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Another approach to Planar Cover Conjecture focusing on rotation systems

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Abstract. We shall propose a new proof scheme for Planar Cover Conjecture, focusing on the rotation systems of planar coverings of connected graphs. We shall introduce the notion of "rotation compatible coverings" and show that a rotation compatible covering of G embedded on the sphere can be covered by a regular covering of G embedded on an orientable closed surface on which its covering transformation group acts. The surface may not be homeomorphic to the sphere in general, but its quotient becomes either the sphere or the projective plane which contains G. As an application of our theory, we shall prove that if a 3-connected graph G has a 3-connected finite planar covering such that the pre-images of each vertex has sufficiently large distance, then G can be embedded on the projective plane.

Introduction

A connected graph $\tilde{G}$ or a projection $p : V(\tilde{G}) \to V(G)$ is called a *covering* of a simple graph G if p induces a bijection between the neighbors of corresponding vertices. In particular, if there is a group action over $\tilde{G}$ such that the vertices equivalent under the action project to the same vertex, then $\tilde{G}$ is said to be *regular*. A covering may be an infinite graph in general, but we shall deal with finite coverings mainly in this paper.

In 1986, the author [21] proved that a connected graph G has a finite regular planar covering if and only if G can be embedded on the projective plane, applying the notion of "faithfulness of embeddings" developed in [17] and his thesis [18], and proposed the conjecture that a connected graph has a finite planar covering if and only if it can be embedded on the projective plane. The sufficiency holds obviously since the sphere covers the projective plane doubly while the necessity is still open for more than 30 years. This has been called *Planar Cover Conjecture* as one of famous unsolved problems in topological graph theory.

In the first paper [21] written for the conjecture, the author presented a proof scheme as follows. Any possible counterexample to the conjecture cannot be embedded on the projective plane and hence if all "minimal" non-projective-planar graphs have no finite planar covering, then the conjecture is true. He himself considered the minimal ones related to inclusion of homeomorphic images, called *irreducible graphs* for the projective plane, and concluded that it suffices to check up the 103 irreducible graphs classified in [1, 5]. By subsequent discussions, we have already known that if $K_{1,2,2,2}$ has no finite planar covering, then the conjecture is true. There have been written a lot of

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papers [2–4, 7–14, 19–31, 34, 36, 39] related to the conjecture and many of them follow this scheme. In particular, the first 20 years of the conjecture have been summarized in [11].

It may be said that most of such papers have focused on how to exclude possible counterexamples. In this paper, we shall give another proof scheme in a different direction, discussing the rotation systems of planar coverings of graphs. In Section 2, we shall introduce a new notion called "rotation compatible coverings". This is strictly wider than the notion of "regular coverings". In fact, if a planar covering of G is regular and 3-connected, then it can be embedded on the sphere as a rotation compatible one, and there exist those 3-connect planar coverings that are rotation compatible, but not regular, as shown in Section 7.

We shall modify or extend the proof scheme given in [21] so as to work for graphs which have rotation compatible planar coverings. After developing such a general theory according to the scheme, we shall show the following theorems as applications of the theory. We say here that a covering is *sufficiently large* if the pre-images of each vertex have sufficiently large distance in the covering. The precise definition of sufficiently large coverings will be given in Section 3.

THEOREM 1. *If a connected graph G has a finite planar covering that is* 3*-connected and sufficiently large, then G can be embedded on the projective plane.*

A graph embedded on the sphere is said to be *maximal planar* if it triangulates the sphere, that is, if its faces are all triangles:

THEOREM 2. *If a connected graph G has a finite maximal planar covering, then G can be embedded on the projective plane.*

Our arguments on coverings in this paper are not only combinatorial but also topological. We shall outline the general theory of coverings spaces in algebraic topology modified for our use and describe the connection to "voltage graphs" in Section 1. On the other hand, we shall develop a combinatorial way to control planar coverings embedded on the sphere with rotation systems and introduce "rotation compatible coverings" in Section 2. In Section 3, we shall give a precise definition of sufficiently large coverings and show that a sufficiently large covering with suitable conditions is rotation compatible. As shown in our later arguments, the 3-connectedness of coverings will be important to discuss the existence of planar coverings of 3-connetced graphs. So we shall prepare several arguments on the connectivity of coverings in Section 4, which will be useful also for general use.

In Section 5, we shall describe our proof scheme, reviewing the original idea given in [21], and prove that if a connected graph has a rotation compatible covering, then it can be embedded on the projective plane, as Theorem 11, which implies Theorem 1. As our conclusion, the maximal planar covering is rotation compatible and hence Theorem 2 follows. However, it is difficult to show it directly. We shall discuss it, using several known facts in Section 6. Finally, we shall illustrate our theory with examples of coverings of $K_{3,3}$ and suggest further studies on this topic in Section 7.

1. Classification of covering spaces

Let $\tilde{X}$ and X be two topological spaces, graphs or surfaces here. If there exists a local homeomorphism $p : \tilde{X} \to X$, that is, a surjective continuous map which induces a homeomorphism between suitable open neighborhoods of corresponding points in $\tilde{X}$ and X, then $\tilde{X}$ or the pair $(\tilde{X}, p)$ is called a *covering space* of X with *covering projection* p. If the projection p is an n-to-1 map for a finite number $n > 0$, then $\tilde{X}$ is called an *n-fold covering* of X. In particular, if $\tilde{G} = \tilde{X}$ and $G = X$ are simple graphs, then the covering projection can be regarded as a surjective map $p : V(\tilde{G}) \to V(G)$ such that p induces a bijection between the neighbors of any vertex $v \in V(G)$ and those of its pre-image $\tilde{v} \in p^{-1}(v)$.

There has been a general theory of covering spaces, which is strongly connected to the fundamental groups of topological spaces. The *fundamental group* $\pi_1(X)$ of a topological space X is defined as a group consisting of closed curves based at a fixed point in X up to continuous deformation (or up to *homotopy*) and the product of two elements in $\pi_1(X)$ corresponds to the closed curve going along the first one and next along the second one. We can find the theory described below in [38] for example. However, the notations have been modified suitably for our use in graph theory.

It has been known that any covering space $\tilde{X}$ of X with projection p corresponds to a subgroup $H < \pi_1(X)$ unique up to conjugation, and that its projection $p : \tilde{X} \to X$ induces an isomorphism $p_\# : \pi_1(\tilde{X}) \to H$. In this situation, we denote such a covering space by $\tilde{X}_H$ and its projection by $p_H : \tilde{X}_H \to X$. Each pre-image $\tilde{x} \in p_H^{-1}(x)$ of any point $x \in X$ corresponds bijectively to a right coset Hg of H in $\pi_1(X)$ and hence $|p_H^{-1}(x)| = (\pi_1(X) : H)$. This value, say n, does not depend on a point $x \in X$ and is often called the *covering index* of p_H and $\tilde{X}$ is called an *n-fold* covering of X.

Furthermore, if H is normal in $\pi_1(X)$, then the quotient group $\pi_1(X)/H$ acts on the set of pre-images $p_H^{-1}(x)$ for each point $x \in X$ naturally, and hence it acts on $\tilde{X}_H$ as the *covering transformation group*. That is, all points in $\tilde{X}_H$ equivalent under this group action project to the same point in X. In particular, the covering space $\tilde{X}_{\{1\}}$ corresponding to the trivial subgroup $\{1\}$ in $\pi_1(X)$ is called the *universal covering space* of X and $\pi_1(X)$ itself acts on $\tilde{X}_{\{1\}}$, where "1" stands for the identity element in $\pi_1(X)$.

If a subgroup N in $\pi_1(X)$ is contained in H, denoted by $N < H$, then the covering $p_N : \tilde{X}_N \to X$ *factors through* p_H, that is, there exists a covering $q : \tilde{X}_N \to \tilde{X}_H$ with $p_N = p_H \circ q$. This covering projection q maps a point in $\tilde{X}_N$ corresponding to a right coset Ng to a point corresponding to Hg. Since $\{1\}$ is contained in any subgroup H, the universal covering $p_{\{1\}} : \tilde{X}_{\{1\}} \to X$ factors through any covering of X.

We shall refer to the total of these facts described in the above as "the classification of covering spaces". We denote the homomorphism between π_1's induced by a continuous function $f : Y \to X$ by $f_\# : \pi_1(Y) \to \pi_1(X)$ in general.

The following lemma presents a well-known fact in group theory, but we shall note its proof for the reader unfamiliar to group theory:

LEMMA 3. *Let Γ be a group and let H be a subgroup in Γ of finite index. Then there is a normal subgroup N in Γ of finite index which H contains.*

PROOF. Suppose that H has index $k < \infty$. Then Γ has a right coset decomposition $\Gamma = Hg_1 \cup Hg_2 \cup \cdots \cup Hg_k$, where $g_1 = 1$ is the identity element in Γ. Any element

$g \in \Gamma$ induces a permutation over the cosets Hg_i's and Γ acts on the coset decomposition as $Hg_ig = Hg_j$ and hence it corresponds to a permutation over $\{1, 2, \ldots, k\}$. Let σ_g denote the corresponding permutation in the symmetry group S_k of degree k. Then the correspondence $\sigma : \Gamma \to S_k$ which maps $g \in \Gamma$ to $\sigma(g) = \sigma_g$ becomes a group homomorphism.

Let N be the kernel of σ. Then N is necessarily a normal subgroup in Γ and is contained in H since $Hg = H$ for any element $g \in N$. Its index is equal to $(\Gamma : N) = |\mathrm{Im}\,\sigma|$ and must be finite since S_k is a finite group. $\square$

We can conclude the following corollary immediately from the above lemma. It will be very important for our later arguments.

COROLLARY 4. *Given a finite covering $p_H : \tilde{G}_H \to G$ of a connected graph G associated with a subgroup H in $\pi_1(G)$, there exists a finite regular covering $p_N : \tilde{G}_N \to G$ which factors through p_H.*

PROOF. Take the normal subgroup N in $\pi_1(G)$ which Lemma 3 guarantees. The covering associated with N is the desired one. $\square$

A combinatorial method to construct coverings of a given graph has been developed as "voltage graphs", which has been used to solve "Map Color Theorem" [37] and described in [6]. The paper [27] has shown a way to translate the notions given above into those in the theory of voltage graphs, as follows.

Let G be a connected graph and let T be a spanning tree of G. Assign a permutation ρ_e over $\{1, \ldots, n\}$ to each edge $e \in E(G)$ not belonging to T. The set of such permutations $\rho = \{\rho_e : e \in E(G) - E(T)\}$ is called a *permutation voltage* of G. Make n copies of T, say $T_1, \ldots T_n$, and let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be the sets of copies of the two ends u and v of $e = uv$ such that u_i and v_i belong to T_i. Add the edges $u_iv_{\rho_e(i)}$ to the collection of $T_1, \ldots, T_n$ for all $i = 1, \ldots, n$. Then the resulting graph G_ρ becomes an n-fold covering of G, called the *covering of G derived by* a permutation voltage ρ.

It has been known that any finite covering of G can be obtained as a covering G_ρ of G derived by a suitable permutation voltage ρ. On the other hand, G_ρ must be associated with a suitable subgroup H in $\pi_1(G)$. Since the edges e correspond to a set of generators of $\pi_1(G)$, the assignment ρ naturally induces a homomorphism $\rho : \pi_1(G) \to S_{\{1,\ldots,n\}}$, where S_X stands for the permutation group over a finite set X in general. By the theory presented in [27], we have known that H is conjugate to $\rho^{-1}(S_{\{2,\ldots,n\}})$.

Now let A be a finite group of order $|A|$ and let $\alpha : E(G) - E(T) \to A$ be an assignment of elements in A to the edges $e = uv$ of G not belonging to T. Make copies $T \times x$ of T for all elements $x \in A$ and add the edges $u_xv_{\alpha(x)}$, where u_x and v_x are the copies of u and v lying on $T \times x$, respectively. Then we obtain another covering G^α of G, called the *covering of G derived by a voltage α*. This is very similar to G_ρ, but G^α is a regular covering of G on which A acts as its covering transformation group.

Let $\langle\rho\rangle$ denote the subgroup generated by $\{\rho_e : e \in E(G) - E(T)\}$ in $S_{\{1,\ldots,n\}}$. Then we can define a voltage $\alpha : E(G) - E(T) \to \langle\rho\rangle$ by $\alpha(e) = \rho_e$. It has been shown in [30,33] that G^α becomes the regular covering $\tilde{G}_N$ of G which factors through $\tilde{G}_H = G_\rho$, given in Corollary 4.

2. Coverings with rotation systems

Let G be a connected graph and let $p : \tilde{G} \to G$ be its covering. Suppose that $\tilde{G}$ is embedded on an oriented closed surface F^2, that is, an orientable closed surface with a fixed orientation over the surface, say "clockwise". Then the embedded $\tilde{G}$ admits naturally a *rotation system* $\sigma = \{\sigma_v : v \in V(\tilde{G})\}$, where $\sigma_v : N(v) \to N(v)$ denotes a cyclic permutation over the neighborhood $N(v)$ of each vertex v of $\tilde{G}$ which indicates the cyclic alignment of neighbors around v on the surface F^2. Thus, $\sigma_v(u)$ is the successor of u in the clockwise list of neighbors of v. If $\tilde{G}$ is 2*-cell embedded* on F^2, that is, if the interior of each face of $\tilde{G}$ is homeomorphic to an open 2-cell, then F^2 is nothing but the oriented closed surface derived by this rotation system σ of $\tilde{G}$, as defined in [6].

Here we focus on the pair $(\tilde{G}, \sigma)$ of the covering $\tilde{G}$ and its rotation system σ. Since the projection $p : V(\tilde{G}) \to V(G)$ maps the neighbors of each vertex v of $\tilde{G}$ bijectively to those of $p(v)$, it induces a rotation $\bar{\sigma}_{p(v)}$ around $p(v)$ as a copy of the rotation σ_v. However, the system of these rotations does not work as a well-defined rotation system of G itself in general since one vertex of G may receive different copies of rotations from its pre-images.

If we obtain a well-defined rotation system of G, then we can embed G on an orientable closed surface. However, we would like to construct an embedding of G on the projective plane, which is not orientable, as our final goal in this paper and hence we need to relax the conditions on the induced rotations, as follows.

> **Assumption 1:** The copies of rotations around the vertices of $\tilde{G}$ projecting to a vertex v of G induce a common rotation around v or its reverse.

Under this assumption, the pre-images of v can be classified into two groups so that the vertices belonging to one group induce the same rotation around v, clockwise or anti-clockwise. To distinguish these two groups, we assign "black" and "white" to the vertices in each group and fix this coloring for all v's. Read the alignment of neighbors of a black vertex in $\tilde{G}$ projecting to v, according to the clockwise orientation over F^2. Then their corresponding vertices around a white vertex projecting to v lie around it anticlockwise.

Since $\tilde{G}$ is assumed to be a simple graph, each edge uv of $\tilde{G}$ joins two distinct vertices u and v. If u and v have the same color, then the edge uv is said to be *synchronous*; otherwise, to be *anti-synchronous.* Note that this definition depends on the colorings of vertices with black and white.

> **Assumption 2:** The edges of $\tilde{G}$ projecting to the same edge of G are either all synchronous or all anti-synchronous.

The pair $(\tilde{G}, \sigma)$ or simply $\tilde{G}$ with a rotation system σ assumed implicitly is said to be *rotation compatible* if both Assumptions 1 and 2 hold. Switching black and white among the vertices projecting to one vertex yields switching synchronous and ani-synchronous edges. However, it is easy to see that the rotation compatibleness of $\tilde{G}$ does not change even if the color assignment of vertices was changed.

Note that if there exists a group action on the surface F^2 which acts on $\tilde{G}$ as its covering transformation group, then the pair $(\tilde{G}, \sigma)$ becomes rotation compatible. Any covering transformation γ carries a local area around each vertex of $\tilde{G}$ homeomorphically, which corresponds to Assumption 1. The transformation γ carries a local area containing each edge $e = uv$ of $\tilde{G}$ homeomorphically, too. If γ were just an automorphism over $\tilde{G}$ which cannot extend over F^2, then γ might put the local part around e to that around $\gamma(e)$ with twisting. In such a case, the rotation around one end of e, say v, corresponds to the rotation around $\gamma(v)$ but the rotation around u corresponds to the inverse of the rotation around $\gamma(u)$ and hence if e is synchronous, then $\gamma(e)$ is not. This means that Assumption 2 would not hold for $(\tilde{G}, \sigma)$.

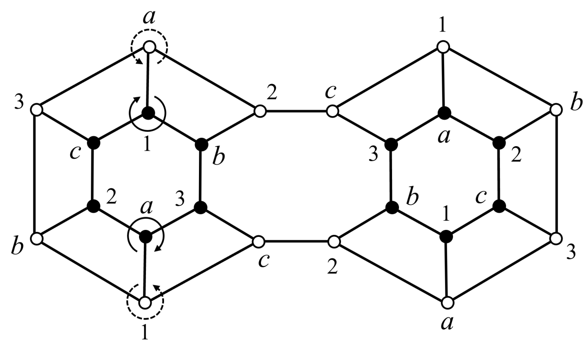

Figure 1. A rotation compatible planar covering of $K_{3,3}$

Figure 1 presents an easy example of a 4-fold planar covering of $K_{3,3}$ with partite sets $\{1, 2, 3\}$ and $\{a, b, c\}$. This is not a regular covering since there does not exist a transformation which carries the set of black vertices to that of white vertices; the former forms two hexagons, but the latter does not.

Each of black vertices labeled by numbers has a rotation $(a\ b\ c)$ while each of those labeled by alphabets has a rotation $(1\ 2\ 3)$. The rotations around white vertices are the inverses of these. It is easy to see that this covering is rotation compatible. Since this has a 2-cut, we can flip out the right half to reverse the rotations within that part. That is, it switches black and white partially and the two edges which form the 2-cut turn to be anti-synchronous. However, the edge labeled $2c$ lying in the left half is still synchronous although they project to the same edge $2c$ in $K_{3,3}$. Thus, Assumption 2 does not hold for the planar covering of $K_{3,3}$ so re-embedded and it is not rotation compatible.

As this example suggests, it depends on embeddings whether or not a planar covering of a graph is rotation compatible. On the other hand, it is well-known that any 3-connected planar graph is uniquely embeddable on the sphere [41, 42], which implies that it has a unique rotation system for its planar embedding, up to reversing. Furthermore, if a 3-connected nonplanar graph has a planar covering, then it has a 3-connected one, as we shall prove in Corollary 10. Therefore, it will be worthy to discuss the rotation compatibleness of 3-connected planar covering.

Many arguments in topological graph theory work with graph minor arguments. For our future research, we shall note the following lemma here.

LEMMA 5. *Let G be a connected graph and let H be another connected graph obtained from G by either deleting or contracting an edge e in G. If G has a rotation compatible covering 2-cell embedded on an orientable closed surface, then so does H.*

PROOF. Let $e = uv$ be an edge in G and assume that H can be obtained from G by deleting or contracting e, that is, $H = G - e$ or $H = G/e$. Let $p : \tilde{G} \to G$ be a rotation compatible covering of G embedded on an orientable closed surface F^2. Delete or contract all edges in $\tilde{G}$ projecting to e over the surface F^2. Then we obtain a covering $\tilde{H}$ of H embedded on F^2 and define the rotation of $\tilde{H}$ as one derived from the orientation of F^2.

If $H = G - e$, then the rotation around each vertex in $\tilde{H}$ can be obtained from that in $\tilde{G}$ by skipping the appearances of the pre-images of u and v. It is clear that Assumptions 1 and 2 hold for $\tilde{H}$ since the coloring of vertices in $\tilde{G}$ by black and white works also as that for $\tilde{H}$. However, we should notice that $\tilde{H}$ may not be 2-cell embedded on F^2 and also may not be connected. Each component of $\tilde{H}$ with the rotation system defined as above derives an orientable closed surface where it is 2-cell embedded.

On the other hand, if $H = G/e$, then the rotations around two ends of each pre-image $\tilde{e}$ of e are unified to one rotation around the vertex that $\tilde{e}$ shrinks to. Since the rotation compatibleness does not depend on the coloring with black and white, we may assume that the two ends of $\tilde{e}$ have the same color, black or white, and the unified rotation also has the same color as they have. This implies that the colors of two ends of any edge not projecting to e do not change after contracting all $\tilde{e}$'s, and hence Assumptions 1 and 2 for $\tilde{G}$ guarantee those for $\tilde{H}$ in this case, too. $\square$

As we have pointed out in the previous proof, the surfaces which contain the 2-cell embeddings of components of $\tilde{H}$ may be different from that for $\tilde{G}$. However, if $\tilde{G}$ is embedded on the sphere, then all surfaces for $\tilde{H}$ also are homeomorphic to the sphere.

3. Sufficiently large coverings

Let G be a 3-connected graph in general and suppose that G is embedded on the sphere. Then each face of G is bounded by a cycle. Look at a vertex v and consider the region given as the union of faces meeting at v. It is easy to see that the region also is bounded by a cycle. The cycle surrounding v is called the *link* of v and is denoted by $\mathrm{lk}(v)$. Thus, the link $\mathrm{lk}(v)$ bounds a polygonal disk containing v at its center and the edges incident to v are placed radially around v to subdivide the disk into the faces meeting at v.

Define $lk_{\max}(G)$ as the maximum of the lengths of $\mathrm{lk}(v)$'s taken over all vertices $v \in V(G)$ and call it the *maximum link length* of G. If $lk_{\max}(G) = 3$, then any vertex v of G is surrounded by a triangle and v has degree 3. This implies that G is isomorphic to K_4. Therefore, we conclude that $lk_{\max}(G) \geq 4$ for any 3-connected planar graph G except K_4.

More generally, if a 3-connected graph G is 2-cell embedded on a closed surface F^2, we need to assume that G is 3-*representative*, that is, any essential closed curve on F^2 meets G in at least three points. Under such an assumption, a local part around each vertex v has a wheel-like structure bounded by a cycle $\mathrm{lk}(v)$ and hence we can define $lk_{\max}(G)$ for such G as well as a 3-connected graph G embedded on the sphere. (It is well-known that the notion of *representativity* of a graph embedded on a closed surface has been introduced in [35]. Slightly earlier than it, the author has already introduced the notion combining the connectivity and representativity, called the *incompressibility* in [18].)

Now let $\tilde{G}$ be a finite covering of a 3-connected graph G and assume that $\tilde{G}$ is 3-connected, too. If $\tilde{G}$ is embedded on the sphere or is 3-representative on a closed surface other than the sphere, then we can define the maximum link length of this $\tilde{G}$. If the distance between any pair of the pre-images of each vertex v of G in $\tilde{G}$ is bigger than $lk_{\max}(\tilde{G})$, then $\tilde{G}$ is said to be *sufficiently large*. Recall that a sufficiently large covering has been defined as one such that the pre-igames of each vertex have sufficiently large distance in introduction. The above definition gives its precise meaning.

LEMMA 6. *Let $\tilde{G}$ be a 3-connected covering of a connected graph G and suppose that $\tilde{G}$ is embedded on the sphere or is 3-representative on an orientable closed surface. If $\tilde{G}$ is sufficiently large, then it is rotation compatible.*

PROOF. Let $p : \tilde{G} \to G$ be the covering projection of $\tilde{G}$ and suppose that it is embedded on the oriented closed surface with rotation system σ and is sufficiently large. First consider the cycle $C = \mathrm{lk}(v)$ given as the link of each vertex v to confirm Assumption 1. Since $\tilde{G}$ is sufficiently large, any two vertices lying on C do not project to the same vertex of G; otherwise, their distance would be less than $|\mathrm{lk}(v)|$ and hence less than $lk_{\max}(\tilde{G})$. Therefore, the cycle C projects isomorphically to a cycle in G.

Let $p^{-1}(p(v)) = \{v_1, \dots, v_n\}$ be the set of pre-images of $p(v)$. Since the cycle $\bar{C} = p(C)$ is 2-regular, the pull-back $p^{-1}(\bar{C})$ of $\bar{C}$ in $\tilde{G}$ also is 2-regular and hence each of its components is a cycle. Suppose that one of such cycles, say C', is not isomorphic to C. This implies that C' contains some neighbors of at least two of $v_1, \dots, v_n$. Choose the nearest pair of such neighbors, say u_1 and u_2. We may assume that u_i is adjacent to v_i for $i = 1, 2$ after relabeling. Then u_1 and u_2 are joined by a segment of C' which contains no other neighbors of $v_1, \dots, v_n$ and projects isomorphically to a segment of $\bar{C}$. Since $p(v)$ has degree at least 3, the length of the segment of $\bar{C}$ does not exceed $|\bar{C}| - 2 = |\mathrm{lk}(v)| - 2$ and hence v_1 and v_2 would be joined by a path of length at most $|\mathrm{lk}(v)|$, which is bounded by $lk_{\max}(\tilde{G})$. This contradicts that $\tilde{G}$ is sufficiently large.

Therefore, we conclude that $p^{-1}(\bar{C})$ consists of mutually disjoint cycles $C_1, \dots, C_n$ each of which projects isomorphically to $\bar{C}$ and only one among those cycles contains the neighbors of v_i, say C_i, for each $i = 1, \dots, n$. Then the cycle C_i, the vertex v_i and the edges incident to v_i form together a subgraph homeomorphic to a wheel and the neighbors of v_i lie along C_i in the cyclic order indicated by σ_{v_i} or its inverse. Since this wheel-like structure projects isomorphically into the part around $p(v)$ in G, this projection copies the rotation around v_i as the rotation around $p(v)$ or its reverse. Thus, Assumption 1 holds for the covering $\tilde{G}$.

Take any edge uv of $\tilde{G}$ to confirm Assumption 2. Let A_1 and A_2 be two faces of $\tilde{G}$ meeting along uv. Then there are two paths P_1 and P_2 such that P_i goes from u to v along the boundary of A_i, missing uv and they form a cycle $P_1 \cup P_2$ which bounds $A_1 \cup A_2$. Let Θ be the subgraph of $\tilde{G}$ consisting of two paths P_1, P_2 and the edge uv. Since this is a part of the wheel-like structure around v discussed in the previous, Θ projects isomorphically into G and can be pulled back isomorphically into the regions around each v_i in $\tilde{G}$.

This implies that the correspondence between two edges uv and $u'v'$ in $\tilde{G}$ projecting the same edge in G preserves whether or not they are synchronous. If uv is synchronous and $u'v'$ is anti-synchronous, then the Θ structure around uv must be mapped to that around $u'v'$ after flipping the part of Θ around u or v. However, if such a thing happened, then the tubular neighborhood of the Θ structure for $u'v'$ would contain a Möbius band, which is contrary to the closed surface being orientable. Therefore, Assumption 2 also holds and hence $\tilde{G}$ is rotation compatible. $\square$

A connected graph G embedded on a closed surface F^2 is called a *triangulation* on F^2 if each face is a triangle. In particular, a triangulation on the sphere is often called a *maximal planar* graph. Note that the link of a vertex in such a triangulation is a cycle consisting of the only neighbors of the vertex and hence $lk_{\max}(G)$ is nothing but the maximum degree of G. Although a covering of a connected graph which can be regarded as a triangulation on a closed surface may not be sufficiently large in general, we can conclude the same as in the previous lemma for such a covering with some conditions.

LEMMA 7. *Let $\tilde{G}$ be a triangulation on an orientable closed surface F^2 which covers a connected graph G. If the distance between any two vertices of $\tilde{G}$ projecting to the same vertex in G is greater than 3 and if any cycle of length 3 in $\tilde{G}$ bounds a 2-cell region on F^2, then the covering $p : \tilde{G} \to G$ is rotation compatible.*

PROOF. First suppose that Assumption 1 does not hold for $\tilde{G}$. Then there are two distinct vertices v_1 and v_2 which project to the same vertex v in G and which induce different rotations around v. Let C_i be the link of v_i in $\tilde{G}$. Then these two cycles C_1 and C_2 project to different cycles around v. This implies that there are four vertices u_1, u_2, u_3 and u_4 lying along $p(C_1)$ in this cyclic order and that two edges $e_1 = u_1u_3$ and $e_2 = u_2u_4$ are contained in $p(C_2)$ but not in $p(C_1)$.

Consider the pre-images of e_1 and e_2 incident to C_1, say $\tilde{e}_1$ and $\tilde{e}_2$. If one end of $\tilde{e}_1$ does not meet C_1, it must be a vertex lying on the link of another vertex v_3 projecting to v and hence there is a path of length 3 joining v_1 to v_3. This contradicts the first condition in the lemma. Thus, both ends of $\tilde{e}_1$ meet C_1 and so do those of $\tilde{e}_2$. However, those and v form a cycle of length 3 for each of $\tilde{e}_1$ and $\tilde{e}_2$. By the second condition, each of the two cycles must bounds a 2-cell region, but it is impossible for them to do it together, a contradiction. Therefore, we conclude that the links of all vertices projecting to v are mapped to the same cycle which consists of the neighbors of v. This implies that Assumption 1 holds for $\tilde{G}$.

Consider any edge uv of $\tilde{G}$ and the two faces uvw_1 and uvw_2 meeting along uv. The Θ structure with two cycles uvw_1 and uvw_2 works as the Θ in the previous proof and we can conclude that Assumption 2 holds. Therefore, $\tilde{G}$ is rotation compatible. $\square$

A natural question arises; is any 3-connected planar covering of a nonplanar graph always rotation compatible? Unfortunately, the answer is negative in general. We shall show such an example for the negative answer to this question in Section 7. However, we do not know yet whether or not a graph which has a planar covering has a rotation compatible planar covering. If it were affirmative, then Planar Cover Conjecture would be solved affirmatively by our later arguments.

4. Connectivity of coverings

Let G be a connected graph and let $p : \tilde{G} \to G$ be its covering. Suppose that $\tilde{G}$ has a 2-cut of vertices, say u and v, that is, $\tilde{G}$ splits into two subgraphs H_1 and H_2 which meet at two vertices u and v. Let $\tilde{G}_i$ be the graph obtained from H_i by identifying u and v. If

$p(u) = p(v)$ and if each of $\tilde{G}_1$ and $\tilde{G}_2$ is a covering of G with the natural projection, then we say that G *splits* into two coverings $\tilde{G}_1$ and $\tilde{G}_2$ at two vertices u and v. For example, the 4-fold covering of $K_{3,3}$ given in Figure 1 splits into two 2-fold planar coverings at the two white vertices labeled by c.

LEMMA 8. *If a covering $\tilde{G}$ of a 3-connected graph G is not 3-connected, then $\tilde{G}$ is 2-connected and splits into two coverings of G at two vertices.*

PROOF. Suppose that $\tilde{G}$ splits into two subgraphs H_1 and H_2 at two distinct vertices u and v, that is, $\tilde{G} = H_1 \cup H_2$ and $H_1 \cap H_2 = \{u, v\}$ and assume that $p(u) \neq p(v)$. Take two distinct neighbors x_1 and x_2 of u such that x_i belongs to H_i for $i = 1, 2$.

Consider the projections of x_1 and x_2, that is, $p(x_1)$ and $p(x_2)$. They are two distinct vertices of G since p projects the neighbors of u bijectively into G. Since G is 3-connected, $p(x_1)$ and $p(x_2)$ can be joined by a path Q missing $p(u)$ and $p(v)$ in G. Thus Q and $p(u)$ with two edges $p(ux_1)$ and $p(ux_2)$ form a cycle in G, say C_u. The pull-back $p^{-1}(C_u)$ of the cycle C_u in $\tilde{G}$ consists of mutually disjoint cycles missing v. However, one of such cycles passing through u cannot be closed since it does not pass through v, a contradiction. This denies the assumption of $p(u) \neq p(v)$, and hence we have $p(u) = p(v)$.

We can consider a similar cycle C_u under the assumption of $p(u) = p(v)$ and $p^{-1}(C_u)$ consists of mutually disjoint cycles, too. Take one of them which passes through u, say C. Then the copies of x_1ux_2 lie separately along C and we find one of those copies such that "u" is placed at v and that "x_1" and "x_2" are placed in H_2 and H_1, respectively. This implies that the neighbors of u lying in H_i and those of v lying in H_j project to the same set of neighbors of $p(u)\,(= p(v))$ for $\{i, j\} = \{1, 2\}$.

Let $N_i(u)$ and $N_i(v)$ be the sets of neighbors of u and v lying in H_i, respectively. Then $p(N_i(u) \cup N_i(v))$ forms the whole neighborhood of $p(u)$ in G. Furthermore, any path $w_1w_2 \cdots w_k$ going from $w_1 = p(u)$ to any other vertex $w_k = x \in V(G)$ can be lifted to a path $\tilde{Q}$ in H_i. If $w_2 \in p(N_i(u))$ (or $p(N_i(v))$), then $\tilde{Q}$ starts from u (or v) and terminates at a vertex $\tilde{x}$ in H_i. Since $p(\tilde{x}) = x$, H_i projects onto G with $p(u) = p(v)$ and hence the graph obtained from H_i with u and v identified to one vertex can be regarded as a covering of G for $i = 1, 2$.

Suppose that $\tilde{G}$ splits into two subgraphs H_1 and H_2 which meets at a cut vertex u in turn. Take two neighbors x_1 and x_2 of u lying in H_1 and H_2 separately. Since G is 3-connected, we can find a cycle C in G passing through $p(x_1)$, $p(u)$ and $p(x_2)$ in this order. The pull-back $p^{-1}(C)$ of C in $\tilde{G}$ must be closed, but it is impossible since x_1 and x_2 lie in different blocks. Therefore, this is not the case and hence $\tilde{G}$ is 2-connected. $\square$

It has been shown in [21] that if $\tilde{G}$ is a regular covering of a 3-connected graph G and is not 3-connected, then $\tilde{G}$ can be obtained as a *cyclic chain* of mutually isomorphic parts. That is, those parts are placed in a cyclic order, and each of them meets the next one at a point. If we take one half of $\tilde{G}$ given in Lemma 8 so as to minimize its size, then it will be one part in the cyclic chain.

COROLLARY 9. *If a planar covering $\tilde{G}$ of a 3-connected graph G is not 3-connected, then $\tilde{G}$ splits into two planar coverings at two vertices.*

PROOF. Embed a planar covering $\tilde{G}$ on the plane (or the sphere). By Lemma 8, if

$\tilde{G}$ is not 3-connected, it splits into two subgraphs H_1 and H_2 meeting at a 2-cut $\{u, v\}$ and H_i with u and v identified is a covering of G, say $\tilde{G}_i$ for $i = 1, 2$. Let j be the number different from i with $\{i, j\} = \{1, 2\}$. Shrink H_j to a point continuously on the plane. Then the vertices u and v are identified to one vertex and we obtain the planar embedding of $\tilde{G}_i$. That is, $\tilde{G}_i$ is a planar covering of G. □

Corollary 10. *If a 3-connected nonplanar G has a planar covering, then it has a 3-connected planar covering.*

Proof. If a planar covering $\tilde{G}$ of G is not 3-connected, G has another planar covering smaller than $\tilde{G}$ by Corollary 9. Therefore, the smallest planar coverings of a 3-connected graph G must be 3-connected. □

5. Proof scheme

First, we shall review the proof scheme given in [21]. Suppose that a connected graph G has a finite regular planar covering $\tilde{G}$. For some technical reasons, we assume that $\tilde{G}$ is 3-connected. Then $\tilde{G}$ can be *faithfully embedded* on the sphere, that is, any automorphism of $\tilde{G}$ extends to an auto-homeomorphism over the sphere [17]. This implies that its covering transformation group, say Γ, acts on the sphere S^2 and hence G can be embedded on its quotient space $O^2 = S^2/\Gamma$. This can be regarded as one of "elliptic 2-orbifolds".

The notion of *orbifolds* has been introduced in Thurston's Lecture Note [40], Chapter 13 and the 2-orbifolds have been classified completely. In particular, the elliptic 2-orbifolds can be regarded as the quotient spaces of the sphere by suitable group actions and their underlying space are homeomorphic to the disk, the sphere or the projective plane. Therefore, our O^2 must be one of those surfaces and G can be embedded on it and hence also on the projective plane. When our technical assumption does not hold, we carry out a kind of reduction, considering a suitable decomposition of coverings.

$$
\begin{tikzcd}
\tilde{G}_N \arrow[rrr] \arrow[dd, "p_N"'] \arrow[dr, "q"] & & & F^2 \arrow[dl, "\bar{q}"'] \arrow[dd, "\bar{p}"] \\
& \tilde{G}_H \arrow[r] \arrow[dl, "p_H"] & S^2 \arrow[dr, "r"'] & \\
G \arrow[rrr] & & & O^2
\end{tikzcd}
$$

Figure 2. Proof scheme diagram

Here we would like to deal with planar coverings which admit no group action. To do this, we need to modify the proof scheme as the diagram in Figure 2 suggests. We shall explain its details in our proof of the theorem below. In particular, if $\tilde{G}_H$ is a regular planar covering of G, then $\tilde{G}_N$ and F^2 coincide with $\tilde{G}_H$ and S^2, respectively and the diagram shrinks into that for the original proof scheme given in [21].

Theorem 11. *A connected graph G has a planar covering which is rotation compatible if and only if G can be embedded on the projective plane.*

PROOF. The sufficiency is clear since we can construct a 2-fold covering of G embedded on the sphere as the pull-back of an embedding of G on the projective plane which the sphere covers doubly. The covering of G so constructed is necessarily rotation compatible since the antipodal map over the sphere acts on it as its covering transformation.

To show the necessity, suppose that G has an n-fold planar covering $\tilde{G}$ and that $\tilde{G}$ is embedded on the oriented sphere S^2. Then $\tilde{G}$ has a unique rotation system σ to exhibit its embedding. Assume that the pair $(\tilde{G}, \sigma)$ is rotation compatible. By the classification of covering spaces, this covering $\tilde{G}$ is associated with a suitable subgroup H in $\pi_1(G)$ of index n. Put $\tilde{G}_H = \tilde{G}$ and denote its projection by $p_H : \tilde{G}_H \to G$, according to our notation.

Consider a tubular neighborhood $U(\tilde{G}_H)$ of $\tilde{G}_H$ embedded on S^2. This looks like a fattened $\tilde{G}_H$ and can be regarded as the union of small disks and rectangular bands joining these disks. Each of the disks contains a vertex v of $\tilde{G}_H$ at its center and each edge e of $\tilde{G}_H$ runs along the centerline of one of the bands. Let D_v and B_e denote them, respectively. The boundary of $U(\tilde{G}_H)$ decomposes into several mutually disjoint closed curves, each of which lies near along the boundary walk of a face of $\tilde{G}_H$ on the sphere S^2. It should be noticed that if a vertex v of $\tilde{G}_H$ has a rotation $\sigma_v = (u_1 u_2 \cdots u_k)$ with $k = \deg v$, then the edges $vu_1, vu_2, \ldots, vu_k$ are placed radially around v, according to the rotation σ_v.

By Corollary 4, there exists a finite regular covering $p_N : \tilde{G}_N \to G$ which factors through p_H and is associated with a normal subgroup N in $\pi_1(G)$. Put $\Gamma = \pi_1(G)/N$. This is the quotient group of $\pi_1(G)$ by N and acts on $\tilde{G}_N$ as the covering transformation group of p_N. That is, if two vertices $\tilde{v}$ and $\tilde{u}$ project to the same vertex of G, then there exists a covering transformation $\gamma \in \Gamma$ such that $\gamma(\tilde{v}) = \tilde{u}$.

Let $q : \tilde{G}_N \to \tilde{G}_H$ be the covering projection with $p_N = p_H \circ q$. Then we can define a rotation around each vertex $\tilde{v} \in q^{-1}(v)$ by pulling back the rotation around $v \in V(\tilde{G}_H)$. Let F^2 be the oriented closed surface derived from the rotation system $\tilde{\sigma}$ so defined. That is, $\tilde{G}_N$ is embedded on F^2 to realize $\tilde{\sigma}$. Then $\tilde{G}_N$ is contained in its tubular neighborhood $U(\tilde{G}_N)$ on F^2 and $U(\tilde{G}_N)$ decomposes into small disks $D_{\tilde{v}}$ and rectangular bands $B_{\tilde{e}}$, as well as $U(\tilde{G}_H)$ does. We may assume that $q(D_{\tilde{v}}) = D_{q(\tilde{v})}$ and $q(B_{\tilde{e}}) = B_{q(\tilde{e})}$. It is clear that $\tilde{G}_N$ with the rotation system $\tilde{\sigma}$ is rotation compatible since the rotation around each vertex $\tilde{v}$ is just a copy of that around $q(\tilde{v})$ and the same color is assigned to both $\tilde{v}$ and $q(\tilde{v})$.

Here we should check up whether or not a covering transformation $\gamma : \tilde{G}_N \to \tilde{G}_N$ extends over the tubular neighborhood $U(\tilde{G}_N)$. Since Assumption 1 holds for $(\tilde{G}_N, \tilde{\sigma})$, it is clear that γ extends naturally over the disk $D_{\tilde{v}}$ for each vertex $\tilde{v}$ of $\tilde{G}_N$. Assumption 2 also holds for the edges of $\tilde{G}_N$. Thus, any edge e of $\tilde{G}_N$ and $\gamma(e)$ are both synchronous or both non-synchronous for any covering transformation $\gamma \in \Gamma$.

Look at an edge $\tilde{u}_1\tilde{v}_1$ of $\tilde{G}_N$. Since the rotation compatibleness of $\tilde{G}_N$ does not depend on the assignment of black and white to its vertices, we may assume that $\tilde{u}_1\tilde{v}_1$ is synchronous and that the two ends $\tilde{u}_1$ and $\tilde{v}_1$ are both black after re-assigning colors. Let $\tilde{v}_1, \tilde{v}_2, \ldots \tilde{v}_k$ be the neighbors of $\tilde{u_1}$ and let $\tilde{u}_1, \tilde{u}_2 \ldots \tilde{u}_h$ be those of $\tilde{v}_1$ and suppose that they lie around $\tilde{u}_1$ and $\tilde{v}_1$ in these cyclic orders according to the rotation $\tilde{\sigma}$. Then the edges $\tilde{u}_1\tilde{v}_1, \ldots, \tilde{u}_1\tilde{v}_k$ and $\tilde{v}_1\tilde{u}_1, \ldots, \tilde{v}_1\tilde{u}_h$ are placed radially around $\tilde{v}_1$ and $\tilde{u}_1$, respectively. The boundary curve of $U(\tilde{G}_N)$ goes first along $\tilde{v}_k\tilde{u}_1$, turns around $\tilde{u}_1$, runs along the band containing the edge $\tilde{u}_1\tilde{v}_1$, turns again around $\tilde{v}_1$ and finally goes away along $\tilde{v_1}\tilde{u_2}$.

Under our assumption here, the edge $\gamma(\tilde{u}_1\tilde{v}_1) = \gamma(\tilde{u}_1)\gamma(\tilde{v}_1)$ is synchronous as well as $\tilde{u}_1\tilde{v}_1$. If $\gamma(\tilde{u_1})$ is black, then so is $\gamma(\tilde{v_1})$ and the same local structure arises around $\gamma(\tilde{u}_1\tilde{v}_1)$ as we have seen around $\tilde{u}_1\tilde{v}_1$ in the above. Then we can extend γ so as to map the band containing $\tilde{u}_1\tilde{v}_1$ onto the band containing $\gamma(\tilde{u}_1\tilde{v}_1)$. If $\gamma(\tilde{u}_1)$ and $\gamma(\tilde{v}_1)$ are white, then the local structure around $\gamma(\tilde{u}_1\tilde{v}_1)$ can be obtained from that around $\tilde{u}_1\tilde{v}_1$ by turning it over, and hence γ carries the band containing $\tilde{u}_1\tilde{v}_1$ to the corresponding band, as well as in the previous case.

Therefore, γ acts on $U(\tilde{G}_N)$ as its auto-homeomorphism which carries the disks to disks and the bands to bands and hence it carries the boundary curves of $U(\tilde{G}_N)$ onto those. This implies that each of those boundary curves, say ℓ, projects onto one of the boundary curves of $U(\tilde{G}_H)$. The closed curve ℓ bounds a 2-cell region $\tilde{R}$ on the closed surface F^2 by the way of construction and its projection $q(\ell)$ also bounds a 2-cell region R on the sphere S^2. Since ℓ wraps around its image several times naturally, we can define a branched covering between two 2-cells $\tilde{R}$ and R with exactly one branch point at their center and obtain a branched covering $\bar{q} : F^2 \to S^2$ between the surfaces which maps $\tilde{G}_N$ onto $\tilde{G}_H$ as an unbranched covering, that is, $\bar{q}|_{\tilde{G}_N} = q$.

On the other hand, the action on $U(\tilde{G}_N)$ by the covering transformation group Γ extends to the action over the whole surface F^2. If a covering transformation $\gamma \in \Gamma$ sends a 2-cell region $\tilde{R}$ onto itself, then the branch point of $\tilde{R}$ discussed in the above becomes a fixed point of γ. Under this situation, we can consider the quotient space O^2 of the surface F^2 by the group action of Γ. Let $\bar{p}_N : F^2 \to O^2$ be the natural projection from F^2 to O^2. Let x be any point on S^2 and choose one of its pre-images by $\bar{q}$, say $\tilde{x}$, that is, $\bar{q}(\tilde{x}) = x$. Define a continuous map $r : S^2 \to O^2$ by $r(x) = \bar{p}_N(\tilde{x})$ for any point $x \in S^2$. It is clear that r is well-defined since all pre-images of x on F^2 are equivalent under the action of Γ.

Since the action of Γ on F^2 has only isolated branch points, its quotient space O^2 becomes a closed surface. Let α be a non-trivial simple closed curve on the surface O^2 missing the branch points. Then $\tilde{\alpha} = r^{-1}(\alpha)$ is a simple closed curve on S^2 as a set and wraps around α several times, say m times, via the continuous map r. Since the sphere S^2 is simply connected, $\tilde{\alpha}$ is trivial in $\pi_1(S^2) = \{1\}$ and hence its continuous image α^m also is trivial in $\pi_1(O^2)$. However, there is no closed surface, other than the sphere and the projective plane, whose fundamental group π_1 contains a non-trivial element of finite order.

Therefore, O^2 must be homeomorphic to either the sphere or the projective plane since $\alpha^m = 1 \in \pi_1(O^2)$. The projection $\bar{p}_N(\tilde{G}_N)$ is nothing but G itself by construction and hence we conclude that G can be embedded on the projective plane in particular. □

We have discussed only rotation systems of coverings to construct an embedding of G on the projective plane. However, the reader might know the method to construct a 2-cell embedding on a non-orientable closed surface using not only rotations around vertices but also twisted edges. We put a rectangular band along an untwisted edge flatly, but place the band along a twisted edge after twisting it through 180°. Notice that each anti-synchromous edge in a rotation compatible covering of G projects to a twisted edge in the 2-cell embedding of G on the projective plane. Assumption 2 guarantees it.

We might be able to describe a proof of Theorem 11 more simply without topological arguments. However, our proof scheme includes an argument on the existence of a regular covering of an orientable closed surface which covers the sphere containing a given rotation compatible covering of a graph. For our further research, it will be worthy to clip it off, as the following corollary:

Corollary 12. *Let G be a connected graph and suppose that G has a finite covering $p_H : \tilde{G}_H \to G$ 2-cell embedded on an orientable closed surface F^2. Then p_H is rotation compatible if and only if there is a regular covering $p_N : \tilde{G}_N \to G$ 2-cell embedded on an orientable closed surface $\tilde{F}^2$ such that p_N factors through p_H and that any covering transformation of $\tilde{G}_N$ extends to an auto-homeomorphism over $\tilde{F}^2$.*

Proof. We can read the second to the eighth paragraphs in the previous proof as a proof of the necessity, replacing S^2 and F^2 with F^2 and $\tilde{F}^2$ in order.

To show the sufficiency, assume the existence of such a regular covering $p_N : \tilde{G}_N \to G$ as in the corollary. This is the same situation as described in the eighth paragraph in the previous proof and the covering transformation group Γ of $\tilde{G}_N$ acts on $\tilde{F}^2$. This implies that p_N itself must be rotation compatible. Since the rotations around vertices in $\tilde{G}_H$ can be regarded as copies of those in $\tilde{G}_N$ via the projection $\bar{q} : \tilde{F}^2 \to F^2$, we conclude that p_H also becomes rotation compatible. $\square$

Since any sufficiently large 3-connected planar covering is rotation compatible by Lemma 6, we can conclude Theorem 1 as an immediate corollary of Theorem 11. On the other hand, Theorem 2 cannot be regarded as a corollary of the theorem unfortunately at this stage since Lemma 7 includes two additional conditions for triangulations.

6. Planar coverings with small faces

Let $p : \tilde{G} \to G$ be a 3-connected planar covering of a connected graph G and suppose that $\tilde{G}$ is embedded on the sphere. If each face of $\tilde{G}$ is bounded by a short cycle, then the maximum link length of $\tilde{G}$ also will be short. This might suggest that such a planar covering tends to be sufficiently large or rotation compatible. So we shall discuss here those planar coverings that have only faces of small size. We can show the following lemma, repeating our arguments in the proof of Theorem 11 carefully:

Lemma 13. *Let G be a connected graph and let $\tilde{G}$ be a finite planar covering of G embedded on the sphere. If $\tilde{G}$ is rotation compatible for the rotation system exhibiting its embedding and if each face is bounded by a cycle of length at most 5, then $\tilde{G}$ is a 1- or 2-fold covering of G and G can be embedded on the projective plane.*

Proof. We have constructed a continuous map $r : S^2 \to O^2$ in the proof of Theorem 11. This is locally homeomorphic except the centers of faces of $\tilde{G}$ at most. The map r works as a branched covering from such an exceptional face A of $\tilde{G}$ to $r(A)$ with exactly one branch point. Therefore, if the face A of $\tilde{G}$ is bounded by a cycle C of length m, then the boundary of $r(A)$, which is $r(C)$ as a set, becomes a cycle and its length $|r(C)|$ is a divisor of m.

Since $r(C)$ is contained in a simple graph $G = r(\tilde{G})$, we have $|r(C)| \geq 3$. If C wraps around $r(C)$ twice or more, then we have $|C| \geq 6$. Since the assumption of the lemma excludes such a case, the cycle C projects to $r(C)$ isomorphically and the face A projects to $r(A)$ homeomorphically. Therefore, the whole r becomes a unbranched covering projection from the sphere S^2 to the closed surface O^2. This happens only when O^2 is homeomorphic to the sphere or the projective plane, and r is 1-fold in the former case and is 2-fold in the latter case. □

If we could prove that any maximal planar covering of G is rotation-compatible, then Theorem 2 would be just a corollary of the above lemma. Unfortunately, we have never found such a proof yet, but can prove the following theorem, using several known facts:

THEOREM 14. *Any maximal planar covering $\tilde{G}$ of a connected graph G is either G itself or a 2-fold covering of G. In the latter case, G is nonplanar and can be embedded on the projective plane as its triangulation.*

PROOF. Let $\tilde{G}$ be an n-fold maximal planar covering of G and suppose that $\tilde{G}$ is embedded on the sphere. Then each face of $\tilde{G}$ is triangular and we have $3|V(\tilde{G})| - |E(\tilde{G})| = 6$ by Euler's formula $V - E + F = 2$ with $2E = 3F$. Assigning $|V(\tilde{G})| = n|V(G)|$ and $|E(\tilde{G})| = n|E(G)|$ to this, we obtain the following equality:

$$n \cdot (3|V(G)| - |E(G)|) = 6$$

Since all variables in the above are integers, n must be one of the divisors of 6.

If G itself is planar, then we have $3|V(G)| - |E(G)| \geq 6$ in general, and hence $n = 1$ and $3|V(G)| - |E(G)| = 6$. This implies that $\tilde{G}$ must coincide with G, which is the first option in the theorem. Otherwise, that is, if G is not planar, then n must be even; it has been known that any planar covering of a nonplanar graph is even-fold by [2]. This implies that n is equal to either 2 or 6.

Suppose that $n = 6$. Then we have $3|V(G)| - |E(G)| = 1$. If G could be embedded on the projective plane, then we would have $3|V(G)| - |E(G)| \geq 3$ by Euler's formula for the projective plane. Thus, G cannot be embedded on the projective plane although it has a planar covering.

This implies that G is one of possible counterexamples to Planar Cover Conjecture, which have been listed in [12]. The list consists of sixteen types of those graphs including $K_{1,2,2,2}$ and it has been known that if one of them has an n-fold planar covering, then so does $K_{1,2,2,2}$. On the other hand, it has been proved in [34, 39] that $K_{1,2,2,2}$ has no n-fold planar covering for $n \leq 10$, and hence all possible counterexamples do not have any 6-fold planar covering in particular. So does not G, a contradiction.

Therefore, $\tilde{G}$ is a 2-fold planar covering of G and hence it is a regular covering. Since any triangulation on a closed surface, except K_3, is 3-connected in general, $\tilde{G}$ can be embedded faithfully on the sphere and its covering transformation group acts on the sphere. Thus, $\tilde{G}$ becomes rotation compatible. By Lemma 13, G can be embedded on the projective plane and each triangular face of $\tilde{G}$ on the sphere projects homeomorphically to a face of G on the projective plane. Therefore, G triangulates the projective plane. □

We used Ota's result [34] on possible planar coverings of $K_{1,2,2,2}$ to prove the previous theorem. However, we can show that the sixteen graphs listed as possible counterexample to Planar Cover Conjecture in [12] and those obtained from them by adding "3-patches" cannot become candidates for G, directly evaluating the value of $3|V(G)| - |E(G)|$; it will not be equal to 1 for all of them.

A *quadrangulation* on a closed surface is a simple graph G embedded on the surface such that each face is bounded by a cycle of length 4. In particular, it is easy to see that any quadrangulation on the sphere is a bipartite graph. Arguments on bipartite planar coverings in a more general situation can be found in [31]. Here we shall consider only those coverings that quadrangulate the sphere.

Theorem 15. *Let G be a 2-connected graph which is not planar and is not biparite. If G has a planar covering $p : \tilde{G} \to G$ such that $\tilde{G}$ is a 3-connected and quadrangulates the sphere, then G can be embedded on the projective plane as a quadrangulation and $\tilde{G}$ is 2-fold.*

Proof. Suppose that $\tilde{G}$ is an n-fold covering of G and embed $\tilde{G}$ on the sphere as a quadrangulation. Then we have $2|V(\tilde{G})| - |E(\tilde{G})| = 4$ by Euler's formula and hence $n \cdot (2|V(G)| - |E(G)|) = 4$. This implies that $n = 2$ or 4 since G is nonplanar. If $n = 2$, then the embedding of $\tilde{G}$ on the sphere is faithful since $\tilde{G}$ is 3-connected and its covering transformation group Γ of order 2 acts on the sphere, leaving $\tilde{G}$ invariant.

Let $u_1v_1u_2v_2$ be any cycle of length 4 which bound a quadrilateral face of $\tilde{G}$. If $p(u_1) = p(u_2)$, then the two edges $p(u_1v_1)$ and $p(u_2v_1)$ would be a pair of multiple edges. This is contrary to our assumption that any quadrangulation is simple. Therefore, we have $p(u_1) \neq p(u_2)$ and similarly $p(v_1) \neq p(v_2)$, and hence the cycle $u_1v_1u_2v_2$ projects isomorphically to a cycle of length 4 in G. This implies that each quadrilateral face of $\tilde{G}$ does not contain a branch point by the action of Γ and hence the projection p extends to an unbranched covering projection $\bar{p} : S^2 \to S^2/\Gamma$. Then S^2/Γ is homeomorphic to the projective plane, and the theorem follows.

Now suppose that $n = 4$. Since G is not bipartite, the covering $p : \tilde{G} \to G$ factors through the *canonical bipartite covering* $b : B(G) \to G$, which is 2-fold, as shown in [20]. That is, there is a covering projection $p' : \tilde{G} \to B(G)$ with $p = b \circ p'$. Since p' is 2-fold, we can carry out the same argument as in the previous, replacing G with $B(G)$, and conclude that $B(G)$ can be embedded on the projective plane as a quadrangulation.

Therefore, the bipartite graph $B(G)$ is a projective-planar 2-fold covering of the 2-connected nonplanar graph G. By the theorem proved in [29], $B(G)$ must be planar. The structures of embeddings of planar graphs on the projective plane have been classified into three types in [15] and hence $B(G)$ must have one of the three structures. Each of the structures has some ambiguous parts depicted as shaded regions and each of those regions should be replaced with a concrete planar graph.

Since $B(G)$ is embedded on the projective plane as a quadrangulation, each face must be quadrilateral. The pictures shown in [15] contains several white regions and they correspond to some faces of embeddings on the projective plane. However, it is easy to see that we cannot subdivide the boundaries of shaded parts to obtain a quadrangulation having no self-loops and no multiple edges and hence the quadrangulation $B(G)$ cannot be obtained by such a way, a contradiction. Therefore, n is not equal to 4. $\square$

7. Examples

We shall illustrate our theory with an example to understand it visually. See the picture depicted in Figure 3, which consists of a big disk with one hole looking like a CD and one of its quarters.

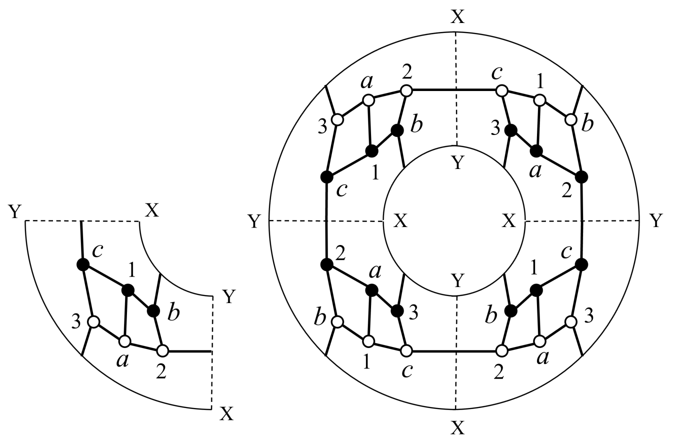

Figure 3. An 8-fold regular covering of $K_{3,3}$ on the torus

First look at the quarter placed in the left. If we deform the two circular arcs along its boundary to be straight line segments of the same length in parallel, then it looks like a square with four corners labeled by two X's and two Y's alternately. Furthermore, deform it into a disk homeomorphically. Then we obtain the usual picture to present $K_{3,3}$ embedded on the projective plane, where each antipodal pair of the points lying along the boundary of the disk should be identified.

The third quadrant of the disk with a hole placed in the right is the same one as the left piece except the labels of vertices and hence it can be deformed into a square. Now we have two squares. Bend each of these, pushing them to the front and to the back, and identify two pairs of their vertical sides to form a cylinder standing vertically which has two holes at its top and bottom. Then the vertices having the same labels are placed in point-symmetrical positions, but they have different colors, black and white. (This is not a coloring to indicate the bipartition of $K_{3,3}$.) If we close up or shrink each of the two ends of the cylinder, then we obtain a 2-fold covering of $K_{3,3}$ embedded on the sphere.

Now make the cylinder suitably longer and bend it to form a shape like a water pipe joint. Place it in the position of the third quadrant piece. Then we can see the same picture on the front face as one drawn in the piece and the picture drawn in the left piece goes to the back face.

Find the straight line passing through one pair of X and Y vertically. Make a copy of the pipe joint, rotating the pipe joint around the vertical line through 180° and join the original and its copy to form a longer pipe joint whose two holes look up. It should be noticed that we can see the same picture as drawn in the lower half of the disk with a hole from our side. If we close the two holes of our longer pipe joint, identifying each pair of horizontal arcs joining X and Y along their holes, then we obtain a closed surface homeomorphic to the sphere and it includes the 4-fold planar covering of $K_{3,3}$ given in

Figure 1. Find two cycles $2a3b1c$ of length 6 consisting of black vertices and one cycle of length 12 passing through only white vertices.

Finally rotate our longer pipe joint upward around the horizontal line passing through two X's and two Y's through $180°$ to form the torus together with the original lower longer pipe joint. Then the torus so constructed includes an 8-fold covering $\tilde{K}$ of $K_{3,3}$. It is clear that the cyclic group of order 2 generated by the rotation around the horizontal axis acts on $\tilde{K}$ to derive the 4-fold covering of $K_{3,3}$ given in Figure 1 as its quotient.

Since the torus has been decomposed into eight fundamental pieces, if any pair of such pieces can be transfered by an auto-homeomorphism over the torus, then we can conclude that $\tilde{K}$ is a regular covering of $K_{3,3}$. Actually, we can find two generators of its covering transformation group, as follows. One is the rotation around the vertical lines passing through X's and Y's, say a and we have $a^2 = 1$. The other one comes from the antipodal map over the cylinder corresponding to the third quadrant. Extend it to the whose torus. Then it switches the second and the fourth quadrants of the torus and induces the antipodal map over the cylinder corresponding to the first quadrant.

It is easy to see that a suitable composition of the two generators carries any one of the eight fundamental pieces to another and that they generate the dihedral group D_4 of order 8. Notice that the first generator preserves the colors of vertices while the second one switches black and white.

Therefore, the 8-fold covering of $K_{3,3}$ on the tours depicted in Figure 3 is a regular covering of $K_{3,3}$ which factors through the rotation compatible 4-fold planar covering of $K_{3,3}$ given in Figure 1. This is the one constructed in our proof of Theorem 11 for this example and the 2-orbifold $O^2 = F^2/\Gamma$ is the projective plane which we have constructed from one of the quarters of the CD at the beginning. Note that the 2-orbfold O^2 for this example has two branch points of index 2 corresponding to X and Y in the figure.

The *Euler number* $\chi(O^2)$ of a 2-orbifold O^2 has been introduced in [40], Chapter 13, under more general situation. Suppose that O^2 is decomposed into cellular pieces $\{c_1, c_2, \ldots\}$ like a CW complex. Assign a natural number $\mathrm{idx}(c_i)$ to each cell c_i to represent the characteristic of its singularity which c_i contains. For example, if c_i contains only one branch point which corresponds to a $1/n$ rotation, then $\mathrm{idx}(c_i) = n$, and if it contains no singularity, then $\mathrm{idx}(c_i) = 1$. The Euler number of O^2 is defined by:

$$\chi(O^2) = \sum_{c_i} (-1)^{\dim c_i} \frac{1}{\mathrm{idx}(c_i)}$$

If O^2 has no singularity, then $\chi(O^2)$ coincides with its Euler number as a surface. It is important that if $\tilde{O}^2$ is an n-fold covering of O^2 in the sense of orbifolds, then we have $\chi(\tilde{O}^2) = n\chi(O^2)$ (Proposition 13.3.4 in [40]), which corresponds to what is called "Hurwitz's formula".

Our 2-orbifold O^2 is homeomorphic to the projective plane and is decomposed into four 2-cells by the embedding of $K_{3,3}$ which has 6 vertices and 9 edges. The hexagonal face $c1b3a2$ and the quadrilateral face $c3b2$ contain branch points X and Y of index 2, respectively and two other faces contain no singularity. Thus, we can calculate the Euler number of O^2 as follows:

$$\chi(O^2) = 6 - 9 + \left(\frac{1}{2} + \frac{1}{2} + 1 + 1 \right) = 0$$

Furthermore, we have $\chi(F^2) = 8\chi(O^2) = 0$. Since the only orientable closed surface with Euler number 0 is the torus, this is consistent with what we have observed above.

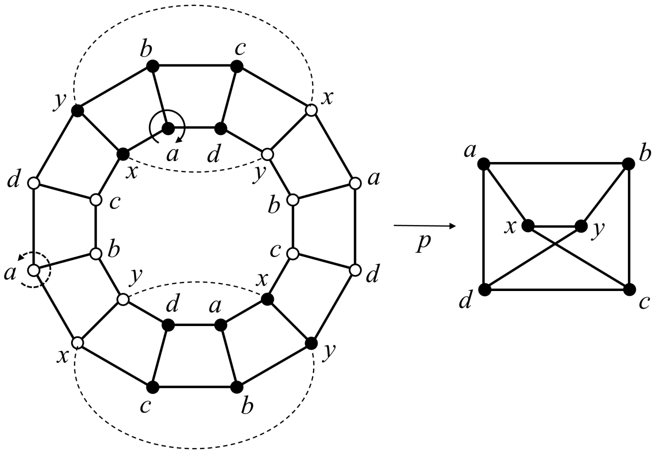

Figure 4. A 4-fold cyclic covering of $K_{3,3}$

Figure 4 presents another example of a 4-fold planar covering of $K_{3,3}$. The labels of vertices are different from those in Figure 1 and suggest that $K_{3,3}$ decomposes into the cycle *abcd* of length 4 and one bridge containing two vertices x and y. This covering is regular since the cyclic group $\mathbb{Z}_4$ of order 4 acts on it. A generator of this action rotates the figure through 90° and exchanges the inner and outer dodecagons. The vertices are colored by black and white in the same rule as in Figure 1 and this also is rotation compatible.

Consider the replacement of the two edges labeled xy in the upper half of the figure with two dashed lines forming an ellipse together. This yields another 4-fold covering of $K_{3,3}$ and switches the colors of x's and y's joined by the replaced edges. The new edges join black and white while the edges xy in the lower half still join the same colors. Thus, the new 4-fold covering of $K_{3,3}$ is not rotation compatible. Similarly, we can replace the edges xy with those along the lower ellipse in addition to obtain one more 4-fold covering of $K_{3,3}$ and it will turn back to be a rotation compatible one.

It is well-known that any 3-connected nonplanar graph other than K_5 contains a subdivision of $K_{3,3}$ and such a nonplanar graph can be obtained from $K_{3,3}$ by adding paths in suitable places in order, preserving the 3-connectedness of graphs. This deformation has been introduced and called a *bridging* in [16] where the author proved "the splitter theorem" for 3-connected graphs. If we add a path to $K_{3,3}$ drawn in Figure 4 to join the midpoints of ab and xy, and also add the four corresponding paths to the 4-fold covering, then the four edges labeled by xy in the covering cannot be flipped out so well as in the previous. Furthermore, adding enough many paths to $K_{3,3}$ to make the covering be a fine mesh will yield a sufficiently large covering, which is rotation compatible.

What we should do in the next might be to discuss whether or not we can deform a given planar covering into a rotation compatible one, focusing on the structure of $K_{3,3}$ with added paths.

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