ such that $\gamma_n(\Gamma)$ is the trivial group for $n > n_\Gamma$, and $\gamma_{n_\Gamma}(\Gamma)$ is non-trivial. It follows that every element in $\gamma_{n_\Gamma}(\Gamma)$ commutes with every element of Γ; that is, $\gamma_{n_\Gamma}(\Gamma)$ is contained in the center of Γ.

Denote by $\Phi_n \colon \gamma_n(\Gamma) \times \mathfrak{X} \to \mathfrak{X}$ the restriction of the action Φ to the subgroup $\gamma_n(\Gamma)$.

DEFINITION 4.1. *A minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, with unique ergodic invariant probability measure μ, is said to have* essential holonomy at depth n_t *if the restricted action $(\mathfrak{X}, \gamma_{n_t}(\Gamma), \Phi_{n_t})$ has essential holonomy, but $(\mathfrak{X}, \gamma_n(\Gamma), \Phi_n)$ has no essential holonomy for $n > n_t$. The action has* essential holonomy at infinite depth *if for all $n \geq 1$, the restricted action $(\mathfrak{X}, \gamma_n(\Gamma), \Phi_n)$ has essential holonomy.*

Here is our main technical result.

PROPOSITION 4.2. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action which is locally quasi-analytic. If the action has essential holonomy, then it has essential holonomy of infinite depth.*

Proof. Suppose that $(\mathfrak{X}, \Gamma, \Phi)$ has essential holonomy at depth n_t. We show that this leads to a contradiction by localizing the action to a sufficiently small adapted set. Recall that μ denotes the unique ergodic invariant probability measure for the action.

The set $\mathfrak{X}^{hol}_{\gamma_{n_t}(\Gamma)}$ has positive μ-measure, while the set $\mathfrak{X}^{hol}_{\gamma_{(n_t+1)}(\Gamma)}$ has μ-measure zero. Thus, there exists $g \in \gamma_{n_t}(\Gamma)$ such that $\mu(\mathfrak{X}^{hol}_g) > 0$. Moreover, for all $\tau \in \gamma_{(n_t+1)}(\Gamma)$ we have $\mu(\mathfrak{X}^{hol}_\tau) = 0$.

Let $x \in \mathfrak{X}$ be such that $g \cdot x = x$, and $\mathfrak{X}^{hol}_g$ has Lebesgue density 1 at x. Let $U \subset \mathfrak{X}$ be an adapted clopen set with $x \in U$, and so $g \in \Gamma_U$, and sufficiently small diameter such that the restricted action $\Phi_U \colon \Gamma_U \times U \to U$ is quasi-analytic, and we have $\mu(\mathfrak{X}^{hol}_g \cap U) \geq 3/4 \cdot \mu(U)$, as in Lemma 2.5.

By the assumption that $x \in \mathfrak{X}^{hol}_g$, there exists $y \in U$ so that $z = g \cdot y \neq y$. Note that $z \neq x$ as $g \cdot x = x$. Let $V \subset U$ with $y \in V$ be a sufficiently small clopen set such that $(g \cdot V) \cap V = \emptyset$. Then choose $k_y \in \Gamma_U$ such that $k_y \cdot x \in V$. Then replace y with $k_y \cdot x$ and we have $g \cdot y \neq y$.

Let $\tau = [g, k_y]$ be the commutator. Then $g \in \gamma_{n_t}(\Gamma)$ implies that $\tau \in \gamma_{(n_t+1)}(\Gamma) \cap \Gamma_U$. By definition we have $g \cdot k_y = \tau \cdot k_y \cdot g$. Then for $w \in \mathfrak{X}^{hol}_g \cap U$ calculate

$$g \cdot (k_y \cdot w) = \tau \cdot k_y \cdot g \cdot w = \tau \cdot (k_y \cdot w) \ . \tag{12}$$

We use the identity (12) to prove the key observation:

LEMMA 4.3. *The sets $k_y \cdot (\mathfrak{X}^{hol}_g \cap U)$ and $(\mathfrak{X}^{hol}_g \cap U)$ are disjoint. In particular,*

$$\mu\left(k_y \cdot (\mathfrak{X}^{hol}_g \cap U) \cap (\mathfrak{X}^{hol}_g \cap U)\right) = 0.$$

Proof. Suppose that $w \in \mathfrak{X}^{hol}_g \cap U$ satisfies $k_y \cdot w \in \mathfrak{X}^{hol}_g \cap U$. Then $k_y \cdot w$ is a fixed point for the action of g, and so by (12) we have $k_y \cdot w$ is a fixed point for the action of $\tau = [g, k_y]$.

As $g \in \gamma_{n_t}(\Gamma)$, we have that $\tau \in \gamma_{(n_t+1)}(\Gamma)$. Then by the definition of the index n_t, for μ-almost all $w \in \mathfrak{X}_g^{hol} \cap U$, we have that $\Phi_U(\tau)$ has trivial holonomy at $k_y \cdot w$, as the action of $\Phi(k_y)$ is a measure preserving homeomorphism.

Thus, there exists a clopen set W_w with $k_y \cdot w \in W_w \subset U$ such that $\Phi_U(\tau)$ acts as the identity on W_w. As we chose U so that the action Φ_U is quasi-analytic on U, this implies that $\Phi_U(\tau)$ acts as the identity on U. However, we also have $g \cdot y \neq y$, so using the identity (12) again, we have $\tau \cdot y = \tau \cdot (k_y \cdot x) \neq k_y \cdot x$, and thus $\Phi_U(\tau)$ does not act as the identity on U, which is a contradiction. Therefore, $k_y \cdot (\mathfrak{X}_g^{hol} \cap U) \cap (\mathfrak{X}_g^{hol} \cap U) = \emptyset$. □

We now complete the proof of Proposition 4.2. As μ is invariant under the action of Φ_U we have $\mu\left(k_y \cdot (\mathfrak{X}_g^{hol} \cap U)\right) = \mu\left(\mathfrak{X}_g^{hol} \cap U\right) \geq 3/4 \cdot \mu(U)$. But then by Lemma 4.3 we obtain the contradiction

$$\mu(U) \geq \mu\left(k_y \cdot (\mathfrak{X}_g^{hol} \cap U)\right) + \mu\left(\mathfrak{X}_g^{hol} \cap U\right) \geq (3/4 + 3/4)\mu(U) > \mu(U) \ . \tag{13}$$

Thus, the action $(\mathfrak{X}, \Gamma, \Phi)$ cannot have essential holonomy at finite depth. □

COROLLARY 4.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action, and suppose that Γ is a finitely generated nilpotent group. Then the set of points with non-trivial holonomy has measure 0; that is, $(\mathfrak{X}, \Gamma, \Phi)$ does not have essential holonomy.*

PROOF. By Theorem 2.7, the action $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic. As Γ is nilpotent, the commutator group $\gamma_{n_\Gamma}(\Gamma)$ is in the center of Γ, hence the set of points with non-trivial holonomy for the action of $\gamma_{n_\Gamma}(\Gamma)$ on $\mathfrak{X}$ has measure 0; that is, it has no essential holonomy. Thus by Proposition 4.2 the action of Γ on $\mathfrak{X}$ has no essential holonomy. □

5. Commutators and not locally quasi-analytic actions

In this section, we exhibit a family of examples to show that the assumption that a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic is essential for the conclusions of Theorem 1.5 and Corollary 1.6.

THEOREM 5.1. *Let $\{n_\ell\}_{\ell \geq 1}$, $n_\ell \geq 5$ be a sequence of positive numbers, such that $n_\ell > 2n_{\ell-1}$. Let S_ℓ be a set with n_ℓ elements, and let $\mathfrak{X} = \prod_{\ell \geq 1} S_\ell$. Let $\mathrm{Sym}(S_\ell)$ be the symmetric group on n_ℓ symbols. There exists a countably generated subgroup $\Gamma \subset \prod_{\ell \geq 1} \mathrm{Sym}(S_\ell)$ with the following properties.*

1. *The lower central series of Γ stabilizes, i.e. $\gamma_n(\Gamma) = [\Gamma, \Gamma]$ for $n \geq 2$.*
2. *The minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$ is not locally quasi-analytic and has essential holonomy, i.e. the set of points with non-trivial holonomy has full measure.*
3. *The induced action $(\mathfrak{X}, [\Gamma, \Gamma], \Phi)$ of the commutator subgroup is minimal, not locally quasi-analytic and it has no essential holonomy.*

At the moment we are not aware of an action of a finitely generated group which exhibits similar properties for the commutator action.

The family of examples in Theorem 5.1 is an amalgam of the family considered in [**2**, Theorem 1.5] and [**16**, Theorem 6.1]. The idea to use actions of alternating groups comes from [**2**, Theorem 1.5], and the construction of an element with positive measure set of points with holonomy is the same as in [**16**, Theorem 6.1]. When constructing the commutator subgroup, we have to restrict to using the direct sum of symmetric groups instead of the direct product to ensure that the action of the commutator in our family has no essential holonomy (in fact, no points with non-trivial holonomy at all), and consequently the group we construct is not finitely generated. Merging these constructions yields the actions in Theorem 5.1.

Proof. The demonstration of Theorem 5.1 follows from Lemmas 5.2, 5.3 and 5.4 below. Denote by A_ℓ the alternating group on n_ℓ symbols.

LEMMA 5.2. *Consider the direct sum*

$$G = \bigoplus_{\ell \geq 1} A_\ell = \{g = (g_1, g_2, \ldots) \mid g_\ell \in A_\ell, \quad g_\ell = e_\ell \textit{ for all but finitely many } \ell\} \ , \quad (14)$$

where e_ℓ is the identity element in A_ℓ. Then the component-wise action of G on $\mathfrak{X}$ is minimal, equicontinuous, not locally quasi-analytic and has no essential holonomy.

Proof. The group G acts on the direct product space $\mathfrak{X}$ minimally, since the action of every A_ℓ on S_ℓ is transitive. Define the metric on $\mathfrak{X}$ by setting $d_{\mathfrak{X}}(x,y) = 1$ for all $x = (x_\ell), y = (y_\ell) \in \mathfrak{X}$ if $x_1 \neq y_1$, and otherwise

$$d_{\mathfrak{X}}(x,y) = 2^{-K}, \quad K = \max\{k \geq 1 \mid x_\ell = y_\ell \text{ for all } 1 \leq \ell \leq k\}.$$

Since G acts on each component of $\mathfrak{X}$ by permutations, its action is equicontinuous.

If $g \in G$ has a fixed point, then this point has a clopen neighborhood entirely fixed by the action of g, since g acts non-trivially on at most a finite number of factors in the direct product $\mathfrak{X}$. Thus the action of G on $\mathfrak{X}$ has no essential holonomy.

We show that the action of G is not locally quasi-analytic. Label the elements in S_ℓ by $\{1, 2, 3, \ldots, n_\ell\}$, and let $a_\ell = (123) \in A_\ell$, in particular, a_ℓ fixes the symbols $\{4, \ldots, n_\ell\} \subset S_\ell$. Let $\epsilon > 0$, and choose an open set V of diameter less than ϵ. Then there exists $m_V > 0$ such that

$$V \supset \prod_{1 \leq \ell < m_V} \{e_\ell\} \times \prod_{\ell \geq m_V} S_\ell.$$

Choose $n_V > m_V$, and define $g_V \in G$ as follows: set $g_\ell = a_\ell$ for $m_V \leq \ell \leq n_V$, and otherwise set $g_\ell = e_\ell$. Then both g and g^2 are the identity on the clopen set

$$W = \prod_{1 \leq \ell < m_V} \{e_\ell\} \times \prod_{m_V \leq \ell \leq n_V} \{4, \ldots, n_\ell\} \times \prod_{\ell > n_V} S_\ell \subset V,$$

while $g|V \neq g^2|V$. Since ϵ is arbitrary, this shows that the action of G on $\mathfrak{X}$ is not locally quasi-analytic. □

For sequences $\{n_\ell\}_{\ell \geq 1}$ with the additional property that $n_{\ell+1} > 2n_\ell$ for $\ell \geq 1$ we now realize G as the commutator of a discrete group Γ whose action has essential holonomy.

For $\ell \geq 1$, define a permutation $\gamma_\ell = (n_\ell - 1\, n_\ell)$, i.e. γ_ℓ fixes every vertex in S_ℓ except the last two, and let $\gamma = (\gamma_1, \gamma_2, \ldots) \in \prod_{\ell \geq 1} \mathrm{Sym}(S_\ell)$ be an element in the *direct product* of symmetric groups. Define

$$\Gamma = \langle \gamma, G \rangle \subset \prod_{\ell \geq 1} \mathrm{Sym}(S_\ell)\ . \tag{15}$$

LEMMA 5.3. *The commutators* $[G, G] = [\Gamma, G] = [\Gamma, \Gamma] = G$.

PROOF. For each $\ell \geq 1$, we have $n_\ell \geq 5$, hence the alternating group A_ℓ is simple and thus perfect, and thus G is also perfect; that is, $G = [G, G]$.

Next we show that $[\Gamma, \Gamma] = G$. Note that the restriction $\gamma|S_\ell = \gamma_\ell$ is an odd permutation, and for $\ell \neq k$ the actions of γ_ℓ on S_ℓ, and of γ_k on S_k, commute. So for any $g \in G$ the commutator $[\gamma, g]|S_\ell$ is an even permutation. For $g \in G$, let $m \geq 1$ be such that $g|S_\ell = e_\ell$ for $\ell \geq m$. Then for $\ell \geq m$

$$[\gamma, g]|S_\ell = \gamma_\ell \gamma_\ell^{-1} = e_\ell,$$

hence $[\gamma, g] \in G$. It follows that $G = [G, G] \subset [\Gamma, G] \subset [\Gamma, \Gamma] \subset G$. □

Using the metric $d_{\mathfrak{X}}$, defined above, it is convenient to think of the finite products $\prod_{1 \leq \ell \leq k} S_\ell$ as sets of vertices of a rooted tree at level $k \geq 1$, where the root is a single vertex at level 0 (it is omitted from $\mathfrak{X}$). In such a tree, each vertex at level ℓ is connected to $n_{\ell+1} = |S_{\ell+1}|$ vertices at level $\ell + 1$, and so one can think of vertices in $\prod_{1 \leq \ell \leq k} S_\ell$ as labeled by finite words $s_1 \cdots s_k$, where $s_\ell \in S_\ell$. An element of $\mathfrak{X}$ is an infinite sequence $s_1 s_2 \cdots$, where each truncated word $s_1 \cdots s_k$ corresponds to a vertex in $\prod_{1 \leq \ell \leq k} S_\ell$. An element of the direct product $\gamma \in \prod_{\ell \geq 1} \mathrm{Sym}(S_\ell)$ acts on the tree so that, for a given $s_\ell \in S_\ell$, $\gamma \cdot s_\ell = \gamma_\ell \cdot s_\ell$, and this action depends only on s_ℓ and not on the preceding or subsequent symbols in the sequence.

Open balls of diameter 2^{-K} in the metric $d_{\mathfrak{X}}$ are the sets of infinite words in $\mathfrak{X}$ which coincide for their first K symbols. The measure of each such open ball is

$$\frac{1}{|\prod_{1 \leq \ell \leq K} S_\ell|} = \frac{1}{n_1 \cdots n_K}.$$

We now show that γ has positive measure set of points with non-trivial holonomy by explicitly computing the Lebesgue density of this set at each fixed point of γ. The argument is the same as in [**16**, Theorem 6.1] and we give it here for completeness and for the convenience of the reader.

LEMMA 5.4. *Let* $\mathrm{Fix}(\gamma)$ *be the set of fixed points of the element* $\gamma \in \prod_{\ell \geq 1} \mathrm{Sym}(S_\ell)$ *defined above. Then every point in* $\mathrm{Fix}(\gamma)$ *has non-trivial holonomy, and* $\mu(\mathrm{Fix}(\gamma)) > 0$.

PROOF. First note that γ fixes an infinite path $s = s_1 s_2 \cdots \in \mathfrak{X}$ if and only if for all $\ell \geq 1$ we have $s_\ell \neq n_\ell$ and $s_\ell \neq n_\ell - 1$. We claim that each such point has non-trivial holonomy. Indeed, let V be an open neighborhood of s. Then there is an $m_V \geq 1$ such that

$$V \cap \left(\prod_{1 \leq \ell \leq m_V} \{s_\ell\} \times \prod_{\ell > m_V} S_\ell \right) = \prod_{1 \leq \ell \leq m_V} \{s_\ell\} \times \prod_{\ell > m_V} S_\ell,$$

and so V contains points which are moved by the action of γ.

For each clopen ball U_ℓ around a fixed point s, the action of γ permutes two clopen balls in U_ℓ consisting of sequences starting with $s_1 \cdots s_\ell(n_{\ell+1} - 1)$ and $s_1 \cdots s_\ell n_{\ell+1}$, which have the total measure $2/(n_1 \cdots n_{\ell+1})$. Each of the remaining $n_{\ell+1} - 2$ clopen balls determined by words of length $\ell + 1$ contains 2 subsets of sequences starting with $s_1 \cdots s_{\ell+1}(n_{\ell+2} - 1)$ and $s_1 \cdots s_{\ell+1}n_{\ell+2}$ permuted by the action, whose total measure is $2(n_{\ell+1} - 2)/(n_1 \cdots n_{\ell+1}n_{\ell+2})$. Continuing by induction, we compute the upper bound on the measure of the complement of the set $\mathrm{Fix}(\gamma)$:

$$\begin{aligned} \mu(U_\ell - \mathrm{Fix}(\gamma)) &= \frac{1}{n_1 \cdots n_\ell} \left(\frac{2}{n_{\ell+1}} + \frac{2(n_{\ell+1} - 2)}{n_{\ell+1}n_{\ell+2}} + \frac{2(n_{\ell+1} - 2)(n_{\ell+2} - 2)}{n_{\ell+1}n_{\ell+2}n_{\ell+3}} + \cdots \right) \\ &< \frac{2}{n_1 \cdots n_\ell} \sum_{i \geq 1} \frac{1}{n_{\ell+i}}. \end{aligned}$$

Since we assume that $n_{\ell+i} > 2n_{\ell+i-1} > 2^{i-1}n_{\ell+1}$, we obtain that

$$\mu(U_\ell - \mathrm{Fix}(\gamma)) < \frac{1}{n_1 \cdots n_\ell} \frac{4}{n_{\ell+1}},$$

and so

$$\mu(U_\ell \cap \mathrm{Fix}(\gamma)) > \frac{1}{n_1 \cdots n_\ell} - \frac{4}{n_1 \cdots n_{\ell+1}}.$$

It follows that for every point in $\mathrm{Fix}(\gamma)$ the Lebesgue density is 1, namely

$$1 = \lim_{\ell \to \infty} \left(1 - \frac{4}{n_{\ell+1}} \right) \leq \lim_{\ell \to \infty} \frac{\mu(U_\ell \cap \mathrm{Fix}(\gamma))}{\mu(U_\ell)} \leq 1 \ . \tag{16}$$

Thus the set of points with non-trivial holonomy for $\gamma \in \Gamma$ has positive measure. □

We have shown that the action of Γ on $\mathfrak{X}$ has essential holonomy, while the action of its commutator $[\Gamma, \Gamma] = G$ has no essential holonomy, which proves the assertions of Theorem 5.1. □

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