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An interpolation of the generalized duality formula for the Schur multiple zeta values to complex functions

By Maki NAKASUJI, Yasuo OHNO and Wataru TAKEDA

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Abstract. One of the important research subjects in the study of multiple zeta functions is to clarify the linear relations and functional equations among them. The Schur multiple zeta functions are a generalization of the multiple zeta functions of Euler-Zagier type. Among many relations, the duality formula and its generalization are important families for both Euler-Zagier type and Schur type multiple zeta values. In this paper, following the method of previous works for multiple zeta values of Euler-Zagier type, we give an interpolation of the sums in the generalized duality formula, called Ohno relation, for Schur multiple zeta values. Moreover, we prove that the Ohno relation for Schur multiple zeta values is valid for complex numbers.

1. Introduction

For positive integers r, $k_1, k_2, \dots, k_r$ with $k_r \geq 2$, a multiple zeta value of Euler-Zagier type is defined by

$$\zeta(k_1, \dots, k_r) = \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

where the summation runs over all the size r sets of ordered positive integers. One can confirm that the above series converges for r-tuples $(k_1, \dots, k_r)$ of positive integers with $k_r \geq 2$. These r-tuples $(k_1, \dots, k_r)$ are called admissible. Many $\mathbb{Q}$-linear relations among multiple zeta values are known. Especially, the duality formula and its generalization are important relations. To state the generalized duality formula, we denote a string $\underbrace{1, \dots, 1}_{r}$ of 1's by $\{1\}^r$. Then, for an admissible index

$$\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \{1\}^{a_2-1}, b_2+1, \dots, \{1\}^{a_m-1}, b_m+1) \tag{1.1}$$

with $a_1, b_1, a_2, b_2, \cdots, a_m, b_m \in \mathbb{Z}_{\geq 1}$, the following index is called a *dual* index of $\mathbf{k}$:

$$\mathbf{k}^\dagger = (\{1\}^{b_m-1}, a_m+1, \{1\}^{b_{m-1}-1}, a_{m-1}+1, \dots, \{1\}^{b_1-1}, a_1+1).$$

The generalized duality formula, called Ohno relation in some literature, can then be described as follows:

THEOREM 1.2 (THE GENERALIZED DUALITY FORMULA. [8]). For any $\ell \in \mathbb{Z}_{\geq 0}$ and

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any admissible index $\mathbf{k} = (k_1, \ldots, k_r)$, and its dual index $\mathbf{k}^\dagger = (k_1^\dagger, \ldots, k_s^\dagger)$,

$$\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_r=\ell \\ \varepsilon_i \geq 0}} \zeta(k_1+\varepsilon_1, \ldots, k_r+\varepsilon_r) = \sum_{\substack{\varepsilon'_1+\cdots+\varepsilon'_s=\ell \\ \varepsilon'_i \geq 0}} \zeta(k_1^\dagger+\varepsilon'_1, \ldots, k_s^\dagger+\varepsilon'_s). \tag{1.3}$$

In Theorem 1.2, when $\ell = 0$, we obtain the duality formula for multiple zeta values of Euler-Zagier type. We may write the left-hand side of (1.3) as $\mathcal{O}(\mathbf{k} : \ell)$ and call $\mathcal{O}$-sum, then (1.3) can be written as

$$\mathcal{O}(\mathbf{k} : \ell) = \mathcal{O}(\mathbf{k}^\dagger : \ell). \tag{1.4}$$

In [5], the first and the second authors generalized Theorem 1.2 to the Schur multiple zeta values under some conditions. In the following, we review their setup:

For any partition λ, i.e., a non-increasing sequence $(\lambda_1, \ldots, \lambda_n)$ of positive integers, we associate the Young diagram $D_\lambda = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}$ depicted as a collection of square boxes with the i-th row having λ_i boxes. For a partition λ, a Young tableau $T = (t_{ij})$ of shape λ over a set X is obtained by filling the boxes of D_λ with $t_{ij} \in X$. We denote by $T_\lambda(X)$ the set of all Young tableaux of shape λ over X and denote by $SSYT_\lambda$ the set of semi-standard Young tableaux $(t_{ij}) \in T_\lambda(\mathbb{N})$ which satisfies the condition of weakly increasing from left to right in each row i, and strictly increasing from top to bottom in each column j. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be two partitions such that $\lambda_i \geq \mu_i$ for all i and $r \geq s$, and let $\delta = \lambda/\mu$ be a partition of skew shape. Then we define $D_\delta = D_\lambda \setminus D_\mu$ and sets of their fillings $T_\delta(X)$, $SSYT_\delta$ in the same way as above. Then, for a given tableau index $\boldsymbol{k} = (k_{ij}) \in T_\delta(\mathbb{Z})$, *Schur multiple zeta value* of shape δ is defined as

$$\zeta_\delta(\boldsymbol{k}) = \sum_{M \in SSYT_\delta} \frac{1}{M^{\boldsymbol{k}}},$$

where $M^{\boldsymbol{k}} = \prod_{(i,j)\in D_\delta} m_{ij}^{k_{ij}}$ for $M = (m_{ij}) \in SSYT_\delta$. The function $\zeta_\delta(\boldsymbol{k})$ absolutely converges in

$$W_\delta = \left\{ \boldsymbol{k} = (k_{ij}) \in T_\delta(\mathbb{Z}) \;\middle|\; \begin{array}{l} k_{ij} \geq 1 \text{ for all } (i,j) \in D_\delta \setminus C_\delta \\ k_{ij} \geq 2 \text{ for all } (i,j) \in C_\delta \end{array} \right\},$$

where C_δ is the set of all corners of δ. Here, we say that $(i,j) \in D_\delta$ is a corner of δ if $(i+1,j) \notin D_\delta$ and $(i,j+1) \notin D_\delta$; for example, if $\delta = (4,3,3,2) \setminus (3,2,1)$, $C_\delta = \{(1,4),(3,3),(4,2)\}$. In this article, we assume that all tableau indices of ζ_δ are elements of W_δ.

The first and the second authors [5] defined dual tableau for $\boldsymbol{k} \in T_\delta(\mathbb{Z})$ under some conditions as follows. First, we denote a finer piece of index $\{1\}^{a-1}, b+1$ as $A(a,b)$ and call it an *admissible piece*. If we write $A_i = A(a_i, b_i)$, its dual can be written as $A_i^\dagger = A(b_i, a_i)$. Then, the above admissible index $\mathbf{k}$ and its dual $\mathbf{k}^\dagger$ can be written in terms of admissible pieces:

$$\mathbf{k} = (A(a_1,b_1), A(a_2,b_2), \ldots, A(a_m,b_m)) = (A_1, A_2, \ldots, A_m)$$

and

$$\mathbf{k}^{\dagger}=(A(b_m,a_m),A(b_{m-1},a_{m-1}),\dots,A(b_1,a_1))=(A_m^{\dagger},A_{m-1}^{\dagger},\dots,A_1^{\dagger}).$$

We now write $\boldsymbol{k}\in T_{\delta}(\mathbb{Z})$ as

$$\boldsymbol{k}=\boldsymbol{k}_1^{\mathrm{col}}\cdots\boldsymbol{k}_{\lambda_1}^{\mathrm{col}}, \tag{1.5}$$

where $\boldsymbol{k}_j^{\mathrm{col}}$ is the j-th column tableau of $\boldsymbol{k}$. For example, when $\lambda=(3,2,1)$ and

$$\boldsymbol{k}=\begin{array}{|c|c|c|}\hline k_{11}&k_{12}&k_{13}\\\hline k_{21}&k_{22}&\\\cline{1-2} k_{31}&&\\\cline{1-1}\end{array},$$

then $\boldsymbol{k}_1^{\mathrm{col}}=\begin{array}{|c|}\hline k_{11}\\\hline k_{21}\\\hline k_{31}\\\hline\end{array}$, $\boldsymbol{k}_2^{\mathrm{col}}=\begin{array}{|c|}\hline k_{12}\\\hline k_{22}\\\hline\end{array}$ and $\boldsymbol{k}_3^{\mathrm{col}}=\begin{array}{|c|}\hline k_{13}\\\hline\end{array}$. Let $T_{\delta}^{\mathrm{diag}}(\mathbb{Z})=\{\boldsymbol{k}\in T_{\delta}(\mathbb{Z})\mid k_{ij}=k_{pq}$ if $j-i=q-p\}$. Let I_{δ}^{D} be the set of elements in $T_{\delta}^{\mathrm{diag}}(\mathbb{Z})$ consisting of admissible pieces such that the right side of the top element in each column is not 1. For $\boldsymbol{k}\in I_{\delta}^{D}$, in terms of admissible pieces, the row that has the topmost component is identified as the first row. In terms of admissible pieces, we can write as A_{ij} the component in the i-th row and j-th column. Note that the component in the upper-right corner is $A_{1\lambda_1}$ and that $A_{ij}=A_{k\ell}$ if $j-i=\ell-k$ when they are not empty. Further, we note that, in terms of tableaux, the top element in A_{ij} and the bottom element in $A_{i(j+1)}$ are located side by side. If the j-th column tableau $\boldsymbol{k}_j^{\mathrm{col}}$ starts A_{nj} for some n and has $m+1$ admissible pieces, then $\boldsymbol{k}_j^{\mathrm{col}}={}^{t}(A_{nj},\dots A_{(n+m)j})$. Then the dual tableau is $\boldsymbol{k}_j^{\mathrm{col},\dagger}={}^{t}(A_{(n+m)j}^{\dagger},\dots A_{nj}^{\dagger})$. We define $\boldsymbol{k}^{\dagger}$ by arranging $\boldsymbol{k}_{\lambda_1}^{\mathrm{col},\dagger},\dots,\boldsymbol{k}_1^{\mathrm{col},\dagger}$ in this order from left to right, where we put the top element in $A_{ij}^{\dagger}$ and the bottom element in $A_{i(j-1)}^{\dagger}$ side by side for $2\le j\le\lambda_1$ if both $A_{ij}^{\dagger}$ and $A_{i(j-1)}^{\dagger}$ are not empty.

For $\boldsymbol{k}=(k_{ij})\in W_{\delta}$, $\boldsymbol{\varepsilon}=(\varepsilon_{ij})\in T_{\delta}(\mathbb{Z}_{\ge 0})$, and $\ell\in\mathbb{Z}_{\ge0}$, we denote by

$$\mathcal{O}(\boldsymbol{k}:\ell)=\sum_{|\boldsymbol{\varepsilon}|=\ell}\zeta_{\delta}(\boldsymbol{k}+\boldsymbol{\varepsilon}),$$

where $\boldsymbol{k}+\boldsymbol{\varepsilon}=(k_{ij}+\varepsilon_{ij})\in T_{\delta}(\mathbb{Z})$ and $|\boldsymbol{\varepsilon}|=\sum_{(i,j)\in D_{\delta}}\varepsilon_{ij}$. Combining the extended Jacobi-Trudi formula for the Schur multiple zeta functions [7] with the Ohno relation for the classical one, Nakasuji and Ohno proved the following Ohno relation for the Schur multiple zeta values.

Theorem 1.6 ([5]). Let λ and μ be partitions and let $\delta=\lambda/\mu$. If $\boldsymbol{k}^{\dagger}$ is the dual tableau of $\boldsymbol{k}\in I_{\delta}^{D}$ and $\ell\in\mathbb{Z}_{\ge0}$, we have

$$\mathcal{O}(\boldsymbol{k}:\ell)=\mathcal{O}(\boldsymbol{k}^{\dagger}:\ell). \tag{1.7}$$

We may regard (1.7) as a generalization of (1.4). Identities (1.4) and (1.7) are based on the addition of positive integers. On the other hand, in [3], Hirose, Murahara and

Onozuka gave an interpolation of (1.4) to complex functions. For an admissible index $\mathbf{k}=(k_1,\dots,k_r)$ and $s\in\mathbb{C}$ with $\Re(s)>-1$, they defined the function $I_{\mathbf{k}}(s)$, called Ohno function in [2], by

$$I_{\mathbf{k}}(s)=\sum_{i=1}^{r}\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}\cdot\frac{1}{n_i^s}\prod_{j\neq i}\frac{n_j}{n_j-n_i}. \tag{1.8}$$

In [3, Lemma 2.2], it is proved that if s is a non-negative integer $m\in\mathbb{Z}_{\geq 0}$, the function $I_{\mathbf{k}}(s)$ is the same as $\mathcal{O}$-sum, that is,

$$\begin{aligned}I_{\mathbf{k}}(m)&=\sum_{\substack{\varepsilon_1+\cdots+\varepsilon_r=m\\ \varepsilon_i\geq 0\,(1\leq i\leq r)}}\zeta(k_1+\varepsilon_1,\dots,k_r+\varepsilon_r)\\ &=\mathcal{O}(\mathbf{k}:m).\end{aligned}$$

Thus, by Theorem 1.2, we have $I_{\mathbf{k}}(m)=I_{\mathbf{k}^\dagger}(m)$. More generally, they gave an interpolation of the Ohno relation to complex numbers.

Theorem 1.9 ([3]). For an admissible index $\mathbf{k}$ and $s\in\mathbb{C}$, we have

$$I_{\mathbf{k}}(s)=I_{\mathbf{k}^\dagger}(s).$$

Subsequently, Kamano and Onozuka introduced two kinds of integral representations of (1.8):

Theorem 1.10 ([2]). For any admissible index $\mathbf{k}$ represented as (1.1) and $s\in\mathbb{C}$ with $\Re(s)>-1$, we have

$$\begin{aligned}I_{\mathbf{k}}(s)=&\frac{1}{(a_1-1)!(b_1-1)!\cdots(a_m-1)!(b_m-1)!\Gamma(s+1)}\\ &\times\int_{0<t_1<\cdots<t_{2m}<1}\frac{dt_1\cdots dt_{2m}}{(1-t_1)t_2\cdots(1-t_{2m-1})t_{2m}}\left(\log\frac{t_2\cdots t_{2m}}{t_1\cdots t_{2m-1}}\right)^s\\ &\times\left(\log\frac{1-t_1}{1-t_2}\right)^{a_1-1}\left(\log\frac{t_3}{t_2}\right)^{b_1-1}\cdots\left(\log\frac{1-t_{2m-1}}{1-t_{2m}}\right)^{a_m-1}\left(\log\frac{1}{t_{2m}}\right)^{b_m-1}.\end{aligned}$$

Theorem 1.11 ([2]). For any admissible index $\mathbf{k}=(k_1,\dots,k_r)$ and $s\in\mathbb{C}$ with $\max_{1\leq j\leq r}\{r-2j+2-(k_j+\cdots+k_r)\}<\Re(s)<0$, we have

$$I_{\mathbf{k}}(s)=-\frac{\sin(\pi s)}{\pi}\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{k_1-1}\cdots n_r^{k_r-1}}\int_0^\infty\frac{w^{-s-1}}{(w+n_1)\cdots(w+n_r)}dw. \tag{1.12}$$

In this paper, we generalize the integral representation given in Theorem 1.11 to the Schur multiple zeta values. In other words, we consider the function

$$I_{\boldsymbol{k}}(s)=-\frac{\sin(\pi s)}{\pi}\sum_{(n_{ij})\in SSYT_\delta}\prod_{(i,j)\in D_\delta}\frac{1}{n_{ij}^{k_{ij}-1}}\int_0^\infty w^{-s-1}\prod_{(i,j)\in D_\delta}\frac{1}{w+n_{ij}}dw$$

in Section 2 and show that this function actually interpolates $\mathcal{O}$-sum for the Schur multiple zeta values in Section 3. Moreover, we prove

THEOREM 1.13 (THEOREM 3.8). Let λ and μ be partitions. Put $\delta = \lambda/\mu$ and $\boldsymbol{k} \in I_\delta^{\mathrm{D}}$ and let $\boldsymbol{k}^\dagger$ be the dual tableau of $\boldsymbol{k}$, for $s \in \mathbb{C}$ we have

$$I_{\boldsymbol{k}}(s) = I_{\boldsymbol{k}^\dagger}(s).$$

2. Integral representation and series expansion

In this section, we make preparations for constructing the function $I_{\boldsymbol{k}}(s)$ as a generalization of (1.12). As in introduction, taking Theorem 1.11 into account, proved by Kamano and Onozuka [2, Theorem 1.6], we can expect that $I_{\boldsymbol{k}}(s)$ can be defined as follows:

$$I_{\boldsymbol{k}}(s) = -\frac{\sin(\pi s)}{\pi} \sum_{(n_{ij})\in SSYT_\delta} \prod_{(i,j)\in D_\delta} \frac{1}{n_{ij}^{k_{ij}-1}} \int_0^\infty w^{-s-1} \prod_{(i,j)\in D_\delta} \frac{1}{w+n_{ij}} dw.$$

We first prove the following lemma for the calculation of this integral:

LEMMA 2.1. For any positive integers r, n and $s \in \mathbb{C}$ with $-r < \Re(s) < 0$,

$$\int_0^\infty \frac{w^{-s-1}}{(w+n)^r} dw = -\frac{\pi}{\sin(\pi s)} \frac{1}{n^{s+r}} \prod_{\ell=1}^{r-1} \frac{s+\ell}{\ell}.$$

PROOF. Changing the variable by $w = nv$ leads to

$$\int_0^\infty \frac{w^{-s-1}}{(w+n)^r} dw = n^{-s-r} \int_0^\infty \frac{v^{-s-1}}{(v+1)^r} dv = \frac{1}{n^{s+r}} B(-s, s+r),$$

where B is the beta function. By a recurrence relation for beta functions and the reflection formula, we have

$$\begin{aligned}\int_0^\infty \frac{w^{-s-1}}{(w+n)^r} dw &= \frac{1}{n^{s+r}} \prod_{\ell=1}^{r-1} \frac{s+r-\ell}{r-\ell} B(-s, s+1) \\ &= -\frac{\pi}{\sin(\pi s)} \frac{1}{n^{s+r}} \prod_{\ell=1}^{r-1} \frac{s+r-\ell}{r-\ell} \\ &= -\frac{\pi}{\sin(\pi s)} \frac{1}{n^{s+r}} \prod_{\ell=1}^{r-1} \frac{s+\ell}{\ell}.\end{aligned}$$

This proves the lemma. □

We next consider, as an example, the case of $\lambda = (2,1)$ and show that the function produced by our calculation interpolates $\mathcal{O}$-sum with respect to $\lambda = (2,1)$. In $-1 < \Re(s) < 0$, by arranging the order of the running indices n_{11}, n_{12} and n_{21}, we compute

$$\sum_{(n_{ij})\in SSYT_\lambda} \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})(w+n_{12})(w+n_{21})} dw$$

$$
\begin{aligned}
&= \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})^2(w+n_{21})} dw \\
&\quad + \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})(w+n_{21})^2} dw \\
&\quad + \left(\sum_{n_{11}<n_{12}<n_{21}} + \sum_{n_{11}<n_{21}<n_{12}} \right) \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})(w+n_{12})(w+n_{21})} dw \\
&= \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})^2(w+n_{21})} dw \\
&\quad + \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})(w+n_{21})^2} dw \\
&\quad - \frac{\pi}{\sin(\pi s)} \left(\sum_{n_{11}<n_{12}<n_{21}} + \sum_{n_{11}<n_{21}<n_{12}} \right) \sum_{(i,j)\in D_\lambda} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{ij}^s} \prod_{(i,j)\neq(i',j')} \frac{n_{i'j'}}{n_{i'j'}-n_{ij}}.
\end{aligned}
$$

The second and fourth terms are obtained by the same procedure as in [2]. We consider the integral

$$
\int_0^\infty \frac{w^{-s-1}}{(w+n_{11})^2(w+n_{21})} dw.
$$

The partial fraction decomposition and Lemma 2.1 lead to

$$
\begin{aligned}
&\int_0^\infty \frac{w^{-s-1}}{(w+n_{11})^2(w+n_{21})} dw \\
&= \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})^2} \frac{1}{(n_{21}-n_{11})} - \frac{w^{-s-1}}{w+n_{11}} \frac{1}{(n_{21}-n_{11})^2} + \frac{w^{-s-1}}{w+n_{21}} \frac{1}{(n_{11}-n_{21})^2} dw \\
&= -\frac{\pi}{\sin(\pi s)} \left(\frac{(1+s)}{n_{11}^{s+2}} \frac{1}{(n_{21}-n_{11})} - \frac{1}{n_{11}^{s+1}} \frac{1}{(n_{21}-n_{11})^2} + \frac{1}{n_{21}^{s+1}} \frac{1}{(n_{11}-n_{21})^2} \right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&- \frac{\sin(\pi s)}{\pi} \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}-1} n_{12}^{k_{12}-1} n_{21}^{k_{21}-1}} \int_0^\infty \frac{w^{-s-1}}{(w+n_{11})^2(w+n_{21})} dw \\
&= (1+s) \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^s} \frac{n_{21}}{(n_{21}-n_{11})} - \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^s} \frac{n_{11}n_{21}}{(n_{21}-n_{11})^2} \\
&\quad + \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^s} \frac{n_{11}^2}{(n_{11}-n_{21})^2}.
\end{aligned}
$$

By changing the role of n_{11} and n_{21} in the above, we have a similar formula for the case $n_{11} < n_{12} = n_{21}$. Combining these calculations, we have

$$
I_{\boldsymbol{k}}(s) = (1+s) \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^s} \frac{n_{21}}{(n_{21}-n_{11})}
$$

$$
\begin{aligned}
&- \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^{s}} \frac{n_{11}n_{21}}{(n_{21}-n_{11})^2} + \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^{s}} \frac{n_{11}^2}{(n_{11}-n_{21})^2} \\
&+ (1+s) \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^{s}} \frac{n_{11}}{(n_{11}-n_{21})} \\
&- \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^{s}} \frac{n_{11}n_{21}}{(n_{11}-n_{21})^2} + \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^{s}} \frac{n_{21}^2}{(n_{21}-n_{11})^2} \\
&+ \left(\sum_{n_{11}<n_{12}<n_{21}} + \sum_{n_{11}<n_{21}<n_{12}} \right) \sum_{(i,j)\in D_\lambda} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{ij}^{s}} \prod_{(i,j)\neq(i',j')} \frac{n_{i'j'}}{n_{i'j'}-n_{ij}}.
\end{aligned} \tag{2.2}
$$

Since we expect this $I_{\boldsymbol{k}}(s)$ to interpolate $\mathcal{O}$-sum for $\lambda=(2,1)$, we substitute non-negative integers for s. At this stage, although non-negative integers are outside of the domain of $I_{\boldsymbol{k}}(s)$ given by the integral, we can consider $I_{\boldsymbol{k}}(s)$ to be analytically continued to the half plane $\Re(s) > -1$ since the series on the right-hand side converges. Substituting $s=0$, the right-hand side becomes

$$
\begin{aligned}
I_{\boldsymbol{k}}(0) = &\sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{n_{21}}{(n_{21}-n_{11})} - \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{n_{11}n_{21}}{(n_{21}-n_{11})^2} \\
&+ \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{n_{11}^2}{(n_{11}-n_{21})^2} \\
&+ \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{n_{11}}{(n_{11}-n_{21})} - \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{n_{11}n_{21}}{(n_{11}-n_{21})^2} \\
&+ \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{n_{21}^2}{(n_{21}-n_{11})^2} \\
&+ \sum_{n_{11}<n_{21}<n_{12}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} + \sum_{n_{11}<n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \\
= &\,\zeta_\lambda(\boldsymbol{k}).
\end{aligned}
$$

Substituting $s=m\in\mathbb{Z}_{\geq 0}$, the right-hand side becomes

$$
\begin{aligned}
I_{\boldsymbol{k}}(m) = &(1+m) \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^{m}} \frac{n_{21}}{(n_{21}-n_{11})} \\
&- \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^{m}} \frac{n_{11}n_{21}}{(n_{21}-n_{11})^2} + \sum_{n_{11}=n_{12}<n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^{m}} \frac{n_{11}^2}{(n_{11}-n_{21})^2} \\
&+ (1+m) \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^{m}} \frac{n_{11}}{(n_{11}-n_{21})} \\
&- \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{21}^{m}} \frac{n_{11}n_{21}}{(n_{11}-n_{21})^2} + \sum_{n_{11}<n_{12}=n_{21}} \frac{1}{n_{11}^{k_{11}} n_{12}^{k_{12}} n_{21}^{k_{21}}} \frac{1}{n_{11}^{m}} \frac{n_{21}^2}{(n_{21}-n_{11})^2} \\
&+ \sum_{e_1+e_2+e_3=m} \sum_{n_{11}<n_{21}<n_{12}} \frac{1}{n_{11}^{k_{11}+e_1} n_{12}^{k_{12}+e_2} n_{21}^{k_{21}+e_3}}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{e_1+e_2+e_3=m}\sum_{n_{11}<n_{12}<n_{21}}\frac{1}{n_{11}^{k_{11}+e_1}n_{12}^{k_{12}+e_2}n_{21}^{k_{21}+e_3}}\\
=&\sum_{n_{11}=n_{12}<n_{21}}\frac{1}{n_{11}^{k_{11}}n_{12}^{k_{12}}n_{21}^{k_{21}}}\left(\frac{m+1}{n_{11}^m}+\frac{m}{n_{11}^{m-1}n_{21}}+\cdots+\frac{1}{n_{21}^m}\right)\\
&+\sum_{n_{11}<n_{12}=n_{21}}\frac{1}{n_{11}^{k_{11}}n_{12}^{k_{12}}n_{21}^{k_{21}}}\left(\frac{m+1}{n_{21}^m}+\frac{m}{n_{21}^{m-1}n_{11}}+\cdots+\frac{1}{n_{11}^m}\right)\\
&+\left(\sum_{n_{11}<n_{21}<n_{12}}+\sum_{n_{11}<n_{12}<n_{21}}\right)\sum_{e_1+e_2+e_3=m}\frac{1}{n_{11}^{k_{11}+e_1}n_{12}^{k_{12}+e_2}n_{21}^{k_{21}+e_3}}\\
=&\sum_{|\boldsymbol{\varepsilon}|=m}\zeta_\lambda(\boldsymbol{k}+\boldsymbol{\varepsilon}).
\end{aligned}
$$

The above two calculations ensure that our $I_{\boldsymbol{k}}(s)$ interpolates $\mathcal{O}$-sum associated with $\lambda=(2,1)$. Based on this, we would like to produce a series expression of $I_{\boldsymbol{k}}(s)$ for the general case, as well. In preparation for that, we offer the following lemma, which gives explicitly the coefficients of the partial fraction decomposition:

LEMMA 2.3. Let

$$
P(N)=\prod_{\alpha=1}^{R_N}\frac{1}{(w+n_\alpha)^{r_\alpha}}
$$

with distinct integers $n_1,\dots,n_{R_N}$. Then

$$
P(N)=\sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha}\frac{1}{(w+n_\alpha)^\ell}\frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}}\left(\frac{1}{(r_\alpha-\ell)!}\prod_{\beta\neq\alpha}\frac{1}{(w+n_\beta)^{r_\beta}}\right)\Bigg|_{w=-n_\alpha}.
$$

PROOF. This lemma follows from the uniqueness of the Laurent series expansion. □

We apply Lemma 2.3 with Lemma 2.1, then it holds that

$$
\begin{aligned}
&-\frac{\sin(\pi s)}{\pi}\int_0^\infty P(N)w^{-s-1}\,dw\\
&=\sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha}\frac{1}{n_\alpha^{s+\ell}}\prod_{p=1}^{\ell-1}\frac{s+p}{p}\frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}}\left(\frac{1}{(r_\alpha-\ell)!}\prod_{\beta\neq\alpha}\frac{1}{(w+n_\beta)^{r_\beta}}\right)\Bigg|_{w=-n_\alpha}.
\end{aligned}
\tag{2.4}
$$

For $N=(n_{ij})\in SSYT_\delta$, we rewrite

$$
\prod_{(i,j)\in D_\delta}\frac{1}{w+n_{ij}}=\prod_{\alpha=1}^{R_N}\frac{1}{(w+n_\alpha)^{r_\alpha}}
$$

by summarizing the same n_{ij}. Identity (2.4) then leads to

$I_{\boldsymbol{k}}(s)$

$$= -\frac{\sin(\pi s)}{\pi} \sum_{N \in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}} \int_0^\infty w^{-s-1} \prod_{(i,j)\in D_\delta} \frac{1}{w+n_{ij}} dw$$

$$= \sum_{N \in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}} \sum_{\alpha=1}^{R_N} \sum_{\ell=1}^{r_\alpha} \frac{1}{n_\alpha^{s+\ell}} \prod_{p=1}^{\ell-1} \frac{s+p}{p} \frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}} \left(\frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(w+n_\beta)^{r_\beta}} \right)\Bigg|_{w=-n_\alpha}.$$

We can now summarize the above as follows.

Lemma 2.5 (Explicit series form of $I_{\boldsymbol{k}}(s)$). For $-1 < \Re(s) < 0$,

$$I_{\boldsymbol{k}}(s) = \sum_{N \in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}} \sum_{\alpha=1}^{R_N} \sum_{\ell=1}^{r_\alpha} \frac{1}{n_\alpha^{s+\ell}} \prod_{p=1}^{\ell-1} \frac{s+p}{p} \frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}} \left(\frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(w+n_\beta)^{r_\beta}} \right)\Bigg|_{w=-n_\alpha}.$$

Substituting $s=0$, we have

$$\begin{aligned} I_{\boldsymbol{k}}(0) &= \sum_{N \in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}} \sum_{\alpha=1}^{R_N} \sum_{\ell=1}^{r_\alpha} \frac{1}{n_\alpha^{\ell}} \frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}} \left(\frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(w+n_\beta)^{r_\beta}} \right)\Bigg|_{w=-n_\alpha} \\ &= \sum_{N \in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}} \prod_{\alpha=1}^{R_N} \frac{1}{n_\alpha^{r_\alpha}} \\ &= \zeta_\delta(\boldsymbol{k}). \end{aligned}$$

This ensures that the series expansion converges in $\Re(s) \geq 0$, which gives the analytic continuation of $I_{\boldsymbol{k}}(s)$ in $\Re(s) > -1$. Furthermore, the series expansion given in Lemma 2.5 is a sum of products of the polynomial and zeta functions associated with a root system of type A. Therefore, $I_{\boldsymbol{k}}(s)$ can be meromorphically continued to the whole space of $\mathbb{C}$ (see [4, Section 2]).

3. Interpolation of the generalized duality formula

In this section, we revisit and generalize [3, Lemma 2.1].

Lemma 3.1 ([3, Lemma 2.1]). For $m \in \mathbb{Z}_{\geq 0}$ and $a_1, \ldots, a_r \in \mathbb{R}$ with $a_i \neq a_j$ for $i \neq j$, we have

$$\sum_{\substack{e_1+\cdots+e_r=m \\ e_i\geq 0\,(1\leq i\leq r)}} a_1^{e_1}\cdots a_r^{e_r} = \sum_{i=1}^r a_i^{m+r-1} \prod_{j\neq i} (a_i-a_j)^{-1}.$$

We note that if $a_1 = a_2$, then for each $i = 1, 2$ the product $\prod_{j\neq i}(a_i - a_j)^{-1}$ is not defined. On the other hand, following the way to the proof of Lemma 3.1 in [3], we can obtain the partial fraction decomposition form formally. For example, letting

$$\mathcal{A}_i = a_i^{r-1} \prod_{j\neq i} (a_i-a_j)^{-1},$$

we have

$$\begin{aligned}\frac{1}{1-a_1x}\frac{1}{1-a_2x}\frac{1}{1-a_3x} &= \frac{\mathcal{A}_1}{1-a_1x}+\frac{\mathcal{A}_2}{1-a_2x}+\frac{\mathcal{A}_3}{1-a_3x}\\ &= \frac{\mathcal{A}_1(1-a_2x)+\mathcal{A}_2(1-a_1x)}{(1-a_1x)(1-a_2x)}+\frac{\mathcal{A}_3}{1-a_3x}.\end{aligned}$$

We note that

$$\mathcal{A}_1(1-a_2x)+\mathcal{A}_2(1-a_1x)=\frac{a_1a_2+a_1a_2a_3x-a_3(a_1+a_2)}{(a_1-a_3)(a_2-a_3)}.$$

Therefore,

$$\frac{1}{1-a_1x}\frac{1}{1-a_2x}\frac{1}{1-a_3x}=\frac{a_1a_2+a_1a_2a_3x-a_3(a_1+a_2)}{(1-a_1x)(1-a_2x)(a_1-a_3)(a_2-a_3)}+\frac{\mathcal{A}_3}{1-a_3x}.$$

Substituting $a_1=a_2$, we have

$$\begin{aligned}\frac{1}{(1-a_1x)^2}\frac{1}{1-a_3x} &= \frac{a_1^2+a_1^2a_3x-2a_3a_1}{(1-a_1x)^2(a_1-a_3)^2}+\frac{\mathcal{A}_3}{1-a_3x}\\ &= \frac{a_1}{(1-a_1x)^2(a_1-a_3)}-\frac{a_1a_3}{(1-a_1x)(a_1-a_3)^2}+\frac{\mathcal{A}_3}{1-a_3x}.\end{aligned}$$

Using the above identity, we have

$$\sum_{\substack{e_1+e_2+e_3=m\\ e_i\geq 0\,(1\leq i\leq 3)}} a_1^{e_1}a_2^{e_2}a_3^{e_3}=(m+1)a_1^m\frac{a_1}{a_1-a_3}+a_1^m\frac{a_1a_3}{(a_1-a_3)^2}+a_3^mA_3. \tag{3.2}$$

Substituting $a_1=n_{11}^{-1}, a_2=n_{12}^{-1}$ and $a_3=n_{21}^{-1}$ into (3.2) and making a simple calculation, we obtain formula (2.2) interpolating $\mathcal{O}$-sum for Schur multiple zeta values of shape $(2,1)$. Indeed, keeping $a_1=a_2$ and $n_{11}=n_{12}$ in mind, we have

$$\begin{aligned}&\sum_{\substack{e_1+e_2+e_3=m\\ e_i\geq 0\,(1\leq i\leq 3)}}\frac{1}{n_{11}^{e_1}}\frac{1}{n_{12}^{e_2}}\frac{1}{n_{21}^{e_3}}\\ &=(m+1)\frac{1}{n_{11}^{m+1}}\left(\frac{1}{n_{11}}-\frac{1}{n_{21}}\right)^{-1}+\frac{1}{n_{11}^{m+1}}\frac{1}{n_{21}}\left(\frac{1}{n_{11}}-\frac{1}{n_{21}}\right)^{-2}+\frac{1}{n_{21}^{m+2}}\left(\frac{1}{n_{21}}-\frac{1}{n_{11}}\right)^{-2}\\ &=(m+1)\frac{1}{n_{11}^{m}}\frac{n_{21}}{n_{21}-n_{11}}+\frac{1}{n_{11}^{m}}\frac{n_{11}n_{21}}{(n_{21}-n_{11})^2}+\frac{1}{n_{21}^{m}}\frac{n_{11}^2}{(n_{11}-n_{21})^2}.\end{aligned} \tag{3.3}$$

This calculation corresponds to the terms of (2.2) with $n_{11}=n_{12}$. Swapping the role of n_{11} and n_{21}, we obtain the identity corresponding to $n_{12}=n_{21}$. Thus, it holds that

$$I_{\boldsymbol{k}}(m)=\left(\sum_{n_{11}<n_{12}<n_{21}}+\sum_{n_{11}<n_{21}<n_{12}}\right)\frac{1}{n_{11}^{k_{11}}n_{12}^{k_{12}}n_{21}^{k_{21}}}\sum_{\substack{e_1+e_2+e_3=m\\ e_i\geq 0\,(1\leq i\leq 3)}}\frac{1}{n_{11}^{e_1}}\frac{1}{n_{12}^{e_2}}\frac{1}{n_{21}^{e_3}}$$

$$+\left(\sum_{n_{11}=n_{12}<n_{21}}+\sum_{n_{11}<n_{12}=n_{21}}\right)\frac{1}{n_{11}^{k_{11}}n_{12}^{k_{12}}n_{21}^{k_{21}}}\sum_{\substack{e_1+e_2+e_3=m\\ e_i\geq 0\,(1\leq i\leq 3)}}\frac{1}{n_{11}^{e_1}}\frac{1}{n_{12}^{e_2}}\frac{1}{n_{21}^{e_3}}$$
$$=\sum_{|\boldsymbol{\varepsilon}|=m}\zeta_\lambda(\boldsymbol{k}+\boldsymbol{\varepsilon}).$$

The first two terms are obtained by the same calculation as in [3]; the last two are obtained via computation (3.3).

More generally, for fixed $\mathcal{A}=(a_i)$, we can define

$$P(\mathcal{A})=\prod_{i=1}^{r}\frac{1}{1-a_ix}=\prod_{\alpha=1}^{R_{\mathcal{A}}}\frac{1}{(1-b_\alpha x)^{r_\alpha}}$$

where b_α is one of a_i's with distinct $b_1,\dots,b_{R_{\mathcal{A}}}$, and r_α is the multiplicity of $(1-b_\alpha x)$. Then, as in the proof of Lemma 2.3, the uniqueness of the Laurent series expansion gives

$$P(\mathcal{A})=\sum_{\alpha=1}^{R_{\mathcal{A}}}\sum_{\ell=1}^{r_\alpha}\frac{1}{(1-b_\alpha x)^{\ell}}\frac{d^{r_\alpha-\ell}}{dx^{r_\alpha-\ell}}\left(\frac{1}{(-b_\alpha)^{r_\alpha-\ell}(r_\alpha-\ell)!}\prod_{\beta\neq\alpha}\frac{1}{(1-b_\beta x)^{r_\beta}}\right)\Bigg|_{x=b_\alpha^{-1}}.$$

By expanding into the geometric series, we have the following Lemma:

Lemma 3.4.

$$\sum_{\substack{e_1+\cdots+e_r=m\\ e_i\geq 0\,(1\leq i\leq r)}}a_1^{e_1}\cdots a_r^{e_r}=\sum_{\alpha=1}^{R_{\mathcal{A}}}\sum_{\ell=1}^{r_\alpha}\binom{m+\ell-1}{\ell-1}b_\alpha^m\mathcal{A}_\alpha^{(\ell)},$$

where

$$\mathcal{A}_\alpha^{(\ell)}=\frac{d^{r_\alpha-\ell}}{dx^{r_\alpha-\ell}}\left(\frac{1}{(-b_\alpha)^{r_\alpha-\ell}(r_\alpha-\ell)!}\prod_{\beta\neq\alpha}\frac{1}{(1-b_\beta x)^{r_\beta}}\right)\Bigg|_{x=b_\alpha^{-1}}.$$

Theorem 3.5. The function $I_{\boldsymbol{k}}(s)$, defined by

$$I_{\boldsymbol{k}}(s)=-\frac{\sin(\pi s)}{\pi}\sum_{(n_{ij})\in SSYT_\delta}\prod_{(i,j)\in D_\delta}\frac{1}{n_{ij}^{k_{ij}-1}}\int_0^\infty w^{-s-1}\prod_{(i,j)\in D_\delta}\frac{1}{w+n_{ij}}dw,$$

or

$$I_{\boldsymbol{k}}(s)=\sum_{N\in SSYT_\delta}\frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}}\sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha}\frac{1}{n_\alpha^{s+\ell}}\prod_{p=1}^{\ell-1}\frac{s+p}{p}\frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}}\left(\frac{1}{(r_\alpha-\ell)!}\prod_{\beta\neq\alpha}\frac{1}{(w+n_\beta)^{r_\beta}}\right)\Bigg|_{w=-n_\alpha},$$

interpolates $\mathcal{O}$-sum for the Schur multiple zeta values of shape δ.

Proof. As above, substituting $b_\alpha = n_\alpha^{-1}$ into Lemma 3.4, we can compute

$$\sum_{\substack{\sum e_{ij}=m \\ e_{ij}\geq 0}} \prod_{(i,j)\in D_\delta} \left(\frac{1}{n_{ij}}\right)^{e_{ij}}$$
$$= \sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha} \binom{m+\ell-1}{\ell-1} \frac{1}{n_\alpha^m} \frac{d^{r_\alpha-\ell}}{dx^{r_\alpha-\ell}} \left((-n_\alpha)^{r_\alpha-\ell} \frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(1-n_\beta^{-1}x)^{r_\beta}} \right)\Bigg|_{x=n_\alpha}$$
$$= N^{\mathbf{1}} \sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha} \binom{m+\ell-1}{\ell-1} \frac{1}{n_\alpha^{m+\ell}} \frac{d^{r_\alpha-\ell}}{dx^{r_\alpha-\ell}} \left((-1)^{r_\alpha-\ell} \frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(n_\beta-x)^{r_\beta}} \right)\Bigg|_{x=n_\alpha},$$

where $N^{\mathbf{1}} = \prod n_{ij} = \prod n_\alpha^{r_\alpha}$. Changing the variable by $w=-x$, we obtain

$$N^{\mathbf{1}} \sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha} \binom{m+\ell-1}{\ell-1} \frac{1}{n_\alpha^{m+\ell}} \frac{d^{r_\alpha-\ell}}{dx^{r_\alpha-\ell}} \left(\frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(n_\beta+w)^{r_\beta}} \right)\Bigg|_{w=-n_\alpha}.$$

Therefore, we have

$$I_{\boldsymbol{k}}(m) = \sum_{N\in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\mathbf{1}}} \sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha} \frac{1}{n_\alpha^{m+\ell}} \prod_{p=1}^{\ell-1} \frac{m+p}{p} \frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}} \left(\frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(w+n_\beta)^{r_\beta}} \right)\Bigg|_{w=-n_\alpha}$$
$$= \sum_{N\in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}-\mathbf{1}}} \sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha} \frac{1}{n_\alpha^{m+\ell}} \binom{m+\ell-1}{\ell-1} \frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}} \left(\frac{1}{(r_\alpha-\ell)!} \prod_{\beta\neq\alpha} \frac{1}{(w+n_\beta)^{r_\beta}} \right)\Bigg|_{w=-n_\alpha}$$
$$= \sum_{N\in SSYT_\delta} \frac{1}{N^{\boldsymbol{k}}} \sum_{\substack{\sum e_{ij}=m \\ e_{ij}\geq 0}} \prod_{(i,j)\in D_\delta} \left(\frac{1}{n_{ij}}\right)^{e_{ij}}.$$

This ensures that $I_{\boldsymbol{k}}(s)$ interpolates $\mathcal{O}$-sum for the Schur multiple zeta values of shape δ. $\square$

Remark 3.6. We may prove Theorem 3.5 more directly, without Lemma 3.4, by using the Hankel contour C with radius r depicted as following Figure 1.

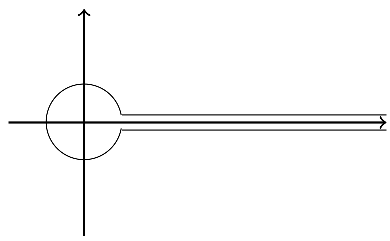

Figure 1. Hankel contour C

In fact, let

$$f(w)=\prod_{(i,j)\in D_\delta}\frac{n_{ij}}{w+n_{ij}}.$$

Then, by the Cauchy residue theorem, we have

$$\left(-\frac{\sin\pi s}{\pi}\int_0^\infty w^{-s-1}f(w)dw\right)\bigg|_{s=m}=(-1)^m\operatorname*{Res}_{w=0}w^{-m-1}f(w).$$

Since

$$f(w)=\prod_{(i,j)\in D_\delta}\frac{n_{ij}}{w+n_{ij}}=\prod_{(i,j)\in D_\delta}\sum_{e_{ij}=0}^{\infty}\left(\frac{-w}{n_{ij}}\right)^{e_{ij}},$$

we have

$$\operatorname*{Res}_{w=0}w^{-m-1}f(w)=(-1)^m\sum_{\substack{\sum e_{ij}=m\\ e_{ij}\geq 0}}\prod_{(i,j)\in D_\delta}\frac{1}{n_{ij}^{e_{ij}}}.$$

Thus, it holds that

$$\begin{aligned}I_{\boldsymbol{k}}(m)&=-\frac{\sin(\pi m)}{\pi}\sum_{(n_{ij})\in SSYT_\delta}\prod_{(i,j)\in D_\delta}\frac{1}{n_{ij}^{k_{ij}-1}}\int_0^\infty w^{-m-1}\prod_{(i,j)\in D_\delta}\frac{1}{w+n_{ij}}dw\\&=\sum_{N\in SSYT_\delta}\frac{1}{N^{\boldsymbol{k}}}\sum_{\substack{\sum e_{ij}=m\\ e_{ij}\geq 0}}\prod_{(i,j)\in D_\delta}\left(\frac{1}{n_{ij}}\right)^{e_{ij}}.\end{aligned}$$

This shows that $I_{\boldsymbol{k}}(s)$ interpolates $\mathcal{O}$-sum for the Schur multiple zeta values of shape δ.

Finally, we obtain the duality formula for $I_{\boldsymbol{k}}(s)$ as complex functions. To show this, we generalize the uniqueness theorem for the Dirichlet series (see [1, Theorem 11.3]).

THEOREM 3.7. Let $p(n,s)$ and $q(n,s)$ be polynomials of s with arithmetic function $a_i(n)$, that is,

$$p(n,s)=\sum_{i=0}^{d}a_i(n)s^i.$$

We assume that the two series

$$F_p(s)=\sum_{n=1}^{\infty}\frac{p(n,s)}{n^s}\text{ and }F_q(s)=\sum_{n=1}^{\infty}\frac{q(n,s)}{n^s}$$

both absolutely converge for $\Re(s)>M$ for some constant M. If $F_p(s)=F_q(s)$ for each $s=s_k$ in an infinite sequence (s_k) such that $\Re(s_k)=\sigma_k\to\infty$ as $k\to\infty$, then $p(n,s)=q(n,s)$ for every n and s.

PROOF. Let $h(n,s)=p(n,s)-q(n,s)$ and let $H(s)=F_p(s)-F_q(s)$. Then $H(s_k)=0$

for each k. To prove that $h(n,s)=0$ for all n and s, we assume that $h(n,s)\neq 0$ for some (n,s) and obtain a contradiction. Let N be the smallest integer for which $h(n,s)$ is not identically 0. Then, we have

$$H(s)=\frac{h(N,s)}{N^s}+\sum_{n=N+1}^{\infty}\frac{h(n,s)}{n^s}.$$

Since $H(s_k)=0$, it holds that

$$|h(N,s_k)|\leq N^{\sigma_k}\sum_{n=N+1}^{\infty}\frac{|h(n,s_k)|}{n^{\sigma_k}}.$$

We take an integer k so that $\sigma_k>c$ where $c>M$, then

$$\begin{aligned}|h(N,s_k)|&\leq N^{\sigma_k}\sum_{n=N+1}^{\infty}\frac{|h(n,s_k)|}{n^{\sigma_k-c}n^c}\\&\leq N^{\sigma_k}\frac{1}{(N+1)^{\sigma_k}}(N+1)^c\sum_{n=N+1}^{\infty}\frac{|h(n,s_k)|}{n^c}.\end{aligned}$$

Since $c>M$, the series converges. Therefore, this implies that

$$|h(N,s_k)|\leq\left(\frac{N}{N+1}\right)^{\sigma_k}J(s_k).$$

We note that $J(s_k)$ is at most of polylnomial order of growth in s_k, hence the right-hand side converges to 0 as $k\to\infty$, so $h(N,s)=0$. This leads to a contradiction and complete the proof. $\square$

Theorem 3.8. Let λ and μ be two partitions such that $\lambda_i\geq\mu_i$ for all i, and let $\delta=\lambda/\mu$. Let $\boldsymbol{k}^\dagger$ be the dual tableau of $\boldsymbol{k}\in I_\delta^D$. Then, for $s\in\mathbb{C}$ we have

$$I_{\boldsymbol{k}}(s)=I_{\boldsymbol{k}^\dagger}(s).$$

Proof. By Theorem 1.6 and Theorem 3.5, we have $I_{\boldsymbol{k}}(s)=I_{\boldsymbol{k}^\dagger}(s)$ for $s\in\mathbb{Z}_{\geq 0}$. As we can express the function $I_{\boldsymbol{k}}(s)$ as

$$\begin{aligned}&\sum_{N\in SSYT_\delta}\frac{1}{N^{\boldsymbol{k}-\boldsymbol{1}}}\left(\sum_{\alpha=1}^{R_N}\sum_{\ell=1}^{r_\alpha}\frac{1}{n_\alpha^{s+\ell}}\binom{s+\ell-1}{\ell-1}\frac{d^{r_\alpha-\ell}}{dw^{r_\alpha-\ell}}\left(\frac{1}{(r_\alpha-\ell)!}\prod_{\beta\neq\alpha}\frac{1}{(w+n_\beta)^{r_\beta}}\right)\Bigg|_{w=-n_\alpha}\right)\\&=\sum_{N\in SSYT_\delta}\frac{1}{N^{\boldsymbol{k}}}\left(\sum_{\alpha=1}^{R_N}\frac{p(n_\alpha,s)}{n_\alpha^s}\right)\\&=\sum_{N\in SSYT_\delta}\frac{1}{N^{\boldsymbol{k}}}\left(\sum_{(i,j)\in D_\delta}\frac{1}{\#\{(k,\ell)\in D_\delta\mid n_{k\ell}=n_{ij}\}}\frac{p(n_\alpha,s)}{n_\alpha^s}\right)\end{aligned}$$

$$= \sum_{(i,j)\in D_\delta} \sum_{n_{ij}=1}^{\infty} \frac{p(n_{ij},s)}{n_{ij}^s} \sum_{\substack{(n_{ab})=N\in SSYT_\delta \\ n_{ab}=n_{ij}}} \frac{1}{N^{\boldsymbol{k}}} \frac{1}{\#\{(k,\ell)\in D_\delta \mid n_{k\ell}=n_{ab}\}},$$

by applying Theorem 3.7, we have $I_{\boldsymbol{k}}(s) = I_{\boldsymbol{k}^\dagger}(s)$ for $\Re(s) > -1$ and $\boldsymbol{k} \in I_\delta^{\mathrm{D}}$. Moreover, $I_{\boldsymbol{k}}(s)$ is meromorphically continued to the whole space of $\mathbb{C}$. Thus, the assertion is proved. $\square$

Remark 3.9. When we put $\lambda = (\{1\}^r)$ and $\mu = \varnothing$ for a positive integer r, then Theorem 3.8 implies Theorem 1.9. Furthermore, substituting non-negative integers for s, we obtain Theorem 1.2.

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Maki Nakasuji
Department of Information and Communication Science, Faculty of Science, Sophia University, 7-1 Kioi-cho, Chiyoda-ku, Tokyo, 102-8554, Japan
Mathematical Institute, Tohoku University, 6-3 Aramaki Aza-Aoba, Aoba-ku, Sendai, 980-8578, Japan
E-mail: nakasuji@sophia.ac.jp

Yasuo Ohno
Mathematical Institute, Tohoku University, 6-3 Aramaki Aza-Aoba,
Aoba-ku, Sendai, 980-8578, Japan
E-mail: ohno.y@m.tohoku.ac.jp

Wataru Takeda
Department of Applied Mathematics, Tokyo University of Science, 1-3
Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan.
E-mail: w.takeda@rs.tus.ac.jp