CONSTRUCTION OF ONE-FIXED-POINT ACTIONS ON SPHERES OF NONSOLVABLE GROUPS II

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Dedicated to Professor Toshio Sumi on the occasion of his 60th birthday

Abstract. Let G be a finite group. If $n \leq 5$ then any n-dimensional homotopy sphere never admits a smooth action of G with exactly one fixed point. Let A_n and S_n denote the alternating group and the symmetric group on some n letters. If $n \geq 6$ then the n-dimensional sphere possesses a smooth action of A_5 with exactly one fixed point. Let V be an n-dimensional real G-representation with exactly one fixed point. It is interesting to ask whether there exists a smooth G-action with exactly one fixed point on the n-dimensional sphere such that the associated tangential G-representation is isomorphic to V. In this paper, we study this problem for nonsolvable groups G and real G-representations V satisfying certain hypotheses. Applying a theory developed in this paper, we can prove that the n-dimensional sphere has an effective smooth action of S_5 with exactly one fixed point if and only if $n = 6$, 10, 11, 12, or $n \geq 14$ and that the n-dimensional sphere has an effective smooth action of $A_5 \times Z$ with exactly one fixed point if n satisfies $n \geq 6$ and $n \neq 9$, where Z is a group of order 2.

1. Introduction

Throughout this paper, manifolds and group actions on manifolds are considered in the smooth category. We denote by S^n the (standard) sphere of dimension n. Let $\mathbb{Z}$ and $\mathbb{N}$ denote the ring of integers and the set of natural numbers. For integers a and b, let $[a..b]$ denote the set $\{n \in \mathbb{Z} \mid a \leq n \leq b\}$ and $[a..\infty)$ the set $\bigcup_{b\in\mathbb{N}} [a..b]$. Let G be a finite group and let $\mathcal{S}(G)$ denote the set of subgroups of G. For a natural number m, we call a G-action on a manifold M an *m-fixed-point action* if the G-fixed-point set M^G of M consists of exactly m points. Let V be a real G-representation (of finite dimension) with the trivial G-fixed-point set, i.e. $V^G = \{0\}$, and let $\mathbb{R}$ be the real G-representation of dimension 1 with the trivial G-action. Let $D(V)$ (resp. $S(V)$) be the unit disk (resp. sphere) of V with respect to a G-invariant inner-product on V. The unit sphere $S(\mathbb{R} \oplus V)$ of $\mathbb{R} \oplus V$ has two G-fixed points. In 1946, D. Montgomery and H. Samelson [16] gave a comment that if a G-action on a sphere has a G-fixed point then it would have a second G-fixed point. Since then, we have been interested in one-fixed-point actions on spheres.

We refer to a closed manifold which is homotopy equivalent to a sphere as a *homotopy sphere*. This raises the question whether there is a one-fixed-point G-action on S^n for a group G possessing a one-fixed-point G-action on an n-dimensional homotopy sphere. Owing to E. Laitinen–P. Traczyk

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[14], M. Furuta [9], [18], S. Demichelis [7], N.P. Buchdahl–S. Kwasik–R. Schultz [6], and S. Kwasik–R. Schultz [11], there are no one-fixed-point actions of finite groups on n-dimensional homotopy spheres with $n \leq 5$. We call a G-action on a disk (resp. sphere) *linear* if it is G-diffeomorphic to $D(V)$ (resp. $S(V)$) for some G-representation V. There exists a one-fixed-point G-action on a homotopy sphere of dimension n if and only if there exists a fixed-point-free G-action on D^n of which the restriction to the boundary ∂D^n is linear. Therefore, the study of one-fixed-point G-actions on spheres is closely related to the study of fixed-point-free G-actions on disks with G-linear boundary.

For a principal ideal domain R, we call a closed manifold M an *R-homology sphere* if the homology groups of M with coefficients in R are isomorphic to those of the sphere of the same dimension. By a *homology sphere*, we mean a $\mathbb{Z}$-homology sphere. If a homology sphere has a one-fixed-point action of G then by R. Oliver [25, 26], G is not a mod-$\mathcal{P}$ hyper-elementary group, i.e. G dose not admit a normal series $P \trianglelefteq H \trianglelefteq G$ such that P and G/H are of prime-power order and H/P is cyclic, cf. [22, Proposition 2.1]. Hereafter we refer to a finite group which is not a mod-$\mathcal{P}$ hyper-elementary group as an *Oliver group*. Clearly, any (finite) nonsolvable group is an Oliver group. For the first time, E. Stein [28] found examples of one-fixed-point actions on spheres, namely he proved that the 7-dimensional sphere admits one-fixed-point actions of the groups $\mathrm{SL}(2,5) \times C_r$ with $(r, 30) = 1$, i.e. r is a natural number prime to 30, where C_r is a cyclic group of order r. T. Petrie [27] also constructed one-fixed-point actions on high-dimensional spheres of finite abelian Oliver groups of odd order (these groups have necessarily at least three noncyclic Sylow subgroups). We showed in E. Laitinen–M. Morimoto [12] with help by [21] that for every Oliver group G, there are one-fixed-point G-actions on high-dimensional spheres. (The case that G is a nonsolvable group such that $|G/G^{\mathrm{sol}}|$ is an odd integer follows from E. Laitinen–M. Morimoto–K. Pawałowski [13, Theorem A], too. Here G^{sol} stands for the smallest normal subgroup N of G such that G/N is solvable.) By [17, 19, 20] and A. Bak–M. Morimoto [2], there exists a one-fixed-point action of A_5 on S^n if and only if $n \geq 6$. On the other hand, A. Borowiecka [4, Theorem 1.1] showed that any 8-dimensional homology sphere does not admit effective one-fixed-point actions of $\mathrm{SL}(2,5)$. A. Borowiecka–P. Mizerka [5] studied some examples of pairs (G, n) of finite groups G with $|G| \leq 216$ and natural numbers $n \leq 10$ such that there are no one-fixed-point G-actions on n-dimensional homotopy spheres. S. Tamura and the author [24] also showed that any n-dimensional homology sphere does not admit one-fixed-point actions of S_5 if $n \in \{7, 8, 9, 13\}$. S. Tamura showed the non-existence of effective one-fixed-point G-actions on S^n for $G = A_6$, $\mathrm{SL}(2,9)$, S_6, $\mathrm{PGL}(2,9)$, M_{10} and $\mathrm{Aut}(A_6)$, and $n \in T_G$, where M_{10} is the Mathieu group of degree 10 and T_G is a certain set of natural numbers depending on G, see [29, Theorems 1.1 and 1.2]. In addition, P. Mizerka [15] and the author [22] showed the non-existence of effective one-fixed-point G-actions on S^n for $G = \mathrm{TL}(2,5)$ and $n \in [0..13] \cup \{15, 16, 17, 21\}$, where

$\mathrm{TL}(2,5)$ is the group SmallGroup$(240,89)$ in GAP [10]. Recently we showed the results that S^6 has effective one-fixed-point actions of A_5, $A_5 \times C_2$ and S_5, that S^7 has effective one-fixed-point actions of A_5 and $A_5 \times C_2$, and that for all natural numbers k and r with $(r, 30) = 1$, the spheres S^{3+4k} and S^{14+8k} have effective one-fixed-point actions of $\mathrm{SL}(2,5) \times C_r$ and $\mathrm{TL}(2,5) \times C_r$, respectively, see [23, Theorem 1.3].

For a G-manifold X and a G-fixed point x_0 of X, the tangent space $T_{x_0}(X)$ of X at x_0 is a real G-representation and we call $T_{x_0}(X)$ the *tangential G-representation* of X at x_0. For an Oliver group G and a real G-representation V of dimension n, it is interesting to ask whether there exists a one-fixed-point G-action on S^n such that the tangential G-representation of S^n and V are isomorphic as real G-representations. In this paper we will give a construction theorem of one-fixed-point actions on spheres for finite nonsolvable groups G and real G-representations V, i.e. Theorem 2.3. Keys to proving the theorem are the reflection method, i.e. Lemma 6.1 with Theorem 5.12, and the equivariant surgery theory under the modified weak gap condition, see Definition 2.4 and [23, Lemma 8.1]. As applications of the theorem, we obtain the following two theorems.

Theorem 1.1. *Let G be the symmetric group S_5. Then there exists an effective one-fixed-point G-action on S^n if and only if $n = 6$, 10, 11, 12, or $n \geq 14$.*

In Theorem 1.1, the necessity follows from the results quoted above, and the sufficiency will be given in Section 3.

Henceforth, the trivial subgroup of G is denoted by E. We call a G-action on a manifold X *m-pseudofree* if $\dim X^H \leq m$ for all $H \in \mathcal{S}(G) \smallsetminus \{E\}$. We call an m-pseudofree G-action on X *properly m-pseudofree* if there is a subgroup $H \in \mathcal{S}(G) \smallsetminus \{E\}$ such that $\dim X^H = m$. We remark that the one-fixed-point actions on S^n for $n = 6$, 10 and 11, obtained in the proof of Theorem 1.1 are properly 3-pseudofree, properly 4-pseudofree and properly 5-pseudofree, respectively.

Theorem 1.2. *Let Z be a group of order 2 and let G be the cartesian product $A_5 \times Z$. Then there exists an effective one-fixed-point G-action on S^n if n satisfies $n \geq 6$ and $n \neq 9$.*

The proof of Theorem 1.2 will be given in Section 4.

We conjecture that there is a one-fixed-point action on S^9 of $G = A_5 \times Z$, where $|Z| = 2$, such that $(S^9)^Z$ is diffeomorphic to S^6. We remark that the one-fixed-point actions on S^n for $n = 6$, 7, 8 and 10, obtained in the proof of Theorem 1.2 are properly 3-pseudofree, properly 3-pseudofree, properly 4-pseudofree and properly 5-pseudofree, respectively.

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2. Construction Theorem of one-fixed-point actions on spheres

For a finite group G, the set $\mathcal{S}(G)$ is an ordered set, i.e. for H, $K \in \mathcal{S}(G)$, we say $H < K$ if H is a proper subgroup of K. For a subset $\mathcal{A}$ of $\mathcal{S}(G)$, let $\max(\mathcal{A})$ (resp. $\min(\mathcal{A})$) denote the set of maximal (resp. minimal) elements of $\mathcal{A}$ with respect to the order on $\mathcal{A}$ inherited from $\mathcal{S}(G)$. For a real G-representation V (resp. a G-manifold X), let $V(\mathcal{A})$ (resp. $X(\mathcal{A})$) denote the union $\bigcup_K V^K$ (resp. $\bigcup_K X^K$) where K ranges over $\mathcal{A}$. We mean by $\dim V(\mathcal{A})$ (resp. $\dim X(\mathcal{A})$) the maximum of $\dim V^K$ (resp. $\dim X^K$), where K ranges over $\mathcal{A}$. Let $\mathcal{S}(G)_{\mathrm{sol}}$ denote the set of solvable subgroups of G and set $\mathcal{S}(G)_{\mathrm{nonsol}} = \mathcal{S}(G) \smallsetminus \mathcal{S}(G)_{\mathrm{sol}}$. For a subset $\mathcal{F}$ of $\mathcal{S}(G)$, let $\mathcal{F}_{\mathrm{sol}}$ denote the set $\mathcal{F} \cap \mathcal{S}(G)_{\mathrm{sol}}$. In the case where G is nonsolvable, by [8, (1.3.2), (1.3.3) and Proposition 1.3.5], there is a unique element β_G of the Burnside ring $\Omega(G)$ of G such that $\chi_L(\beta_G) = 0$ for all $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$ and $\chi_H(\beta_G) = 1$ for all $H \in \mathcal{S}(G)_{\mathrm{sol}}$. Let V be a real G-representation. We say that V is $\mathcal{S}(G)_{\mathrm{nonsol}}$-*free* if $V^L = 0$ for all $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$. For the G-connected-sum operation associated with $[G/G] - \beta_G$ on G-framed maps with the target manifold $D(V)$ or $S(\mathbb{R} \oplus V)$, we need the next definition.

Definition 2.1. Let V be an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation. We say that V is *ample* for β_G if

$$\mathrm{Iso}(G, \beta_G) \smallsetminus \max(\mathcal{S}(G)_{\mathrm{sol}}) \subset \mathrm{Iso}(G, V \smallsetminus \{0\}).$$

Let M, H and K be subgroups of G. We say that H is M-*conjugate* (resp. M-*subconjugate*) to K if there is $g \in M$ such that $H = gKg^{-1}$ (resp. $H \subset gKg^{-1}$). We denote by $(H)_{G,M}$ the M-conjugacy class of H in $\mathcal{S}(G)$, i.e.

$$(H)_{G,M} = \{gHg^{-1} \mid g \in M\}.$$

In the case $G = M$, we set $(H)_G = (H)_{G,M}$. We write $(K)_G \leq (H)_G$ if K is G-subconjugate to H. For H and $M \in \mathcal{S}(G)$, define $\mathcal{U}_G(H)$, $\mathcal{V}_G(H)$, and $\mathcal{V}_{M,G}(H)$ by

$$\mathcal{U}_G(H) = \{K \in \mathcal{S}(G) \mid H < K\},$$

$$\mathcal{V}_G(H) = \{K \in \mathcal{S}(G) \mid K \text{ is not } G\text{-subconjugate to } H\}, \text{ and}$$

$$\mathcal{V}_{M,G}(H) = \mathcal{S}(M) \smallsetminus \bigcup_{K \in (H)_G} \mathcal{S}(K \cap M).$$

The next proposition will be used in Sections 3 and 4.

Proposition 2.1. *Let V and W be $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representations. If V is ample for β_G and $V \subset W$ then W is ample for β_G.*

Proof. Since $V \smallsetminus \{0\} \subset W \smallsetminus \{0\}$, we get

$$\mathrm{Iso}(G, \beta_G) \smallsetminus \max(\mathcal{S}(G)_{\mathrm{sol}}) \subset \mathrm{Iso}(G, V \smallsetminus \{0\}) \subset \mathrm{Iso}(G, W \smallsetminus \{0\}).$$

□

Let $\mathcal{F}$ and $\mathcal{H}$ be sets of subgroups of G such that $\mathcal{F} \subset \mathcal{H}$. We say that $\mathcal{F}$ is *upwardly closed* in $\mathcal{H}$ or that $\mathcal{F}$ is an *upwardly closed subset* of $\mathcal{H}$, if K belongs to $\mathcal{F}$ whenever $H \in \mathcal{F}$, $K \in \mathcal{H}$ and $H \subset K$. In the case where a complete set $\mathcal{F}^*$ of representatives of G-conjugacy classes of subgroups in $\mathcal{F}$ and a subset $\mathcal{K}$ of $\mathcal{F}$ are specified, let $\mathcal{K}^*$ denote the set $\mathcal{K} \cap \mathcal{F}^*$.

Definition 2.2. Let G be a nonsolvable group and let $\mathcal{F}$ and $\mathcal{F}'$ be G-conjugation-invariant, upwardly closed subsets of $\mathcal{S}(G)_{\mathrm{sol}}$ satisfying

(1) $\max(\mathcal{S}(G)_{\mathrm{sol}}) \subset \mathcal{F}' \subset \mathcal{F}$ and $\mathcal{F} \smallsetminus \mathcal{F}' \subset \min(\mathcal{F})$.

We say that $(\mathcal{F}, \mathcal{F}')$ is *G-simply organized* if there are a complete set $\mathcal{F}^*$ of representatives of G-conjugacy classes contained in $\mathcal{F}$, i.e. $\mathcal{F} = \coprod_{H \in \mathcal{F}^*} (H)_G$, and a map $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ satisfying the next conditions (2) and (3).

(2) $N_G(H) \subset \rho_{\max}(H)$ for any $H \in \mathcal{F}^*$.

(3) $(H)_G \cap \mathcal{S}(\rho_{\max}(H)) = (H)_{\rho_{\max}(H)}$ for any $H \in {\mathcal{F}'}^*$.

Let $\overline{\rho}_{\max} : \mathcal{F} \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ denote the G-conjugation-invariant extension of the map $\rho_{\max}$ above, i.e. the equality $\overline{\rho}_{\max}(K) = \rho_{\max}(H)$ holds if K is G-conjugate to a subgroup H in $\mathcal{F}^*$. For $H \in \mathcal{F}^*$, we define the subset $\mathcal{X}(G, \rho_{\max}, H)$ of $\mathcal{U}_M(H)$, where $M = \rho_{\max}(H)$, by

$$\mathcal{X}(G, \rho_{\max}, H) = \{K \in \mathcal{U}_M(H) \mid \overline{\rho}_{\max}(K) \neq M\}. \tag{2.1}$$

We set

$$\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \bigcup_{H \in \mathcal{F}^*} \mathcal{X}(G, \rho_{\max}, H). \tag{2.2}$$

For use of G-surgery theory, we quote the notions of 'weak gap condition' and 'modified weak gap condition' from [23, Section 7]. Let V be a real G-representation and H a subgroup of G.

Definition 2.3. We say that V satisfies the *weak gap condition* at H if

$$2 \dim V^K \leq \dim V^H \tag{2.3}$$

holds for all $K \in \mathcal{U}_G(H)$.

Definition 2.4. We say that V satisfies the *modified weak gap condition* at H if the following conditions (1)–(3) are fulfilled.

(1) V satisfies the weak gap condition at H.

(2) If $\dim V^H > 0$, $K \in \mathcal{U}_G(H)$, and $2 \dim V^K = \dim V^H$, then

(i) $K \subset N_G(H)$,

(ii) K/H contains at most one element of order 2, and

(iii) $\dim V^L + 1 < \dim V^K$ for all $L \in \mathcal{U}_G(K)_{\mathrm{sol}}$.

(3) If $K_1 \in \mathcal{U}_G(H)_{\mathrm{sol}}$, $K_2 \in \mathcal{U}_G(H)_{\mathrm{sol}}$ and $2 \dim V^{K_1} = 2 \dim V^{K_2} = \dim V^H > 0$, then the smallest subgroup $\langle K_1, K_2 \rangle$ of G containing $K_1 \cup K_2$ is solvable.

For a non-negative integer k, we set

$$\begin{aligned} &\mathcal{H}(G, V, k) = \{K \in \mathcal{S}(G)_{\mathrm{sol}} \mid \dim V^K = k\}, \\ &\mathcal{H}(G, V, \leq k) = \{K \in \mathcal{S}(G)_{\mathrm{sol}} \mid \dim V^K \leq k\}, \text{ and} \\ &\mathcal{F}(0) = \max(\mathcal{S}(G)_{\mathrm{sol}}) \cup \mathcal{H}(G, V, 0). \end{aligned} \tag{2.4}$$

Let H and M be solvable subgroups of G such that $H \subset M$. Then define $\mathcal{Y}(G, M, H)$ by

$$\mathcal{Y}(G, M, H) = \{K \in \mathcal{U}_G(H)_{\mathrm{sol}} \mid K \cap M = H\}. \tag{2.5}$$

Let $\mathcal{Z}(G, V, M, H)$ denote the set of pairs (K, L) consisting of $K \in \mathcal{Y}(G, M, H) \smallsetminus \mathcal{H}(G, V, 0)$ and $L \in \mathcal{U}_M(H)$ such that $\dim V^K + \dim V^L + 1 = \dim V^H$, and set

$$\begin{aligned} &\mathcal{Z}(G, V, M, H)_1 = \{K \mid (K, L) \in \mathcal{Z}(G, V, M, H)\}, \text{ and} \\ &\mathcal{Z}(G, V, M, H)_2 = \{L \mid (K, L) \in \mathcal{Z}(G, V, M, H)\}. \end{aligned} \tag{2.6}$$

Definition 2.5. Let V be an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation, and let H and M be solvable subgroups of G such that $H \subset M$. We say that V satisfies the (G, M)-*cobordism gap condition* at H if the following conditions (1)–(3) are fulfilled.

(1) The following (A1) or (A2) holds.

(A1) (i) $2 \dim V^K + 1 < \dim V^H$ for all $K \in \mathcal{Y}(G, M, H) \smallsetminus \mathcal{H}(G, V, 0)$, and

(ii) $\dim V^K + \dim V^L + 1 \leq \dim V^H$ for all $K \in \mathcal{Y}(G, M, H) \smallsetminus \mathcal{H}(G, V, 0)$ and $L \in \mathcal{U}_M(H)$.

(A2) (i) $\dim V^H = 3$,

(ii) $\mathcal{Y}(G, M, H) \subset \mathcal{H}(G, V, \leq 1)$,

(iii) $\mathcal{U}_M(H) \subset \mathcal{H}(G, V, 0)$,

(iv) $N_G(K) \cap M = H$ for all $K \in \mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$, and

(v) $(K)_{G,M} = (K')_{G,M}$ for all K, $K' \in \mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$.

(2) The following (B1) and (B2) both hold for all $K \in \mathcal{X}(G, \rho_{\max}, H) \smallsetminus \mathcal{H}(G, V, 0)$.

(B1) $\dim V^K = 1$, and

(B2) $N_G(K) \cap M = K$.

(3) In the case $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$, the following (C1) or (C2) holds.

(C1) (i) $\dim V^H \geq 5$,

(ii) $\dim V^L + 2 < \dim V^H$ for all $L \in \mathcal{U}_M(H)$,

(iii) $\dim V^L \geq 2$ for all $L \in \mathcal{Z}(G, V, M, H)_2$, and

(iv) $\dim V^{\langle L_1, L_2 \rangle} + 1 < \dim V^{L_1}$ for all L_1, $L_2 \in \mathcal{Z}(G, V, M, H)_2$ with $L_1 \neq L_2$.

(C2) (i) $\dim V^H \geq 4$,

(ii) $\mathcal{Z}(G, V, M, H)_1 \subset \mathcal{H}(G, V, 1)$, and

(iii) $N_G(K) \cap M = H$ for all $K \in \mathcal{Z}(G, V, M, H)_1 \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$.

Proposition 2.2. *Let V, H and M be as in Definition 2.5. Suppose V satisfies the (G, M)-cobordism gap condition at H. If $\dim V^H = 4$ and $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$ then $\mathcal{Y}(G, M, H) \smallsetminus \mathcal{H}(G, V, 0) \subset \mathcal{Z}(G, V, M, H)_1$. Therefore if $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$ and $\mathcal{Y}(G, M, H) \smallsetminus (\mathcal{H}(G, V, 0) \cup \mathcal{Z}(G, V, M, H)_1) \neq \emptyset$ then $\dim V^H \geq 5$.*

Proof. To prove the first claim, we suppose $\dim V^H = 4$. By Definition 2.5 (1) (A1) (i), we have $2 \dim V^K < 3$ for all $K \in \mathcal{Y}(G, M, H)$, which means $\dim V^K \leq 1$ for all $K \in \mathcal{Y}(G, M, H)$. Since $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$, we get $\dim V(\mathcal{Y}(G, M, H)) = 1$. By Definition 2.5 (1) (A1) (ii), we have $1 + \dim V^L + 1 \leq 4$ for all $L \in \mathcal{U}_M(H)$, which means $\dim V^L \leq 2$ for all $L \in \mathcal{U}_M(H)$. Since $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$, we get $\dim V(\mathcal{U}_M(H)) = 2$. For $K' \in \mathcal{Y}(G, M, H) \smallsetminus \mathcal{Z}(G, V, M, H)_1$, it must hold that $\dim V^{K'} + \dim V(\mathcal{U}_M(H)) + 1 < \dim V^H = 4$, which implies $\dim V^{K'} = 0$ and hence $K' \in \mathcal{H}(G, V, 0)$.

The second claim immediately follows from the first claim. □

Now we are ready to state a construction result of one-fixed-point G-actions on spheres for a given nonsolvable group G and a given real G-representation V.

Theorem 2.3 (cf. [23, Theorem 11.2])**.** *Let G be a nonsolvable group and V an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation of dimension $n > 5$ which is ample for β_G. Let $(\mathcal{F}, \mathcal{F}')$ be a G-simply organized pair with $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$, where $\mathcal{F}' \subset \mathcal{F}$ are upwardly closed G-conjugation-invariant subsets of $\mathcal{S}(G)_{\mathrm{sol}}$. Suppose V satisfies the following conditions* (D1)–(D4)*.*

(D1) *For $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$, if an element $\overline{H} \in \mathcal{U}_G(H)_{\mathrm{sol}} \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ satisfies $V^H = V^{\overline{H}}$ then $\mathcal{F} \cap \mathcal{U}_G(H) \subset \mathcal{S}(\rho_{max}(H))$ and $\overline{\rho}_{\max}(\overline{H}) = \rho_{\max}(H)$.*

(D2) *The $(G, \rho_{\max}(H))$-cobordism gap condition at H for all $H \in (\mathcal{F}^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus \mathcal{F}(0)$.*

(D3) $\dim V^H = 3$ *or* $\dim V^H \geq 5$ *for all $H \in \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F}$.*

(D4) *The modified weak gap condition at H for all $H \in \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F}$.*

Then there exists a one-fixed-point G-action on the standard sphere S of the same dimension as V, say $S^G = \{x_0\}$, possessing the following properties (1)–(4).

(1) $T_{x_0}(S) \cong V$ as real G-representations.

(2) $S^L = \{x_0\}$ *for all* $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$.

(3) S^H *is* $N_G(H)$*-diffeomorphic to a standard sphere for each* $H \in \mathcal{F}$.

(4) S^H *is a homotopy (resp. homology) sphere for each* $H \in \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F}$ *with* $\dim V^H \geq 5$ *(resp.* $\dim V^H = 3$*).*

By the same argument as the proof of [23, Theorem 11.2], Theorem 2.3 follows from Theorem 2.4 below. In this paper, let I denote the closed interval $[0, 1]$. We call a homotopy $\Xi : (X, \partial X) \times I \to (Y, \partial Y)$ a *homotopy rel.* ∂ if $\Xi(x, t) = \Xi(x, 0)$ for all $x \in \partial X$ and $t \in I$.

Theorem 2.4 (cf. [23, Theorem 11.1])**.** *Let* G*,* V*,* $(\mathcal{F}, \mathcal{F}')$ *and* $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ *be those in Theorem 2.3. Then there exist a* G*-action on the disk* D *of the same dimension as* V *with* $D^G = \emptyset$ *and a* G*-map* $\eta : (D, \partial D) \to (D(V), \partial D(V))$ *possessing the following properties (1)–(4).*

(1) $\eta|_{\partial D} : \partial D \to \partial D(V)$ *is the identity map.*

(2) $D^L = \emptyset$ *for all* $L \in \mathcal{S}(G)_{\mathrm{nonsol}}$.

(3) $\eta^H : D^H \to D(V)^H$ *is* $N_G(H)$*-homotopic rel.* ∂ *to a diffeomorphism for each* $H \in \mathcal{F}$.

(4) $\eta^H : D^H \to D(V)^H$ *is a homotopy equivalence (resp. homology equivalence) rel.* ∂ *for each* $H \in \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F}$ *with* $\dim V^H \geq 5$ *(resp.* $\dim V^H = 3$*).*

This theorem will be proved in Section 6. The next proposition will be used in Sections 3 and 4.

Proposition 2.5. *Let* G*,* $\mathcal{F}$*,* $\mathcal{F}^*$ *and* $\rho_{\max}$ *be those in Theorem 2.3. Let* V *be a real* G*-representation having the property:*

(D1′) $K \subset \rho_{\max}(H)$ *and* $\overline{\rho}_{\max}(K) = \rho_{\max}(H)$ *for all* $H \in \mathcal{F}^*$ *and* $K \in \mathcal{U}_G(H)_{\mathrm{sol}}$ *such that* $V^H = V^K$.

Then an arbitrary real G*-representation* W *containing* V *inherits the property (D1′) from* V*.*

Proof. Let $H \in \mathcal{F}^*$ and $K \in \mathcal{U}_G(H)_{\mathrm{sol}}$ and suppose $W^H = W^K$. Let $W = V \oplus U$ be a direct-sum decomposition of W into two real G-representations V and U. It is clear that $W^H = V^H \oplus U^H$, $W^K = V^K \oplus U^K$, $V^H \supset V^K$ and $U^H \supset U^K$. Therefore we get $V^H = V^K$, which concludes $K \subset \rho_{\max}(H)$ and $\overline{\rho}_{\max}(K) = \rho_{\max}(H)$. □

3. Proof of Theorem 1.1

Let S_5 (resp. A_5) denote the symmetric group (resp. the alternating group) on the five letters 1, 2, …, 5. Throughout the current section, we set $G = S_5$. We fix subgroups of S_5 as follows.

S_4 (resp. A_4) the symmetric group (resp. the alternating group) on the letters 2, 3, 4, 5.

S_3 the symmetric group on the letters 1, 2, 3.

$\mathfrak{C}_2 = \langle(4,5)\rangle$, $\mathfrak{C}_4 = \langle(2,4,3,5)\rangle$, and $\mathfrak{C}_6 = \langle(1,2,3)(4,5)\rangle$ (cyclic groups).

$\mathfrak{S}_3\mathfrak{C}_2 = \langle(1,2),\ (1,2,3),\ (4,5)\rangle$ $(\cong S_3 \times \mathfrak{C}_2)$.

$C_2 = \langle(2,3)(4,5)\rangle$, $C_3 = \langle(1,2,3)\rangle$, and $C_5 = \langle(1,2,3,4,5)\rangle$ (cyclic groups).

$D_4 = \langle(2,3)(4,5),\ (2,4)(3,5)\rangle$, $D_6 = \langle(1,2,3),\ (2,3)(4,5)\rangle$, and

$D_{10} = \langle(1,2,3,4,5),\ (2,5)(3,4)\rangle$ (dihedral groups).

$\mathfrak{D}_4 = \langle(2,3),\ (2,3)(4,5)\rangle$, and $\mathfrak{D}_8 = \langle(2,4,3,5),\ (2,3)\rangle$ (dihedral groups).

$\mathfrak{F}_{20} = \langle(1,2,3,4,5),\ (2,3,5,4)\rangle \qquad ((2,3,5,4)^2 = (2,5)(3,4)$ and $\mathrm{ord}(\mathfrak{F}_{20}) = 20)$.

We tabulate the normalizers of subgroups of S_5 in Table 3.1.

H	A_5	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
$N_G(H)$	G	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4	$\mathfrak{F}_{20}$	$\mathfrak{D}_8$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E
$N_G(H)$	$\mathfrak{F}_{20}$	$\mathfrak{D}_8$	S_4	$\mathfrak{D}_8$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{D}_8$	G

TABLE 3.1

The Hasse diagram of subgroups of S_5 (up to conjugations) is as in Diagram 3.1.

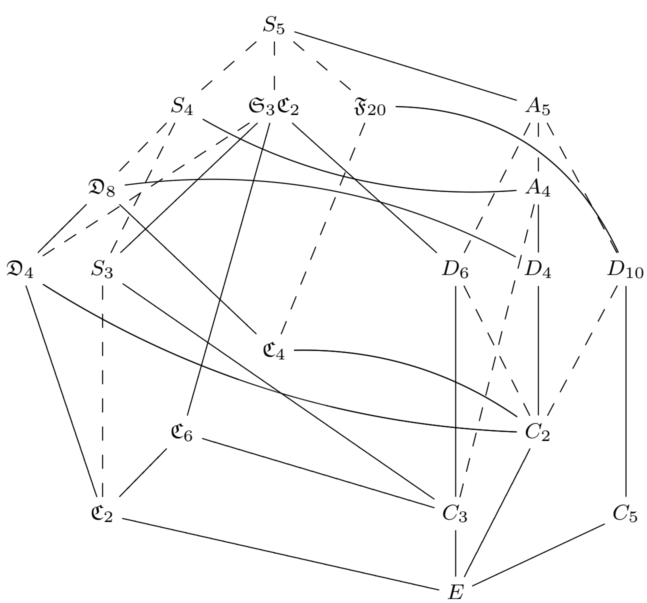

Diagram 3.1

Here a real (resp. dotted) line from a lower subgroup H to an upper subgroup K indicates $gHg^{-1} \triangleleft K$ (resp. $gHg^{-1} < K$) for some $g \in G$. We assign $\rho_{\max}(H)$ to H as in Table 3.2.

Let $\mathcal{F}_{\max}$ be $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \{E\}$, let $\mathcal{F}'_{\max}$ be $\mathcal{F}_{\max} \smallsetminus (\mathfrak{C}_2)_G$, let $\mathcal{F}_{\max}{}^*$ be the set of subgroups listed as H in Table 3.2, and let $\rho_{\max} : \mathcal{F}_{\max}{}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ be the map given by Table 3.2. In the

H	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
$\rho_{\max}(H)$	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4	$\mathfrak{F}_{20}$	S_4	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2
$\rho_{\max}(H)$	$\mathfrak{F}_{20}$	S_4	S_4	S_4	$\mathfrak{S}_3\mathfrak{C}_2$	$\mathfrak{S}_3\mathfrak{C}_2$	S_4

Table 3.2

remainder of this section, restrictions of $\rho_{\max}$ to subsets of ${\mathcal{F}_{\max}}^*$ will be denoted by $\rho_{\max}$, too. We give Diagram 3.2 below to grasp inductive steps of S_5-surgeries on S_5-framed maps.

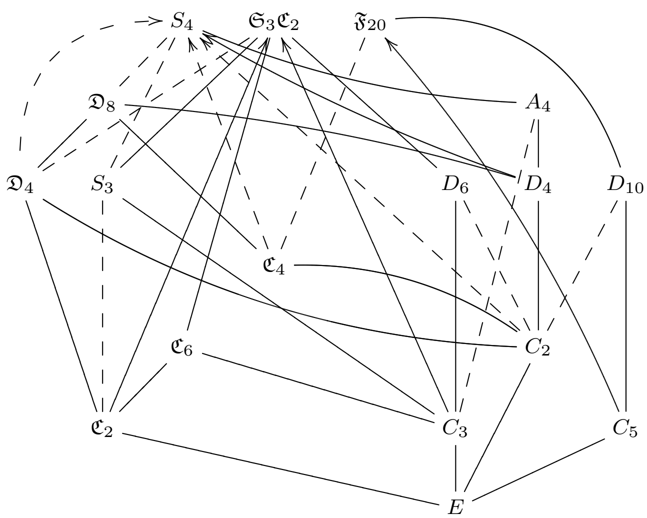

Diagram 3.2

In the diagram above, an arrow from a lower subgroup H to an upper subgroup K indicates $\rho_{\max}(H) = K$ and $K \triangleleft \rho_{\max}(H)$, and a dotted arrow from a lower subgroup H to an upper subgroup K indicates $\rho_{\max}(H) = K$ and $K \not\triangleleft \rho_{\max}(H)$. We can check straightforwardly the next proposition.

Proposition 3.1. *Let $\mathcal{F} = \mathcal{F}_{\max}$, $\mathcal{F}^* = {\mathcal{F}_{\max}}^*$ and $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ be those given above. Let $H \in \mathcal{F}^*$ and $M = \rho_{\max}(H)$. Then $(H)_G \cap \mathcal{S}(M) = (H)_M$ (resp. $(H)_G \cap \mathcal{S}(M) \neq (H)_M$) if $H \neq \mathfrak{C}_2$ (resp. $H = \mathfrak{C}_2$).*

Therefore we have the next fact.

Proposition 3.2. *The pair $(\mathcal{F}_{\max}, \mathcal{F}'_{\max})$ is G-simply organized with respect to $\rho_{\max} : {\mathcal{F}_{\max}}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ given above.*

We can check straightforwardly the next proposition.

Proposition 3.3. *Let $\mathcal{F} = \mathcal{F}_{\max}$, $\mathcal{F}^* = {\mathcal{F}_{\max}}^*$ and $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ be those in Proposition 3.1. Then the following holds.*

(1) *In the case* $H = A_4$ *and* $M = S_4$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(2) *In the case* $H = D_{10}$ *and* $M = \mathfrak{F}_{20}$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(3) *In the case* $H = \mathfrak{D}_8$ *and* $M = S_4$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(4) *In the case* $H = D_6$ *and* $M = \mathfrak{S}_3\mathfrak{C}_2$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(5) *In the case* $H = \mathfrak{C}_6$ *and* $M = \mathfrak{S}_3\mathfrak{C}_2$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(6) *In the case* $H = S_3$ *and* $M = \mathfrak{S}_3\mathfrak{C}_2$,

  (i) $\mathcal{X}(G, \rho_{\max}, H) = \emptyset$,

  (ii) $\mathcal{Y}(G, M, H) = \{S_4, {S_4}'\}$, *where* ${S_4}'$ *is* M*-conjugate to* S_4, *and*

  (iii) $\langle S_4, {S_4}' \rangle = G$.

(7) *In the case* $H = C_5$ *and* $M = \mathfrak{F}_{20}$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(8) *In the case* $H = D_4$ *and* $M = S_4$, $\mathcal{X}(G, \rho_{\max}, H) = \mathcal{Y}(G, M, H) = \emptyset$.

(9) *In the case* $H = \mathfrak{D}_4$ *and* $M = S_4$,

  (i) $\mathcal{X}(G, \rho_{\max}, H) = \emptyset$,

  (ii) $\mathcal{Y}(G, M, H) = \{\mathfrak{S}_3\mathfrak{C}_2, \mathfrak{S}_3{\mathfrak{C}_2}'\}$, *where* $\mathfrak{S}_3{\mathfrak{C}_2}'$ *is* M*-conjugate to* $\mathfrak{S}_3\mathfrak{C}_2$, *and*

  (iii) $\langle \mathfrak{S}_3\mathfrak{C}_2, \mathfrak{S}_3{\mathfrak{C}_2}' \rangle = G$.

(10) *In the case* $H = \mathfrak{C}_4$ *and* $M = S_4$,

  (i) $\mathcal{X}(G, \rho_{\max}, H) = \emptyset$, *and*

  (ii) $\mathcal{Y}(G, M, H) = \{{\mathfrak{F}_{20}}', {\mathfrak{F}_{20}}''\}$, *where* ${\mathfrak{F}_{20}}'$, ${\mathfrak{F}_{20}}''$ *are mutually* M*-conjugate subgroups being* G*-conjugate to* $\mathfrak{F}_{20}$, *and*

  (iii) $\langle {\mathfrak{F}_{20}}', {\mathfrak{F}_{20}}'' \rangle = G$.

(11) *In the case* $H = C_3$ *and* $M = \mathfrak{S}_3\mathfrak{C}_2$,

  (i) $\mathcal{X}(G, \rho_{\max}, H) = \emptyset$,

  (ii) $\mathcal{Y}(G, M, H) = \{{A_4}', {A_4}''\}$, *where* ${A_4}'$, ${A_4}''$ *are mutually* M*-conjugate subgroups being* G*-conjugate to* A_4, *and*

  (iii) $\langle {A_4}', {A_4}'' \rangle = A_5$.

(12) *In the case* $H = C_2$ *and* $M = S_4$,

  (i) $\mathcal{X}(G, \rho_{\max}, H) = \emptyset$,

  (ii) $\mathcal{Y}(G, M, H) = \{{D_6}', {D_6}'', {D_{10}}', {D_{10}}''\}$, *where* ${D_6}'$, ${D_6}''$ *(resp.* ${D_{10}}'$, ${D_{10}}''$*) are mutually* M*-conjugate subgroups being* G*-conjugate to* D_6 *(resp.* D_{10}*), and*

  (iii) $\langle K_1, K_2 \rangle = A_5$ *for* K_1, $K_2 \in \mathcal{Y}(G, M, H)$ *with* $K_1 \neq K_2$.

(13) *In the case* $H = \mathfrak{C}_2$ *and* $M = S_3\mathfrak{C}_2$,

  (i) $\mathcal{X}(G, \rho_{\max}, H) = \{\mathfrak{D}_4, {\mathfrak{D}_4}', {\mathfrak{D}_4}''\}$, *where* ${\mathfrak{D}_4}'$ ${\mathfrak{D}_4}''$ *are* M*-conjugate to* $\mathfrak{D}_4$ *(*$(\mathfrak{D}_4)_G \cap \mathcal{S}(M) = (\mathfrak{D}_4)_M$*)*,

  (ii) $N_G(K) \cap M = H$ *for all* $K \in \mathcal{X}(G, \rho_{\max}, H)$.

(iii) $\mathcal{Y}(G, M, H) = \{S_3{}', S_3{}'', S_3{}'''\}$, *where* $S_3{}'$, $S_3{}''$, $S_3{}'''$ *are mutually* M*-conjugate subgroups being* G*-conjugate to* S_3, *and*

(iv) $\langle K_1, K_2 \rangle \in (S_4)_G$ *for all* K_1, $K_2 \in \mathcal{Y}(G, M, H)$ *with* $K_1 \neq K_2$.

The proposition above indicates that for the ambient group $G = S_5$, there may arise difficulties in G-surgeries of isotropy types $(H)_G$ for $H = \mathfrak{D}_4$, $\mathfrak{C}_4$, C_3, C_2, and $\mathfrak{C}_2$.

Lemma 3.4 ([23, Proposition 3.2])**.** *The idempotent* β_G *in the Burnside ring* $\Omega(G)$ *is given by the formula*

$$\beta_G = [S_5/S_4] + [S_5/\mathfrak{F}_{20}] + [S_5/(\mathfrak{S}_3\mathfrak{C}_2)] - [S_5/S_3] - [S_5/\mathfrak{D}_4] - [S_5/\mathfrak{C}_4] + [S_5/\mathfrak{C}_2]. \tag{3.1}$$

Therefore $\mathrm{Iso}(G, \beta_G)$ *is the union of* $(S_4)_G$, $(\mathfrak{F}_{20})_G$, $(\mathfrak{S}_3\mathfrak{C}_2)_G$, $(S_3)_G$, $(\mathfrak{D}_4)_G$, $(\mathfrak{C}_4)_G$, *and* $(\mathfrak{C}_2)_G$.

There are 7 irreducible real S_5-representations $\mathbb{R}$, V_1, V_4, W_4, V_5, W_5, and V_6, up to isomorphisms, with characters in Table 3.3.

	e	$(4,5)$	$(1,2)(4,5)$	$(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4,5)$	$(1,2,3)(4,5)$
$\mathbb{R}$	1	1	1	1	1	1	1
V_1	1	-1	1	1	-1	1	-1
V_4	4	-2	0	1	0	-1	1
W_4	4	2	0	1	0	-1	-1
V_5	5	-1	1	-1	1	0	-1
W_5	5	1	1	-1	-1	0	1
V_6	6	0	-2	0	0	1	0

TABLE 3.3

Using this character table, we can compute the fixed-point-set dimensions $\dim V^H$ of the irreducible real G-representations V for subgroups H of G. The result is tabulated in Table 3.4.

	S_5	A_5	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3
$\mathbb{R}$	1	1	1	1	1	1	1	1	1
V_1	0	1	0	0	0	1	1	0	0
V_4	0	0	0	0	0	1	0	0	0
W_4	0	0	1	0	1	1	0	1	2
V_5	0	0	0	1	0	0	1	1	0
W_5	0	0	0	0	1	0	1	1	1
V_6	0	0	0	0	0	0	0	0	1

	D_6	$\mathfrak{C}_6$	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E
$\mathbb{R}$	1	1	1	1	1	1	1	1	1	1
V_1	1	0	1	0	1	0	1	0	1	1
V_4	1	1	0	0	1	1	2	1	2	4
W_4	1	1	0	2	1	1	2	3	2	4
V_5	1	0	1	1	2	2	1	2	3	5
W_5	1	1	1	2	2	1	1	3	3	5
V_6	0	1	2	1	0	1	2	3	2	6

TABLE 3.4

We draw the diagram of the fixed-point-set dimensions of $V = V_6$.

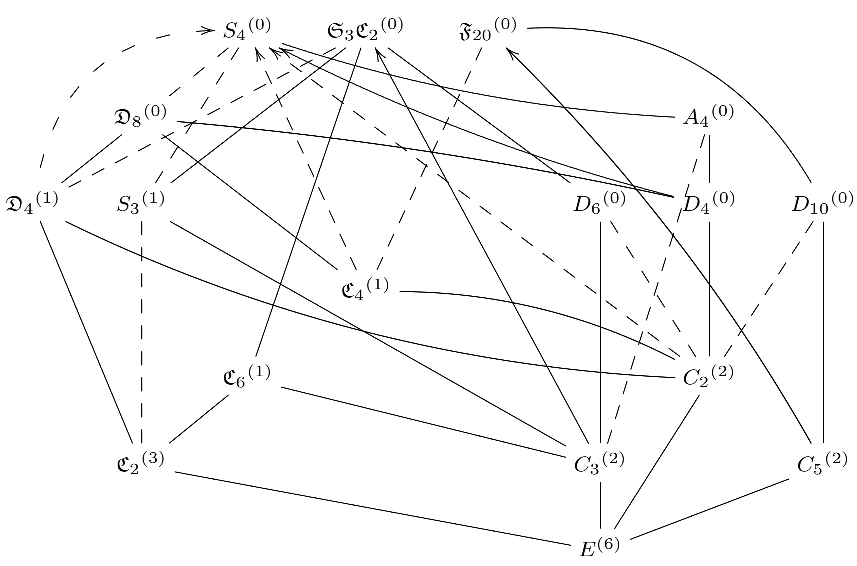

Diagram 3.3

In the diagram above, $H^{(k)}$ indicates $\dim V^H = k$.

Proposition 3.5. *Let V be an $\mathcal{S}(G)_{\text{nonsol}}$-free real G-representation. If V contains a G-subrepresentation isomorphic to V_6 then V is ample for β_G.*

Proof. We obtain the equality

$$\text{Iso}(G, V_6 \smallsetminus \{0\}) = (S_3)_G \cup (\mathfrak{C}_6)_G \cup (C_5)_G \cup (\mathfrak{D}_4)_G \cup (\mathfrak{C}_4)_G \cup (C_3)_G \cup (\mathfrak{C}_2)_G \cup (C_2)_G \cup \{E\} \tag{3.2}$$

from Diagram 3.3. This and Lemma 3.4 imply that V_6 is ample for β_G. By Proposition 2.1, V is ample for β_G. □

Proposition 3.6. *Let $\mathcal{F} = \mathcal{F}_{\max}$, $\mathcal{F}^* = {\mathcal{F}_{\max}}^*$ and $\rho_{\max}$ be those given in Proposition 3.1. If an $\mathcal{S}(G)_{\text{nonsol}}$-free real G-representation V contains a subrepresentation isomorphic to V_6 then V has the property (D1′).*

Proof. By Proposition 2.5, it suffices to prove that V_6 has the property (D1′). Let $H \in \mathcal{F}^*$ and $K \in \mathcal{U}_G(H)_{\text{sol}}$ such that $V^H = V^K$. Observing Diagram 3.3, we can see that $\dim V_6^H = \dim V_6^K = 0$, $K \subset \rho_{\max}(H)$ and $\overline{\rho}_{\max}(K) = \rho_{\max}(H)$. □

We can readily obtain Table 3.5 from Table 3.4.

H	A_5	S_4	$\mathfrak{F}_{20}$	$\mathfrak{S}_3\mathfrak{C}_2$	A_4	D_{10}	$\mathfrak{D}_8$	S_3	D_6	$\mathfrak{C}_6$
${V_6}^{\oplus k}$	0	0	0	0	0	0	0	k	0	k
${V_6}^{\oplus k} \oplus V_4$	0	0	0	0	1	0	0	k	1	$k+1$
${V_6}^{\oplus k} \oplus V_5$	0	0	1	0	0	1	1	k	1	k
${V_6}^{\oplus k} \oplus W_5$	0	0	0	1	0	1	1	$k+1$	1	$k+1$
${V_6}^{\oplus k} \oplus {V_4}^{\oplus 2}$	0	0	0	0	2	0	0	k	2	$k+2$
${V_6}^{\oplus k} \oplus V_4 \oplus V_5$	0	0	1	0	1	1	1	k	2	$k+1$
${V_6}^{\oplus k} \oplus {V_4}^{\oplus 2} \oplus V_5$	0	0	1	0	2	1	1	k	3	$k+2$

H	C_5	$\mathfrak{D}_4$	D_4	$\mathfrak{C}_4$	C_3	$\mathfrak{C}_2$	C_2	E
${V_6}^{\oplus k}$	$2k$	k	0	k	$2k$	$3k$	$2k$	$6k$
${V_6}^{\oplus k} \oplus V_4$	$2k$	k	1	$k+1$	$2k+2$	$3k+1$	$2k+2$	$6k+4$
${V_6}^{\oplus k} \oplus V_5$	$2k+1$	$k+1$	2	$k+2$	$2k+1$	$3k+2$	$2k+3$	$6k+5$
${V_6}^{\oplus k} \oplus W_5$	$2k+1$	$k+2$	2	$k+1$	$2k+1$	$3k+3$	$2k+3$	$6k+5$
${V_6}^{\oplus k} \oplus {V_4}^{\oplus 2}$	$2k$	k	2	$k+2$	$2k+4$	$3k+2$	$2k+4$	$6k+8$
${V_6}^{\oplus k} \oplus V_4 \oplus V_5$	$2k+1$	$k+1$	3	$k+3$	$2k+3$	$3k+3$	$2k+5$	$6k+9$
${V_6}^{\oplus k} \oplus {V_4}^{\oplus 2} \oplus V_5$	$2k+1$	$k+1$	4	$k+4$	$2k+5$	$3k+4$	$2k+7$	$6k+13$

TABLE 3.5

For each $n \in \{6, 10, 11, 12\} \cup [14..\infty)$, let $V(n)$ be the real G-representations of dimension n defined by

$$V(n) = \begin{cases} {V_6}^{\oplus k} & \text{for } n = 6k \text{ with } k \geq 1 \\ {V_6}^{\oplus k} \oplus V_4 & \text{for } n = 6k+4 \text{ with } k \geq 1 \\ {V_6}^{\oplus k} \oplus V_5 & \text{for } n = 6k+5 \text{ with } k \geq 1 \\ {V_6}^{\oplus k} \oplus {V_4}^{\oplus 2} & \text{for } n = 6k+8 \text{ with } k \geq 1 \\ {V_6}^{\oplus k} \oplus V_4 \oplus V_5 & \text{for } n = 6k+9 \text{ with } k \geq 1 \\ {V_6}^{\oplus k} \oplus {V_4}^{\oplus 2} \oplus V_5 & \text{for } n = 6k+13 \text{ with } k \geq 1. \end{cases} \tag{3.3}$$

In the rest of this section, we give $\mathcal{F}$ as follows.

$$
\mathcal{F} = \begin{cases}
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G) & (n = 6k \text{ with } k \geq 1) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \{E\} & (n = 10) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G) & (n = 6k+4 \text{ with } k \geq 2) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G) & (n = 6k+5 \text{ with } k \geq 1) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (C_2)_G \cup (C_3)_G) & (n = 14) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G \cup (C_2)_G \cup (C_3)_G) & (n = 6k+8 \text{ with } k \geq 2) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G) & (n = 6k+9 \text{ with } k \geq 1) \\
\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G) & (n = 6k+13 \text{ with } k \geq 1).
\end{cases} \tag{3.4}
$$

and set $\mathcal{F}' = \mathcal{F} \smallsetminus (\mathfrak{C}_2)_G$. Further let $\mathcal{F}^*$ be the set of subgroups H in Table 3.2 satisfying $H \in \mathcal{F}$, and let $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ be the map given by Table 3.2. Note that

$$
(S_4)_G \cup (\mathfrak{S}_3\mathfrak{C}_2)_G \subset \mathcal{H}(G, V(n), 0). \tag{3.5}
$$

Proposition 3.3 implies

$$
\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) \smallsetminus \mathcal{H}(G, V(n), 0) \subset (\mathfrak{D}_4)_G \tag{3.6}
$$

and

$$
\bigcup_{H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)} \mathcal{Y}(G, \rho_{\max}(H), H) \smallsetminus \mathcal{H}(G, V(n), 0) \subset (\mathfrak{F}_{20})_G \cup (A_4)_G \cup (S_3)_G \cup (D_6)_G \cup (D_{10})_G. \tag{3.7}
$$

It is helpful in the following arguments to keep (3.6) and (3.7) in mind.

Case $n = 6k$ ($k \geq 1$). The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 3.4.

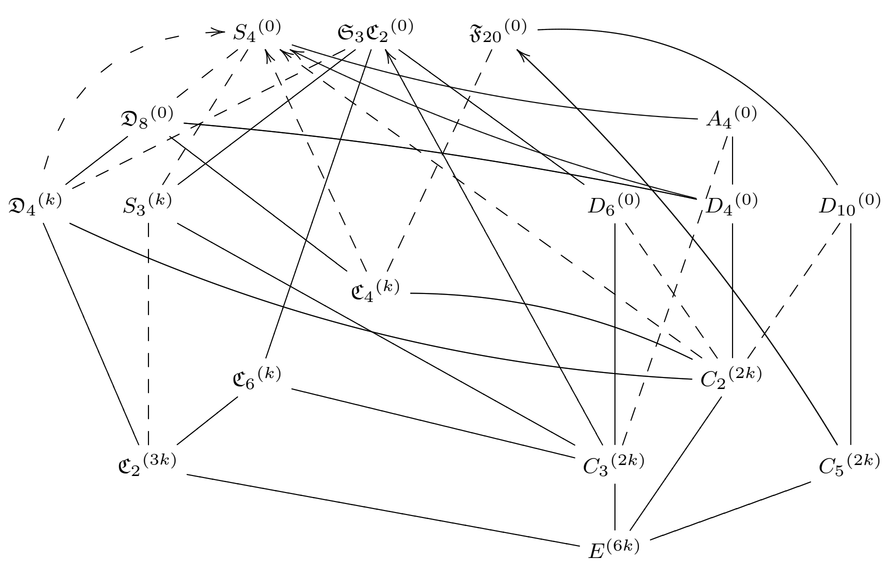

Diagram 3.4

Clearly, we have

$$\mathrm{Iso}(G, V \smallsetminus \{0\}) = \mathrm{Iso}(G, V_6 \smallsetminus \{0\}). \tag{3.8}$$

Recall that in this case, $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{F}' = \mathcal{F}$. There are no pairs (H, K) such that $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$, $K \in \mathcal{U}_G(H)_{\mathrm{sol}} \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ and $\dim V^H = \dim V^K$. The condition (D1) of Theorem 2.3 is obviously fulfilled. We have $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \emptyset$, $\mathcal{Y}(G, \rho_{\max}(H), H) \smallsetminus \mathcal{H}(G, V, 0) = \emptyset$ and $\mathcal{Z}(G, V, \rho_{\max}(H), H) = \emptyset$ for all $H \in (\mathcal{F}^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus \mathcal{F}(0)$. Therefore (D2) in Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G$. Observing Diagram 3.4, we can easily see that (D3) and (D4) in Theorem 2.3 are fulfilled.

The fixed-point-set dimensions of $V = V(6k + 4)$ are as in Diagram 3.5.

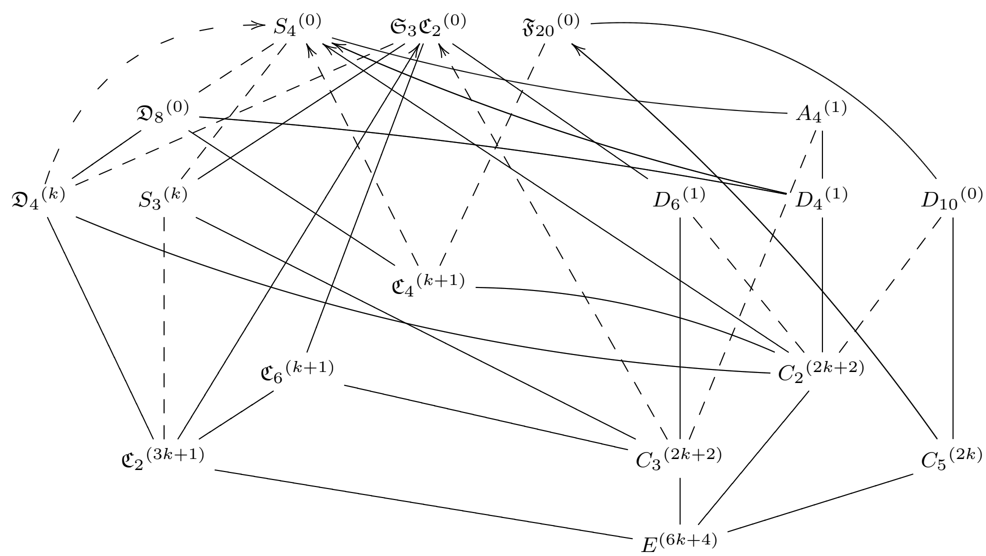

Diagram 3.5

Observing the diagram above, we get

$$\mathrm{Iso}(G, V \smallsetminus \{0\}) = \mathrm{Iso}(G, V_6 \smallsetminus \{0\}) \cup (A_4)_G \cup (D_6)_G. \tag{3.9}$$

We remark that $\mathcal{U}_G(D_4) \cap \mathcal{S}(G)_{\mathrm{sol}} \subset S_4$.

Case $n = 10$. In this case, $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \{E\}$ and $\mathcal{F}' = \mathcal{F} \smallsetminus (\mathfrak{C}_2)_G$. Diagram 3.5 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H)_{\mathrm{sol}} \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$, then $H = D_4$ and $K = A_4$. Therefore (D1) of Theorem 2.3 is fulfilled. It holds that $\dim V^{\mathfrak{D}_4} = 1$ and $N_G(K) \cap \mathfrak{S}_3\mathfrak{C}_2 = K$ for all $K \in (\mathfrak{D}_4)_G \cap \mathcal{U}_{\mathfrak{S}_3\mathfrak{C}_2}(\mathfrak{C}_2)$. By Diagram 3.5, we get $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \le 1)$ for all $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$. If $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$ and $K \in \mathcal{Y}(G, \rho_{\max}(H), H) \cap \mathcal{H}(G, V, 1)$ then $(H, K) \in \{C_3\} \times (A_4)_G$, $\{C_2\} \times (D_6)_G$ or $\{\mathfrak{C}_2\} \times (S_3)_G$. Note that $\dim V^H = 4$ and $\dim V(\mathcal{U}_{\rho_{\max}(H)}(H)) = 2$ for $H = C_3$, C_2 and $\mathfrak{C}_2$. By Proposition 3.3, the conditions (A1) and (C2) in Definition 2.5 are fulfilled at $H = C_3$, C_2 and $\mathfrak{C}_2$ with $M = \rho_{\max}(H)$. Recall $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) \subset (\mathfrak{D}_4)_G$. The conditions (B1), (B2) in Definition 2.5 (2) are fulfilled for $H = \mathfrak{C}_2$ and $K \in \mathcal{X}(G, \rho_{\max}, \mathfrak{C}_2) \smallsetminus \mathcal{H}(G, V, 0)$. Now it is easy to see that V satisfies the $(G, \rho_{\max}(H))$-cobordism gap condition at H for all $H \in (\mathcal{F}^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus \mathcal{F}(0)$, i.e. (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E\}$. It is also clear that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 4$ ($k \geq 2$). In this case, $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{F}' = \mathcal{F}$. Diagram 3.5 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H)_{\mathrm{sol}} \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$, then $H = D_4$ and $K = A_4$. Therefore (D1) of Theorem 2.3 is fulfilled. Since $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \emptyset$, there is no need to check Definition 2.5 (1). Diagram 3.5 shows $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 1)$ for all $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$. If $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$ and $K \in \mathcal{Y}(G, \rho_{\max}(H), H) \cap \mathcal{H}(G, V, 1)$ then $(H, K) \in \{C_3\} \times (A_4)_G$ or $(H, K) \in \{C_2\} \times (D_6)_G$. Note that $\dim V^H = 2k + 2$ and $\dim V(\mathcal{U}_{\rho_{\max}(H)}(H)) = k + 1$ for $H = C_3$, C_2. We have

$$2 \dim V^K + 1 = 3 < 6 \leq \dim V^H \text{ and}$$
$$\dim V^K + \dim V(\mathcal{U}_{\rho_{\max}(H)}(H)) + 1 = 1 + (k + 1) + 1 = k + 3 < 2k + 2 = \dim V^H$$

for $(H, K) \in (\{C_3\} \times (A_4)_G) \cup (\{C_2\} \times (D_6)_G)$. Therefore the condition (A1) of Definition 2.5 (1) is fulfilled and there is no need to check Definition 2.5 (3). Observing Diagram 3.5, we can see without difficulties that (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

The fixed-point-set dimensions of $V = V(6k + 5)$ ($k \geq 1$) are as in Diagram 3.6.

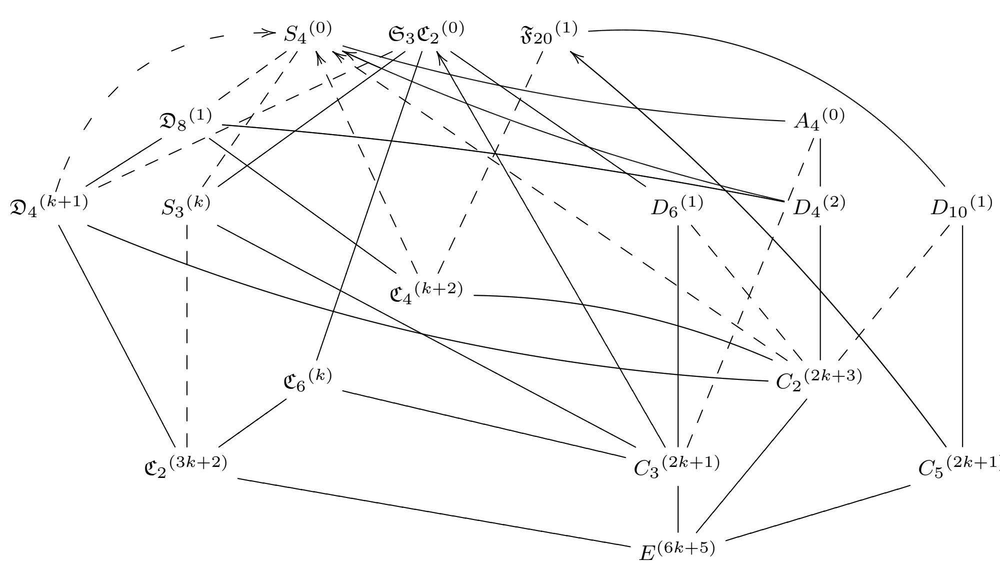

Diagram 3.6

Observing the diagram above, we obtain

$$\operatorname{Iso}(G, V \smallsetminus \{0\}) = \operatorname{Iso}(G, V_6 \smallsetminus \{0\}) \cup (\mathfrak{F}_{20})_G \cup (\mathfrak{D}_8)_G \cup (D_6)_G \cup (D_4)_G. \tag{3.10}$$

For $n = 6k + 5$ $(k \geq 1)$, $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{F}' = \mathcal{F}$. Diagram 3.6 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H)_{\rm sol} \cap \operatorname{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$ then $H = D_{10}$ and $K = \mathfrak{F}_{20}$. Therefore (D1) of Theorem 2.3 is fulfilled. The same diagram shows $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 1)$ for all $H \in \mathcal{F}^*$ as well as $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \emptyset$. If $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$ and $K \in \mathcal{Y}(G, \rho_{\max}(H), H) \cap \mathcal{H}(G, V, 1)$ then $(H, K) \in \{\mathfrak{C}_4\} \times (\mathfrak{F}_{20})_G$ or $(H, K) \in \{C_2\} \times ((D_6)_G \cup (D_{10})_G)$.

Case $n = 11$. Let $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$ and $K \in \mathcal{Y}(G, \rho_{\max}(H), H) \cap \mathcal{H}(G, V, 1)$. Note that $\dim V^H = 3$ (resp. 5) and $\dim V(\mathcal{U}_{S_4}(H)) = 0$ (resp. 2) for $H = \mathfrak{C}_4$ (resp. C_2). Recall Proposition 3.3 (10). In the case where $H = \mathfrak{C}_4$ and $K \in \mathcal{U}_G(H) \cap (\mathfrak{F}_{20})_G$, it holds that $\dim V^K = 1$ and $N_G(K) \cap S_4 = \mathfrak{C}_4$, and therefore (A2) in Definition 2.5 (1) is fulfilled. In the case where $H = C_2$ and $K \in \mathcal{U}_G(H) \cap ((C_{10})_G \cup (C_6)_G)$, $\dim V^K = 1$ and $\dim V^K + \dim V(\mathcal{U}_{S_4}(H)) + 1 < \dim V^H$, and therefore (A1) in Definition 2.5 (1) is fulfilled. Thus (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\rm sol} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 5$ $(k \geq 2)$. In this case, $\mathcal{F} = \mathcal{S}(G)_{\rm sol} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{F}' = \mathcal{F}$. If $H = \mathfrak{C}_4$ and $K \in \mathcal{U}_G(H) \cap (\mathfrak{F}_{20})_G$ then $2 \dim V^K + 1 = 3 < 4 \leq k + 2 = \dim V^H$ and

$$\dim V^K + \dim V(\mathcal{U}_{S_4}(H)) + 1 = 1 + 1 + 1 = 3 < 4 \leq k + 2 \dim V^H.$$

If $H = C_2$ and $K \in \mathcal{U}_G(H) \cap ((D_6)_G \cup (D_{10})_G)$ then we have $2 \dim V^K + 1 = 3 < 7 \leq 2k + 3 = \dim V^H$ and

$$\dim V^K + \dim V(\mathcal{U}_{S_4}(H)) + 1 = 1 + (k + 2) + 1 = k + 4 < 2k + 3 = \dim V^H.$$

Therefore (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\rm sol} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

The fixed-point-set dimensions of $V = V(6k + 8)$ are as in Diagram 3.7.

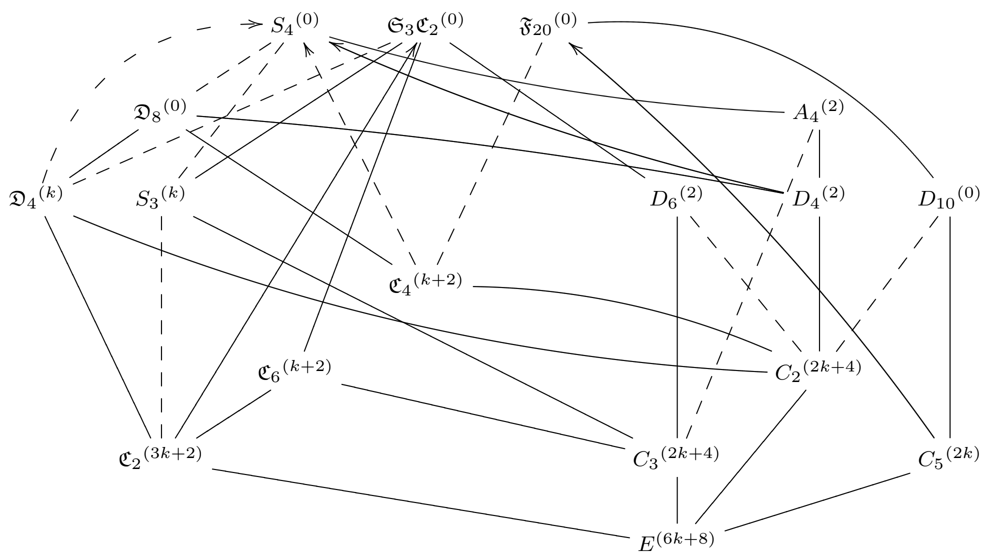

Diagram 3.7

Observing the diagram above, we get

$$\text{Iso}(G, V \smallsetminus \{0\}) = \text{Iso}(G, V_6 \smallsetminus \{0\}) \cup (A_4)_G \cup (D_6)_G. \tag{3.11}$$

Case $n = 14$. In this case, $\mathcal{F} = \mathcal{S}(G)_{\text{sol}} \smallsetminus (\{E\} \cup (C_2)_G \cup (C_3)_G)$ and $\mathcal{F}' = \mathcal{F} \smallsetminus (\mathfrak{C}_2)_G$. Diagram 3.7 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H) \cap \text{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$ then $H = D_4$ and $K = A_4$. Therefore (D1) of Theorem 2.3 is fulfilled. Recall $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) \subset (\mathfrak{D}_4)_G$. Observing Diagram 3.7, we can easily see that $\dim V^{\mathfrak{D}_4} = 1$ and $N_G(K) \cap \mathfrak{S}_3\mathfrak{C}_2 = K$ for all $K \in (\mathfrak{D}_4)_G \cap \mathcal{U}_{\mathfrak{S}_3\mathfrak{C}_2}(\mathfrak{C}_2)$. Diagram 3.7 shows that $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, 0)$ for all $H \in \mathcal{F}^* \smallsetminus (\mathcal{F}(0) \cup (S_3)_G)$. Firstly note $\dim V^K = 1$ for all $K \in \mathcal{X}(G, \rho_{\max}, \mathfrak{C}_2)$. Secondly note $\dim V^{S_3} = 1$, $\dim V^{\mathfrak{C}_2} = 5$, and

$$\dim V^{S_3} + \dim V(\mathcal{U}_{\mathfrak{S}_3\mathfrak{C}_2}(\mathfrak{C}_2)) + 1 = 1 + 3 + 1 = 5 = \dim V^{\mathfrak{C}_2}$$

as well as $2 \dim V^{S_3} + 1 = 3 < \dim V^{\mathfrak{C}_2}$. Therefore (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\text{sol}} \smallsetminus \mathcal{F} = \{E\} \cup (C_2)_G \cup (C_3)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 8$ *(*$k \geq 2$*).* In this case, $\mathcal{F} = \mathcal{S}(G)_{\text{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G \cup (C_2)_G \cup (C_3)_G)$ and $\mathcal{F}' = \mathcal{F}$. Diagram 3.7 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H) \cap \text{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$ then $H = D_4$ and $K = A_4$. Therefore (D1) of Theorem 2.3 is fulfilled. Note that $\mathcal{X}(G, \rho_{\max}, H) = \emptyset$ and $\mathcal{Y}(G, \rho_{\max}(H), H) = \emptyset$ for all $H \in \mathcal{F}^* \smallsetminus \mathcal{F}(0)$. Therefore (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\text{sol}} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G \cup (C_2)_G \cup (C_3)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 9$ $(k \geq 1)$. The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 3.8.

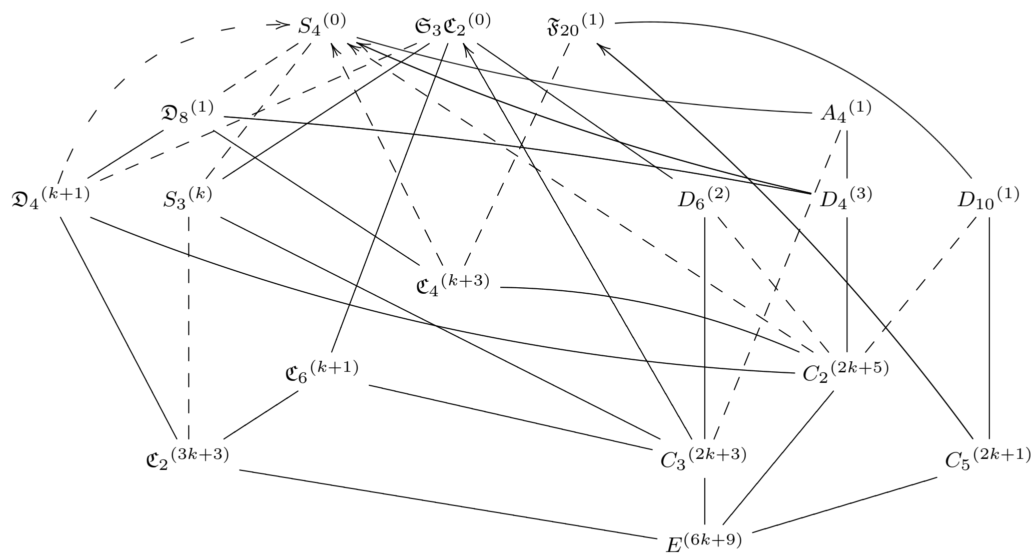

Diagram 3.8

Observing the diagram above, we get

$$\mathrm{Iso}(G, V \smallsetminus \{0\}) = \mathrm{Iso}(G, V_6 \smallsetminus \{0\}) \cup (\mathfrak{F}_{20})_G \cup (A_4)_G \cup (\mathfrak{D}_8)_G \cup (D_6)_G \cup (D_4)_G. \tag{3.12}$$

In the case, we have $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{F}' = \mathcal{F}$. Diagram 3.8 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$ then $H = D_{10}$ and $K = \mathfrak{F}_{20}$. Therefore (D1) of Theorem 2.3 is fulfilled. We clearly get $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \emptyset$. By Diagram 3.8, we get $\bigcup_H \mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 2)$, $\bigcup_H \mathcal{Y}(G, \rho_{\max}(H), H) \cap \mathcal{H}(G, V, 1) \subset (A_4)_G \cup (D_{10})_G$ and $\bigcup_H \mathcal{Y}(G, \rho_{\max}(H), H) \cap \mathcal{H}(G, V, 2) \subset (D_6)_G$, where H runs over $\mathcal{F}^* \smallsetminus \mathcal{F}(0)$. Since $\dim V^{C_3} \geq 5$, V satisfies the $(G, \mathfrak{S}_3\mathfrak{C}_2)$-cobordism gap condition at C_3. Note $\dim V^{C_2} \geq 7$, $2 \dim V^{D_6} + 1 = 5 < 7 \leq \dim V^{C_2}$, and

$$\dim V(\mathcal{U}_{S_4}(C_2)) + \dim V^{D_6} + 1 = (k+3) + 2 + 1 = k + 6 \leq 2k + 5 = \dim V_2^C,$$

where the equality $k + 6 = 2k + 5$ holds only in the case $k = 1$. If $k = 1$ then the codimension condition $\dim V^{C_2} - \dim V(\mathcal{U}_{S_4}(C_2)) \geq 3$ is fulfilled. Observing Diagram 3.8, we can readily see that V satisfies the (G, S_4)-cobordism gap condition at C_2. Therefore (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 13$ $(k \geq 1)$. The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 3.9.

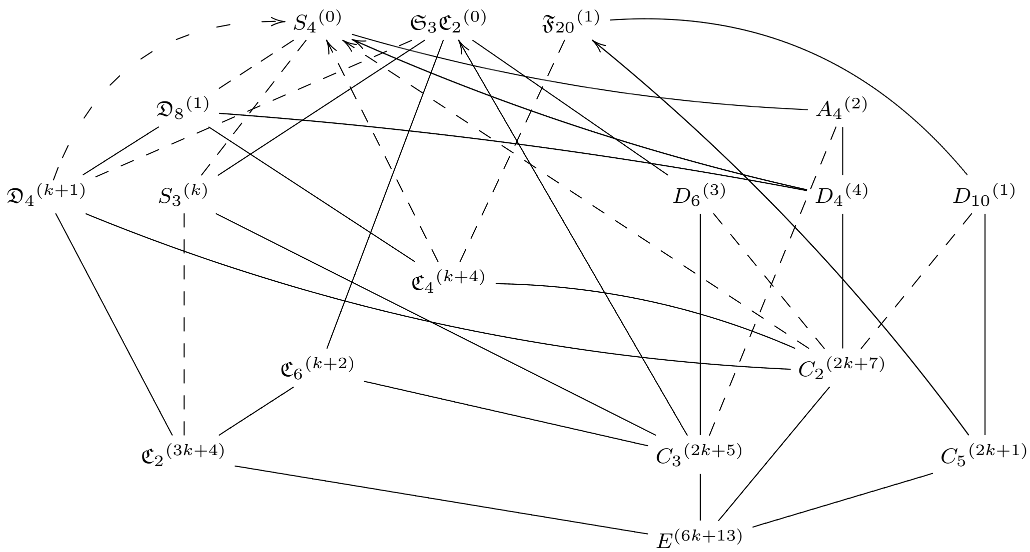

Diagram 3.9

Observing the diagram above, we obtain

$$\operatorname{Iso}(G, V \smallsetminus \{0\}) = \operatorname{Iso}(G, V_6 \smallsetminus \{0\}) \cup (\mathfrak{F}_{20})_G \cup (A_4)_G \cup (\mathfrak{D}_8)_G \cup (D_6)_G \cup (D_4)_G. \tag{3.13}$$

In the case, we have $\mathcal{F} = \mathcal{S}(G)_{\text{sol}} \smallsetminus (\{E\} \cup (\mathfrak{C}_2)_G)$ and $\mathcal{F}' = \mathcal{F}$. Diagram 3.9 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H) \cap \operatorname{Iso}(G, V \smallsetminus \{0\})$ satisfies $\dim V^H = \dim V^K$ then $H = D_{10}$ and $K = \mathfrak{F}_{20}$. Therefore (D1) of Theorem 2.3 is fulfilled. Note that $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \emptyset$ and $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 3)$. We have $\dim V^{C_3} = 2k+5 \geq 7$, $2\dim V^{A_4} + 1 = 4+1 = 5 < \dim V^{C_3}$, and

$$\dim V(\mathcal{U}_{\mathfrak{S}_3\mathfrak{C}_2}(C_3)) + \dim V^{A_4} = (k+2) + 2 < 2k+5 = \dim V^{C_3}.$$

Therefore V satisfies the $(G, \mathfrak{S}_3\mathfrak{C}_2)$-cobordism gap condition at C_3. We have $\dim V^{C_2} = 2k+7 \geq 9$, $2\dim V^{D_6} + 1 = 6+1 = 7 < \dim V^{C_2}$, and

$$\dim V(\mathcal{U}_{S_4}(C_2)) + \dim V^{D_6} + 1 = (k+4) + 3 + 1 = k+8 \leq 2k+7 = \dim V^{C_2},$$

where the equality $k+8 = 2k+7$ holds only in the case $k=1$. If $k=1$ then the codimension condition $\dim V^{C_2} - \dim V(\mathcal{U}_{S_4}(C_2)) \geq 3$ is fulfilled. Observing Diagram 3.9, we can see that V satisfies the (G, S_4)-cobordism gap condition at C_2. Therefore (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\text{sol}} \smallsetminus \mathcal{F} = \{E\} \cup (\mathfrak{C}_2)_G$. It is easy to see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Putting the arguments above together, we have shown that the data $(G, V(n), \mathcal{F}, \mathcal{F}', \mathcal{F}^*, \rho_{\max})$ specified in this section satisfy the conditions required in Theorem 2.3. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Throughout this section, let Z be a group of order 2 and $G = A_5 \times Z$. As it is in [23, Section 7], we identify subgroups $H \in \mathcal{S}(A_5)$ with $H \times \{e\} \in \mathcal{S}(G)$, respectively, and Z with $\{e\} \times Z \in \mathcal{S}(G)$. Let $\mathcal{C}_2$ be the subgroup of order 2 belonging to $\mathcal{S}(C_2Z) \smallsetminus \{C_2, Z\}$. Let $\mathcal{D}_{2n}$ be the dihedral subgroup of order $2n$ generated by C_n and $\mathcal{C}_2$. Table 4.1 below shows the subgroups H giving a complete set of representatives of conjugacy classes of subgroups of G and the normalizers of H.

H	G	A_5	A_4Z	$D_{10}Z$	D_6Z	A_4	$\mathcal{D}_{10}$	D_{10}	C_5Z	D_4Z	C_3Z
$N_G(H)$	G	G	A_4Z	$D_{10}Z$	D_6Z	A_4Z	$D_{10}Z$	$D_{10}Z$	$D_{10}Z$	A_4Z	D_6Z

H	$\mathcal{D}_6$	D_6	C_5	$\mathcal{D}_4$	C_2Z	D_4	C_3	$\mathcal{C}_2$	C_2	Z	E
$N_G(H)$	D_6Z	D_6Z	$D_{10}Z$	A_4Z	D_4Z	A_4Z	D_6Z	D_4Z	D_4Z	G	G

Table 4.1

The Hasse diagram of subgroups (up to conjugations) of G is as follows.

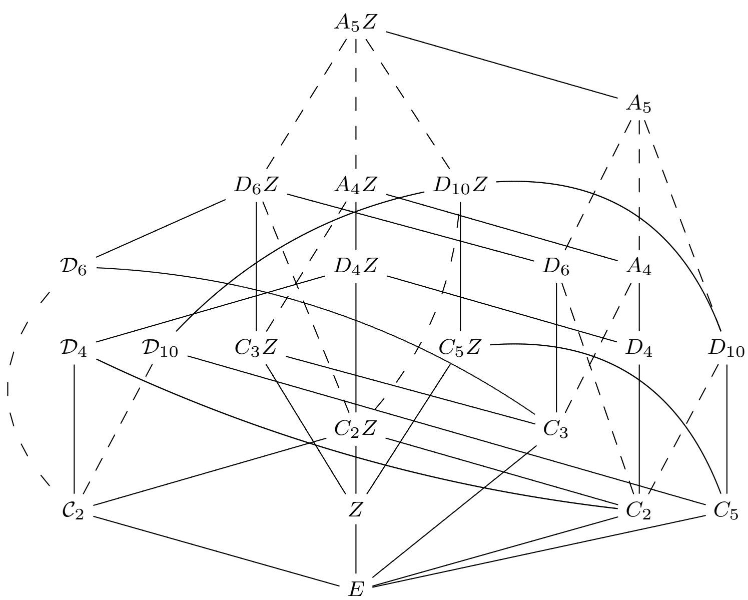

Diagram 4.1

Assign $\rho_{\max}(H)$ to H as in Table 4.2.

H	A_4Z	$D_{10}Z$	D_6Z	A_4	$\mathcal{D}_{10}$	D_{10}	C_5Z	D_4Z	C_3Z
$\rho_{\max}(H)$	A_4Z	$D_{10}Z$	D_6Z	A_4Z	$D_{10}Z$	$D_{10}Z$	$D_{10}Z$	A_4Z	D_6Z

H	$\mathcal{D}_6$	D_6	C_5	$\mathcal{D}_4$	C_2Z	D_4	C_3	C_2	$\mathcal{C}_2$
$\rho_{\max}(H)$	D_6Z	D_6Z	$D_{10}Z$	A_4Z	A_4Z	A_4Z	D_6Z	A_4Z	A_4Z

Table 4.2

We can grasp the correspondence $H \longmapsto \rho_{\max}(H)$ from Diagram 4.2.

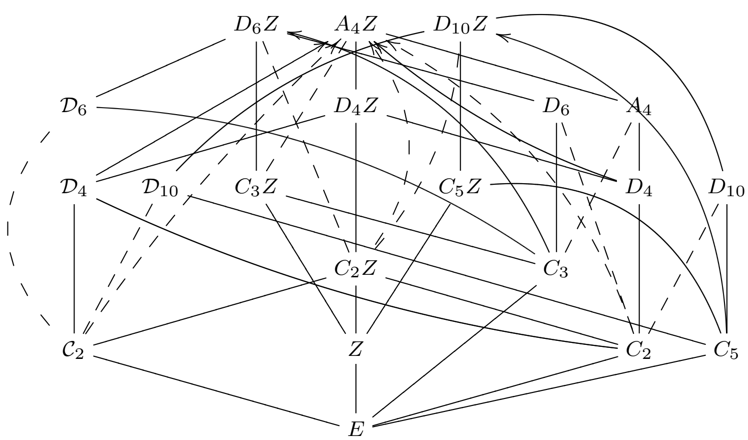

Diagram 4.2

By [23, Proposition 3.1 and Remark 3.1], the idempotent β_G in $\Omega(G)$ has the form

$$\beta_G = [G/A_4Z] + [G/D_{10}Z] + [G/D_6Z] - [G/C_3Z] - 2[G/C_2Z] + [G/Z], \tag{4.1}$$

and therefore

$$\mathrm{Iso}(G, \beta_G) = (A_4Z)_G \cup (D_{10}Z)_G \cup (D_6Z)_G \cup (C_3Z)_G \cup (C_2Z)_G \cup (Z)_G. \tag{4.2}$$

Let W_3, W_4 and W_5 be irreducible real A_5-representations of dimension 3, 4 and 5, respectively. We obtain irreducible real G-representations $V_{3,1}$, $V_{3,2}$, $V_{4,2}$ and $V_{5,2}$ by $V_{3,1} = W_3 \otimes \mathbb{R}$, $V_{3,2} = W_3 \otimes \mathbb{R}_\pm$, $V_{4,2} = W_4 \otimes \mathbb{R}_\pm$ and $V_{5,2} = W_5 \otimes \mathbb{R}_\pm$, respectively, where $\mathbb{R}_\pm$ stands for the 1-dimensional real Z-representation with nontrivial Z-action. The H-fixed-point-set dimensions of these G-representations are as in Table 4.3.

H	E	Z	C_2	$\mathcal{C}_2$	C_3	D_4	C_2Z	$\mathcal{D}_4$	C_5	C_3Z	D_6	$\mathcal{D}_6$	D_4Z	C_5Z	D_{10}	$\mathcal{D}_{10}$	A_4	K
$V_{3,1}$	3	3	1	1	1	0	1	0	1	1	0	0	0	1	0	0	0	0
$V_{3,2}$	3	0	1	2	1	0	0	1	1	0	0	1	0	0	0	1	0	0
$V_{4,2}$	4	0	2	2	2	1	0	1	0	0	1	1	0	0	0	0	1	0
$V_{5,2}$	5	0	3	2	1	2	0	1	1	0	1	0	0	0	1	0	0	0

TABLE 4.3

where K ranges over $\{A_4Z, D_6Z, D_{10}Z\}$. We draw the diagram of H-fixed-point-set dimensions of $V_{3,1}$.

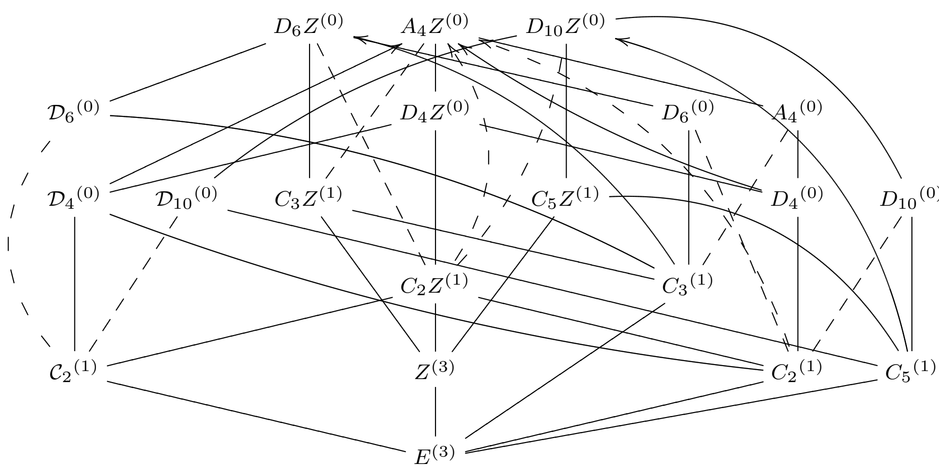

Diagram 4.3

Observing the diagram above, we obtain

(4.3) $$\mathrm{Iso}(G, V_{3,1} \smallsetminus \{0\}) = (C_5Z)_G \cup (C_3Z)_G \cup (C_2Z)_G \cup \{Z\}.$$

Note that

(4.4) $$\max(\mathcal{S}(G)_{\mathrm{sol}}) = (A_4Z)_G \cup (D_{10}Z)_G \cup (D_6Z)_G.$$

Comparing these with (4.2), we get

(4.5) $$\mathrm{Iso}(G, \beta_G) \subset \max(\mathcal{S}(G)_{\mathrm{sol}}) \cup \mathrm{Iso}(G, V_{3,1} \smallsetminus \{0\}).$$

Therefore $V_{3,1}$ is ample for β_G.

Proposition 4.1. *Let V be an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation. If V contains a G-subrepresentation isomorphic to $V_{3,1}$ then V is ample for β_G.*

Proof. This result follows from Proposition 2.1. □

Proposition 4.2. *Let $\mathcal{F} = \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E, Z\} \cup (\mathcal{C}_2)_G)$. Let $\mathcal{F}^*$ be the set of H appearing in Table 4.1 such that $H \in \mathcal{F}$ and let $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ be the map given by Table 4.2. If an $\mathcal{S}(G)_{\mathrm{nonsol}}$-free real G-representation V contains a subrepresentation isomorphic to $V_{3,1}$ then V has the property (D1′).*

Proof. By Proposition 2.5, it suffices to prove the proposition for the case $V = V_{3,1}$. Let $H \in \mathcal{F}^*$ and $K \in \mathcal{U}_G(H)_{\mathrm{sol}}$ such that ${V_{3,1}}^H = {V_{3,1}}^K$. Observing Diagram 4.3, we see that $\dim {V_{3,1}}^H = \dim {V_{3,1}}^K = 0$, or $(H, K) = (C_2, C_2Z)$, (C_3, C_3Z), (C_5, C_5Z). Therefore we can readily see that $V_{3,1}$ has the property (D1′). □

In this section, we set $V_6 = V_{3,1} \oplus V_{3,2}$, $V_7 = V_{3,1} \oplus V_{4,2}$, $V_8 = V_{3,1} \oplus V_{5,2}$ and $V_{9,2} = V_{4,2} \oplus V_{5,2}$. Further define $V(n)$ for $n \in [6..\infty)$ as follows.

$$V(n) = \begin{cases} {V_6}^{\oplus k} & (n = 6k \text{ with } k \in \mathbb{N}) \\ V_7 \oplus {V_6}^{\oplus k} & (n = 6k + 7 \text{ with } k \in \mathbb{N} \cup \{0\}) \\ V_8 \oplus {V_6}^{\oplus k} & (n = 6k + 8 \text{ with } k \in \mathbb{N} \cup \{0\}) \\ V_{3,2} \oplus V_6 & (n = 9) \\ V_{9,2} \oplus {V_6}^{\oplus k} & (n = 6k + 9 \text{ with } k \in \mathbb{N}) \\ V_{4,2} \oplus {V_6}^{\oplus k} & (n = 6k + 4 \text{ with } k \in \mathbb{N}) \\ V_{5,2} \oplus {V_6}^{\oplus k} & (n = 6k + 5 \text{ with } k \in \mathbb{N}) \end{cases} \tag{4.6}$$

The H-fixed-point-set dimensions of the real G-representations above are as in Table 4.4.

H	E	Z	C_2	$\mathcal{C}_2$	C_3	D_4	C_2Z	$\mathcal{D}_4$	C_5
V_6	6	3	2	3	2	0	1	1	2
$V(7)$	7	3	3	3	3	1	1	1	1
$V(8)$	8	3	4	3	2	2	1	1	2
$V(9)$	9	3	3	5	3	0	1	2	3
$V_{9,0}$	9	0	5	4	3	3	0	2	1
$V(10)$	10	3	4	5	4	1	1	2	2
$V(11)$	11	3	5	5	3	2	1	2	3
$V(6k+6)$	$6k+6$	$3k+3$	$2k+2$	$3k+3$	$2k+2$	0	$k+1$	$k+1$	$2k+2$
$V(6k+7)$	$6k+7$	$3k+3$	$2k+3$	$3k+3$	$2k+3$	1	$k+1$	$k+1$	$2k+1$
$V(6k+8)$	$6k+8$	$3k+3$	$2k+4$	$3k+3$	$2k+2$	2	$k+1$	$k+1$	$2k+2$
$V(6k+9)$	$6k+9$	$3k$	$2k+5$	$3k+4$	$2k+3$	3	k	$k+2$	$2k+1$
$V(6k+10)$	$6k+10$	$3k+3$	$2k+4$	$3k+5$	$2k+4$	1	$k+1$	$k+2$	$2k+2$
$V(6k+11)$	$6k+11$	$3k+3$	$2k+5$	$3k+5$	$2k+3$	2	$k+1$	$k+2$	$2k+3$

H	C_3Z	D_6	$\mathcal{D}_6$	D_4Z	C_5Z	D_{10}	$\mathcal{D}_{10}$	A_4	K
V_6	1	0	1	0	1	0	1	0	0
$V(7)$	1	1	1	0	1	0	0	1	0
$V(8)$	1	1	0	0	1	1	0	0	0
$V(9)$	1	0	2	0	1	0	2	0	0
$V_{9,0}$	0	2	1	0	0	1	0	1	0
$V(10)$	1	1	2	0	1	0	1	1	0
$V(11)$	1	1	1	0	1	1	1	0	0
$V(6k+6)$	$k+1$	0	$k+1$	0	$k+1$	0	$k+1$	0	0
$V(6k+7)$	$k+1$	1	$k+1$	0	$k+1$	0	k	1	0
$V(6k+8)$	$k+1$	1	k	0	$k+1$	1	k	0	0
$V(6k+9)$	k	2	$k+1$	0	k	1	k	1	0
$V(6k+10)$	$k+1$	1	$k+2$	0	$k+1$	0	$k+1$	1	0
$V(6k+11)$	$k+1$	1	$k+1$	0	$k+1$	1	$k+1$	0	0

TABLE 4.4

where K ranges over $\{A_4Z, D_6Z, D_{10}Z\}$. The table shows $(A_4Z)_G \cup (D_{10}Z)_G \cup (D_6Z)_G \subset \mathcal{H}(G, V(n), 0)$.

We remark that Cases $n = 6$ and $n = 7$ of Theorem 1.2 are already proved in [23, Section 12]. In the rest of this section, we give $\mathcal{F}$ as follows

$$(4.7) \qquad \mathcal{F} = \begin{cases} \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E, Z\} \cup (\mathcal{C}_2)_G \cup (C_2)_G \cup (C_3)_G) & (n = 7) \\ \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus (\{E, Z\} \cup (\mathcal{C}_2)_G) & (n \in \{6, 8\} \cup [10, ..\infty)). \end{cases}$$

We set $\mathcal{F}' = \mathcal{F}$. The set $\mathcal{F}^*$ consists of the subgroups H in Table 4.2 such that $H \in \mathcal{F}$. The map $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ is given by Table 4.2. Therefore, by [23, Proposition 7.6], the pair $(\mathcal{F}, \mathcal{F}')$ is G-simply organized and $\mathcal{X}(G, \rho_{\max}, \mathcal{F}^*) = \emptyset$.

Case $n = 6k$ ($k \geq 1$). The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 4.4.

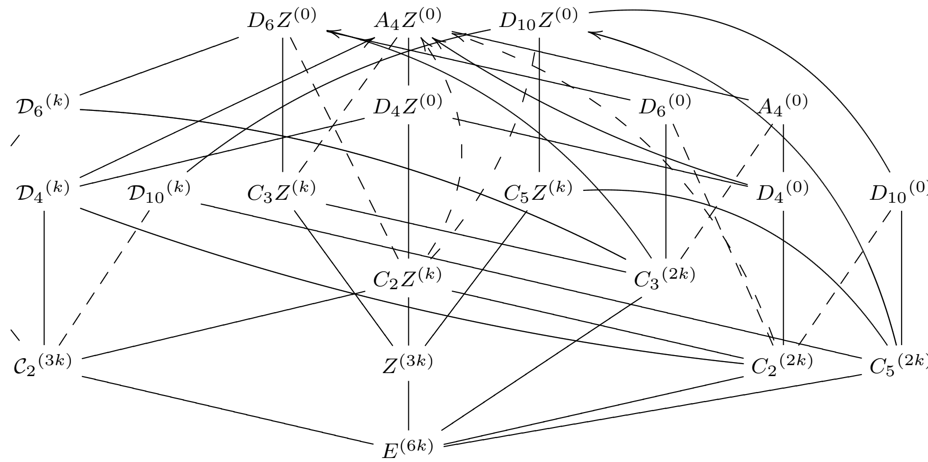

Diagram 4.4

Observing the diagram above, we get

$$\begin{aligned}\operatorname{Iso}(G, V_6 \smallsetminus \{0\}) = (\mathfrak{D}_{10})_G \cup (C_5Z)_G \cup (\mathfrak{D}_6)_G \cup (C_3Z)_G \cup (C_5)_G \cup (\mathfrak{D}_4)_G \\ \cup (C_2Z)_G \cup (C_3)_G \cup (\mathfrak{C}_2)_G \cup \{Z\} \cup (C_2)_G \cup \{E\}\end{aligned} \tag{4.8}$$

and $\operatorname{Iso}(G, V \smallsetminus \{0\}) = \operatorname{Iso}(G, V_6 \smallsetminus \{0\})$. Diagram 4.4 shows that there is no pair (H, K) such that $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$, $K \in \mathcal{U}_G(H) \cap \operatorname{Iso}(G, V \smallsetminus \{0\})$ and $\dim V^H = \dim V^K$. Therefore (D1) of Theorem 2.3 is fulfilled. The same diagram shows $\mathcal{Y}(G, \rho_{\max}(H), H) \smallsetminus \mathcal{H}(G, V, 0) = \emptyset$. It shows that (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E, Z\} \cup (\mathcal{C}_2)_G$. We can readily see that (D3) and (D4) of Theorem 2.3 are fulfilled.

For $n = 6k + 7$ ($k \geq 0$), the fixed-point-set dimensions of $V = V(n)$ are as in Diagram 4.5.

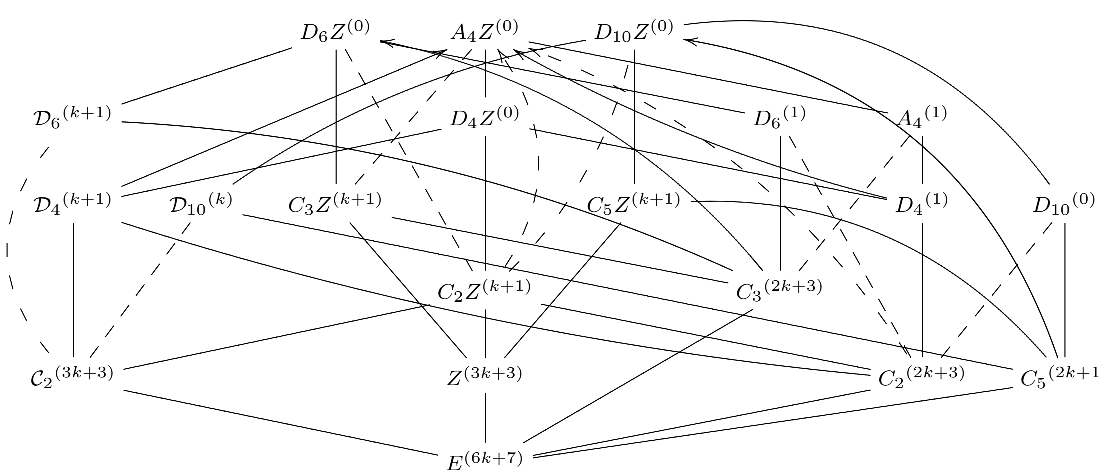

Diagram 4.5

Observing the diagram above, we obtain

$$\mathrm{Iso}(G, V \smallsetminus \{0\}) = \mathrm{Iso}(G, V_6 \smallsetminus \{0\}) \cup (A_4)_G \cup (D_6)_G. \tag{4.9}$$

Diagram 4.5 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ satisfy $\dim V^H = \dim V^K$ then $H = D_4$ and $K = A_4$. Therefore (D1) of Theorem 2.3 is fulfilled.

Case $n = 7$. Diagram 4.5 shows $\mathcal{Y}(G, M, H) \subset \mathcal{H}(G, V, 0)$ for all $H \in \mathcal{F}$. The condition (D2) of Theorem 2.3 is clearly fulfilled. We can readily see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 7$ $(k \geq 1)$. Diagram 4.5 shows $\mathcal{Y}(G, M, H) \subset \mathcal{H}(G, V, 0)$ (resp. $\mathcal{Y}(G, M, H) \subset \mathcal{H}(G, V, \leq 1)$) for all $H \in \mathcal{F} \smallsetminus ((C_2)_G \cup (C_3)_G)$ (resp. $H \in \mathcal{F}$) and $M = \rho_{\max}(H)$. By the same diagram, we have

$$2 \dim V^{A_4} + 1 = 2 + 1 = 3 < 5 \leq \dim V^{C_3},$$

$$\dim V^{A_4} + \dim V(\mathcal{U}_{D_6 Z}(C_3)) + 1 \leq 1 + (k + 1) + 1 = k + 3 < 2k + 3 \leq \dim V^{C_3},$$

$$2 \dim V^{D_6} + 1 = 2 + 1 = 3 < 5 \leq \dim V^{C_2}, \text{ and}$$

$$\dim V^{D_6} + \dim V(\mathcal{U}_{A_4 Z}(C_2)) + 1 = 1 + (k + 1) + 1 = k + 3 < 2k + 3 \leq \dim V^{C_2}.$$

It is easy to see that (D2) of Theorem 2.3 is fulfilled. Recalling $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E, Z\} \cup (\mathcal{C}_2)_G$, we can readily see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 8$ $(k \geq 0)$. The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 4.6.

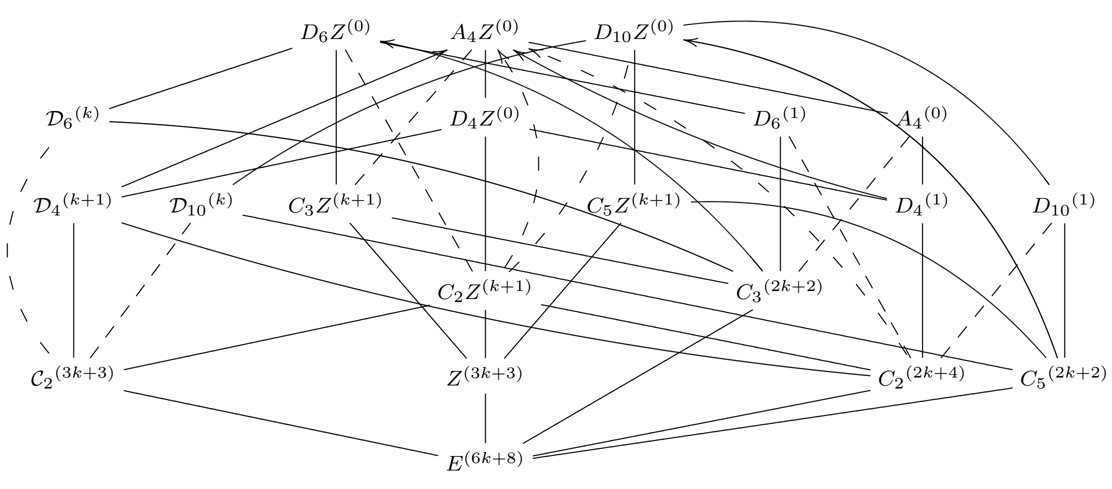

Diagram 4.6

Observing the diagram above, we get

$$\mathrm{Iso}(G, V \smallsetminus \{0\}) = \mathrm{Iso}(G, V_6 \smallsetminus \{0\}) \cup (D_{10})_G \cup (D_6)_G \cup (D_4)_G. \tag{4.10}$$

Diagram 4.6 shows that there is no pair (H, K) such that $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$, $K \in \mathcal{U}_G(H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ and $\dim V^H = \dim V^K$. Therefore (D1) of Theorem 2.3 is fulfilled. The diagram shows $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, 0)$ (resp. $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 1)$) for all $H \in \mathcal{F}^* \smallsetminus (C_2)_G$ (resp. $H \in \mathcal{F}$). In this case, we have

$$2 \dim V^{D_s} + 1 = 2 + 1 = 3 < 4 \leq \dim V^{C_2} \quad \text{and}$$

$$\dim V^{D_s} + \dim V(\mathcal{U}_{A_4 Z}(C_2)) + 1 = 1 + (k+1) + 1 = k + 3 < 2k + 4 = \dim V^{C_2},$$

where $s = 6$, 10. It is easy to see that (D2) of Theorem 2.3 is fulfilled. By virtue of $\mathcal{S}(G)_{\text{sol}} \smallsetminus \mathcal{F} = \{E, Z\} \cup (\mathcal{C}_2)_G$, we can readily see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 9$ *($k \geq 1$).* The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 4.7.

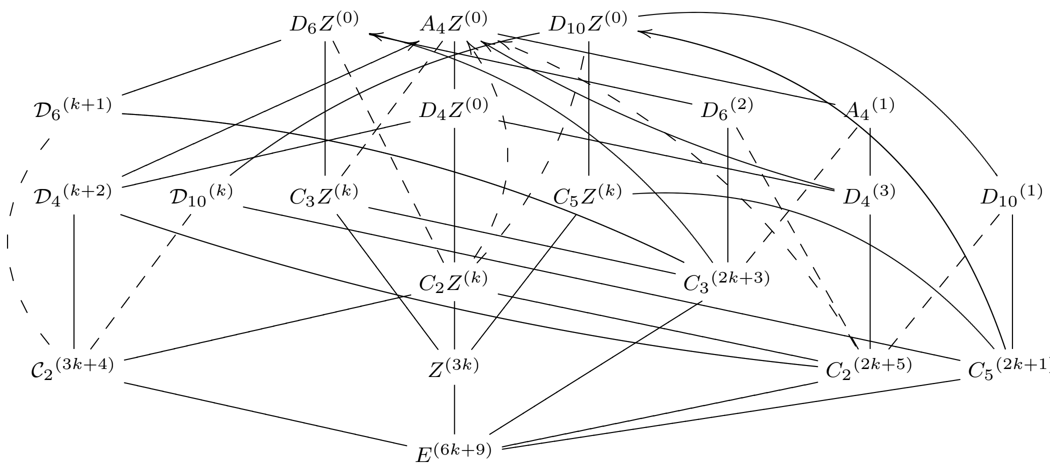

Diagram 4.7

Observing the diagram above, we get

$$\text{Iso}(G, V \smallsetminus \{0\}) = \text{Iso}(G, V_6 \smallsetminus \{0\}) \cup (A_4)_G \cup (D_{10})_G \cup (D_6)_G \cup (D_4)_G. \tag{4.11}$$

Diagram 4.7 shows that there is no pair (H, K) such that $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$, $K \in \mathcal{U}_G(H) \cap \text{Iso}(G, V \smallsetminus \{0\})$ and $\dim V^H = \dim V^K$. Therefore (D1) of Theorem 2.3 is fulfilled. The diagram shows $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, 0)$ (resp. $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 1)$, $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 2)$) for all $H \in \mathcal{F} \smallsetminus ((C_2)_G \cup (C_3)_G)$ (resp. $H \in \mathcal{F} \smallsetminus (C_2)_G$, $H \in \mathcal{F}$). We have

$$2 \dim V^{A_4} + 1 = 2 + 1 = 3 < 5 \leq \dim V^{C_3},$$

$$\dim V^{A_4} + \dim V(\mathcal{U}_{D_6 Z}(C_3)) + 1 = 1 + (k+1) + 1 = k + 3 < 2k + 3 = \dim V^{C_3},$$

$$2 \dim V^{D_6} + 1 = 4 + 1 = 5 < 7 \leq \dim V^{C_2}, \text{ and}$$

$$\dim V^{D_6} + \dim V(\mathcal{U}_{A_4 Z}(C_2)) + 1 = 1 + (k+2) + 1 = k + 4 < 2k + 5 = \dim V^{C_2}.$$

It is easy to see that (D2) of Theorem 2.3 is fulfilled. Recall $\mathcal{S}(G)_{\text{sol}} \smallsetminus \mathcal{F} = \{E, Z\} \cup (\mathcal{C}_2)_G$. We can check without difficulties that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 10$ *($k \geq 0$).* The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 4.8.

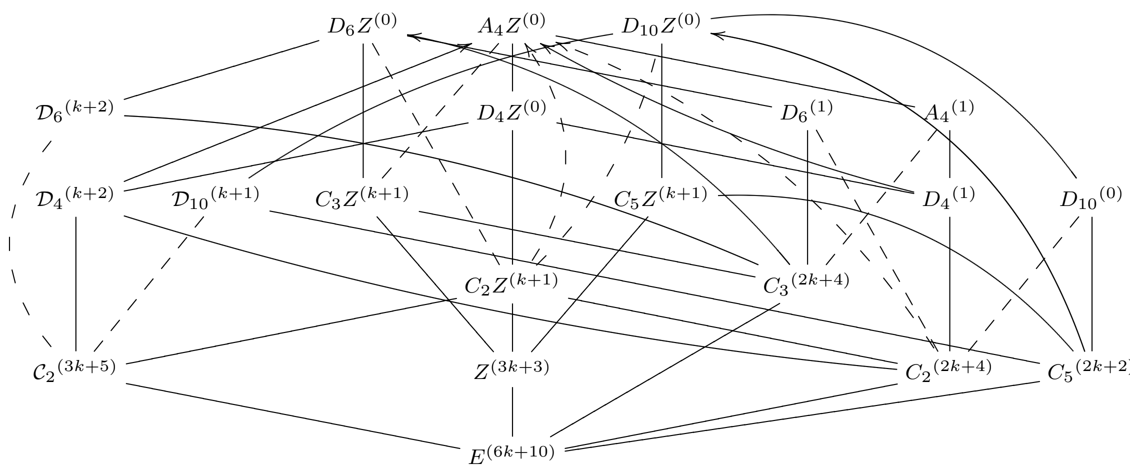

Diagram 4.8

Observing the diagram above, we obtain

$$\operatorname{Iso}(G, V \smallsetminus \{0\}) = \operatorname{Iso}(G, V_6 \smallsetminus \{0\}) \cup (A_4)_G \cup (D_6)_G. \tag{4.12}$$

Diagram 4.8 shows that if $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$ and $K \in \mathcal{U}_G(H) \cap \operatorname{Iso}(G, V \smallsetminus \{0\})$ satisfy $\dim V^H = \dim V^K$ then $H = D_4$ and $K = A_4$. Therefore (D1) of Theorem 2.3 is fulfilled. The diagram shows $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, 0)$ (resp. $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 1)$) for all $H \in \mathcal{F} \smallsetminus ((C_2)_G \cup (C_3)_G)$ (resp. $H \in (C_2)_G \cup (C_3)_G$). We have

$$\begin{aligned}
&2 \dim V^{A_4} + 1 = 2 + 1 = 3 < 4 \leq \dim V^{C_3}, \\
&\dim V^{A_4} + \dim V(\mathcal{U}_{D_6 Z}(C_3)) + 1 = 1 + (k+2) + 1 = k + 4 \leq 2k + 4 = \dim V^{C_3}, \\
&2 \dim V^{D_6} + 1 = 2 + 1 = 3 < 4 \leq \dim V^{C_2}, \text{ and} \\
&\dim V^{D_6} + \dim V(\mathcal{U}_{A_4 Z}(C_2)) + 1 = 1 + (k+2) + 1 = k + 4 \leq 2k + 4 = \dim V^{C_2}.
\end{aligned}$$

Here the equality $k + 4 = 2k + 4$ holds only in the case $k = 0$. Note that for $H = C_2$ and C_3, the subgroup $\langle K_1, K_2 \rangle$ coincides with A_5 whenever K_1, $K_2 \in (\mathcal{U}_G(H) \smallsetminus \mathcal{U}_{\rho_{\max}(H)}(H)) \cap \mathcal{H}(G, V, 1)$ with $K_1 \neq K_2$. In the case $k = 0$, the condition (C2) of Definition 2.5 (4) is satisfied for $H = C_2$, C_3 and $M = \rho_{\max}(H)$. It is easy to see that (D2) of Theorem 2.3 is fulfilled. Recalling $\mathcal{S}(G)_{\rm sol} \smallsetminus \mathcal{F} = \{E, Z\} \cup (\mathcal{C}_2)_G$, we can readily see that (D3) and (D4) of Theorem 2.3 are fulfilled.

Case $n = 6k + 11$ ($k \geq 0$). The fixed-point-set dimensions of $V = V(n)$ are as in Diagram 4.9.

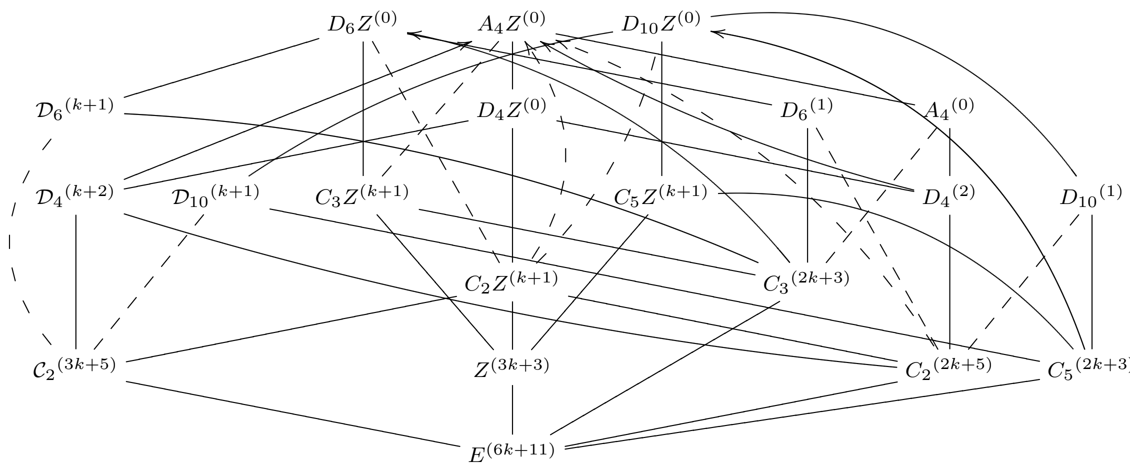

Diagram 4.9

Observing the diagram above, we get

$$\mathrm{Iso}(G, V \smallsetminus \{0\}) = \mathrm{Iso}(G, V_6 \smallsetminus \{0\}) \cup (D_{10})_G \cup (D_6)_G \cup (D_4)_G. \tag{4.13}$$

Diagram 4.9 shows that there is no pair (H, K) such that $H \in \mathcal{F}^* \smallsetminus \mathcal{H}(G, V, 0)$, $K \in \mathcal{U}_G(H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ and $\dim V^H = \dim V^K$. Therefore (D1) of Theorem 2.3 is fulfilled. The diagram shows $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, 0)$ (resp. $\mathcal{Y}(G, \rho_{\max}(H), H) \subset \mathcal{H}(G, V, \leq 1)$) for all $H \in \mathcal{F} \smallsetminus (C_2)_G$ (resp. $H \in \mathcal{F}$). We have

$$2 \dim V^{D_s} + 1 = 2 + 1 = 3 < 5 \leq \dim V^{C_2}, \text{ and}$$

$$\dim V^{D_s} + \dim V(\mathcal{U}_{A_4Z}(C_2)) + 1 = 1 + (k+2) + 1 = k + 4 < 2k + 5 = \dim V^{C_2},$$

where $s = 6$, 10. It is easy to see that (D2) of Theorem 2.3 is fulfilled. By virtue of $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F} = \{E, Z\} \cup (\mathcal{C}_2)_G$, we can check without difficulties that (D3) and (D4) of Theorem 2.3 are fulfilled.

Putting the arguments above together, we have shown that the data $(G, V(n), \mathcal{F}, \mathcal{F}', \mathcal{F}^*, \rho_{\max})$ specified in this section satisfy the conditions required in Theorem 2.3. This completes the proof of Theorem 1.2.

5. Extension of a product M-embedding Ψ_M

In the remainder of the current article, let G, $(\mathcal{F}, \mathcal{F}')$, $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)^*_{\mathrm{sol}})$, and V be those stated in Theorem 2.3, let Y be the unit disk $D(V)$ of V, and let $\boldsymbol{f} = (f, b)$ and $\boldsymbol{F}_L = (F_L, B_L)$, $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})$, be a G-framed map rel. ∂ and L-framed cobordisms from $\mathrm{res}^G_L \boldsymbol{f}$ to $\mathrm{res}^G_L \boldsymbol{id}_Y$ rel. ∂, respectively, obtained in [23, Section 9]. Therefore $\mathcal{F}$ and $\mathcal{F}'$ contain $\max(\mathcal{S}(G)_{\mathrm{sol}})$, cf. Definition 2.2, $f : (X, \partial X) \to (Y, \partial Y)$ is a G-map,

$$b : \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus \varepsilon_X(\mathbb{R}^\ell) \to \varepsilon_X(\mathbb{R} \oplus V \oplus \mathbb{R}^\ell)$$

is a G-bundle isomorphism,

$$F_L : (W_L, \partial_0 W_L, \partial_1 W_L, \partial_{01} W_L) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

are L-maps, $\partial_0 W_L = \{0\} \times X$, $\partial_1 W_L = \{1\} \times Y$, $\partial_{01} W_L = I \times \partial Y$, $Z = I \times Y$, $\partial_0 Z = \{0\} \times Y$, $\partial_1 Z = \{1\} \times Y$, and $\partial_{01} Z = I \times \partial Y$,

$$B_L : T(W_L) \oplus \varepsilon_{W_L}(\mathbb{R}^\ell) \to \varepsilon_{W_L}(\mathbb{R} \oplus V \oplus \mathbb{R}^\ell)$$

are L-bundle isomorphisms. By the construction, $X^K = \emptyset$ for all $K \in \mathcal{S}(G)_{\text{nonsol}}$ and $f^K : (X^K, \partial X^K) \to (Y^K, \partial Y^K)$ is a map of degree 1 whenever $\dim V^K > 0$, see [23, Lemma 9.1]. When we refer to a G-framed map $\boldsymbol{f}'$ (resp. an L-framed cobordism $\boldsymbol{F}'_L$), $\boldsymbol{f}'$ is a pair (f', b') consisting of a G-map $f' : (X', \partial X) \to (Y, \partial Y)$ and G-bundle isomorphism

$$b' : \varepsilon_{X'}(\mathbb{R}) \oplus T(X') \oplus \varepsilon_{X'}(\mathbb{R}^\ell) \to \varepsilon_{X'}(\mathbb{R} \oplus V \oplus \mathbb{R}^\ell)$$

(resp. $\boldsymbol{F}'_L$ is a pair (F'_L, B'_L) consisting of an L-map

$$F'_L : (W'_L, \partial_0 W'_L, \partial_1 W'_L, \partial_{01} W'_L) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

and an L-bundle isomorphism

$$B'_L : T(W'_L) \oplus \varepsilon_{W'_L}(\mathbb{R}^\ell) \to \varepsilon_{W'_L}(\mathbb{R} \oplus V \oplus \mathbb{R}^\ell)).$$

We use $\boldsymbol{f}''$, $\boldsymbol{F}''_L$, and etc. in a similar way.

Let $\mathcal{H} \subset \mathcal{S}(G)$. For $L \in \mathcal{S}(G)$ we set

$$\begin{aligned} &\mathcal{H}|_L = \mathcal{H} \cap \mathcal{S}(L) \text{ and} \\ &[L, \mathcal{H}] = \{gKg^{-1} \mid g \in L,\ K \in \mathcal{H}\}. \end{aligned} \tag{5.1}$$

Therefore $[L, \mathcal{H}]$ is the L-invariant closure of $\mathcal{H}$ with respect to the conjugation L-action on $\mathcal{S}(G)$.

Proposition 5.1. *Let $H \in \mathcal{F}'^*$, where $\mathcal{F}'^* = \mathcal{F}' \cap \mathcal{F}^*$, and $M = \rho_{\max}(H)$. Then $(H)_G|_M = (H)_M$.*

Proof. Since $(\mathcal{F}, \mathcal{F}')$ is G-simply organized, see Definition 2.2 (3), we have $(H)_G|_M = (H)_M$. □

Let $X(\mathcal{H})$ denote the simplicial subcomplex of X defined by

$$X(\mathcal{H}) = \bigcup_{K \in \mathcal{H}} X^K.$$

For a G-simplicial subcomplex A of X with respect to some smooth G-triangulation of X such that A is a union of smooth submanifolds A_i of X, let $N_G(A, X)$ denote a G-regular neighborhood of A in X which is the union of some tubular neighborhoods of A_i in X. For a subgroup H of G, V has the form of direct sum $V = V^H \oplus V_H$ as real $N_G(H)$-representations. By virtue of the bundle data b and B_L, we have the next property which will be used without mentioning.

Proposition 5.2. *Let H be a solvable subgroup of G. Then the tubular neighborhood $N_G(X^H, X)$ is $N_G(H)$-diffeomorphic to $X^H \times D(V_H)$, where $D(V_H)$ is the unit disk of V_H. Furthermore if $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ and $H \leq L$, then $N_L({W_L}^H, W_L)$ is $N_L(H)$-diffeomorphic to ${W_L}^H \times D(V_H)$.*

For a submanifold X_0 of X and a smooth embedding $\Psi : I \times X_0 \to W_L$, where $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$, we call Ψ a *product embedding* if

(1) $\Psi(t, x) = (t, x)$ in $\partial_{01} W_L$ for all $x \in X_0 \cap \partial X$ and $t \in I$,
(2) $\Psi(t, x) = (t, x)$ in a collar neighborhood $C_X = [0, \delta] \times X$ of $\{0\} \times X$ in W_L for all $t \in [0, \delta]$ and $x \in X_0$, and
(3) $\Psi(1-t, x) = (1-t, \psi(x))$ in a collar neighborhood $C_Y = [1-\delta, 1] \times Y$ of $\{1\} \times Y$ in W_L for all $t \in [0, \delta]$ and $x \in X_0$, for some embedding $\psi : X_0 \to Y$.

Here δ is a small positive real number, and the sets $[0, \delta]$, $[1-\delta, 1]$ are the closed intervals $\subset \mathbb{R}$. For a simplicial subcomplex A of X and a topological embedding $\Psi_0 : I \times A \to W_L$, we call Ψ_0 a *product embedding* if there are a manifold neighborhood X_0 of A and a product embedding $\Psi : I \times X_0 \to W_L$ extending Ψ_0.

Let $\mathcal{K}$ be a subset of $\mathcal{F}$ which is G-conjugation invariant and upwardly closed in $\mathcal{S}(G)_{\mathrm{sol}}$. We readily obtain the next proposition.

Proposition 5.3. *Let $H \in \mathcal{F}'^* \smallsetminus \mathcal{K}$ and $M = \rho_{\max}(H)$. Then $(\mathcal{K} \cup (H)_G)|_M = \mathcal{K}|_M \cup (H)_M$.*

For a G-space A, we set $A^{>H} = A(\mathcal{U}_G(H))$ and $A^{=H} = A^H \smallsetminus A^{>H}$.

Definition 5.1. Let $M \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ and let H be a subgroup of G satisfying $N_G(H) \subset M$. We say that (X, Y, W_M) has the (G, M)-*tame singular set at* H (or $X^{>H}$ is (G, M)-*tame in* (X, W_M)) if there is a product M-embedding $\Phi : I \times N_M(M \cdot X^{>H}, X) \to W_M$ such that $\mathrm{Image}(\Phi)^{>H} = {W_M}^{>H}$, where $M \cdot X^{>H} = \{gx \mid g \in M, x \in X^{>H}\}$, $\mathrm{Image}(\Phi)^{>H} = \mathrm{Image}(\Phi)(\mathcal{U}_M(H))$ and ${W_M}^{>H} = W_M(\mathcal{U}_M(H))$.

For $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$, we set

$$\mathcal{K}_L = [L, \mathcal{K} \cap (\rho_{\max}^{-1}(L) \cup \mathcal{U}_L(\rho_{\max}^{-1}(L)))], \tag{5.2}$$

where $\rho_{\max} : \mathcal{F}^* \to \max(\mathcal{S}(G)_{\mathrm{sol}})^*$ and

$$\mathcal{U}_L(\rho_{\max}^{-1}(L)) = \bigcup_{H_0 \in \rho_{\max}^{-1}(L)} \mathcal{U}_L(H_0).$$

Note that $\mathcal{K} \cap \rho_{\max}^{-1}(L) \subset \mathcal{K} \cap \mathcal{F}^* \cap \mathcal{S}(L)$ and $\mathcal{K} \cap \mathcal{U}_L(\rho_{\max}^{-1}(L)) \subset \mathcal{K} \cap \mathcal{F}' \cap \mathcal{S}(L)$. In the case where $H \in \mathcal{F}^*$ and $M = \rho_{\max}(H)$, we have $\mathcal{K}_M = \mathcal{K}|_M$.

Proposition 5.4. *Let $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^*$, $M = \rho_{\max}(H)$ and $L \in \max(\mathcal{S}(G)_{\rm sol})^*$. Then the following holds.*

(1) $\mathcal{K}_M \cap \mathcal{U}_G(H)_{\rm sol} = \mathcal{U}_M(H)$.

(2) $(\mathcal{K} \cup (H)_G)_L$

$$= \begin{cases} \mathcal{K}_M \cup (H)_M & (H \in \mathcal{F}' \text{ and } L = M) \\ \mathcal{K}_L \cup [L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))] & (H \in \mathcal{F}' \text{ and } L \neq M) \\ \mathcal{K}_M \cup (H)_M & (H \notin \mathcal{F}' \text{ and } L = M) \\ \mathcal{K}_L & (H \notin \mathcal{F}' \text{ and } L \neq M). \end{cases}$$

Therefore $(\mathcal{K} \cup (H)_G)_L \subset \mathcal{K}_L \cup ((H)_G \cap \mathcal{S}(L))$.

Proof. The definition of $\mathcal{K}_M$ implies $\mathcal{K}_M \cap \mathcal{U}_G(H)_{\rm sol} \subset \mathcal{U}_M(H)$. It suffices to prove $\mathcal{U}_M(H) \subset \mathcal{K}_M$. Let $K \in \mathcal{U}_M(H)$. By the definition, It holds that $H < K \leq M$. The condition $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^*$ and the hypothesis that $\mathcal{K}$ is upwardly closed in $\mathcal{S}(G)_{\rm sol}$ imply $K \in \mathcal{K}$. Therefore, we see

$$K \in \mathcal{K} \cap \mathcal{U}_M(H) \subset \mathcal{K} \cap \mathcal{U}_M(\rho_{\max}^{-1}(M)) \subset \mathcal{K}_M.$$

We have completed the proof of the claim (1).

We have the equalities

$$\begin{aligned} (\mathcal{K} \cup (H)_G)_L &= [L, (\mathcal{K} \cup (H)_G) \cap (\rho_{\max}^{-1}(L) \cup \mathcal{U}_L(\rho_{\max}^{-1}(L)))] \\ &= \mathcal{K}_L \cup [L, (H)_G \cap (\rho_{\max}^{-1}(L) \cup \mathcal{U}_L(\rho_{\max}^{-1}(L)))] \\ &= \begin{cases} \mathcal{K}_M \cup [M, \{H\}] \cup [M, (H)_G \cap \mathcal{U}_M(\rho_{\max}^{-1}(M))] & (L = M) \\ \mathcal{K}_L \cup [L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))] & (L \neq M) \end{cases} \\ &= \begin{cases} \mathcal{K}_M \cup (H)_M \cup [M, (H)_G \cap \mathcal{U}_M(\rho_{\max}^{-1}(M))] & (L = M) \\ \mathcal{K}_L \cup [L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))] & (L \neq M) \end{cases} \end{aligned} \tag{5.3}$$

The claim (2) follows from (5.3). □

Definition 5.2. Let $\mathcal{H}$ be a subset of $\mathcal{S}(G)_{\rm sol}$ which is upwardly closed in $\mathcal{S}(G)_{\rm sol}$ and G-conjugation invariant. We say that $(\boldsymbol{f}, \{\boldsymbol{F}_L\}_L)$ (or $(X, \{W_L\}_L)$), where L runs over $\max(\mathcal{S}(G)_{\rm sol})^*$, is *adjusted on* $(\mathcal{H}, \mathcal{K})$ if there are

- L-regular neighborhoods $N_L(X(\mathcal{H} \cup \mathcal{K}_L), X)$ of $X(\mathcal{H} \cup \mathcal{K}_L)$ in X,
- product L-embeddings $\Psi_L : I \times N_L(X(\mathcal{H} \cup \mathcal{K}_L), X) \to W_L$, and
- L-homotopies $\mathbb{H}_L : (W_L, \partial_0 W_L) \times I \to (I \times Y, \{0\} \times Y)$ from F_L to $F_{L,1}$ rel. $\partial_1 W_L \cup \partial_{01} W_L$,

for all $L \in \max(\mathcal{S}(G)_{\rm sol})^*$, satisfying the condition that for each $K \in \mathcal{K}^*$ $(= \mathcal{K} \cap \mathcal{F}^*)$ and $L = \rho_{\max}(K)$, the restriction

$$F_{L,1}|_{\mathrm{Image}(\Psi_L)} : \mathrm{Image}(\Psi_L) \to F_{L,1}(\mathrm{Image}(\Psi_L)) \ \ (\subset I \times Y)$$

is an L-diffeomorphism. (Hence

$$F_{L,1}|_{\Psi_L(\{0\} \times N)} : \Psi_L(\{0\} \times N) \to F_{L,1}(\Psi_L(\{0\} \times N)) \ \ (\subset \{0\} \times Y),$$

where $N = N_L(X(\mathcal{H} \cup \mathcal{K}_L), X)$, is also an L-diffeomorphism.)

If $(\boldsymbol{f}, \{\boldsymbol{F}_L\}_L)$ is adjusted on $(\emptyset, \mathcal{K})$ then we say that $(\boldsymbol{f}, \{\boldsymbol{F}_L\}_L)$ is adjusted on $\mathcal{K}$.

By the construction of $\boldsymbol{f}$ and $\{\boldsymbol{F}_L\}_L$ (see [23, Lemmas 9.1 and 9.2]), we can suppose without any loss of generality that $(\boldsymbol{f}, \{\boldsymbol{F}_L\}_L)$ is adjusted on $(\mathcal{H}(G, V, 0), \mathcal{F}(0))$, where L ranges over $\max(\mathcal{S}(G)_{\rm sol})^*$. In the rest of this section, we suppose that

(K1) $\mathcal{K} \supset \mathcal{F}(0)$ and

(K2) $(\boldsymbol{f}, \{\boldsymbol{F}_L\}_L)$ is adjusted on $(\mathcal{H}(G, V, 0), \mathcal{K})$ with respect to product L-embeddings $\Psi_L : I \times N_L(X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_L), X) \to W_L$ as above.

In the remainder of this section, let $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ and $M = \rho_{\max}(H)$.

Proposition 5.5. *The following equalities hold.*

(1) $X(\mathcal{H}) = X(\mathcal{H} \cap \mathrm{Iso}(G, V \smallsetminus \{0\}))$ *for any subset* $\mathcal{H}$ *of* $\mathcal{K}$ *such that* $\mathcal{H}$ *is upwardly closed in* $\mathcal{S}(G)_{\rm sol}$.

(2) $\mathcal{K}|_M \cap \mathcal{U}_G(H)_{\rm sol} = \mathcal{U}_M(H)$ *and* $W_M(\mathcal{K}|_M)^H = W_M(\mathcal{U}_M(H))$.

(3) $\mathcal{K}_M \cap \mathcal{U}_G(H)_{\rm sol} = \mathcal{U}_M(H)$ *and* $W_M(\mathcal{K}_M)^H = W_M(\mathcal{U}_M(H))$.

(4) $X(\mathcal{Y}(G, M, H)) \smallsetminus X(\mathcal{X}(G, \rho_{\max}, H)) = X(\mathcal{Y}(G, M, H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus X(\mathcal{X}(G, \rho_{\max}, H))$.

Proof. It is easy to show the claims (1) and (2). The claim (3) follows from Proposition 5.4. Here we prove the claim (4). It is obvious that $X(\mathcal{Y}(G, M, H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \subset X(\mathcal{Y}(G, M, H))$. Let K be an element of $\mathcal{Y}(G, M, H) \smallsetminus \mathrm{Iso}(G, V \smallsetminus \{0\})$ and let $\overline{K}$ be the element of $\mathrm{Iso}(G, V \smallsetminus \{0\})$ such that $V^K = V^{\overline{K}}$. By the hypothesis, we have $H < K < \overline{K}$, $K \in \mathcal{K}$ and $\overline{K} \in \mathcal{K}$ as well as $\overline{\rho}_{\max}(K) = \overline{\rho}_{\max}(\overline{K}) \neq M$, and by the hypothesis (K2) we have $X^K = X^{\overline{K}}$.

If $\overline{K} \cap M = H$ then we have $\overline{K} \in \mathcal{Y}(G, M, H)$, moreover $\overline{K} \in \mathcal{Y}(G, M, H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$, and

$$X^K = X^{\overline{K}} \subset X(\mathcal{Y}(G, M, H) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})).$$

Suppose $\overline{K} \cap M > H$. Then $K' = \overline{K} \cap M$ lies in $\mathcal{X}(G, \rho_{\max}, H)$. This shows

$$X^K = X^{\overline{K}} \subset X^{K'} \subset X(\mathcal{X}(G, \rho_{\max}, H)).$$

Therefore we have proved the claim (4). □

Set

$$N_{X,\mathcal{K}} = N_G(X(\mathcal{K}), X), \quad N_{W_M,\mathcal{K}} = N_M(W_M(\mathcal{K}|_M), W_M), \text{ and}$$
$$N_{X,M,\mathcal{K}} = N_{X,\mathcal{K}} \cap N_{W_M,\mathcal{K}},$$

where we choose $N_{X,\mathcal{K}}$ and $N_{W_M,\mathcal{K}}$ so that $N_{X,M,\mathcal{K}} = N_M(X(\mathcal{K}|_M), X)$. For a submanifold N of W_M (resp. X) such that $\mathrm{Closure}(N) = N$ and $\dim N = \dim W_M$ (resp. $\dim N = \dim X$), define $\mathring{N}$ by

$$\mathring{N} = W_M \smallsetminus \mathrm{Closure}(W_M \smallsetminus N)$$

$$(\text{resp. } \overset{\circ}{N} = X \smallsetminus \text{Closure}(X \smallsetminus N)).$$

We set

$$\begin{aligned} &X_0^H = X^H \smallsetminus \overset{\circ}{N} \quad \text{where } N = N_M(X(\mathcal{H}(G,V,0) \cup \mathcal{K}|_M), X), \\ (5.4) \qquad &W_{M,0}^H = {W_M}^H \smallsetminus \overset{\circ}{N} \quad \text{and} \quad Y_0^H = Y^H \smallsetminus \overset{\circ}{N} \\ &\qquad \text{where } Y \text{ is identified with } \{1\} \times Y \ (= \partial_1 W_M) \text{ and } N = \text{Image}(\Psi_M). \end{aligned}$$

In the present situation, it holds that $N_G(H)$ coincides with $N_M(H)$ and the group $N_M(H)/H$ acts freely on X_0^H and $W_{M,0}^H$.

By the hypothesis (K2), there are M-homotopies

(1) $\mathbb{h}_M : X \times I \to Y$ from $\text{res}_M^G f$ to $\text{res}_M^G f_1$ rel. ∂ and

(2) $\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \to (I \times Y, \{0\} \times Y, \{1\} \times Y, I \times \partial Y)$ from F_M to $F_{M,1}$ rel. $\partial_1 W_M \cup \partial_{01} W_M$

such that $\mathbb{H}_M|_{\{0\}\times X} = \mathbb{h}_M$ and $F_{M,1}|_{\text{Image}(\Psi_M)} : \text{Image}(\Psi_M) \to F_{M,1}(\text{Image}(\Psi_M))$ is an M-diffeomorphism. Note $I \times N \cong_M \text{Image}(\Psi_M) \cong_M I \times \text{Image}(\xi)$, where $N = N_M(X(\mathcal{H}(G,V,0) \cup \mathcal{K}_M), X)$, for some M-embedding $\xi : N \to Y$ rel. ∂. We remark that $X^{>H}$ coincides with $X(\mathcal{U}_M(H)) \cup X(\mathcal{Y}(G,M,H))$.

For each $K \in \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,1)$, let $A_{K,i}$, where i ranges over $[1..t_K]$, be the connected components of $(X_0^H)^K$, where $A_{K,i} \neq \emptyset$ for all $i \in [1..t_K]$ and $A_{K,i} \neq A_{K,j}$ for all i, $j \in [1..t_K]$ with $i \neq j$. By the hypothesis (K2), we have $A_{K,i} \cap X^{=K} = \emptyset$ for all $i \in [1..t_K]$ if $K \notin \text{Iso}(G, V \smallsetminus \{0\})$.

Proposition 5.6. *Suppose* $H \in \text{Iso}(G, V \smallsetminus \{0\})$ *and* $\dim V^H \geq 2$. *Let* $K \in \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,1) \cap \text{Iso}(G, V \smallsetminus \{0\})$. *Then the following holds.*

(1) $A_{K,i} \cap A_{K,j} = \emptyset$ *if* i, $j \in [1..t_K]$ *and* $i \neq j$.

(2) $A_{K,i}$ *is diffeomorphic to* D^1 *(*$= [-1,1]$*) for all* $i \in [1..t_K]$.

(3) $\text{Iso}(G, A_{K,i}) = \{K\}$ *for all* $i \in [1..t_K]$.

(4) $A_{K,i} \cap A_{K',j} = \emptyset$ *if* $K' \in \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,1)$, $K' \neq K$, $i \in [1..t_K]$ *and* $j \in [1..t_{K'}]$.

(5) Let $K' \in \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,1)$ *with* $K' \neq K$, $i \in [1..t_K]$, $j \in [1..t_{K'}]$ *and* $g \in N_G(H)$. *If* $A_{K,i} \cap gA_{K',j} \neq \emptyset$ *then* $gK'g^{-1} = K$ *and* $A_{K,i} = gA_{K',j}$.

(6) If $N_G(K) \cap M = H$ *then the group*

$$L_{K,i} = \{g \in N_G(H) \mid gA_{K,i} = A_{K,i}\}$$

coincides with H.

Proof. We prove the proposition by step-by-step basis.

Claim (1). It is clear from the definition of 'connected component'.

Claim (2). It follows from the hypothesis $f^K : X^K \to Y^K \cong D^1$ is homotopic to a diffeomorphism.

Claim (3). Assume there is a point $x \in A_{K,i}$ such that $G_x \neq K$. Then $G_x > K$. If $G_x \cap M \neq H$ then $x \in X(\mathcal{U}_M(H))$, which is a contradiction. Therefore $G_x \in \mathcal{Y}(G, M, H)$. If $\dim {A_{K,i}}^{G_x} = 0$ then $x \in X(\mathcal{H}(G, V, 0))$, which is a contradiction. It says that $G_x \in \mathcal{H}(G, V, 1)$, and therefore $K < G_x$ and $\dim V^K = \dim V^{G_x} = 1$. This contradicts the hypothesis $K \in \mathrm{Iso}(G, V \smallsetminus \{0\})$.

Claim (4). Assume there is a point $x \in A_{K,i} \cap A_{K',j}$. It follows from Claim (3) that $K = G_x = K'$, which contradicts the hypothesis $K \neq K'$.

Claim (5). Let $x \in A_{K,i} \cap gA_{K',j}$. Then $G_x = K$ as well as $G_x = gK'g^{-1}$. Therefore we get $K = gK'g^{-1}$. Note that $gA_{K',j} = A_{gK'g^{-1},j'} = A_{K,j'}$ for some $j' \in [1..t_K]$. Since $A_{K,i} \cap A_{K,j'} \neq \emptyset$, we get $j' = i$ and $gA_{K',j} = A_{K,j'} = A_{K,i}$.

Claim (6). Let $g \in L_{K,i}$. Then, since $gKg^{-1} = K$, we get $g \in N_G(K) \cap N_G(H) \subset N_G(K) \cap M = H$. Therefore Claim (6) is valid. □

Let us consider the case that (A2) in Definition 2.5 (1) is fulfilled.

Proposition 5.7 (Case (A2))**.** *Suppose that the condition (A2) in Definition 2.5 (1) is fulfilled. Then, up to modification of* $\boldsymbol{F}_M$ *by* 1*-dimensional* M*-surgeries rel.* $\partial W_M \cup \mathrm{Image}(\Psi_M)$ *(see (K2)) of isotropy type* $(H)_M$*, there is a product* $N_G(H)$*-embedding* $\phi_{N_M(H),H,\mathcal{U}} : I \times (X_0^H \cap X(\mathcal{U}_G(H)_{\mathrm{sol}})) \to {W_M}^H$ *compatible with* Ψ_M*, i.e.* $\phi_{N_M(H),H,\mathcal{U}} \cup \Psi_M$ *is a well-defined embedding. Therefore there is a product* M*-embedding* $\phi_{M,H,\mathcal{U}} : I \times X([M, \mathcal{U}_G(H)_{\mathrm{sol}}]) \to W_M$ *compatible with* Ψ_M*, and there is a product* M*-embedding* $\Phi_M : I \times X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_M \cup [M, \mathcal{U}_G(H)_{\mathrm{sol}}]) \to W_M$ *compatible with* Ψ_M*.*

In the case of the proposition above, $\mathcal{X}(G, \rho_{\max}, H) \cap \mathcal{H}(G, V, 1) = \emptyset$ and $\mathcal{Z}(G, V, M, H) = \emptyset$.

Proof. If $\mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) = \emptyset$ then we have nothing to prove. Therefore we suppose $\mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \neq \emptyset$. Similarly to Proposition 5.5 (4), we have

$$X(\mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1)) \smallsetminus \mathring{N} = X(\mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus \mathring{N}.$$

Recall Proposition 5.6. We can suppose without loss of generality that ${F_{M,1}}^H$ is transversal on ${W_{M,0}}^K$ to $(I \times Y)^K$ in $(I \times Y)^H$ for all $K \in \mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$.

Let $K \in \mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$. Let $B_{K,i}$, $i \in [1..t_K]$, be the connected components of $(F_{M,1}|_{W^H_{M,0}})^{-1}((I \times Y)^K)$ such that $B_{K,i} \cap X_0 = A_{K,i}$. $B_{K,i}$ is a compact orientable 2-dimensional surface. Since $F_{M,1}|_{\partial_1 W_M} = id_Y$, we see

$$(F_{M,1}|_{Y_0^H})^{-1}((\{1\} \times Y)^K) = (Y_0^H)^K \subset Y^K \cong D^1.$$

This shows that $(Y_0^H)^K$ can not contains circles, which implies $\partial B_{K,i} \cap (Y_0^H)^K \cong D^1$ and $\partial B_{K,i} \cong \partial(I \times D^1)$. It also follows from the transversality construction above that if $B_{K,i} \cap B_{K,j} \neq \emptyset$, for some i, $j \in [1..t_K]$ then $B_{K,i} = B_{K,j}$, i.e. $i = j$. Let $g \in N_M(H)$ such that $B_{K,i} \cap gB_{K,i} \neq \emptyset$. Then

$gB_{K,i} = B_{gKg^{-1},j} = B_{K,j}$ for some $j \in [1..t_K]$. Since $B_{K,i} \cap B_{K,j} \neq \emptyset$, we get $i = j$, and therefore $B_{K,i} = gB_{K,i}$ and $A_{K,i} = gA_{K,i}$. By Proposition 5.6 (6), g is an element of H. We get

$$\{a \in N_M(H) \mid B_{K,i} \cap aB_{K,i} \neq \emptyset\} = H.$$

Since $B_{K,i}$ is cobordant rel. ∂ to $I \times D^1$, we can perform 1-dimensional $N_M(H)/H$-surgeries rel. ∂ on $\coprod_{i\in[1..t_K]} B_{K,i}$ ($\subset W^H_{M,0}$) so that the resulting $\coprod_{i\in[1..t_K]} B'_{K,i}$ is diffeomorphic to $\coprod_{i\in[1..t_K]} I \times D^1$. This says that we can perform 1-dimensional M-surgeries rel. $\partial W_M \cup \mathrm{Image}(\Psi_M)$ on W_M of isotropy type $(H)_M$ so that the resulting surfaces $gB'_{K,i}$ in the resulting M-manifold W'_M are diffeomorphic to $I \times D^1$ for all $g \in N_M(H)$ and $i \in [1..t_K]$. After this modification of $\boldsymbol{F}_M$, there is a product $N_G(H)$-embedding

$$\phi_{N_M(H),H} : I \times (X_0^H \cap X((K)_{G,N_M(H)})) \to W^H_{M,0}$$

compatible with Ψ_M. By the hypotheses, we have $\mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1) \cap \mathrm{Iso}(G, V \smallsetminus \{0\}) = (K)_{G,M}$. Since $X(\mathcal{U}_G(H)_{\mathrm{sol}}) \subset X(\mathcal{U}_M(H) \cup (K)_{G,M} \cup \mathcal{H}(G, V, 0))$, we can obtain the desired product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{U}} : I \times (X_0^H \cap X(\mathcal{U}_G(H)_{\mathrm{sol}})) \to {W_M}^H$ compatible with Ψ_M. □

Next we consider the case that $H \in \mathrm{Iso}(G, V \smallsetminus \{0\})$ and (A1) in Definition 2.5 (1) is fulfilled. Under the hypothesis $\mathcal{Y}(G, M, H) \neq \emptyset$, we set

$$k = \dim V(\mathcal{Y}(G, M, H)) \ \big(= \max\{\dim V^K \mid K \in \mathcal{Y}(G, M, H)\}\big).$$

Here k satisfies the inequality $2k + 1 < \dim V^H$.

In the case $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$, by Theorem 2.3 (D2) and Definition 2.5 (3), we see that either (C1) or (C2) is satisfied. Recall that $\dim V^H \geq 5$ in the case (C1) and $\dim V^H \geq 4$ in the case (C2). If $\mathcal{Z}(G, V, M, H) \neq \emptyset$ and $k \geq 1$ then we can modify $\boldsymbol{f}$ (resp. $\boldsymbol{F}_M$) so that f^H (resp. ${F_M}^H$) is $(k+1)$-connected by G-surgeries of isotropy type $(H)_G$ (resp. M-surgeries of isotropy type $(H)_M$). (In order to make simultaneously f^H and ${F_M}^H$ both $(k+1)$-connected, we need M-surgeries on $\boldsymbol{F}_M$ of isotropy types in $(H)_{M,G}$.) Particularly, in the case where (C1) is satisfied, we can modify $\boldsymbol{F}_M$ so that ${F_M}^H$ is $\max(3, k+1)$-connected.

Proposition 5.8 (Case (A1, C2, $\mathcal{Z}$, 1))**.** *Suppose $H \in \mathrm{Iso}(G, V \smallsetminus \{0\})$. Suppose that the condition (A1) in Definition 2.5 (1) and the condition (C2) in Definition 2.5 (3) both are fulfilled. Further suppose $\mathcal{Z}(G, V, M, H)_1 \neq \emptyset$. Then, up to modification of $\boldsymbol{F}_M$ by 1-dimensional M-surgeries rel. $\partial W_M \cup \mathrm{Image}(\Psi_M)$ of isotropy type $(H)_M$, there is a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Z}}$: $I \times (X_0^H \cap X(\mathcal{Z}(G, V, M, H)_1)) \to W^H_{M,0}$ compatible with Ψ_M, i.e. $\phi_{N_M(H),H,\mathcal{Z}} \cup \Psi_M$ is a well-defined embedding. Therefore there is a product M-embedding $\phi_{M,H,\mathcal{Y}} : I \times X([M, \mathcal{Y}(G, M, H)]) \to W_M$ compatible with Ψ_M, where the equality $\mathcal{Y}(G, M, H) = \mathcal{Z}(G, V, M, H)_1 \cup (\mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 0))$ holds.*

In the case of the proposition above, we have $k = 1$, $\dim V^H - \dim V(\mathcal{U}_M(H)) = 2$,

$$\mathcal{Z}(G, V, M, H)_1 \cap \mathcal{H}(G, V, 1) = \mathcal{Z}(G, V, M, H)_1 = \mathcal{Y}(G, M, H) \cap \mathcal{H}(G, V, 1), \text{ and}$$
$$X(\mathcal{Z}(G, V, M, H)_1) \smallsetminus \mathring{N} = X(\mathcal{Z}(G, V, M, H)_1 \cap \mathrm{Iso}(G, V \smallsetminus \{0\})) \smallsetminus \mathring{N}.$$

Proof. Let $K \in \mathcal{Z}(G, V, M, H)_1 \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$. $N_M(H)/H$ acts freely on $N_M(H) \cdot (X_0^H)^K$. By the hypothesis, $f_1|_{X^K} : X^K \to Y^K$ is a diffeomorphism. We can suppose without any loss of generality that ${F_{M,1}}^H$ is transversal on $W_{M,0}^H$ to $I \times Y^K$ in $I \times Y^H$ (here $Y^K \cong D^1$). Let $B_{K,i}$ be the connected component of $(F_{M,1}|_{W_{M,0}^H})^{-1}(I \times Y^K)$ such that $B_{K,i} \cap X_0^H = A_{K,i}$. Then $B_{K,i} \cap B_{K,j} = \emptyset$ if $i \neq j$.

For $K' \in \mathcal{Z}(G, V, M, H)_1 \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ with $(K)_{G,M} \neq (K')_{G,M}$, the inequality $\dim B_{K,i} + \dim B_{K',j} = 4 < 5 \leq \dim W_{M,0}^H$ holds, and therefore, by the general position argument (up to M-homotopic deformation of $F_{M,1}$), we may suppose

$$B_{K,i} \cap B_{K',j} = \emptyset. \tag{5.5}$$

Let $g \in N_M(H)$ such that $B_{K,i} \cap gB_{K,i} \neq \emptyset$. Then $gB_{K,i} = B_{gKg^{-1},j} = B_{K,j}$ for some $j \in [1..t_K]$. Since $B_{K,i} \cap B_{K,j} \neq \emptyset$, we get $i = j$, and therefore $B_{K,i} = gB_{K,i}$ and $A_{K,i} = gA_{K,i}$. By Proposition 5.6 (6), g is an element of H. It means

$$\{g \in N_M(H) \mid B_{K,i} \cap gB_{K,i} \neq \emptyset\} = H. \tag{5.6}$$

Since $B_{K,i}$ is a compact connected orientable 2-dimensional surface such that $\partial B_{K,i} \cong \partial(I \times D^1)$. $B_{K,i}$ is cobordant rel. ∂ to $I \times D^1$. Therefore by 1-dimensional $N_M(H)$-surgeries on W_M of isotropy type $\{H\}$ rel. $\partial W_M \cup \mathrm{Image}(\Psi_M)$, we can modify the connected components $B_{K,i}$ so that $B_{gKg^{-1},i} \cong I \times D^1$ for all $g \in N_M(H)$. By virtue of (5.6), 1-dimensional M-surgeries on W_M of isotropy type $(H)_M$ rel. $\partial W_M \cup \mathrm{Image}(\Psi_M)$, we can modify $gB_{K,i}$ $(= B_{gKg^{-1},j})$ so that $gB_{K,i} \cong I \times D^1$ for all $g \in M$. It shows that up to the modification above, we can obtain a product $N_M(H)$-embedding $\phi_{N_M(H),H,K} : I \times X_0^H(K) \to W_{M,0}^H$ compatible with Ψ_M. Because of (5.5), there is a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Z}} : I \times (X_0^H \cap X(\mathcal{Z}(G, V, M, H)_1)) \to W_{M,0}^H$ compatible with Ψ_M. Using $\phi_{N_M(H),H,\mathcal{Z}}$ and Ψ_M, we can obtain a product M-embedding $\phi_{M,H,\mathcal{Y}} : I \times X([M, \mathcal{Y}(G, M, H)]) \to W_M$ compatible with Ψ_M. □

Proposition 5.9 (Case (A1, C1, $\mathcal{Z}$, 1))**.** *Suppose $H \in \mathrm{Iso}(G, V \smallsetminus \{0\})$. Suppose that the condition (A1) of Definition 2.5 (1) and the condition (C1) of Definition 2.5 (3) both are fulfilled. Suppose $\mathcal{Z}(G, V, M, H)_1 \cap \mathcal{H}(G, V, 1) \neq \emptyset$. Further suppose that $f^H : X^H \to Y^H$ and ${F_M}^H : {W_M}^H \to I \times Y^H$ are $(k+1)$-connected. Then, there is a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Z}} : I \times (X_0^H \cap X(\mathcal{Z}(G, V, M, H)_1)) \to W_{M,0}^H$ compatible with Ψ_M. Therefore there is a product M-embedding $\phi_{M,H,\mathcal{Y}} : I \times X([M, \mathcal{Y}(G, M, H)]) \to W_M$ compatible with Ψ_M.*

In the case of the proposition above, we have $k = 1$, $\dim V^H - \dim V(\mathcal{U}_M(H)) = 2$,

$$\mathcal{Z}(G,V,M,H)_1 = \mathcal{Z}(G,V,M,H)_1 \cap \mathcal{H}(G,V,1) = \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,1), \text{ and}$$

$$X(\mathcal{Z}(G,V,M,H)_1) \smallsetminus \mathring{N} = X(\mathcal{Z}(G,V,M,H)_1 \cap \mathrm{Iso}(G,V \smallsetminus \{0\})) \smallsetminus \mathring{N}.$$

Proof. By the hypotheses, X^H and ${W_M}^H$ are 1-connected. Since $\dim V^H - \dim V(\mathcal{U}_M(H)) \geq 3$, X_0^H, Y_0^H and $W_{M,0}^H$ are 1-connected, too.

Let K be an element of $\mathcal{Z}(G,V,M,H)_1 \cap \mathcal{H}(G,V,1) \cap \mathrm{Iso}(G,V \smallsetminus \{0\})$. For each $i \in [1..t_K]$, by virtue of the connectedness of Y_0^H, there exists an embedding $\partial\iota_{K,i} : \partial(I \times D^1) \to W_{M,0}^H$ such that $\partial\iota_{K,i}(\{0\} \times D^1) = A_{K,i}$, $\partial\iota_{K,i}(t,x) = \Phi_M(t, \partial\iota_{K,i}(0,x))$ for all $t \in I$ and $x \in \partial D^1$, and $\partial\iota_{K,i}(\{1\} \times D^1) \subset (\{1\} \times Y^K) \cap W_{M,0}^H$. Since $W_{M,0}^H$ is 1-connected and $\dim {W_{M,0}}^H \geq 6$, $\partial\iota_{K,i}$ is bounded by an embedding $\iota_{K,i} : \partial(I \times D^1) \to W_{M,0}^H$. Set $B_{K,i} = \mathrm{Image}(\iota_{K,i})$. Let

$$\pi_{M,0}^H : W_{M,0}^H \to W_{M,0}^H / N_M(H) \tag{5.7}$$

be the canonical projection. Recall $\dim W_{M,0}^H / N_M(H) \geq 6$ and $\dim B_{M,i} = 2$. Applying the general position argument to

$$\{\pi_{M,0}^H \circ \iota_{K,i} \mid K \in \mathcal{Z}(G,V,M,H)_1 \cap \mathcal{H}(G,V,1) \cap \mathrm{Iso}(G,V \smallsetminus \{0\})\},$$

we can suppose without loss of generality that $B_{K,i} \cap B_{K',j} = \emptyset$ for all K, $K' \in \mathcal{Z}(G,V,M,H)_1 \cap \mathcal{H}(G,V,1) \cap \mathrm{Iso}(G,V \smallsetminus \{0\})$, $i \in [1..t_K]$, and $j \in [1..t_{K'}]$ unless $B_{K,i} = B_{K',j}$ (i.e. $K = K'$ and $i = j$). Therefore there is a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Z}} : I \times (X_0^H \cap X(\mathcal{Z}(G,V,M,H)_1)) \to W_{M,0}^H$ compatible with Ψ_M. It yields a product M-embedding $\phi_{M,H,\mathcal{Y}} : I \times X([M, \mathcal{Y}(G,M,H)]) \to W_M$ compatible with Ψ_M. □

Proposition 5.10 (Case (A1, C1, $\mathcal{Y} \smallsetminus \mathcal{Z}$))**.** *Suppose $H \in \mathrm{Iso}(G,V \smallsetminus \{0\})$. Suppose that the condition (A1) of Definition 2.5 (1) and the condition (C1) of Definition 2.5 (3) both are fulfilled. Suppose $\mathcal{Y}(G,M,H) \smallsetminus (\mathcal{Z}(G,V,M,H)_1 \cup \mathcal{H}(G,V,0)) \neq \emptyset$. Further suppose that $f^H : X^H \to Y^H$ and ${F_M}^H : {W_M}^H \to I \times Y^H$ are $(k+1)$-connected. Set $\mathcal{T} = \mathcal{Y}(G,M,H) \smallsetminus \mathcal{Z}(G,V,M,H)_1$. Then, there is a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Y} \smallsetminus \mathcal{Z}} : I \times (X_0^H \cap X(\mathcal{T})) \to W_{M,0}^H$ compatible with Ψ_M. Therefore there is a product M-embedding $\phi_{M,H,\mathcal{Y} \smallsetminus \mathcal{Z}} : I \times X([M, \mathcal{T}]) \to W_M$ compatible with Ψ_M.*

In the proposition above, it holds that $k \geq 1$, $\dim V^H \geq 4$ and

$$\dim V^H - \dim V(\mathcal{U}_M(H)) > \dim V(\mathcal{T}) + 1 \geq 2.$$

Proof. We identify X (resp. Y) as $\{0\} \times X$ (resp. $\{1\} \times Y$) $\subset W_{M,0}$. Set $s = \dim V(\mathcal{T})$. Then $s \geq 1$, $s \in \{k-1, k\}$ and $\dim X_0^H(\mathcal{T}) = s$. By the hypotheses, X^H and ${W_M}^H$ are k-connected. Since $\dim V^H - \dim V(\mathcal{U}_M(H)) > s + 1$, X_0^H, Y_0^H and $W_{M,0}^H$ are s-connected. Recall that $\partial X_0^H \subset$

$\mathrm{Image}(\Psi_M) \cup \partial X$. Therefore there is a product $N_M(H)$-embedding

$$\partial\mu : I \times (\partial X_0^H)(\mathcal{T}) \to W_{M,0}^H \cap (\mathrm{Image}(\Psi_M) \cup \partial X).$$

Since Y_0^H and $W_{M,0}^H$ are s-connected, there is a product embedding $\mu : I \times X_0^H(\mathcal{T}) \to W_{M,0}^H$ extending $\partial\mu$. Consider the canonical covering projection $\pi_{M,0}^H : W_{M,0}^H \to W_{M,0}^H/N_M(H)$. Recall $\dim I \times X_0^H(\mathcal{T}) = s+1$ and

$$\dim W_{M,0}^H/N_M(H) = \dim V^H + 1 > (2k+1) + 1 \geq 2(s+1).$$

Applying the general position argument, we can obtain a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Y}\smallsetminus\mathcal{Z}} : I \times (X_0^H \cap X(\mathcal{T})) \to W_{M,0}^H$ compatible with Ψ_M. □

If $\mathcal{Z}(G,V,M,H)_1 \smallsetminus \mathcal{H}(G,V,1) \neq \emptyset$, then $k \geq 2$ and

$$\mathcal{Z}(G,V,M,H)_1 = \mathcal{Z}(G,V,M,H))_1 \cap \mathcal{H}(G,V,k) = \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,k)$$

(see Theorem 2.3 (D2) and Definition 2.5 (A1) (ii)).

Proposition 5.11 (Case (A1, $\mathcal{Z}$, $k \geq 2$))**.** *Suppose $H \in \mathrm{Iso}(G, V \smallsetminus \{0\})$. Suppose that the condition (A1) of Definition 2.5 (1) is fulfilled, Suppose $\mathcal{Z}(G,V,M,H)_1 \smallsetminus \mathcal{H}(G,V,1) \neq \emptyset$. Further suppose that $f^H : X^H \to Y^H$ and ${F_M}^H : {W_M}^H \to I \times Y^H$ are $(k+1)$-connected. Then, there is a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Z}} : I \times (X_0^H \cap X(\mathcal{Z}(G,V,M,H)_1)) \to W_{M,0}^H$ compatible with Ψ_M and $\phi_{M,H,\mathcal{Y}\smallsetminus\mathcal{Z}}$ in the previous proposition. Therefore there is a product M-embedding $\phi_{M,H,\mathcal{Y}} : I \times X([M, \mathcal{Y}(G,M,H)]) \to W_M$ compatible with Ψ_M.*

In the situation of the proposition, we have

$$\mathcal{Z}(G,V,M,H)_1 = \mathcal{Z}(G,V,M,H)_1 \cap \mathcal{H}(G,V,k) = \mathcal{Y}(G,M,H) \cap \mathcal{H}(G,V,k),$$

$\dim V^H > 2k+1$ (≥ 5), $\dim V^H - \dim V(\mathcal{U}_M(H)) \geq k+1$ (≥ 3), and (C1) of Definition 2.5 is fulfilled. Recall (iii) $\dim V^L \geq 2$ for $L \in \mathcal{Z}(G,V,M,H)_2$ and (iv) $\dim V^{L_1} - \dim V^{\langle L_1, L_2\rangle} \geq 2$ for L_1, $L_2 \in \mathcal{Z}(G,V,M,H)_2$ with $L_1 \neq L_2$.

Proof. The spaces X^H and ${W_M}^H$ are k-connected. Since $\dim V^H - \dim V(\mathcal{U}_M)(H))) = k+1$, X_0^H and $W_{M,0}^H$ are $(k-1)$-connected.

Note $A = X_0^H \cap X(\mathcal{Z}(G,V,M,H)_1)$ is a k-dimensional manifold. There is a product embedding $\partial\iota : I \times \partial A \to W_{M,0}^H$ compatible with Φ_M. Since ${W_M}^H$ is k-connected and $\dim {W_M}^H > 2(k+1)$, there is a product embedding $\iota : I \times A \to {W_M}^H$ extending $\partial\iota$. By the general position argument, we can suppose without loss of generality that

(1) $\mathrm{Image}(\iota) \cap W_M(\mathcal{U}_M(G) \smallsetminus \mathcal{Z}(G,V,M,H)_2)) = \emptyset$,
(2) $|\mathrm{Image}(\iota) \cap W_M(\mathcal{Z}(G,V,M,H)_2)| < \infty$,

(3) $T_z(W_M) = T_z(\mathrm{Image}(\iota)) \oplus T_z({W_M}^L)$ for every $z \in \mathrm{Image}(\iota) \cap W_M(\mathcal{Z}(G,V,M,H)_2)$.

Recall $\dim {W_M}^H \geq 6$ and $\dim {W_M}^H - \dim W_M(\mathcal{Z}(G,V,M,H)_2) \geq 3$. For $L \in \mathcal{Z}(G,V,M,H)_2$ and $z \in \mathrm{Image}(\iota) \cap W_M(\mathcal{Z}(G,V,M,H)_2)$, there is a 2-dimensional disk $\Delta_{L,z}$ in ${W_M}^H$ with $\partial\Delta_{L,z} = I_{01} \cup I_{12} \cup I_{20}$ such that I_{01}, I_{12} and I_{20} are diffeomorphic to $I = [0,1]$, and moreover $I_{01} \cap I_{20} = \{z\}$, $\Delta_{L,z} \cap Y^H = I_{12}$, $I_{20} \subset \mathrm{Image}(\iota)$, and

$$\Delta_{L,z} \smallsetminus \partial\Delta_{L,z} \subset (\mathrm{Int}({W_M}^H) \smallsetminus (\mathrm{Image}(\iota) \cup W_M(\mathcal{U}_M(H)))).$$

Here we may assume $\Delta_{L,z} \cap \Delta_{L,z'} = \emptyset$ for all $z' \in \mathrm{Image}(\iota)^{L'}$ with $z' \neq z$, and $\Delta_{L,z} \cap \Delta_{L',z'} = \emptyset$ for all $L' \in \mathcal{Z}(G,V,M,H)_2$ with $L' \neq L$, $z \in \mathrm{Image}(\iota)^L$ and $z' \in \mathrm{Image}(\iota)^{L'}$. Observe Figure 5.1.

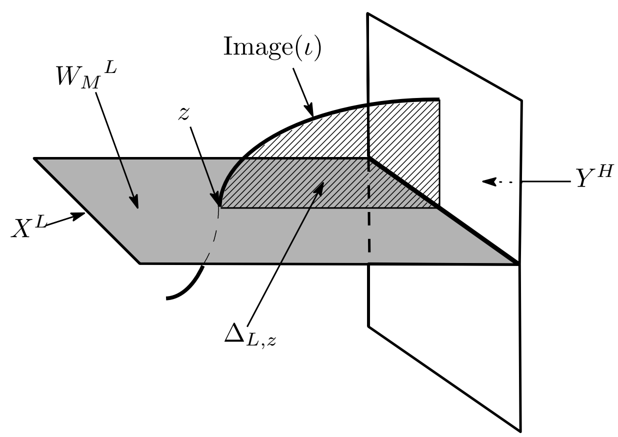

FIGURE 5.1

Via the Whitney trick along the disk $\Delta_{L,z}$, we can remove the intersection point z by an isotopic deformation of ι. Therefore, we can assume without loss of generality that $\mathrm{Image}(\iota) \cap W_M(\mathcal{U}_M(H)) = \emptyset$, and furthermore that $\mathrm{Image}(\iota) \subset W^H_{M,0}$.

Applying the general position argument to $\pi \circ \iota$, where $\pi : {W_M}^{=H} \to {W_M}^{=H}/N_M(H)$ is the canonical projection, we can obtain a product $N_M(H)$-embedding $\phi_{N_M(H),H,\mathcal{Z}} : I \times A \to W^H_{M,0}$ compatible with Ψ_M and $\phi_{M,H,\mathcal{Y} \smallsetminus \mathcal{Z}}$ in the previous proposition. □

Putting Propositions 5.7–5.11 together, we obtain the next theorem.

Theorem 5.12. *Let G, V and $(\mathcal{F}, \mathcal{F}')$ be those in Theorem 2.3. Let $\boldsymbol{f}$ be a G-framed map and let $\boldsymbol{F}_L$ be L-framed cobordisms stated in the first paragraph of this section, where L runs over $\max(\mathcal{S}(G)_{\mathrm{sol}})^*$. Let $\mathcal{K}$ be a G-conjugation-invariant and upwardly closed subset of $\mathcal{S}(G)_{\mathrm{sol}}$ fulfilling the hypotheses (K1) and (K2). Let $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ and $M = \rho_{\max}(H)$. Then, up to G-surgeries* *rel.* ∂ *on* $\boldsymbol{f}$ *of isotropy type* $(H)_G$ *and* M*-surgeries rel.* $\partial_1 W_M \cup \partial_{01} W_M \cup \mathrm{Image}(\Psi_M)$ *on* $\boldsymbol{F}_M$ *of isotropy types in* $(H)_{M,G}$*, there is a product* M*-embedding* $\phi_{M,H,\mathcal{Y}} : I \times X([M, \mathcal{Y}(G,M,H)]) \to W_M$ *compatible with* Ψ_M*. Therefore there is a product* M*-embedding* $\Phi_{M,H,\mathcal{Y}} : I \times X(\mathcal{H}(G,V,0) \cup \mathcal{K}_M \cup [M, \mathcal{Y}(G,M,H)]) \to W_M$ *compatible with* Ψ_M*.*

6. Proof of Theorem 2.4

We prove Theorem 2.4 by induction on the G-conjugacy classes $(H)_G$ contained in $\mathcal{S}(G)_{\mathrm{sol}}$. Let $\boldsymbol{f}$ and $\{\boldsymbol{F}_L\}_L$ be those in the previous section, where L ranges over $\max(\mathcal{S}(G)_{\mathrm{sol}})^*$.

We quote the reflection method in the equivariant surgery theory.

Lemma 6.1 ([23, Lemma 6.1])**.** *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be* G*-conjugation-invariant and upwardly closed subsets of* $\mathcal{S}(G)_{\mathrm{sol}}$ *such that* $\mathcal{K} \subset \mathcal{F}$*, let* M *be an element of* $\max(\mathcal{S}(G)_{\mathrm{sol}})^*$*, and let* H *be an element of* $\mathcal{S}(M) \smallsetminus (\mathcal{H} \cup \mathcal{K})$ *such that* $N_G(H) \subset M$*. Invoke the following two hypotheses.*

(S1) *There is a product* M*-embedding* $\Phi_M : I \times N_M(X(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X^{>H}, X) \to W_M$ *and* (X, Y, W_M) *has the* (G,M)*-tame singular set at* H *with respect to the restriction of* Φ_M *to* $I \times N_M(M \cdot X^{>H}, X)$*.*

(S2) *There is an* M*-homotopy*

$$\mathbb{H}_M : (W_M, \partial_0 W_M, \partial_1 W_M, \partial_{01} W_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z) \quad \textit{(where } Z = I \times Y\textit{)}$$

rel. $\partial_1 W_M \cup \partial_{01} W_M$ *such that* $\mathbb{H}_M|_{W_M \times \{0\}}$ *coincides with* F_M *and*

$$\mathbb{H}_M|_{\mathrm{Image}(\Phi_M)\times\{1\}} : \mathrm{Image}(\Phi_M) \times \{1\} \to \mathbb{H}_M(\mathrm{Image}(\Phi_M) \times \{1\})$$

is a diffeomorphism.

Then there are

- *a* G*-framed map* $\boldsymbol{f}'$ *rel.* ∂*, where (as is described before)* $\boldsymbol{f}'$ *is a pair* (f', b') *of* $f' : (X', \partial X') \to (Y, \partial Y)$ *and* $b' : \varepsilon_{X'}(\mathbb{R}) \oplus T(X') \oplus \varepsilon_{X'}(\mathbb{R}^\ell) \to \varepsilon_{X'}(\mathbb{R} \oplus V \oplus \mathbb{R}^\ell)$*,*
- *a* G*-framed cobordism* $\boldsymbol{F}_G$ *from* $\boldsymbol{f}$ *to* $\boldsymbol{f}'$ *rel.* ∂ *and* $\mathcal{V}_G(H)$*,*
- *an* M*-framed cobordism* $\mathbb{F}_M$ *from* $\mathrm{res}^G_M \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_M \boldsymbol{f}} \boldsymbol{F}_M$ *to* $\boldsymbol{F}'_M$ *rel.* ∂ *and* $\mathcal{V}_{M,G}(H)$*, where* $\boldsymbol{F}'_M = (F'_M, B'_M)$ *with*

$$F'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

is an M*-framed cobordism from* $\mathrm{res}^G_M \boldsymbol{f}'$ *to* $\mathrm{res}^G_M \boldsymbol{id}_Y$ *rel.* ∂ *and* $\mathcal{V}_{M,G}(H)$*,*
- *a natural identification* M*-map* : $N_M(X(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X^{>H}, X) \to N_M(X'(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X'^{>H}, X')$*,*
- *a product* M*-embedding* $\Phi'_M : I \times N_M(X'(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X'^{H}, X') \to W'_M$*,*

- *a natural identification* M*-map* $: \mathrm{Image}(\Phi_M) \to \Phi'_M(I \times N_M(X'(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X'^{>H}, X'))$ *such that the diagram*

$$\begin{array}{ccc} I \times N_M(X(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X^{>H}, X) & \xrightarrow{\Phi_M} & \mathrm{Image}(\Phi_M) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ I \times N_M(X'(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X'^{>H}, X') & \xrightarrow[\Phi'_M|]{} & \Phi'_M(X'(\mathcal{H} \cup \mathcal{K}_M) \cup M \cdot X'^{>H}, X') \end{array}$$

commutes, and

- *an* M*-homotopy*

$$\mathbb{H}'_M : (W'_M, \partial_0 W'_M, \partial_1 W'_M, \partial_{01} W'_M) \times I \to (Z, \partial_0 Z, \partial_1 Z, \partial_{01} Z)$$

rel. $\partial_1 W'_M \cup \partial_{01} W'_M$

possessing the following compatible properties.

(1) $\mathbb{H}'_M|_{W'_M \times \{0\}}$ *coincides with* F'_M,
(2) $\mathbb{H}'_M|_{N_M(M \cdot W'^{H}_M, W'_M) \times \{1\}}$ *is a diffeomorphism, and*
(3) $\mathbb{H}'_M|_{\mathrm{Image}(\Phi_M) \times I}$ *coincides with* $\mathbb{H}_M|_{\mathrm{Image}(\Phi_M) \times I}$.

In particular, X'^H *is* $N_G(H)$*-diffeomorphic rel.* ∂ *to* Y^H *and* $f'^H : X'^H \to Y^H$ *is* $N_G(H)$*-homotopic rel.* ∂ *to a diffeomorphism.*

Proof. Recall

$$\mathcal{K}_M = [M, \mathcal{K} \cap (\rho_{\max}^{-1}(M) \cup \mathcal{U}_M(\rho_{\max}^{-1}(M)))].$$

Since $(\mathcal{H} \cup \mathcal{K}) \cap \mathcal{V}_G(H) = \emptyset$, the lemma follows from the proof of [23, Lemma 6.1]. □

Remark 6.2. If $(H)_G|_M = (H)_M$, where $(H)_G|_M = (H)_G \cap \mathcal{S}(M)$, then we get the conclusions in Lemma 6.1 for H replaced by arbitrary $H' \in (H)_G|_M$.

We can suppose without loss of generality that $(\boldsymbol{f}, \{\boldsymbol{F}_L\}_L)$ is adjusted on $(\mathcal{H}(G, V, 0), \mathcal{F}(0))$. For $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$, we set $\mathcal{T}(L) = \mathcal{H}(G, V, 0) \cup \mathcal{K}_L$.

Proposition 6.3. *Let* $\mathcal{K}$ *be a* G*-conjugation-invariant and upwardly closed subset of* $\mathcal{F}$ *fulfilling the hypotheses (K1) and (K2). Let* $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^* \smallsetminus \mathrm{Iso}(G, V \smallsetminus \{0\})$ *and* $M = \rho_{\max}(H)$*. Then there exist*

- *a* G*-framed cobordism* $\boldsymbol{F}_G = (F_G, B_G)$ *from* $\boldsymbol{f}$ *to* $\boldsymbol{f}'$ *rel.* $N_M(X(\mathcal{T}(M)), X) \cup \partial X$ *and* $\mathcal{V}_G(H)$*, where* $F_G : W_G \to I \times Y$ *and* $\boldsymbol{f}' = (f', b')$ *with* $f' : (X', \partial X') \to (Y, \partial Y)$*, and*
- *a family* $\{\mathbb{F}_L \mid L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*\}$ *consisting of* L*-framed cobordisms* $\mathbb{F}_L$ *from* $\mathrm{res}^G_L \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_L \boldsymbol{f}} \boldsymbol{F}_L$ *to* $\boldsymbol{F}'_L$ *rel.* $(I \times N_L(X'(\mathcal{T}(L)), X'))^\# \cup \partial_1 W_L \cup \partial_{01} W_L$ *and* $\mathcal{V}_{L,G}(H)$*, where* $(I \times N_L(X'(\mathcal{T}(L)), X'))^\#$

stands for

$$(I \times N_L(X(\mathcal{T}(L)), X)) \bigcup_{\{0\}\times N_L(X(\mathcal{T}(L)),X)} \Psi_L(I \times N_L(X(\mathcal{T}(L)), X)),$$

$\boldsymbol{F}'_L$ is obtained by L-surgeries of isotropy types contained in $(H)_{L,G}$ on $\mathrm{res}^G_L \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_L \boldsymbol{f}} \boldsymbol{F}_L$, and $\mathbb{F}_L$ is the trace of the L-surgeries,

such that $(\boldsymbol{f}', \{\boldsymbol{F}'_L\}_L)$ is adjusted on $(\mathcal{H}(G, V, 0), \mathcal{K} \cup (H)_G)$, where L ranges over $\max(\mathcal{S}(G)_{\mathrm{sol}})^*$.

In the proposition above, ${f'}^H : {X'}^H \to Y^H$ is $N_G(H)$-homotopic rel. ∂ to a diffeomorphism, and therefore ${X'}^H$ is $N_G(H)$-diffeomorphic to the disk $D(V)^H$, for the subgroup H.

Proof. By the hypothesis, we have $\mathcal{U}_G(H)_{\mathrm{sol}} \subset \mathcal{K}$ and X^K is diffeomorphic to a disk D^d for all $K \in \mathcal{U}_G(H)_{\mathrm{sol}}$, where $d = \dim V^K$. By Proposition 5.4 we have $\mathcal{U}_M(H) \subset \mathcal{K}_M$, and ${W_M}^K$ is diffeomorphic to $Z^K = I \times D^d$ for all $K \in \mathcal{U}_M(H)$ $(\subset \mathcal{F}')$.

Recall the hypothesis $\dim V^H > 0$. The hypothesis $H \notin \mathrm{Iso}(G, V \smallsetminus \{0\})$ implies that there is a subgroup $\overline{H} \in \mathrm{Iso}(G, V \smallsetminus \{0\}) \cap \mathcal{U}_G(H)_{\mathrm{sol}}$ such that $V^H = V^{\overline{H}}$. By the condition (D1) of Theorem 2.3, we have $\overline{H} \subset M$ and $\overline{\rho}_{\max}(\overline{H}) = \rho_{\max}(H) = M$. Particularly we have $\overline{H} \cap M = \overline{H} > H$. It holds that $X^H = X^{=H} \amalg X^{\overline{H}}$, and $X^{\overline{H}}$ is diffeomorphic to the disk D^d, and that ${W_M}^H = {W_M}^{=H} \amalg {W_M}^{\overline{H}}$, and ${W_M}^{\overline{H}}$ is diffeomorphic to $I \times D^d$, where $d = \dim V^H$. Let W' be a copy of ${W_M}^H$ and observe the $N_M(H)$-cobordism $W'' = W' \bigcup_{X^H} {W_M}^H$. W'' is $N_M(H)$-cobordant to $I \times Y^H$ rel. $\partial W''$. By the reflection method, i.e. Lemma 6.1, we can obtain a G-framed cobordism $\boldsymbol{F}_G$ from $\boldsymbol{f}$ to $\boldsymbol{f}'$ rel. ∂ and $\mathcal{V}_G(H)$, an M-framed cobordism $\mathbb{F}_M$ from $\mathrm{res}^G_M \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_M \boldsymbol{f}} \boldsymbol{F}_M$ to $\boldsymbol{F}'_M$ rel. ∂ and $\mathcal{V}_{M,G}(H)$, and an M-homotopy $\mathbb{H}'_M$ rel. $\partial_1 W'_M \cup \partial_{01} W'_M$ and $\mathcal{U}_M(H)$ from F'_M to $F'_{M,1}$ satisfying the condition that

$${F'_{M,1}}^H : ({W'_M}^H, \partial_0 {W'_M}^H, \partial_1 {W'_M}^H, \partial_{01} {W'_M}^H) \to (I \times Y^H, \{0\} \times Y^H, \{1\} \times Y^H, I \times \partial Y^H)$$

is a diffeomorphism. (Therefore ${f'}^H : {X'}^H \to Y^H$ is $N_M(H)$-homotopic to a diffeomorphism. Recall $N_G(H) = N_M(H)$.) It implies that ${F'_{M,1}}^{H'}$ is an $N_M(H')$-diffeomorphism for all $H' \in (H)_M$ and that ${f'}^{H'} : {X'}^{H'} \to Y^{H'}$ is $N_M(H')$-homotopic to a diffeomorphism for all $H' \in (H)_M$, where the equality $N_M(H') = N_G(H')$ holds.

Next let $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^* \smallsetminus \{M\}$ and observe the L-framed cobordism $\boldsymbol{F}''_L = \mathrm{res}^G_L \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_L \boldsymbol{f}} \boldsymbol{F}_L$ from $\mathrm{res}^G_L \boldsymbol{f}'$ to $\mathrm{res}^G_L \boldsymbol{id}_Y$ rel. ∂. Let $K \in \mathcal{K}^*$ such that $\rho_{\max}(K) = L$. Since $\mathcal{K}$ is G-conjugation invariant as well as upwardly closed in $\mathcal{S}(G)_{\mathrm{sol}}$ and $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^*$, K is not G-subconjugate to H. Therefore we have ${W_G}^K = I \times X^K$ and ${X'}^K = X^K$. If $K \in \mathcal{S}(L)$ then

$$(W_G \cup_X W_L)^K = (I \times X^K) \cup_{X^K} {W_L}^K \cong {W_L}^K \cong I \times Y^K.$$

If $\mathcal{S}(L) \cap (H)_G = \emptyset$, then we can adopt $\boldsymbol{F}''_L$ as $\boldsymbol{F}'_L$ desired in the proposition. Therefore we now suppose $\mathcal{S}(L) \cap (H)_G \neq \emptyset$. We must modify $\boldsymbol{F}''_L$ to achieve the property ${W_L}^{H'} \cong I \times Y^{H'}$ for all $H' \in [L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))]$. Decompose $[L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))]$ to the disjoint sum

$$[L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))] = \coprod_{i \in [1..m]} (H_i)_L$$

such that $H_i \in (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))$ equipped with $H_{i,0} \in \mathcal{U}_L(\rho_{\max}^{-1}(L))$ satisfying $H_{i,0} < H_i \leq L$. By the definition, the group $M_i = \overline{\rho}_{\max}(H_i) = M$ does not coincide with L, and therefore $H_i \in \mathcal{X}(G, \rho_{\max}, H_{i,0})$. By the condition (D2) in Theorem 2.3 and the condition (B1) in Definition 2.5 (2), we get

$$\dim V^{H_i} \leq 1.$$

The hypothesis $H \notin \mathrm{Iso}(G, V \smallsetminus \{0\})$ implies $H_i \notin \mathrm{Iso}(G, V \smallsetminus \{0\})$. Since $\mathcal{K} \supset \mathcal{H}(G, V, 0)$, we get $\dim V^{H_i} = 1$. There is a subgroup $K_i \in \mathcal{U}_G(H_i) \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ such that $V^{K_i} = V^{H_i}$.

First consider the case of i such that $K_i \cap L > H_i$. If $K \in \mathcal{U}_L(H_i)$ then we see $K \in \mathcal{K}_L$ because $\mathcal{U}_L(H_{i,0}) \subset \mathcal{U}_L(\rho_{\max}^{-1}(L))$ and $K \in \mathcal{K}$. By the hypothesis (K2), we get

$$W''^{>H_i}_L = \bigcup_{K \in \mathcal{U}_L(H_i)} {W''_L}^K = {W''_L}^{K_i \cap L} \ \ (\cong I \times X^{K_i \cap L})$$

(recall $X^{H_i} = X^{K_i \cap L} = X^{K_i}$). We remark ${W''_L}^{H_i} = W''^{=H_i}_L \amalg W''^{>H_i}_L$. Each connected component of $W''^{=H_i}_L$ is a closed oriented 2-dimensional surface and hence null-cobordant. By the condition (B2) in Definition 2.5 (2), we have $N_G(H_i) \cap L = H_i$. We can perform L-surgeries on $\boldsymbol{F}''_L$ of isotropy type $(H_i)_L$ rel. ∂ to remove $W''^{=H_i}_L$. This argument allows us to suppose $W''^{=H_i}_L = \emptyset$ whenever $K_i \cap L > H_i$.

Next we consider the case of i such that $K_i \cap L = H_i$. In this case, we have $\dim V^T = 0$ for all $T \in \mathcal{U}_L(H_i)$. Thus we get $Y(\mathcal{U}_L(H_i)) = Y^G = \{0\}$, which implies that $X(\mathcal{U}_L(H_i)) = X^L$ and X^L consists of only one point x_L. In addition, we have $W''_L(\mathcal{U}_L(H_i)) = {W''_L}^L \cong I \times \{0\}$, because $\mathcal{K} \supset \mathcal{H}(G, V, 0)$. Recall that $X^{H_i} \cong Y^{H_i} = D^1$. We have the decomposition ${W''_L}^{H_i} = S \amalg \coprod_j S_j$ consisting of connected components, where S is the component containing ${X'}^{H_i} \cup \partial {W''_L}^{H_i} \cup Y^{H_i}$. Note that $S \supset {W''_L}^L$, that S is a compact orientable 2-dimensional surface with boundary diffeomorphic to $I \times \partial Y^{H_i}$, and that each S_j is a closed orientable 2-dimensional surface. Therefore we can perform surgeries on ${W''_L}^{H_i}$ rel. ∂ so as to achieve ${W''_L}^{H_i} \cong I \times Y^{H_i}$ ($\cong I \times {X'}^{H_i}$). By the condition (B2) in Definition 2.5 (2), we have $N_G(H_i) \cap L = H_i$. We can perform L-surgeries on $\boldsymbol{F}''_L$ of isotropy type $(H_i)_L$ rel. ∂ to obtain $\boldsymbol{F}'_L$ such that ${W'_L}^{H_i} \cong I \times Y^{H_i}$. Since W'_L is an L-cobordism, we see ${W'_L}^K \cong I \times Y^K$ for all $K \in (H_i)_L = [L, \{H_i\}]$.

Putting all this together, we obtain the proposition. □

Proposition 6.4. *Let $\mathcal{K}$ be a G-conjugation-invariant and upwardly closed subset of $\mathcal{F}$ fulfilling the hypotheses (K1) and (K2). Let $H \in \max(\mathcal{F} \smallsetminus \mathcal{K})^* \cap \mathrm{Iso}(G, V \smallsetminus \{0\})$ and $M = \rho_{\max}(H)$. Then the same conclusion as Proposition 6.3 holds.*

Proof. Since $H \in \mathcal{F}^* \smallsetminus \mathcal{K}$, we have $\dim V^H > 0$. Recall the following.

- The map Ψ_L is a product L-embedding $I \times N_L(X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_L), X) \to W_L$ for $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$.
- The map $\Phi_{M,H,\mathcal{Y}}$ in Theorem 5.12 is a product M-embedding $I \times X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_M \cup [M, \mathcal{Y}(G, M, H)]) \to W_M$ compatible with Ψ_M.
- $X^{>H} = X(\mathcal{Y}(G, M, H) \cup \mathcal{U}_M(H))$ and $\mathcal{U}_M(H) = \mathcal{K}_M \cap \mathcal{U}_G(H)_{\mathrm{sol}}$ (see Proposition 5.4).

Therefore $X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_M) \cup MX^{>H}$ coincides with $X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_M \cup [M, \mathcal{Y}(G, M, H)])$. There is a product M-embedding

$$\Phi_M : I \times N_M(X(\mathcal{H}(G, V, 0) \cup \mathcal{K}_M) \cup MX^{>H}, X) \to W_M$$

extending Ψ_M and $\Phi_{M,H,\mathcal{Y}}$. Let $\boldsymbol{f}'$, $\boldsymbol{F}_G$, $\boldsymbol{F}'_M$ and Φ'_M be the resulting maps by Lemma 6.1.

To obtain the desired L-framed cobordism $\boldsymbol{F}'_L$ for $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^* \smallsetminus \{M\}$, we set $\boldsymbol{F}''_L = \mathrm{res}^G_L \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_L \boldsymbol{f}} \boldsymbol{F}_L$. We have to arrange $\boldsymbol{F}''_L$ so that ${W''_L}^K \cong I \times {X'}^K$ for $K \in (\mathcal{K} \cup (H)_G)_L$. By the hypothesis (K2), ${W''_L}^K \cong I \times {X'}^K$ for $K \in \mathcal{K}_L$ and ${X'}^K \cong Y^K = D^1$ for $K \in (H)_G$. By Proposition 5.4, we see

$$(\mathcal{K} \cup (H)_G)_L = \begin{cases} \mathcal{K}_L \cup [L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))] & (H \in \mathcal{F}') \\ \mathcal{K}_L & (H \notin \mathcal{F}') \end{cases}$$

If $H \notin \mathcal{F}'$ then we have nothing to modify on $\boldsymbol{F}''_L$. Therefore we now consider the case $H \in \mathcal{F}'$. Decompose $[L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))]$ to the disjoint union

$$[L, (H)_G \cap \mathcal{U}_L(\rho_{\max}^{-1}(L))] = \coprod_{i \in [1..m]} (H_i)_L$$

with $H_i \in (H)_G$ and $H_{i,0} \in \rho_{\max}^{-1}(L)$ such that $H_{i,0} < H_i \le L$. Since $H_i \notin \mathcal{H}(G, V, 0) \subset \mathcal{K}$ and $H_i \in \mathcal{X}(G, \rho_{\max}, H_{i,0})$, we get $\dim V^{H_i} = 1$. We remark the following.

- $H_i \in \mathrm{Iso}(G, V \smallsetminus \{0\})$.
- $Y^K = \{0\}$, ${X'}^K = \{x_K\}$ and ${W''_L}^K \cong I$ for $K \in \mathcal{U}_G(H_i)_{\mathrm{sol}}$.
- Each connected component of $(W''_L \smallsetminus \mathring{N})^{H_i}$, where $N = \mathrm{Image}(\Psi_L)$, is a 2-dimensional compact orientable surface of which the boundary is empty or diffeomorphic to $\partial(I \times D^1)$.

Therefore we can perform surgeries on ${W''_L}^{H_i}$ rel. ∂ and ${W''_L}^{H_i} \cap N$ so that the resulting manifold ${W''_L}^{H_i}$ is diffeomorphic to $I \times {X'}^{H_i}$. Since $N_G(H_i) \cap L = H_i$, we can perform L-surgeries on W''_L rel. ∂ of isotropy types $(H_i)_L$, $i \in [1..m]$, so that the resulting manifold W'_L satisfies ${W'_L}^{H_i} \cong I \times {X'}^{H_i}$.

Putting all this together, we obtain the lemma above. □

By inductive argument on $\mathcal{K}$ using Propositions 6.3 and 6.4, we can obtain the next proposition.

Proposition 6.5. *There exist*

- *a G-framed cobordism $\boldsymbol{F}_G$ from $\boldsymbol{f}$ to $\boldsymbol{f}'$ rel. ∂ and $\mathcal{S}(G)_{\mathrm{nonsol}}$, and*
- *L-framed cobordisms $\mathbb{F}_L$ from $\mathrm{res}^G_L \boldsymbol{F}_G \bigcup_{\mathrm{res}^G_L \boldsymbol{f}} \boldsymbol{F}_L$ to $\boldsymbol{F}'_L$ rel. $\partial_1 W_L \cup \partial_{01} W_L$, where L ranges over $\max(\mathcal{S}(G)_{\mathrm{sol}})^*$ and $\boldsymbol{F}'_L$ is an L-framed cobordism from $\mathrm{res}^G_L \boldsymbol{f}'$ to $\mathrm{res}^G_L \boldsymbol{id}_Y$ rel. ∂,*

such that $(\boldsymbol{f}', \{\boldsymbol{F}'_L\}_L)$ is adjusted on $(\mathcal{H}(G, V, 0), \mathcal{F})$.

Lastly we consider the case $H \in \mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F}$. For $H \trianglelefteq N \in \mathcal{S}(G)$, let $\mathcal{G}_1(N, H)$ denote the set of all $K \in \mathcal{U}_N(H)$ such that K/H is hyperelementary, i.e. there is a cyclic group $C \trianglelefteq K/H$ such that $|(K/H)/C|$ is a prime power.

Proposition 6.6. *Let H be an element of $\mathcal{S}(G)_{\mathrm{sol}} \smallsetminus \mathcal{F}$ and set $N = N_G(H)$. Suppose that $f^K : X^K \to Y^K$ is a homology equivalence for all $K \in \mathcal{G}_1(N, H)$. Then a G-framed map $\boldsymbol{f}' = (f', b')$ rel. ∂ such that*

(1) $\mathrm{res}^G_H \boldsymbol{f}'$ is H-framed cobordant rel. ∂ to $\mathrm{res}^G_H \boldsymbol{id}_Y$ and

(2) ${f'}^H : {X'}^H \to Y^H$ is a homotopy (resp. homology) equivalence if $\dim V^H \geq 5$ (resp. $\dim V^H = 3$)

is obtainable by G-connected-sum operations associated with $[G/G] - \beta_G$ and G-surgeries of isotropy type $(H)_G$ on $\boldsymbol{f}$.

Proof. First we remark that $\mathcal{G}_1(N, H) \subset \mathcal{S}(G)_{\mathrm{sol}}$. Let $L \in \max(\mathcal{S}(G)_{\mathrm{sol}})^*$. Set $\Sigma(\boldsymbol{f}) = \boldsymbol{f} \bigcup_{\partial \boldsymbol{f}} \boldsymbol{id}_Y$, $\Sigma(\boldsymbol{id}_Y) = \boldsymbol{id}_Y \bigcup_{\partial \boldsymbol{id}_Y} \boldsymbol{id}_Y$, and $\Sigma(\boldsymbol{F}_L) = \boldsymbol{F}_L \bigcup_{I \times \mathrm{res}^G_L \partial \boldsymbol{f}} (I \times \mathrm{res}^G_L \boldsymbol{id}_Y)$. Then $\Sigma(\boldsymbol{F}_L)$ is an L-framed cobordism from $\mathrm{res}^G_L \Sigma(\boldsymbol{f})$ to $\mathrm{res}^G_L \Sigma(\boldsymbol{id}_Y)$. Here we remark that $\Sigma(\boldsymbol{id}_Y) = \boldsymbol{id}_{S(\mathbb{R} \oplus V)}$. Recall that Proposition 9.3 of [23] was obtained by equivariant connected-sum operations associated with $[G/G] - \beta_G$ and G-surgeries of isotropy type $(H)_G$ on $\boldsymbol{f}$. (The keys of the proof were the equivariant surgery theory [1, 3] and the induction theory [21, Theorem 13.5].) Therefore the proposition above follows from [23, Proposition 9.3]. □

Theorem 2.4 follows from Propositions 6.5 and 6.6.

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