The irreducible weak modules for the fixed point subalgebra of the vertex algebra associated to a non-degenerate even lattice by an automorphism of order 2 (Part 2)

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Abstract

Let V_L be the vertex algebra associated to a non-degenerate even lattice L, θ the automorphism of V_L induced from the -1 symmetry of L, and V_L^+ the fixed point subalgebra of V_L under the action of θ. In this series of papers, we classify the irreducible weak V_L^+-modules and show that any irreducible weak V_L^+-module is isomorphic to a weak submodule of some irreducible weak V_L-module or to a submodule of some irreducible θ-twisted V_L-module. Let $M(1)^+$ be the fixed point subalgebra of the Heisenberg vertex operator algebra $M(1)$ under the action of θ. In this paper (Part 2), we show that there exists an irreducible $M(1)^+$-submodule in any non-zero weak V_L^+-module and we compute extension groups for $M(1)^+$.

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Contents

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1 Introduction

Let L be a non-degenerate even lattice of finite rank d, V_L the vertex algebra associated to L, θ the automorphism of V_L induced from the -1 symmetry of L, and V_L^+ the fixed point subalgebra of V_L under the action of θ. The fixed point subalgebras play an important role in the study of vertex algebras. For example, the moonshine vertex algebra $V^\natural$ is constructed as a direct sum of V_Λ^+ and some irreducible V_Λ^+-module in [12] where Λ is the Leech lattice. The moonshine conjecture [7], which is an unexpected connection between the monster group and modular functions, was proved by Borcherds using $V^\natural$ in [6]. The aim of this series of papers is to classify the irreducible weak V_L^+-modules (see Definition 2.1 for the definition). Because of the large number of pages in the original paper [18], we divide the paper into 3 parts in a series for publication. This paper is Part 2 and a continuation of Part 1 [19]. I will write the main result here again, which is stated in [19, Theorem 1.1]:

Theorem 1.1. *Let L be a non-degenerate even lattice of finite rank with a bilinear form $\langle\ ,\ \rangle$. The following is a complete set of representatives of equivalence classes of the irreducible weak V_L^+-modules:*

(1) $V_{\lambda+L}^{\pm}$, $\lambda+L\in L^\perp/L$ with $2\lambda\in L$.

(2) $V_{\lambda+L}\cong V_{-\lambda+L}$, $\lambda+L\in L^\perp/L$ with $2\lambda\notin L$.

(3) $V_L^{T_\chi,\pm}$ for any irreducible $\hat{L}/P$-module T_χ with central character χ.

In the theorem, $L^\perp$ is the dual lattice of L, $V_{\lambda+L}^{\pm}=\{u\in V_{\lambda+L}\mid \theta(u)=\pm u\}$ for $\lambda+L\in L^\perp/L$ with $2\lambda\in L$, $\hat{L}$ is the canonical central extension of L by the cyclic group $\langle\kappa\rangle$ of order 2 with the commutator map $c(\alpha,\beta)=\kappa^{\langle\alpha,\beta\rangle}$ for $\alpha,\beta\in L$, $P=\{\theta(a)a^{-1}\mid a\in\hat{L}\}$, $V_L^{T_\chi}$ is an irreducible θ-twisted V_L-module, and $V_L^{T_\chi,\pm}=\{u\in V_L^{T_\chi}\mid\theta(u)=\pm u\}$. Note that in Theorem 1.1, $V_L^{T_\chi,\pm}$ in (3) are V_L^+-modules, however, if L is not positive definite, then $V_{\lambda+L}^{\pm}$ in (1) and $V_{\lambda+L}$ in (2) are not V_L^+-modules (cf. [19, (2.18)]). See Section 1 of Part 1 [19] for the background and the detailed introduction to Theorem 1.1. We note that since we do not assume any grading in the definition of a weak module, Theorem 1.1 does not follow from [13, Theorem 8.1], which deals with lower-bounded generalized modules for fixed point vertex algebras with some conditions. In fact, if L is not positive definite, then the weak modules listed in (1) and (2) in Theorem 1.1 are not lower-bounded generalized modules.

Let $M(1)$ be the Heisenberg vertex operator algebra associated to $\mathfrak{h}=\mathbb{C}\otimes_{\mathbb{Z}}L$ (see the explanation under (2.35) for the definition) and $M(1)^+$ the fixed point subalgebra of $M(1)$ under the action of θ. The vertex operator algebra $M(1)^+$ is a subalgebra of V_L^+ and, as stated in [19, Section 1], representations of $M(1)^+$ play a crucial role in the proof of Theorem 1.1. The irreducible $M(1)^+$-modules are classified in [9, Theorem 4.5] for the case of $\dim_{\mathbb{C}}\mathfrak{h}=1$ and [10, Theorem 6.2.2] for the general case as follows: any irreducible $M(1)^+$-module is isomorphic to one of

$$M(1)^{\pm}, M(1)(\theta)^{\pm},\ \text{or}\ M(1,\lambda)\cong M(1,-\lambda)\ (0\neq\lambda\in\mathfrak{h}). \tag{1.1}$$

Here $M(1)(\theta)$ is the irreducible θ-twisted $M(1)$-module, $M(1)^{\pm}=\{u\in M(1)\mid\theta u=\pm u\}$, $M(1)(\theta)^{\pm}=\{u\in M(1)(\theta)\mid\theta u=\pm u\}$, and $M(1,\lambda)$ is the irreducible $M(1)$-module generated by the vector e^λ such that $(\alpha(-1)\mathbf{1})_0e^\lambda=\langle\alpha,\lambda\rangle e^\lambda$ and $(\alpha(-1)\mathbf{1})_ne^\lambda=0$ for all $\alpha\in\mathfrak{h}$ and $n\in\mathbb{Z}_{>0}$ (See (2.31)–(2.55) in Section 2 for the precise definitions of these symbols). In the previous paper (Part 1), we showed that when the rank of L is 1, for any non-zero weak V_L^+-module M there exists a non-zero $M(1)^+$-submodule in M. In this paper (Part 2), we first strengthen and generalize this result to L of an arbitrary rank. Precisely, we show that for any non-zero weak V_L^+-module M, there exists an irreducible $M(1)^+$-submodule in M (Proposition 3.5 and Corollary 4.12). We next study extension groups and generalized Verma modules (see [8, Theorem 6.2] for the definition) for $M(1)^+$ (Proposition 4.9, Corollary 4.10, and Lemma 4.11). We shall explain how to use these results in Part 3. Let M be an irreducible weak V_L^+-module. By Corollary 4.12, there exists an irreducible $M(1)^+$-submodule K of M. If $K \cong M(\theta)^{\pm}$, then the same argument as in Section 3 of the present paper shows that M is a V_L^+-module. In this case, [20, Proposition 4.15] shows that M is isomorphic to one of the irreducible weak V_L^+-modules in Theorem 1.1 (3). Assume $K \cong M(1)^{\pm}$ or $M(1,\lambda)$ with $0 \neq \lambda \in \mathfrak{h}$. Since V_L^+ is a direct sum of irreducible $M(1)^+$-modules, for any irreducible $M(1)^+$-submodule N of V_L^+, the V_L^+-module structure of M induces an intertwining operator $I(\ ,x) : N \times K \to M((x))$ for weak $M(1)^+$-modules (see Definition 2.2 for the definition). We denote by Q the weak $M(1)^+$-submodule of M that is the image of $I(\ ,x)$. The same argument as in Section 3 of the present paper shows that there exists an irreducible $M(1)^+$-submodule R of Q. Moreover, If $R \neq Q$, then Q/R is an irreducible $M(1)^+$-module and by Proposition 4.9 the exact sequence $0 \to R \to Q \to Q/R \to 0$ splits, namely $Q \cong R \oplus Q/R$ as $M(1)^+$-modules. Since V_L^+ is a direct sum of irreducible $M(1)^+$-modules, this leads to the result that M is a direct sum of irreducible $M(1)^+$-modules. Moreover, we find that the irreducible $M(1)^+$-modules in the direct sum are pairwise non-isomorphic. Using fusion rules (see the explanation under Definition 2.2) for the irreducible $M(1)^+$-modules obtained in [1, Theorem 5.13] and [4, Theorem 7.7], we can determine the weak V_L^+-module M with such an $M(1)^+$-module structure and thus M is one of the irreducible weak V_L^+-modules in Theorem 1.1 (1) and (2).

Let us briefly explain the basic idea to show Proposition 3.5, the main result in Section 3. Let V be a vertex algebra and M a weak V-module. For $a \in V$ and $u \in M$, we define $\epsilon(a,u) \in \mathbb{Z} \cup \{-\infty\}$ by

$$a_{\epsilon(a,u)}u \neq 0 \text{ and } a_i u = 0 \text{ for all } i > \epsilon(a,u) \tag{1.2}$$

if $Y_M(a,x)u \neq 0$ and $\epsilon(a,u) = -\infty$ if $Y_M(a,x)u = 0$. It is well-known that the vertex operator algebra $M(1)^+$ is generated by homogeneous elements $\omega^{[i]}$ of weight 2, $J^{[i]}$(or $H^{[i]}$) of weight 4, and $S_{lm}(1,r)$ of weight $r+1$ ($1 \leq i \leq d, 1 \leq m < l \leq d, r = 1,2,3$) such that $[\omega^{[i]}_k, \omega^{[j]}_l] = [\omega^{[i]}_k, H^{[j]}_l] = [H^{[i]}_k, H^{[j]}_l] = 0$ for any $k, l \in \mathbb{Z}$ and any pair of distinct elements $i, j \in \{1,\dots,d\}$ (see (2.59) and (2.61) for these symbols). Hence for any non-zero weak V_L^+-module M, it follows from [19, Lemma 3.7] that there exists a simultaneous eigenvector u of $\{\omega^{[i]}_1, H^{[i]}_3\}_{i=1}^d$ in M such that $\epsilon(\omega^{[i]},u) \leq 1$ and $\epsilon(H^{[i]},u) \leq 3$ for all $i = 1,\dots,d$. By induction on $\max\{\epsilon(S_{ij}(1,1),u) \mid i > j\}$, we get a simultaneous eigenvector of $\{\omega^{[i]}_1, H^{[i]}_3\}_{i=1}^d$, which we denote by the same symbol u, such that $\epsilon(\omega^{[i]},u) \leq 1$, $\epsilon(H^{[i]},u) \leq 3$, and $\epsilon(S_{lm}(1,r),u) \leq r$ for all $i = 1,\dots,d, 1 \leq m < l \leq d$, and $r = 1,2,3$. Namely, $u \in \Omega_{M(1)^+}(M)$ (see (2.3) for the definition). Since u is a simultaneous eigenvector of $\{\omega^{[i]}_1, H^{[i]}_3\}_{i=1}^d$, $A(M(1)^+)u$ is of finite dimension, where $A(M(1)^+)$ is the Zhu algebra for $M(1)^+$ (see (2.28)–(2.30) for the definition). Hence, by [8, Theorem 6.2] we have the result.

We next explain the basic idea to show Proposition 4.9, the main result in Section 4. The result shows that in most cases the exact sequence

$$0 \to W \to N \xrightarrow{\pi} M \to 0 \tag{1.3}$$

splits for two irreducible $M(1)^+$-modules $W = \oplus_{i \in \delta + \mathbb{Z}_{\geq 0}} W_i, M = \oplus_{i \in \gamma + \mathbb{Z}_{\geq 0}} M_i$ and a weak $M(1)^+$-module N, where $\omega = \sum_{j=1}^d \omega^{[j]}$ is the conformal vector (Virasoro element) of $M(1)^+$ and $M_i := \{u \in M \mid \omega_1 u = iu\}$ for $i \in \mathbb{C}$. Precisely, in Section 4 we deal with the case where M is a general $M(1)^+$-module in the exact sequence (1.3) in order to show Corollary 4.10, however, here we assume M is irreducible to simplify the argument. As in Part 1 [19], we first find some relations for $\omega^{[i]}, H^{[i]}$ $(i=1,\dots,d), S_{lm}(1,r)$ $(1 \leq m < l \leq d, r=1,2,3)$ in $M(1)^{+}$ with the help of computer algebra system Risa/Asir[16] ((4.12)–(4.15)). For $\zeta=(\zeta^{[1]},\dots,\zeta^{[d]}), \xi=(\xi^{[1]},\dots,\xi^{[d]}) \in \mathbb{C}^{d}$, let $v \in M_{\gamma}$ such that $(\omega^{[i]}_{1}-\zeta^{[i]})v=(H^{[i]}_{3}-\xi^{[i]})v=0$ for all $i=1,\dots,d$. Assume $(W,M_{\gamma}) \not\cong (M(1)^{+},M(1)^{-}_{1}),(M(1)^{-},M(1)^{+}_{0})$. Using the relations (4.16)–(4.22) obtained by (4.12)–(4.15), we can take $u \in N_{\gamma}$ such that $\pi(u)=v$, $(\omega^{[i]}_{1}-\zeta^{[i]})u=(H^{[i]}_{3}-\xi^{[i]})u=0$ for all $i=1,\dots,d$ and $u \in \Omega_{M(1)^{+}}(N_{\gamma})$ (Lemmas 4.4 and 4.6). After studying the following two cases, we have the result:

(1) The case that $M \not\cong W$. Assume that the exact sequence (1.3) does not split. Since the intersection of W and the $M(1)^{+}$-submodule of N generated by u is not trivial, we have $\delta \in \gamma+\mathbb{Z}_{\geq 0}$. By taking the restricted dual of (1.3), the same argument shows that $\gamma \in \delta+\mathbb{Z}_{\geq 0}$ and hence $\delta=\gamma$. Since $M \not\cong W$, we have $N_{\gamma} \cong W_{\gamma} \oplus M_{\gamma}$ as $A(M(1)^{+})$-modules and hence the exact sequence (1.3) splits. This is a contradiction (Lemma 4.7).

(2) The case that $M=W$ and $M \in \{M(1)^{\pm}, M(1)(\theta)^{\pm}\}$. Using the relations (4.16)–(4.22) again, we have $N_{\gamma} \cong M_{\gamma} \oplus W_{\gamma}$ as $A(M(1)^{+})$-modules and hence the exact sequence (1.3) splits (Lemma 4.8).

Complicated computation has been done by a computer algebra system Risa/Asir[16]. Throughout this paper, the word “a direct computation” often means a direct computation with the help of Risa/Asir. Details of computer calculations such as (2.68), (4.12), (4.16), (4.37), etc., and (A2.1)–(A2.36) in Appendix A2 can be found on the internet at [17].

The organization of the paper is as follows. In Section 2 we recall some basic properties of weak modules for a vertex algebra. We also recall the Heisenberg algebra $M(1)$ and its fixed point algebra $M(1)^{+}$. In Section 3 we show that for any non-zero weak V_{L}^{+}-module M there exists a non-zero submodule for $M(1)^{+}$ in M. In Section 4 we study extension groups and generalized Verma modules for $M(1)^{+}$. In Appendix A2 we put computations of $a_{k}b$ for some $a,b \in V_{L}^{+}$ and $k=0,1,\dots$ to find the commutation relation $[a_{i},b_{j}]=\sum_{k=0}^{\infty}\binom{i}{k}(a_{k}b)_{i+j-k}$. In Notation we list some notation.

2 Preliminary

We assume that the reader is familiar with the basic knowledge on vertex algebras as presented in [5, 12, 14, 15].

Throughout this paper, V is a vertex algebra and we always assume that V has an element ω such that $\omega_{0}a=a_{-2}\mathbf{1}$ for all $a \in V$. For a vertex operator algebra V, this condition automatically holds since V has the conformal vector (Virasoro element). Throughout this paper, we follow the notation and terminology of [19]. We will explain some of them. We note that if V is a vertex operator algebra, then the notion of a module for V viewed as a vertex algebra is different from the notion of a module for V viewed as a vertex operator algebra (cf. [14, Definitions 4.1.1 and 4.1.6]). To avoid confusion, throughout this paper, we refer to a module for a vertex algebra defined in [14, Definition 4.1.1] as a *weak module*. Here we write down the definition of a weak V-module:

Definition 2.1. A *weak V-module* M is a vector space over $\mathbb{C}$ equipped with a linear map

$$\begin{aligned} Y_{M}(\ ,x) &: V \otimes_{\mathbb{C}} M \to M((x)) \\ a \otimes u &\mapsto Y_{M}(a,x)u=\sum_{n\in\mathbb{Z}} a_{n}ux^{-n-1} \end{aligned} \tag{2.1}$$

such that the following conditions are satisfied:

(1) $Y_M(\mathbf{1},x)=\mathrm{id}_M$.

(2) For $a,b\in V$ and $u\in M$,

$$x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y_M(a,x_1)Y_M(b,x_2)u-x_0^{-1}\delta(\frac{x_2-x_1}{-x_0})Y_M(b,x_2)Y_M(a,x_1)u$$
$$=x_1^{-1}\delta(\frac{x_2+x_0}{x_1})Y_M(Y(a,x_0)b,x_2)u. \tag{2.2}$$

For $n\in\mathbb{C}$ and a weak V-module M, we define $M_n=\{u\in V\mid \omega_1 u=nu\}$. For $a\in V_n$ $(n\in\mathbb{C})$, $\mathrm{wt}\,a$ denotes n. For a vertex algebra V which admits a decomposition $V=\oplus_{n\in\mathbb{Z}}V_n$ and a subset U of a weak V-module, we define

$$\Omega_V(U)=\left\{u\in U\ \middle|\ \begin{array}{l} a_iu=0 \text{ for all homogeneous elements } a\in V\\ \text{and } i>\mathrm{wt}\,a-1.\end{array}\right\}. \tag{2.3}$$

For a vertex algebra V which admits a decomposition $V=\oplus_{n\in\mathbb{Z}}V_n$, a weak V-module N is called $\mathbb{N}$-*graded* if N admits a decomposition $N=\oplus_{n=0}^{\infty}N(n)$ such that $a_iN(n)\subset N(\mathrm{wt}\,a-i-1+n)$ for all homogeneous elements $a\in V$, $i\in\mathbb{Z}$, and $n\in\mathbb{Z}_{\geq 0}$, where we define $N(n)=0$ for all $n<0$. For a vertex algebra V which admits a decomposition $V=\oplus_{n\in\mathbb{Z}}V_n$, a weak V-module N is called a *V-module* if N admits a decomposition $N=\oplus_{n\in\mathbb{C}}N_n$ such that $\dim_{\mathbb{C}}N_n<\infty$ for all $n\in\mathbb{C}$ and $N_n=0$ for n whose real part is sufficiently negative. We recall the definition of an intertwining operator from [11, Definition 5.4.1].

Definition 2.2. Let V be a vertex algebra and let M,W, and N be three weak V-modules. An *intertwining operator* of type $\binom{N}{M\ W}$ is a linear map

$$I(\ ,x):M\otimes_{\mathbb{C}}W\to N\{x\}$$
$$I(u,x)v=\sum_{\alpha\in\mathbb{C}}u_\alpha v x^{-\alpha-1},$$
$$u\in M, v\in W, \text{ and } u_\alpha\in\mathrm{Hom}_{\mathbb{C}}(W,N), \tag{2.4}$$

such that the following conditions are satisfied:

(1) For $u\in M, v\in W$, and $\alpha\in\mathbb{C}$,

$$u_{\alpha+m}v=0 \text{ for sufficiently large } m\in\mathbb{N}. \tag{2.5}$$

(2) For $u\in M$ and $a\in V$,

$$x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y(a,x_1)I(u,x_2)-x_0^{-1}\delta(\frac{x_2-x_1}{-x_0})I(u,x_2)Y(a,x_1)$$
$$=x_1^{-1}\delta(\frac{x_2+x_0}{x_1})I(Y(a,x_0)u,x_2). \tag{2.6}$$

(3) For $u\in M$,

$$I(\omega_0u,x)=\frac{d}{dx}I(u,x). \tag{2.7}$$

For irreducible weak V-modules M, W, and N, $I\binom{N}{M\ W}$ denotes the space of all intertwining operators of type $\binom{N}{M\ W}$ and we call its dimension the *fusion rule* of type $\binom{N}{M\ W}$. In this paper, for an intertwining operator $I(\ ,x)$ from $M \times W$ to N, we consider only the case that the image of $I(\ ,x)$ is contained in $N((x))$. For $A \subset M$ and $B \subset W$,

$$A \cdot B \text{ denotes } \mathrm{Span}_{\mathbb{C}}\{a_i b \mid a \in A, i \in \mathbb{Z}, b \in B\} \subset N. \tag{2.8}$$

For an intertwining operator $I(\ ,x) : M \times W \to N((x))$, $u \in M$, and $v \in W$, we define $\epsilon_I(u,v) = \epsilon(u,v) \in \mathbb{Z} \cup \{-\infty\}$ by

$$u_{\epsilon_I(u,v)} v \neq 0 \text{ and } u_i v = 0 \text{ for all } i > \epsilon_I(u,v) \tag{2.9}$$

if $I(u,x)v \neq 0$ and $\epsilon_I(u,v) = -\infty$ if $I(u,x)v = 0$. For a subset A of V and a subset B of a weak V-module M, let

$$A_{-}B := \mathrm{Span}_{\mathbb{C}}\{a_{-i} b \mid a \in A, b \in B, \text{ and } i \in \mathbb{Z}_{>0}\} \subset M \tag{2.10}$$

and

$$\langle A_{-}\rangle B := \mathrm{Span}_{\mathbb{C}}\left\{a^{(1)}_{-i_1} \cdots a^{(n)}_{-i_n} b \;\middle|\; \begin{array}{l} n \in \mathbb{Z}_{\geq 0}, a^{(1)}, \ldots, a^{(n)} \in A, b \in B, \\ i_1, \ldots, i_n \in \mathbb{Z}_{>0} \end{array}\right\} \subset M. \tag{2.11}$$

When $B = \{b\}$, we will simply write $A_{-}B$ and $\langle A_{-}\rangle B$ as $A_{-}b$ and $\langle A_{-}\rangle b$, respectively.

Lemma 2.3. *Let $V = \oplus_{n=0}^{\infty} V_n$ be a vertex operator algebra, A and B finite subsets of $\bigcup_{n=1}^{\infty} V_n$, M a weak V-module, and U a finite dimensional subspace of M. Assume $A \cdot \langle A_{-}\rangle \mathbf{1} \subset \langle A_{-}\rangle \mathbf{1}$. For each $a \in A$ and $b \in B$, we choose $\epsilon(a), \epsilon(b) \in \mathbb{Z}$ so that $\epsilon(a) \geq \max\{\{\epsilon(a,u) \mid u \in U\} \cup \{-1\}\}$ and $\epsilon(b) \geq \max\{\{\epsilon(b,u) \mid u \in U\} \cup \{-1\}\}$. For $j \in \mathbb{Z}_{\geq 0}$, we define*

$$\delta(j) := \max\left\{ \sum_{i=1}^{n} (\epsilon(a^{(i)}) - \mathrm{wt}(a^{(i)}) + 1) \;\middle|\; \begin{array}{l} n \in \mathbb{Z}_{\geq 0}, a^{(1)}, \ldots, a^{(n-1)} \in A, a^{(n)} \in B, \\ \sum_{i=1}^{n} \mathrm{wt}(a^{(i)}) \leq j. \end{array} \right\} - 1 + j, \tag{2.12}$$

where $\sum_{i=1}^{n}(\epsilon(a^{(i)}) - \mathrm{wt}(a^{(i)}) + 1) := 0$ for $n = 0$.

(1) Assume $A \cdot (\langle A_{-}\rangle B_{-}\mathbf{1}) \subset \langle A_{-}\rangle B_{-}\mathbf{1}$. For each $a \in A$ and $b \in B$, we choose $\gamma(a), \gamma(b) \in \mathbb{Z}_{\geq -1}$. Then, for a homogeneous element $c \in \langle A_{-}\rangle B_{-}\mathbf{1}$, $u \in U$, and $k \in \mathbb{Z}$, $c_k u$ is a linear combination of elements of the form

$$p^{(1)}_{\sigma_1} \cdots p^{(l)}_{\sigma_l} q_{\tau} p^{(l+1)}_{\sigma_{l+1}} \cdots p^{(m)}_{\sigma_m} u \tag{2.13}$$

where $l, m \in \mathbb{Z}_{\geq 0}$ with $0 \leq l \leq m$, $p^{(1)}, \ldots, p^{(m)} \in A$, $\sigma_i \in \mathbb{Z}_{\leq \gamma(p^{(i)})}$ $(i = 1, \ldots, l)$, $\sigma_i \in \mathbb{Z}_{\geq \gamma(p^{(i)})+1}$ $(i = l+1, \ldots, m)$, $q \in B, \tau \in \mathbb{Z}$ such that

$$\sum_{i=1}^{m} \mathrm{wt}(p^{(i)}) + \mathrm{wt}(q) \leq \mathrm{wt}(c) \text{ and} \tag{2.14}$$

$$\mathrm{wt}(c) - k - 1 = \sum_{i=1}^{m} (\mathrm{wt}(p^{(i)}) - \sigma_i - 1) + \mathrm{wt}(q) - \tau - 1. \tag{2.15}$$

In particular, $c_{\delta(\mathrm{wt}(c))}u$ is a linear combination of elements of the form

$$p^{(1)}_{\epsilon(p^{(1)})}\cdots p^{(m)}_{\epsilon(p^{(m)})}q_{\epsilon(q)}u \tag{2.16}$$

where $m\in\mathbb{Z}_{\geq 0}$, $p^{(1)},\dots,p^{(m)}\in A$, and $q\in B$. Moreover, for $k>\delta(\mathrm{wt}(c))$,

$$c_k u=0. \tag{2.17}$$

(2) Assume $B\cdot\langle A_-\rangle\mathbf{1}\subset B_-\langle A_-\rangle\mathbf{1}$. Then, for a homogeneous element $c\in B_-\langle A_-\rangle\mathbf{1}$ and $u\in U$, $c_{\delta(\mathrm{wt}(c))}u$ is a linear combination of elements of the form

$$q_{\epsilon(q)}p^{(1)}_{\epsilon(p^{(1)})}\cdots p^{(m)}_{\epsilon(p^{(m)})}u \tag{2.18}$$

where $m\in\mathbb{Z}_{\geq 0}$, $p^{(1)},\dots,p^{(m)}\in A$, and $q\in B$. Moreover, for $k>\delta(\mathrm{wt}(c))$,

$$c_k u=0. \tag{2.19}$$

(3) Assume $B\cdot\langle A_-\rangle\mathbf{1}\subset B_-\langle A_-\rangle\mathbf{1}$ and $A\cdot(B_-\langle A_-\rangle\mathbf{1})\subset B_-\langle A_-\rangle\mathbf{1}$. For $a\in A$, we define

$$\zeta(a):=\max\{\{\epsilon(a)\}\cup\{\delta(\mathrm{wt}\,a+\mathrm{wt}\,b-1)-\epsilon(b)\mid b\in B\}\}. \tag{2.20}$$

If $a_{\epsilon(a)}u\in U$ for all $a\in A$ and $u\in U$, then the subspace $W:=\mathrm{Span}_{\mathbb{C}}\{b_{\epsilon(b)}u\mid b\in B,u\in U\}$ of M is stable under the action of $a_{\zeta(a)}$ for all $a\in A$. Moreover, for $a\in A$ and $k>\zeta(a)$, $a_kW=0$.

(4) Assume $B\cdot\langle A_-\rangle\mathbf{1}\subset B_-\langle A_-\rangle\mathbf{1}$, $A\cdot(B_-\langle A_-\rangle\mathbf{1})\subset B_-\langle A_-\rangle\mathbf{1}$, and $B\cdot(B_-\langle A_-\rangle\mathbf{1})\subset B_-\langle A_-\rangle\mathbf{1}$. For any $a\in A$, $b\in B$, and $u\in U$, we assume $\epsilon(a,u)\geq\mathrm{wt}(a)-1$ and the value $\epsilon(b)-\mathrm{wt}(b)+1$ is a constant independent of $b\in B$, which we denote by ρ. We define $W:=\mathrm{Span}_{\mathbb{C}}\{b_{\epsilon(b)}u\mid b\in B,u\in U\}$. If $a_{\mathrm{wt}(a)-1}u\in U$ for all $a\in A$ and $u\in U$, then for any homogeneous element $c\in B_-\langle A_-\rangle\mathbf{1}$, $c_{\mathrm{wt}(c)-1}W\subset W$ and $c_kW=0$ for all $k>\mathrm{wt}(c)-1$.

Proof. (1) Let $c:=a^{(1)}_{-i_1}\cdots a^{(n)}_{-i_n}b_{-i_{n+1}}\mathbf{1}\in\langle A_-\rangle B_-\mathbf{1}$ where $n\in\mathbb{Z}_{\geq 0}$, $a^{(1)},\dots,a^{(n)}\in A$, $b\in B$, and $i_1,\dots,i_n,i_{n+1}\in\mathbb{Z}_{>0}$. We shall show (2.13)–(2.15) by induction on $\mathrm{wt}\,c$. If $c=\mathbf{0}$ or $c=b_{-i}\mathbf{1}$ with $b\in B$ and $i\in\mathbb{Z}_{>0}$, then the results hold. Let $n\geq 1$. We define $f:=a^{(2)}_{-i_2}\cdots a^{(n)}_{-i_n}b_{-i_{n+1}}\mathbf{1}$ and note that $\mathrm{wt}(f)<\mathrm{wt}(c)$. For $k\in\mathbb{Z}$, using [19, Lemma 2.2], we have

$$\begin{aligned}c_ku&=(a^{(1)}_{-i_1}f)_ku\\&=\sum_{\substack{s\leq\gamma(a^{(1)})\\ s+t+i_1=k}}\binom{-s-1}{i_1-1}a^{(1)}_sf_tu+\sum_{\substack{s\geq\gamma(a^{(1)})+1\\ s+t+i_1=k}}\binom{-s-1}{i_1-1}f_ta^{(1)}_su\\&\quad+(-1)^{i_1}\sum_{l=0}^{\infty}\binom{l+i_1-1}{i_1-1}\binom{\gamma(a^{(1)})+i_1}{l+i_1}(a^{(1)}_lf)_{k-i_1-l}u.\end{aligned}\tag{2.21}$$

In the second term in (2.21), by the induction hypothesis in the setting c, k, and u are replaced by f, t, and $a^{(1)}_su$, respectively, we find that $f_ta^{(1)}_su$ is a linear combination of elements of the form (2.13)–(2.15). In the third term in (2.21), since $\mathrm{wt}(a^{(1)}_lf)=\mathrm{wt}(c)-i_1-l<\mathrm{wt}(c)$ for $l\in\mathbb{Z}_{\geq 0}$, by the induction hypothesis in the setting c and k are replaced by $a^{(1)}_lf$ and

$k-i_1-l$, respectively, we find that $(a_l^{(1)}f)_{k-i_1-l}u$ is a linear combination of elements of the form (2.13)–(2.15). Hence (2.13)–(2.15) hold.

We shall show (2.16) and (2.17). Let $k \geq \delta(\mathrm{wt}\, c))$. We set $\gamma(a)=\epsilon(a)$ for $a \in A$ and $\gamma(b)=\epsilon(b)$ for $b\in B$. Assume that in the expansion of $c_k u$, the coefficient of an element of the form (2.13) is not zero. Since $p_j u = 0$ for $p\in A$ and $j>\epsilon(p)$, we have $l=m$ and hence $\tau \leq \epsilon(q)$. By (2.15),

$$\begin{aligned}
0 &= -\mathrm{wt}(c)+k+1+\sum_{i=1}^{m}(\mathrm{wt}(p^{(i)})-\sigma_i-1)+\mathrm{wt}(q)-\tau-1\\
&\geq -\mathrm{wt}(c)+\delta(\mathrm{wt}(c))+1+\sum_{i=1}^{m}(\mathrm{wt}(p^{(i)})-\sigma_i-1)+\mathrm{wt}(q)-\tau-1\\
&\geq -\mathrm{wt}(c)+\big(\sum_{i=1}^{m}(\epsilon(p^{(i)})-\mathrm{wt}(p^{(i)})+1)+(\epsilon(q)-\mathrm{wt}(q)+1)-1+\mathrm{wt}(c)\big)+1\\
&\quad +\sum_{i=1}^{m}(\mathrm{wt}(p^{(i)})-\sigma_i-1)+\mathrm{wt}(q)-\tau-1\\
&= \sum_{i=1}^{m}(\epsilon(p^{(i)})-\sigma_i)+(\epsilon(q)-\tau)\geq 0
\end{aligned}$$

and hence $k=\delta(\mathrm{wt}(c))$, $\epsilon(p^{(i)})=\sigma_i$ for all $i=1,\dots,m$ and $\epsilon(b)=\tau$. Here we have used (2.14) and the definition (2.12) of $\delta(\mathrm{wt}(c))$. Thus, (2.16) and (2.17) hold.

(2) The same argument as in (1) shows the results.

(3) For $j\in\mathbb{Z}_{>0}$, by the definition of $\delta(j)$ we have $\delta(j)-1\geq\delta(j-1)$ and hence $\delta(j)-i\geq\delta(j-i)$ for all $i=0,\dots,j$. Let $u\in U$ and $k\in\mathbb{Z}_{\geq\zeta(a)}$. For $a\in A$ and $b\in B$,

$$a_k b_{\epsilon(b)}u = b_{\epsilon(b)}a_k u+[a_k,b_{\epsilon(b)}]u = b_{\epsilon(b)}a_k u+\sum_{i=0}^{\infty}\binom{k}{i}(a_i b)_{k+\epsilon(b)-i}u. \tag{2.22}$$

For $i\in\mathbb{Z}_{\geq 0}$, since

$$\begin{aligned}
k+\epsilon(b)-i &\geq \zeta(a)+\epsilon(b)-i\geq\delta(\mathrm{wt}\,a+\mathrm{wt}\,b-1)-i\\
&\geq\delta(\mathrm{wt}\,a+\mathrm{wt}\,b-1-i)=\delta(\mathrm{wt}(a_i b)),
\end{aligned}\tag{2.23}$$

we have the results by (2).

(4) For any $a\in A$ and $u\in U$, since $\epsilon(a,u)\geq\mathrm{wt}(a)-1$, we choose $\epsilon(a)=\mathrm{wt}(a)-1$. For a homogeneous element $c\in B_-\langle A_-\rangle\mathbf{1}$, by the definition (2.12) of δ,

$$\delta(\mathrm{wt}(c))=\max\{\epsilon(b)-\mathrm{wt}(b)\mid b\in B\}+\mathrm{wt}(c)=\rho+\mathrm{wt}(c)-1. \tag{2.24}$$

For $a\in A$ and $b\in B$, by (2.24),

$$\begin{aligned}
\delta(\mathrm{wt}(a)+\mathrm{wt}(b)-1)-\epsilon(b) &= \rho+\mathrm{wt}(a)+\mathrm{wt}(b)-2-\epsilon(b)\\
&=\epsilon(b)-\mathrm{wt}(b)+\mathrm{wt}(a)+\mathrm{wt}(b)-1-\epsilon(b)\\
&=\mathrm{wt}(a)-1.
\end{aligned}\tag{2.25}$$

Since we have chosen $\epsilon(a) = \mathrm{wt}(a) - 1$, $\zeta(a) = \mathrm{wt}(a) - 1$ in (2). By (2), the results hold for $a \in A$. For $b, b' \in B$, $u \in U$, and $j \geq \mathrm{wt}(b) - 1$, by the Borcherds identity (cf. [14, (3.1.7)] putting $u = b, v = b', l = -1, m = j+1$, and $n = \epsilon(b')$ in the symbol used there) we have

$$b_j b'_{\epsilon(b')} u = \sum_{i=0}^{\infty} \binom{j+1}{i} (b_{-1+i} b')_{j+1+\epsilon(b')-i} u. \tag{2.26}$$

Since

$$\begin{aligned} \delta(\mathrm{wt}(b_{-1+i} b')) &= \rho + \mathrm{wt}(b) + \mathrm{wt}(b') - i - 1 \\ &= \epsilon(b') - \mathrm{wt}(b') + \mathrm{wt}(b) + \mathrm{wt}(b') - i \\ &= (\mathrm{wt}(b) - 1) + 1 + \epsilon(b') - i \\ &\leq j + 1 + \epsilon(b') - i, \end{aligned} \tag{2.27}$$

the results hold for $b \in B$ by (2). Thus, for any homogeneous element $c \in B_{-}\langle A_{-}\rangle\mathbf{1}$, an inductive argument on $\mathrm{wt}\, c$ shows the results.

□

We recall the *Zhu algebra* $A(V)$ of a vertex operator algebra V from [21, Section 2]. For homogeneous $a \in V$ and $b \in V$, we define

$$a \circ b = \sum_{i=0}^{\infty} \binom{\mathrm{wt}\, a}{i} a_{i-2} b \in V \tag{2.28}$$

and

$$a * b = \sum_{i=0}^{\infty} \binom{\mathrm{wt}\, a}{i} a_{i-1} b \in V. \tag{2.29}$$

We extend (2.28) and (2.29) for an arbitrary $a \in V$ by linearity. We also define $O(V) = \mathrm{Span}_{\mathbb{C}}\{a \circ b \mid a, b \in V\}$. Then, the quotient space

$$A(V) = M/O(V), \tag{2.30}$$

called the *Zhu algebra* of V, is an associative $\mathbb{C}$-algebra with multiplication (2.29) by [21, Theorem 2.1.1]. It is shown in [21, Theorem 2.2.1] that for a vertex operator algebra V there is a one to one correspondence between the set of all isomorphism classes of irreducible $\mathbb{N}$-graded weak V-modules and that of irreducible $A(V)$-modules.

We recall the vertex operator algebra $M(1)$ associated to the Heisenberg algebra and the vertex algebra V_L associated to a non-degenerate even lattice L from [14, Sections 6.3–6.5] and [10, Section 2.2]. Let $\mathfrak{h}$ be a finite dimensional vector space equipped with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$. Set a Lie algebra

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \tag{2.31}$$

with the Lie bracket relations

$$[\beta \otimes t^m, \gamma \otimes t^n] = m\langle \beta, \gamma\rangle \delta_{m+n,0} C, \qquad [C, \hat{\mathfrak{h}}] = 0 \tag{2.32}$$

for $\beta, \gamma \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. For $\beta \in \mathfrak{h}$ and $n \in \mathbb{Z}$, $\beta(n)$ denotes $\beta \otimes t^n \in \widehat{\mathfrak{h}}$. Set two Lie subalgebras of $\mathfrak{h}$:

$$\widehat{\mathfrak{h}}_{\geq 0} = \bigoplus_{n \geq 0} \mathfrak{h} \otimes t^n \oplus \mathbb{C}C \qquad \text{and} \qquad \widehat{\mathfrak{h}}_{<0} = \bigoplus_{n \leq -1} \mathfrak{h} \otimes t^n. \tag{2.33}$$

For $\beta \in \mathfrak{h}$, $\mathbb{C}e^{\beta}$ denotes the one dimensional $\widehat{\mathfrak{h}}_{\geq 0}$-module uniquely determined by the condition that for $\gamma \in \mathfrak{h}$

$$\gamma(i) \cdot e^{\beta} = \begin{cases} \langle \gamma, \beta \rangle e^{\beta} & \text{for } i = 0 \\ 0 & \text{for } i > 0 \end{cases} \qquad \text{and} \qquad C \cdot e^{\beta} = e^{\beta}. \tag{2.34}$$

We take an $\widehat{\mathfrak{h}}$-module

$$M(1, \beta) = \mathscr{U}(\widehat{\mathfrak{h}}) \otimes_{\mathscr{U}(\widehat{\mathfrak{h}}_{\geq 0})} \mathbb{C}e^{\beta} \cong \mathscr{U}(\widehat{\mathfrak{h}}_{<0}) \otimes_{\mathbb{C}} \mathbb{C}e^{\beta} \tag{2.35}$$

where $\mathscr{U}(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then, $M(1) = M(1, 0)$ has a vertex operator algebra structure with the conformal vector

$$\omega = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} h_i(-1) h'_i(-1) \mathbf{1} \tag{2.36}$$

where $\{h_1, \ldots, h_{\dim \mathfrak{h}}\}$ is a basis of $\mathfrak{h}$ and $\{h'_1, \ldots, h'_{\dim \mathfrak{h}}\}$ is its dual basis. Moreover, $M(1, \beta)$ is an irreducible $M(1)$-module for any $\beta \in \mathfrak{h}$. The vertex operator algebra $M(1)$ is called the *vertex operator algebra associated to the Heisenberg algebra* $\oplus_{0 \neq n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}C$.

Let L be a non-degenerate even lattice. We define $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and denote by $L^{\perp}$ the dual of L: $L^{\perp} = \{\gamma \in \mathfrak{h} \mid \langle \beta, \gamma \rangle \in \mathbb{Z} \text{ for all } \beta \in L\}$. Taking $M(1)$ for $\mathfrak{h}$, we define $V_{\lambda+L} = \oplus_{\beta \in \lambda+L} M(1, \beta) \cong \mathscr{U}(\widehat{\mathfrak{h}}_{<0}) \otimes_{\mathbb{C}} (\oplus_{\beta \in \lambda+L} \mathbb{C}e^{\beta})$ for $\lambda + L \in L^{\perp}/L$. Then, $V_L = V_{0+L}$ admits a unique vertex algebra structure compatible with the action of $M(1)$ and is called the *vertex algebra associated to* L (cf. [14, Section 6.5]). Moreover, for each $\lambda + L \in L^{\perp}/L$ the vector space $V_{\lambda+L}$ is an irreducible weak V_L-module which admits the following decomposition:

$$V_{\lambda+L} = \bigoplus_{n \in \langle \lambda, \lambda \rangle/2 + \mathbb{Z}} (V_{\lambda+L})_n \text{ where } (V_{\lambda+L})_n = \{a \in V_{\lambda+L} \mid \omega_1 a = na\}. \tag{2.37}$$

Let $\hat{L}$ be the canonical central extension of L by the cyclic group $\langle \kappa \rangle$ of order 2 with the commutator map $c(\alpha, \beta) = \kappa^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in L$:

$$0 \to \langle \kappa \rangle \to \hat{L} \xrightarrow{-} L \to 0. \tag{2.38}$$

Then, the -1-isometry of L induces an automorphism θ of $\hat{L}$ of order 2 and an automorphism, by abuse of notation we also denote by θ, of V_L of order 2 (see [12, (8.9.22)]). In $M(1)$, we have

$$\theta(h^1(-i_1) \cdots h^n(-i_n)\mathbf{1}) = (-1)^n h^1(-i_1) \cdots h^n(-i_n)\mathbf{1} \tag{2.39}$$

for $n \in \mathbb{Z}_{\geq 0}$, $h^1, \ldots, h^n \in \mathfrak{h}$, and $i_1, \ldots, i_n \in \mathbb{Z}_{>0}$. We set

$$V_L^+ = \{a \in V_L \mid \theta(a) = a\} \text{ and } M(1)^+ = \{a \in M(1) \mid \theta(a) = a\}. \tag{2.40}$$

For a weak V_L-module M, we define a weak V_L-module $(M \circ \theta, Y_{M \circ \theta})$ by $M \circ \theta = M$ and

$$Y_{M \circ \theta}(a, x) = Y_M(\theta(a), x) \tag{2.41}$$

for $a \in V_L$. Then $V_{\lambda+L} \circ \theta \cong V_{-\lambda+L}$ for $\lambda \in L^\perp$. Thus, for $\lambda \in L^\perp$ with $2\lambda \in L$ we define

$$V^{\pm}_{\lambda+L} = \{u \in V_{\lambda+L} \mid \theta(u) = \pm u\}. \tag{2.42}$$

Next, we recall the construction of θ-twisted modules for $M(1)$ and V_L from [12, Section 9]. Set a Lie algebra

$$\hat{\mathfrak{h}}[-1] = \mathfrak{h} \otimes t^{1/2}\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}C \tag{2.43}$$

with the Lie bracket relations

$$[C, \hat{\mathfrak{h}}[-1]] = 0 \qquad \text{and} \qquad [\alpha \otimes t^m, \beta \otimes t^n] = m\langle \alpha, \beta\rangle \delta_{m+n,0} C \tag{2.44}$$

for $\alpha, \beta \in \mathfrak{h}$ and $m, n \in 1/2 + \mathbb{Z}$. For $\alpha \in \mathfrak{h}$ and $n \in 1/2 + \mathbb{Z}$, $\alpha(n)$ denotes $\alpha \otimes t^n \in \widehat{\mathfrak{h}}$. Set two Lie subalgebras of $\hat{\mathfrak{h}}[-1]$:

$$\widehat{\mathfrak{h}}[-1]_{\geq 0} = \bigoplus_{n \in 1/2+\mathbb{N}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}C \qquad \text{and} \qquad \widehat{\mathfrak{h}}[-1]_{<0} = \bigoplus_{n \in 1/2+\mathbb{N}} \mathfrak{h} \otimes t^{-n}. \tag{2.45}$$

Let $\mathbb{C}\mathbf{1}_{\mathrm{tw}}$ denote a unique one dimensional $\widehat{\mathfrak{h}}[-1]_{\geq 0}$-module such that

$$\begin{aligned} h(i) \cdot \mathbf{1}_{\mathrm{tw}} &= 0 \quad \text{ for } h \in \mathfrak{h} \text{ and } i \in \frac{1}{2} + \mathbb{N}, \\ C \cdot \mathbf{1}_{\mathrm{tw}} &= \mathbf{1}_{\mathrm{tw}}. \end{aligned} \tag{2.46}$$

We take an $\widehat{\mathfrak{h}}[-1]$-module

$$M(1)(\theta) = \mathscr{U}(\widehat{\mathfrak{h}}[-1]) \otimes_{\mathscr{U}(\widehat{\mathfrak{h}}[-1]_{\geq 0})} \mathbb{C}u_\zeta \cong \mathscr{U}(\widehat{\mathfrak{h}}[-1]_{<0}) \otimes_{\mathbb{C}} \mathbb{C}u_\zeta. \tag{2.47}$$

We define for $\alpha \in \mathfrak{h}$,

$$\alpha(x) = \sum_{i \in 1/2+\mathbb{Z}} \alpha(i) x^{-i-1} \tag{2.48}$$

and for $u = \alpha_1(-i_1) \cdots \alpha_k(-i_k)\mathbf{1} \in M(1)$,

$$Y_0(u, x) = {}^{\circ}_{\circ} \frac{1}{(i_1 - 1)!}(\frac{d^{i_1-1}}{dx^{i_1-1}}\alpha_1(x)) \cdots \frac{1}{(i_k - 1)!}(\frac{d^{i_k-1}}{dx^{i_k-1}}\alpha_k(x)) {}^{\circ}_{\circ}. \tag{2.49}$$

Here, for $\beta_1, \ldots, \beta_n \in \mathfrak{h}$ and $i_1, \ldots, i_n \in 1/2 + \mathbb{Z}$, we define ${}^{\circ}_{\circ}\beta_1(i_1) \cdots \beta_n(i_n){}^{\circ}_{\circ}$ inductively by

$$\begin{aligned} {}^{\circ}_{\circ}\beta_1(i_1){}^{\circ}_{\circ} &= \beta_1(i_1) \qquad \text{and} \\ {}^{\circ}_{\circ}\beta_1(i_1) \cdots \beta_n(i_n){}^{\circ}_{\circ} &= \begin{cases} {}^{\circ}_{\circ}\beta_2(i_2) \cdots \beta_n(i_n){}^{\circ}_{\circ}\beta_1(i_1) & \text{if } i_1 \geq 0, \\ \beta_1(i_1){}^{\circ}_{\circ}\beta_2(i_2) \cdots \beta_n(i_n){}^{\circ}_{\circ} & \text{if } i_1 < 0. \end{cases} \end{aligned} \tag{2.50}$$

Let $h^{[1]}, \ldots, h^{[\dim \mathfrak{h}]}$ be an orthonormal basis of $\mathfrak{h}$. We define $c_{mn} \in \mathbb{Q}$ for $m, n \in \mathbb{Z}_{\geq 0}$ by

$$\sum_{m,n=0}^{\infty} c_{mn} x^m y^n = -\log(\frac{(1+x)^{1/2} + (1+y)^{1/2}}{2}) \tag{2.51}$$

and

$$\Delta_x = \sum_{m,n=0}^{\infty} c_{mn} \sum_{i=1}^{\dim \mathfrak{h}} h^{[i]}(m)h^{[i]}(n)x^{-m-n}. \tag{2.52}$$

Then, for $u \in M(1)$ we define a vertex operator $Y_{M(1)(\theta)}$ by

$$Y_{M(1)(\theta)}(u,x) = Y_0(e^{\Delta_x}u,x). \tag{2.53}$$

Then, [12, Theorem 9.3.1] shows that $(M(1)(\theta), Y_{M(1)(\theta)})$ is an irreducible θ-twisted $M(1)$-module. We define the action of θ on $M(1)(\theta)$ by

$$\theta(h^1(-i_1)\cdots h^n(-i_n)\mathbf{1}_{\mathrm{tw}} = (-1)^n h^1(-i_1)\cdots h^n(-i_n)\mathbf{1}_{\mathrm{tw}} \tag{2.54}$$

for $n \in \mathbb{Z}_{\geq 0}$, $h^1,\dots,h^n \in \mathfrak{h}$, $i_1,\dots,i_n \in 1/2+\mathbb{Z}_{>0}$ and set

$$M(1)(\theta)^{\pm} = \{u \in M(1)(\theta) \mid \theta u = \pm u\}. \tag{2.55}$$

Set a submodule $P = \{\theta(a)a^{-1} \mid a \in \hat{L}\}$ of $\hat{L}$. Let T_χ be the irreducible $\hat{L}/P$-module associated to a central character χ such that $\chi(\kappa) = -1$. We set

$$V_L^{T_\chi} = M(1)(\theta) \otimes T_\chi. \tag{2.56}$$

Then, [12, Theorem 9.5.3] shows that $V_L^{T_\chi}$ admits an irreducible θ-twisted V_L-module structure compatible with the action of $M(1)$. We define the action of θ on $V_L^{T_\chi}$ by

$$\theta(h^1(-i_1)\cdots h^n(-i_n)u) = (-1)^n h^1(-i_1)\cdots h^n(-i_n)u \tag{2.57}$$

for $n \in \mathbb{Z}_{\geq 0}$, $h^1,\dots,h^n \in \mathfrak{h}$, $i_1,\dots,i_n \in 1/2+\mathbb{Z}_{>0}$, and $u \in T_\chi$ and set

$$V_L^{T_\chi,\pm} = \{u \in V_L^{T_\chi} \mid \theta(u) = \pm u\}. \tag{2.58}$$

Let $h^{[1]},\dots,h^{[d]}$ be an orthonormal basis of $\mathfrak{h}$. For $i = 1,\dots,d$, we define the following elements in $M(1)^+$:

$$\begin{aligned}
\omega^{[i]} &= \frac{1}{2}h^{[i]}(-1)^2\mathbf{1},\\
\omega &= \omega^{[1]}+\cdots+\omega^{[d]},\\
H^{[i]} &= \frac{1}{3}h^{[i]}(-3)h^{[i]}(-1)\mathbf{1} - \frac{1}{3}h^{[i]}(-2)^2\mathbf{1},\\
J^{[i]} &= h^{[i]}(-1)^4\mathbf{1} - 2h^{[i]}(-3)h^{[i]}(-1)\mathbf{1} + \frac{3}{2}h^{[i]}(-2)^2\mathbf{1}\\
&= -9H^{[i]} + 4(\omega^{[i]}_{-1})^2\mathbf{1} - 3\omega^{[i]}_{-3}\mathbf{1}.
\end{aligned} \tag{2.59}$$

For $\alpha \in \mathfrak{h}$, we define

$$E(\alpha) = e^{\alpha} + \theta(e^{\alpha}). \tag{2.60}$$

We recall the following notation and some results from [10, Sections 4 and 5]: for any pair of distinct elements $i, j \in \{1, \dots, d\}$ and $r, s \in \mathbb{Z}_{>0}$,

$$\begin{aligned} S_{ij}(r,s) &= h^{[i]}(-r)h^{[j]}(-s)\mathbf{1}, \\ E^u_{ij} &= 5S_{ij}(1,2) + 25S_{ij}(1,3) + 36S_{ij}(1,4) + 16S_{ij}(1,5), \\ E^t_{ij} &= -16S_{ij}(1,2) + 145S_{ij}(1,3) + 19S_{ij}(1,4) + 8S_{ij}(1,5), \\ \Lambda_{ij} &= 45S_{ij}(1,2) + 190S_{ij}(1,3) + 240S_{ij}(1,4) + 96S_{ij}(1,5). \end{aligned} \tag{2.61}$$

It follows from [10, Proposition 5.3.14] that in the Zhu algebra $(A(M(1)^+), *)$, $A^u = \oplus_{i,j}\mathbb{C}E^u_{ij}$ and $A^t = \oplus_{i,j}\mathbb{C}E^t_{ij}$ are two-sided ideals, each of which is isomorphic to the $d \times d$ matrix algebra and $A^u * A^t = A^t * A^u = 0$. By [10, Proposition 5.3.12], for any pair of distinct elements $i, j \in \{1, \dots, d\}$, we have $A^u * \Lambda_{ij} = \Lambda_{ij} * A^u = A^t * \Lambda_{ij} = \Lambda_{ij} * A^t = 0$. By [10, Proposition 5.3.15], $A(M(1)^+)/(A^u + A^t)$ is a commutative algebra generated by the images of $\omega^{[i]}, H^{[i]}$ and Λ_{jk} where $i = 1, \dots, d$ and $j, k \in \{1, \dots, d\}$ with $j \neq k$.

For $\lambda \in \mathfrak{h}$, $k = 1, \dots, d$, and any pair of distinct elements $i, j \in \{1, \dots, d\}$,

$$\begin{aligned} \omega^{[k]}_1 e^\lambda &= \frac{\langle \lambda, h^{[k]}\rangle^2}{2} e^\lambda, \\ H^{[k]}_3 e^\lambda &= 0, \\ S_{ij}(1,1)_1 e^\lambda &= -S_{ij}(1,2)_2 e^\lambda = S_{ij}(1,3)_3 e^\lambda = \langle \lambda, h^{[i]}\rangle\langle \lambda, h^{[j]}\rangle e^\lambda, \end{aligned} \tag{2.62}$$

$$\begin{aligned} \omega^{[k]}_1 h^{[j]}(-1)\mathbf{1} &= \delta_{jk} h^{[j]}(-1)\mathbf{1}, \\ H^{[k]}_3 h^{[j]}(-1)\mathbf{1} &= \delta_{jk} h^{[j]}(-1)\mathbf{1}, \\ S_{ij}(1,1)_1 h^{[j]}(-1)\mathbf{1} &= h^{[i]}(-1)\mathbf{1}, \\ S_{ij}(1,2)_2 h^{[j]}(-1)\mathbf{1} &= -2h^{[i]}(-1)\mathbf{1}, \\ S_{ij}(1,3)_3 h^{[j]}(-1)\mathbf{1} &= 3h^{[i]}(-1)\mathbf{1}, \end{aligned} \tag{2.63}$$

and

$$\begin{aligned} \omega^{[k]}_1 \mathbf{1}_{\rm tw} &= \frac{1}{16}\mathbf{1}_{\rm tw}, \\ H^{[k]}_3 \mathbf{1}_{\rm tw} &= \frac{-1}{128}\mathbf{1}_{\rm tw}, \\ S_{ij}(1,1)_1 \mathbf{1}_{\rm tw} &= S_{ij}(1,2)_2 \mathbf{1}_{\rm tw} = S_{ij}(1,3)_3 \mathbf{1}_{\rm tw} = 0, \\ \omega^{[k]}_1 h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw} &= \delta_{jk}\frac{9}{16} h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw}, \\ H^{[k]}_3 h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw} &= \delta_{jk}\frac{15}{128} h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw}, \\ S_{ij}(1,1)_1 h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw} &= \frac{1}{2} h^{[i]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw}, \\ S_{ij}(1,2)_2 h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw} &= \frac{-3}{4} h^{[i]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw}, \\ S_{ij}(1,3)_3 h^{[j]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw} &= \frac{15}{16} h^{[i]}(-\tfrac{1}{2})\mathbf{1}_{\rm tw}. \end{aligned} \tag{2.64}$$

For any pair of distinct elements $i, j, k \in \{1, \ldots, d\}$, $l, m \in \mathbb{Z}$, and $r, s \in \mathbb{Z}_{>0}$, a direct computation shows that

$$[h^{[j]}(l), S_{ij}(r,s)_m] = s\binom{l}{s}\binom{-l-m+r+s-2}{r-1}h^{[i]}(l+m-r-s+1), \tag{2.65}$$

$$[\omega_l^{[j]}, S_{ij}(1,r)_m] = rS_{ij}(1,r+1)_{m+l} + lrS_{ij}(1,r)_{m+l-1}, \tag{2.66}$$

$$[S_{kj}(1,1)_l, S_{ij}(1,r)_m] = r\sum_{t=1}^{r+1}\binom{l}{t}S_{ik}(1,t)_{l+m-r-1+t}. \tag{2.67}$$

We also have

$$\begin{aligned}
S_{ij}(2,1) &= \omega_0 S_{ij}(1,1) - S_{ij}(1,2),\\
S_{ij}(3,1) &= \frac{1}{2}\omega_0^2 S_{ij}(1,1) - \omega_0 S_{ij}(1,2) + S_{ij}(1,3),\\
S_{ij}(2,2) &= \omega_0 S_{ij}(1,2) - 2S_{ij}(1,3),\\
S_{ij}(3,2) &= -\omega_{-2}^{[j]} S_{ij}(1,1) + 2\omega_{-1}^{[j]} S_{ij}(1,2) + \frac{1}{2}\omega_0^2 S_{ij}(1,2) - 2\omega_0 S_{ij}(1,3),\\
S_{ij}(3,3) &= \frac{-1}{2}\omega_0\omega_{-2}^{[j]} S_{ij}(1,1) + \frac{3}{2}\omega_{-2}^{[i]} S_{ij}(1,2)\\
&\quad - \omega_0\omega_{-1}^{[i]} S_{ij}(1,2) + \omega_0\omega_{-1}^{[j]} S_{ij}(1,2)\\
&\quad + \frac{1}{4}\omega_0^3 S_{ij}(1,2) + 2\omega_{-1}^{[i]} S_{ij}(1,3) - \omega_0^2 S_{ij}(1,3).
\end{aligned} \tag{2.68}$$

For $P \subset \{1, \ldots, d\}$, we define the subspace

$$M(1)_P := \mathrm{Span}_{\mathbb{C}}\left\{h^{[i_1]}(-j_1)\ldots h^{[i_n]}(-j_n)\mathbf{1} \;\middle|\; \begin{array}{l} n \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_n \in \{1, \ldots, d\}, j_1, \ldots, j_n \in \mathbb{Z}_{>0},\\ \{l \in \{1, \ldots, n\} \mid |\{k \mid i_k = l\}| \text{ is odd}\} = P \end{array}\right\} \tag{2.69}$$

of $M(1)$ and the subspace

$$M(1)_P^+ := M(1)_P \cap M(1)^+ \tag{2.70}$$

of $M(1)^+$. Note that if $|P|$ is odd, then $M(1)_P^+ = \{0\}$.

For $P, P' \subset \{1, \ldots, d\}$, we define $P \ominus P' := (P \cup P') \setminus (P \cap P') \subset \{1, \ldots, d\}$.

Lemma 2.4.

(1) We have $M(1)^+ = \bigcup_{\substack{P \subset \{1, \ldots, d\}\\ |P| \text{ is even}}} M(1)_P^+$.

(2) For $P, P' \subset \{1, \ldots, d\}$,

$$\begin{aligned}
&(M(1)_P) \cdot M(1)_{P'} \subset M(1)_{P \ominus P'} \text{ and}\\
&(M(1)_P^+) \cdot M(1)_{P'}^+ \subset M(1)_{P \ominus P'}^+.
\end{aligned} \tag{2.71}$$

Proof. The result (1) follows from the definition of $M(1)_P^+$. The result (2) follows from the fact that for $i \in \{1, \ldots, d\}$ and $j \in \mathbb{Z}$, $h^{[i]}(j)M(1)_{P'} \subset M(1)_{\{i\} \ominus P'}$. □

For $P=\{p_1,\dots,p_{2t}\}\subset\{1,\dots,d\}$ with $p_1>\cdots>p_{2t}$, we define

$$\begin{aligned}B_P&:=\{h^{[p_1]}(-1)h^{[p_2]}(-r_1)\cdots h^{[p_{2t-1}]}(-1)h^{[p_{2t}]}(-r_t)\mathbf{1}\mid r_1,\dots,r_t\in\{1,2,3\}\}\\&=\{S_{p_1,p_2}(1,r_1)_{-1}\cdots S_{p_{2t-1},p_{2t}}(1,r_t)_{-1}\mathbf{1}\mid r_1,\dots,r_t\in\{1,2,3\}\}.\end{aligned}\tag{2.72}$$

By (2.63), (2.65), (2.68), we know that

$$M(1)^+_P=M(1)^+_{\varnothing}\cdot(B_P)_{-}\mathbf{1}\tag{2.73}$$

and hence by (2.65) again, $M(1)^+_P$ is also spanned by the elements of the form

$$S_{p_1,p_2}(1,r_1)_{-s_1}\cdots S_{p_{2t-1},p_{2t}}(1,r_t)_{-s_t}a,\tag{2.74}$$

where $a\in M(1)^+_{\varnothing}$, $s_1,\dots,s_t\in\mathbb{Z}_{>0}$, and $r_1,\dots,r_t\in\{1,2,3\}$. By (2.63), (2.65), (2.68), and [19, (3.4), (3.5), (3.7), (3.9), (3.10), (3.11)], $M(1)^+_P$ is spanned by the elements of the form

$$a^{(1)}_{-l_1}\dots a^{(m)}_{-l_m}S_{p_1,p_2}(1,r_1)_{-s_1}\cdots S_{p_{2t-1},p_{2t}}(1,r_t)_{-s_t}\mathbf{1},\tag{2.75}$$

where $m\in\mathbb{Z}_{\geq0}$, $a^{(1)},\dots,a^{(m)}\in\{\omega^{[j]},H^{[j]}\mid j\in\{1,\dots,d\}\}$, $l_1,\dots,l_m,s_1,\dots,s_t\in\mathbb{Z}_{>0}$, and $r_1,\dots,r_t\in\{1,2,3\}$. By (2.74), in the same way, we know that $M(1)^+_P$ is spanned by the elements of the form

$$S_{p_1,p_2}(1,r_1)_{-s_1}\cdots S_{p_{2t-1},p_{2t}}(1,r_t)_{-s_t}a^{(1)}_{-l_1}\dots a^{(m)}_{-l_m}\mathbf{1},\tag{2.76}$$

where $m\in\mathbb{Z}_{\geq0}$, $a^{(1)},\dots,a^{(m)}\in\{\omega^{[j]},H^{[j]}\mid j\in\{1,\dots,d\}\}$, $l_1,\dots,l_m,s_1,\dots,s_t\in\mathbb{Z}_{>0}$, and $r_1,\dots,r_t\in\{1,2,3\}$.

Remark 2.5. Let $u,v\in M(1)^+$, $i,j\in\mathbb{Z},\rho,\sigma\in\mathbb{Z}_{\geq-1}$, and p,q a pair of distinct elements in $\{1,\dots,d\}$. Throughout this paper, if $[u_i,v_j]=\sum_{k=0}^{\infty}\binom{i}{k}(u_kv)_{i+j-k}\in M(1)^+_{\{p,q\}}$, then we frequently express this element as a linear combination of elements of the form

$$a^{(1)}_{i_1}\cdots a^{(k)}_{i_k}S_{pq}(1,r)_tb^{(1)}_{j_1}\cdots b^{(l)}_{j_l}\tag{2.77}$$

where $k,l\in\mathbb{Z}_{\geq0}$, $r=1,2,3$, $t\in\mathbb{Z}$, and

$$\begin{aligned}&(a^{(1)},i_1),\dots,(a^{(k)},i_k)\in\{(\omega^{[k]},m)\mid m\leq\rho\}_{k=1}^d\cup\{(H^{[k]},n)\mid n\leq\sigma\}_{k=1}^d,\\&(b^{(1)},j_1),\dots,(b^{(l)},j_l)\in\{(\omega^{[k]},m)\mid m\geq\rho+1\}_{k=1}^d\cup\{(H^{[k]},n)\mid n\geq\sigma+1\}_{k=1}^d.\end{aligned}\tag{2.78}$$

For the calculation, we use [19, Lemma 2.2] and the data a_kb $(k=0,1,\dots)$ in Appendix A2. In most cases, we obtain the explicit expressions of the results by using computer algebra system Risa/Asir[16].

For example, we shall compute $[H^{[j]}_l,S_{ij}(1,1)_n]$ for a pair of distinct elements $i,j\in\{1,\dots,d\}$

and $l, n \in \mathbb{Z}$. For $m \in \mathbb{Z}_{\geq -1}$, by [19, Lemma 2.2], we have

$$\begin{aligned}
(\omega^{[j]}_{-2}S_{ij}(1,1))_n &= \sum_{\substack{r\leq m\\ r+s+2=n}} (-r-1)\omega^{[j]}_r S_{ij}(1,1)_s + \sum_{\substack{r\geq m+1\\ r+s+2=n}} (-r-1)S_{ij}(1,1)_s\omega^{[j]}_r \\
&\quad + \sum_{t=0}^{1}(t+1)\binom{m+2}{t+2}(\omega^{[j]}_t S_{ij}(1,1))_{n-2-t} \\
&= \sum_{\substack{r\leq m\\ r+s+2=n}} (-r-1)\omega^{[j]}_r S_{ij}(1,1)_s + \sum_{\substack{r\geq m+1\\ r+s+2=n}} (-r-1)S_{ij}(1,1)_s\omega^{[j]}_r \\
&\quad + \binom{m+2}{2}S_{ij}(1,2)_{n-2} + 2\binom{m+2}{3}S_{ij}(1,1)_{n-3}, \\
(\omega^{[j]}_{-1}S_{ij}(1,2))_n &= \sum_{\substack{r\leq m\\ r+s+1=n}} \omega^{[j]}_r S_{ij}(1,2)_s + \sum_{\substack{r\geq m+1\\ r+s+1=n}} S_{ij}(1,2)_s\omega^{[j]}_r \\
&\quad - \sum_{t=0}^{1}\binom{t}{0}\binom{m+1}{t+1}(\omega^{[j]}_t S_{ij}(1,2))_{n-1-t} \\
&= \sum_{\substack{r\leq m\\ r+s+1=n}} \omega^{[j]}_r S_{ij}(1,2)_s + \sum_{\substack{r\geq m+1\\ r+s+1=n}} S_{ij}(1,2)_s\omega^{[j]}_r \\
&\quad - 2\binom{m+1}{1}S_{ij}(1,3)_{n-1} - 2\binom{m+1}{2}S_{ij}(1,2)_{n-2}.
\end{aligned} \tag{2.79}$$

By (A2.4) and (2.79), we have

$$\begin{aligned}
&[H^{[j]}_l, S_{ij}(1,1)_n] = \sum_{k=0}^{3}\binom{l}{k}(H^{[j]}_k S_{ij}(1,1))_{l+n-k} \\
&= (-2\omega^{[j]}_{-2}S_{ij}(1,1) + 4\omega^{[j]}_{-1}S_{ij}(1,2))_{l+n} \\
&\quad + 4lS_{ij}(1,3)_{l+n-1} + \binom{l}{2}\frac{7}{3}S_{ij}(1,2)_{l+n-2} + \binom{l}{3}S_{ij}(1,1)_{l+n-3} \\
&= -2\Big(\sum_{\substack{r\leq m\\ r+s+2=l+n}} (-r-1)\omega^{[j]}_r S_{ij}(1,1)_s + \sum_{\substack{r\geq m+1\\ r+s+2=l+n}} (-r-1)S_{ij}(1,1)_s\omega^{[j]}_r \\
&\qquad + \binom{m+2}{2}S_{ij}(1,2)_{l+n-2} + 2\binom{m+2}{3}S_{ij}(1,1)_{l+n-3}\Big) \\
&\quad + 4\Big(\sum_{\substack{r\leq m\\ r+s+1=l+n}} \omega^{[j]}_r S_{ij}(1,2)_s + \sum_{\substack{r\geq m+1\\ r+s+1=l+n}} S_{ij}(1,2)_s\omega^{[j]}_r \\
&\qquad - 2\binom{m+1}{1}S_{ij}(1,3)_{l+n-1} - 2\binom{m+1}{2}S_{ij}(1,2)_{l+n-2}\Big) \\
&\quad + 4lS_{ij}(1,3)_{l+n-1} + \binom{l}{2}\frac{7}{3}S_{ij}(1,2)_{l+n-2} + \binom{l}{3}S_{ij}(1,1)_{l+n-3}.
\end{aligned} \tag{2.80}$$

3 Modules for the Zhu algebra of $M(1)^+$ in a weak V_L^+-module: the general case

Let L be a non-degenerate even lattice of finite rank d and $\mathfrak{h} := \mathbb{C}\otimes_{\mathbb{Z}} L$. In this section, we shall show that there exists an irreducible $M(1)^+$-module in any non-zero weak V_L^+-module (Proposition 3.5). Throughout this section M is a weak V_L^+-module.

Lemma 3.1. *For a non-degenerate even lattice L of finite rank d, there exists a sequence of elements $\beta_1,\dots,\beta_d\in L$ such that $\langle\beta_i,\beta_i\rangle\neq 0$ for $i=1,\dots,d$ and $\langle\beta_j,\beta_k\rangle=0$ for any pair of distinct elements $j,k\in\{1,\dots,d\}$.*

Proof. Let $\gamma_1,\dots,\gamma_d$ be a basis of $\mathbb{Q}\otimes_{\mathbb{Z}} L$ such that $\langle\gamma_i,\gamma_i\rangle\neq 0$ and $\langle\gamma_j,\gamma_k\rangle=0$ for all $i\in\{1,\dots,d\}$ and $j,k\in\{1,\dots,d\}$ with $j\neq k$. . Since $\gamma_1,\dots,\gamma_d\in\mathbb{Q}\otimes_{\mathbb{Z}} L$, there exists a non-zero integer m_i such that $m\gamma_i\in L$ for all $i=1,\dots,d$. Then, the elements $\beta_i=m\gamma_i$ $(i=1,\dots,d)$ satisfy the condition. □

Let $\Lambda=\oplus_{i=1}^d\mathbb{Z}\beta_i$ be a sublattice of L such that $\langle\beta_i,\beta_i\rangle\neq 0$ for $i=1,\dots,d$ and $\langle\beta_j,\beta_k\rangle=0$ for any pair of distinct elements $j,k\in\{1,\dots,d\}$. We have

$$V_{\mathbb{Z}\beta_1}^+\otimes\cdots\otimes V_{\mathbb{Z}\beta_d}^+\subset V_L^+ \tag{3.1}$$

and take the orthonormal basis $h^{[1]},\dots,h^{[d]}$ of $\mathfrak{h}$ defined by

$$h^{[i]}=\frac{1}{\sqrt{\langle\beta^{[i]},\beta^{[i]}\rangle}}\beta^{[i]}\quad(i=1,\dots,d). \tag{3.2}$$

Since $[h^{[i]}_l,h^{[j]}_m]=0$ for any pair of distinct elements $i,j\in\{1,\dots,d\}$ and $l,m\in\mathbb{Z}$, it follows by induction on d using [19, Lemma 3.7] that there exists a simultaneous eigenvector u of $\{\omega^{[i]}_1,H^{[i]}_3\}_{i=1}^d$ in a weak V_L^+-module M such that $\epsilon(\omega^{[i]},u)\leq 1$ and $\epsilon(H^{[i]},u)\leq 3$ for all $i=1,\dots,d$.

Lemma 3.2. *Let U be a subspace of a weak $M(1)^+$-module.*

(1) Let $i,j\in\{1,\dots,d\}$ with $i\neq j$ and $k\in\mathbb{Z}$ such that $k\geq\epsilon(S_{ij}(1,1),u)$ for all $u\in U$. If U is stable under the action of $\omega^{[j]}_1$, then

$$\epsilon(S_{ij}(1,r+1),u)\leq k+r \tag{3.3}$$

for all $r\in\mathbb{Z}_{\geq 0}$.

(2) Assume $\alpha\in\mathbb{C}h^{[1]}$. Let $i\in\{2,\dots,d\}$ and $t\in\mathbb{Z}$ such that $t\geq\epsilon(E(\alpha),u)$ for all $u\in U$. If U is stable under the action of $S_{i1}(1,1)_1$, then

$$\epsilon(S_{i1}(1,1)_0E(\alpha),u)\leq t+1. \tag{3.4}$$

Proof. (1) For $l,m\in\mathbb{Z}$ and $r\in\mathbb{Z}_{>0}$, by (2.66)

$$S_{ij}(1,r+1)_{l+m}=\frac{1}{m}[\omega^{[j]}_l,S_{ij}(1,r)_m]-lS_{ij}(1,r)_{l+m-1}, \tag{3.5}$$

which implies (3.3).

(2) Since $S_{i1}(1,1)_mE(\alpha)=0$ for all $m\in\mathbb{Z}_{>0}$, the same argument as above shows the result. □

Lemma 3.3. *Let U be a finite dimensional subspace of a weak $M(1)^+$-module M such that for all $k=1,\dots,d$ and $u\in U$, $\epsilon(\omega^{[k]},u)\leq 1, \epsilon(H^{[k]},u)\leq 3$, and $\omega^{[k]}_1 u\in U$, $H^{[k]}_3 u\in U$. For any pair of distinct elements $i,j\in\{1,\dots,d\}$ we denote $\max\{\{\epsilon(S_{ij}(1,1),u)\mid u\in U\}\cup\{-1\}\}$ by $\epsilon(S_{ij})$.*

(1) Let i,j be a pair of distinct elements in $\{1,\dots,d\}$. We define $W:=\mathrm{Span}_{\mathbb{C}}\{S_{ij}(1,r)_{\epsilon(S_{ij})+r-1}u\mid r=1,2,3\}$. For any $w\in W$ and $k=1,\dots,d$, we have $\epsilon(\omega^{[k]},w)\leq 1, \epsilon(H^{[k]},w)\leq 3$ and $\omega^{[k]}_1 w, H^{[k]}_3 w\in W$.

(2) Assume $\epsilon(S_{ij})\leq 1$ for any pair of distinct elements $i,j\in\{1,\dots,d\}$, namely $U\subset\Omega_{M(1)^+}(M)$. For $P=\{p_1,\dots,p_{2t}\}\subset\{1,\dots,d\}$ with $p_1>\cdots>p_{2t}$, we define

$$S_PU:=\mathrm{Span}_{\mathbb{C}}\{S_{p_1,p_2}(1,r_1)_{r_1}\cdots S_{p_{2t-1},p_t}(1,r_t)_{r_t}u\mid u\in U \text{ and } r_1,\dots,r_t\in\{1,2,3\}\} \quad (3.6)$$

and $SU:=\sum_{\substack{P\subset\{1,\dots,d\},\\ |P| \text{ is even}}} S_PU$. Then, SU is an $A(M(1)^+)$-submodule of $\Omega_{M(1)^+}(M)$.

Proof. (1) We may take (i,j) to be $(2,1)$. We define $\epsilon(S):=\epsilon(S_{21}(1,1))$, $\epsilon(S_{21}(1,2)):=\epsilon(S)+1$, and $\epsilon(S_{21}(1,3)):=\epsilon(S)+2$. By Lemma 3.2 (1), we have $\epsilon(S_{21}(1,i))\geq\epsilon(S_{21}(1,i),u)$ for all $i=1,2,3$. We define $A:=\{\omega^{[i]},H^{[i]}\}_{i=1}^d$, $B:=\{S_{21}(1,i)\mid i=1,2,3\}$, $\epsilon(\omega^{[i]}):=1=\mathrm{wt}(\omega^{[i]})-1$ and $\epsilon(H^{[i]}):=3=\mathrm{wt}(H^{[i]})-1$ for $i=1,\dots,d$. In order to apply Lemma 2.3 (3) to u, for $a\in A$ we shall compute $\zeta(a)$ defined in (2.20). Note that $M(1)^+_{\varnothing}=\langle A_-\rangle\mathbf{1}$ and $M(1)^+_{\{2,1\}}=B_-\langle A_-\rangle\mathbf{1}=\langle A_-\rangle B_-\mathbf{1}$ (see (2.70) for the definition of $M(1)^+_P$). By Lemma 2.4 (2), we have $A\cdot\langle A_-\rangle\mathbf{1}\subset\langle A_-\rangle\mathbf{1}, A\cdot(\langle A_-\rangle B_-\mathbf{1})\subset\langle A_-\rangle B_-\mathbf{1}, B\cdot\langle A_-\rangle\mathbf{1}\subset B_-\langle A_-\rangle\mathbf{1}$, and $A\cdot(B_-\langle A_-\rangle\mathbf{1})\subset B_-\langle A_-\rangle\mathbf{1}$, where the symbols $\langle A_-\rangle\mathbf{1},\langle A_-\rangle B_-\mathbf{1}$, and $B_-\langle A_-\rangle\mathbf{1}$ are defined in (2.10) and (2.11). For $j\in\mathbb{Z}$ with $j\geq\min\{\mathrm{wt}(b)\mid b\in B\}=2$, by the definition (2.12) of δ ,

$$\delta(j)=\epsilon(S_{21}(1,1))-2+j. \quad (3.7)$$

For $a\in\{\omega^{[i]}_1,H^{[i]}_3\}_{i=1}^d$ and $j=1,2,3$, since $\mathrm{wt}(a)\geq 2$, we have

$$\begin{aligned}&\delta(\mathrm{wt}(a)+\mathrm{wt}(S_{21}(1,j))-1)-\epsilon(S_{21}(1,j))=\delta(\mathrm{wt}(a)+j)-(\epsilon(S_{21}(1,1))+j-1)\\&=\epsilon(S_{21}(1,1))-2+(\mathrm{wt}(a)+j)-(\epsilon(S_{21}(1,1))+j-1)=\mathrm{wt}(a)-1\end{aligned} \quad (3.8)$$

and hence $\zeta(a)=\mathrm{wt}(a)-1$. Applying Lemma 2.3 (3) to u, we have $a_{\mathrm{wt}(a)-1}U\subset U$ and $a_kU=0$ for $a\in A$ and $k>\mathrm{wt}(a)-1$.

(2) For $P\subset\{1,\dots,d\}$ such that $|P|$ is even, by using (1), an inductive argument on $|P|$ shows that $\epsilon(\omega^{[k]},u)\leq 1,\epsilon(H^{[k]},u)\leq 3$, and $\omega^{[k]}_1u\in S_PU$, $H^{[k]}_3u\in S_PU$ for all $k=1,\dots,d$ and $u\in S_PU$. Let $A:=\{\omega^{[i]},H^{[i]}\}_{i=1}^d$ and

$$\begin{aligned}B&:=\left\{h^{[p_1]}(-1)h^{[p_2]}(-r_1)\cdots h^{[p_{2t-1}]}(-1)h^{[p_{2t}]}(-r_n)\mathbf{1}\;\middle|\;\begin{array}{l}t\in\mathbb{Z}_{\geq 0},r_1,\dots,r_t\in\{1,2,3\}\\ p_1,\dots,p_{2t}\in\{1,\dots,d\}\text{ such that}\\ p_1>\cdots>p_{2t}\end{array}\right\}\\&=\left\{S_{p_1,p_2}(1,r_1)_{-1}\cdots S_{p_{2t-1},p_{2t}}(1,r_t)_{-1}\mathbf{1}\;\middle|\;\begin{array}{l}t\in\mathbb{Z}_{\geq 0},r_1,\dots,r_t\in\{1,2,3\}\\ p_1,\dots,p_{2t}\in\{1,\dots,d\}\text{ such that}\\ p_1>\cdots>p_{2t}\end{array}\right\}.\end{aligned} \quad (3.9)$$

By Lemma 2.4, (2.75), and (2.76), we have $A\cdot(B_-\mathbf{1})=\langle A_-\rangle B_-\mathbf{1}=B_-\langle A_-\rangle\mathbf{1}=M(1)^+$. We define $\epsilon(u):=\mathrm{wt}(u)-1$ for a homogeneous element $u\in M(1)^+$. Note that $\mathrm{Span}_{\mathbb{C}}\{b_{\epsilon(b)}u\mid b\in B,u\in U\}=SU$. Since $A\cdot\langle A_-\rangle\mathbf{1}\subset\langle A_-\rangle\mathbf{1}$ and $A\cdot M(1)^+=B\cdot M(1)^+=M(1)^+$, the result follows from Lemma 2.3 (4).

□

Lemma 3.4. *For a non-zero weak V_L^+-module M, there exists an irreducible $A(M(1)^+)$-submodule of $\Omega_{M(1)^+}(M)$.*

Proof. Let $h^{[1]},\dots,h^{[d]}$ be the orthonormal basis of $\mathfrak{h}$ defined by (3.2). By the argument just after (3.2), we can take a simultaneous eigenvector u of $\{\omega_1^{[i]}, H_3^{[i]}\}_{i=1}^d$ such that $\epsilon(\omega^{[i]},u)\leq 1$ and $\epsilon(H^{[i]},u)\leq 3$ for all $i=1,\dots,d$. We take a pair of distinct elements i,j so that $\epsilon(S_{ij}(1,1),u)\geq \epsilon(S_{lm}(1,1),u)$ for any pair of distinct elements $l,m\in\{1,\dots,d\}$. We may assume $(i,j)=(2,1)$. We define $\epsilon(S)=\epsilon(S_{21}(1,1)):=\epsilon(S_{21}(1,1),u)$, $\epsilon(S_{21}(1,2)):=\epsilon(S)+1$, and $\epsilon(S_{21}(1,3)):=\epsilon(S)+2$. By Lemma 3.2 (1), we have $\epsilon(S_{21}(1,i))\geq \epsilon(S_{21}(1,i),u)$ for all $i=1,2,3$. Hence if $\epsilon(S)\leq 0$, then $u\in\Omega_{M(1)^+}(M)$ and $\mathbb{C}u$ is an irreducible $A(M(1)^+)$-module.

From now, we assume $\epsilon(S)\geq 1$. We define the subspace $W:=\sum_{r=1}^3\mathbb{C}S_{21}(1,r)_{\epsilon(S)+r-1}u$ of M. Let $w\in W$ and (j,k) a pair of distinct elements in $\{1,\dots,d\}$. We shall investigate $\epsilon(S_{jk}(1,1),w)$. We note that $\epsilon(S_{jk}(1,r),u)\leq \epsilon(S)+r-1$ for all $r\geq 1$ by the definition of $\epsilon(S)$ and Lemma 3.2. If $\{j,k\}\cap\{1,2\}=\varnothing$, then

$$\epsilon(S_{jk}(1,1),w)\leq \epsilon(S_{jk}(1,1),u) \tag{3.10}$$

since $S_{jk}(1,1)_l S_{21}(1,r)_m=S_{21}(1,r)_m S_{jk}(1,1)_l$ for all $l,m\in\mathbb{Z}$ and $r=1,2,3$. For $j\in\{3,\dots,d\}$, $r\in\{1,2,3\}$, and $l\in\mathbb{Z}$, by (2.67)

$$S_{j1}(1,1)_l S_{21}(1,r)_{\epsilon(S)+r-1}u=S_{21}(1,r)_{\epsilon(S)+r-1}S_{j1}(1,1)_l u+r\sum_{s=1}^{r+1}\binom{l}{s}S_{2j}(1,s)_{\epsilon(S)-2+l+s}u. \tag{3.11}$$

Thus, $\epsilon(S_{j1}(1,1),S_{21}(1,r)_{\epsilon(S)+r-1}u)\leq \max\{\epsilon(S_{j1}(1,1),u),1\}$ for $r=1,2,3$ and hence

$$\epsilon(S_{j1}(1,1),w)\leq \max\{\epsilon(S_{j1}(1,1),u),1\}. \tag{3.12}$$

The same argument shows that for $j\in\{3,\dots,d\}$,

$$\epsilon(S_{j2}(1,1),w)\leq \max\{\epsilon(S_{j2}(1,1),u),1\}. \tag{3.13}$$

We set $(j,k)=(2,1)$. For $i\in\mathbb{Z}$ and $r=1,2,3$, by [14, Proposition 4.5.7] putting $u=S_{21}(1,1), v=S_{21}(1,r), p=i, q=\epsilon(S)+r-1, l=\epsilon(S)+1$, and $m=0$ in the symbol used there, we have

$$S_{21}(1,1)_i S_{21}(1,r)_{\epsilon(S)+r-1}u=\sum_{j=0}^{\epsilon(S)+1}\binom{\epsilon(S)+1}{j}(S_{21}(1,1)_{i-(\epsilon(S)+1)+j}S_{21}(1,r))_{2\epsilon(S)+r-j}u. \tag{3.14}$$

By Lemma 2.4 (2), $S_{21}(1,1)_{i-(\epsilon(S)+1)+j}S_{21}(1,r)$ is an element of $M(1)^+_{\varnothing}$. Since $\epsilon(\omega^{[i]},u)\leq 1$ and $\epsilon(H^{[i]},u)\leq 3$ for all $i=1,\dots,d$, for any homogeneous element $a\in M(1)^+_{\varnothing}$, by using (2.75), an inductive argument on $\mathrm{wt}\,a$ shows that $a_i u=0$ for all $i>\mathrm{wt}(a)-1$. We see that for $j\in\mathbb{Z}$ and $r\in\mathbb{Z}_{>0}$,

$$2\epsilon(S)+r-j-(\mathrm{wt}((S_{21}(1,1)_{i-(\epsilon(S)+1)+j}S_{21}(1,r)))-1)=\epsilon(S)+i-2. \tag{3.15}$$

Thus, if $\epsilon(S)\geq 2$, then by (3.14), $S_{21}(1,1)_i S_{21}(1,r)_{\epsilon(S)+r-1}u=0$ for all $i\geq\epsilon(S)$ and hence

$$\epsilon(S_{21}(1,1),w)\leq \epsilon(S)-1. \tag{3.16}$$

For any $w\in W$ and $i=1,\dots,d$, it follows from Lemma 3.3 (1) that $\omega_1^{[i]}w, H_3^{[i]}w\in W$, and $\epsilon(\omega^{[i]},w)\leq 1$, $\epsilon(H^{[i]},w)\leq 3$. Thus, if $\epsilon(S)\geq 2$, then by (3.10), (3.12), (3.13), and (3.16), we can take a simultaneous eigenvector v of $\{\omega_1^{[i]},H_3^{[i]}\}_{i=1}^d$ in W such that $\epsilon(S_{21}(1,1),v)<\epsilon(S_{21}(1,1),u)$ and $\epsilon(S_{ij}(1,1),v)\leq\max\{\epsilon(S_{ij}(1,1),u),1\}$ for any pair of distinct element $i,j\in\{1,\dots,d\}$ with $\{i,j\}\neq\{1,2\}$. Replacing u by this v repeatedly, we get a non-zero element $u\in\Omega_{M(1)^+}(M)$. Now, the result follows from Lemma 3.3 (2), □

By Lemma 3.4 and [8, Theorem 6.2], we have the following result, which is already shown in [19, Proposition 3.13] when rank $L = 1$:

Proposition 3.5. *Let L be a non-degenerate even lattice of finite rank and M a non-zero weak V_L^+-module. Then, there exists a non-zero $M(1)^+$-submodule of M.*

4 Extension groups for $M(1)^+$

In this section we study some weak modules for $M(1)^+$ with rank d. As stated in Section 1, the irreducible $M(1)^+$-modules are classified in [9, Theorem 4.5] for the case of $\dim_{\mathbb{C}}\mathfrak{h} = 1$ and [10, Theorem 6.2.2] for the general case (see (1.1)). Results in this section will be used in Part 3 of this series of papers to show that every irreducible weak V_L^+-module is a direct sum of irreducible $M(1)^+$-modules. When $d = 1$, some of the results in this section have already been obtained in [2, Section 5]. In some parts of the following argument, we shall use techniques in [2, Section 5]. Throughout this section, M is an $M(1)^+$-module, W is an irreducible $M(1)^+$-module, and N is a weak $M(1)^+$-module. In this section, we consider the following exact sequence

$$0 \to W \to N \xrightarrow{\pi} M \to 0 \tag{4.1}$$

of weak $M(1)^+$-modules. We shall use the symbols in (2.59) and (2.61). We note that $[\omega_1^{[i]}, \omega_1^{[j]}] = [\omega_1^{[i]}, H_3^{[j]}] = [H_3^{[i]}, H_3^{[j]}] = 0$ for all $i, j = 1, \ldots, d$. Let B be an irreducible $A(M(1)^+)$-submodule of $M(0)$. For $\zeta = (\zeta^{[1]}, \ldots, \zeta^{[d]}), \xi = (\xi^{[1]}, \ldots, \xi^{[d]}) \in \mathbb{C}^d$, let $v \in B$ such that

$$(\omega_1^{[i]} - \zeta^{[i]})v = (H_3^{[i]} - \xi^{[i]})v = 0 \tag{4.2}$$

for all $i = 1, \ldots, d$ and we define

$$W_{\zeta,\xi} = \bigcap_{j=1}^{d} \operatorname{Ker}(\omega_1^{[j]} - \zeta^{[j]}) \cap \bigcap_{j=1}^{d} \operatorname{Ker}(H_3^{[j]} - \xi^{[j]}) \cap W. \tag{4.3}$$

Lemma 4.1. *Under the setting above, there exists $u \in N$ such that*

$$\pi(u) = v, \quad (\omega_1^{[i]} - \zeta^{[i]})u, (H_3^{[i]} - \xi^{[i]})u \in W_{\zeta,\xi} \tag{4.4}$$

and

$$(\omega_1^{[i]} - \zeta^{[i]})^2 u = (H_3^{[i]} - \xi^{[i]})^2 u = 0 \tag{4.5}$$

for all $i = 1, \ldots, d$.

Proof. Let $u \in N$ such that $\pi(u) = v$. Since $(\omega_1^{[i]} - \zeta^{[i]})u, (H_3^{[i]} - \xi^{[i]})u \in W$ and the actions of $\omega_1^{[i]}$ and $H_3^{[i]}$ on W are semisimple for all $i = 1, \ldots, d$, the subspace $U := \operatorname{Span}_{\mathbb{C}}\{a_{(1)} \cdots a_{(n)}u \mid n \in \mathbb{Z}_{\geq 0}, a_{(1)}, \ldots, a_{(n)} \in \{\omega_1^{[i]}, H_3^{[i]}\}_{i=1}^d\}$ of N is finite dimensional. For $\rho = (\rho^{[i]})_{i=1}^d, \sigma = (\sigma^{[i]})_{i=1}^d \in \mathbb{C}^d$, we define

$$U_{\rho,\sigma} := \left\{ w \in U \;\middle|\; \begin{array}{l} \text{there exists } n \in \mathbb{Z}_{>0} \text{ such that} \\ (\omega_1^{[i]} - \rho^{[i]})^n w = (H_3^{[i]} - \sigma^{[i]})^n w = 0 \text{ for all } i = 1, \ldots, d. \end{array} \right\} \tag{4.6}$$

and we take a decomposition $U = \oplus_{\rho,\sigma \in \mathbb{C}^d} U_{\rho,\sigma}$. For any $\rho, \sigma \in \mathbb{C}^d$, we also take a linear map $f^{\rho,\sigma} \in \operatorname{Span}_{\mathbb{C}}\{a_{(1)} \cdots a_{(n)} \mid n \in \mathbb{Z}_{\geq 0}, a_{(1)}, \ldots, a_{(n)} \in \{\omega_1^{[i]}, H_3^{[i]}\}_{i=1}^d\}$ such that $f^{\rho,\sigma}|_{U_{\rho,\sigma}} = \mathrm{id}_{U_{\rho,\sigma}}$

and $f^{\rho,\sigma}|_{U_{\mu,\nu}} = 0$ for all $(\mu,\nu) \neq (\rho,\sigma)$. We write $u = \sum_{\rho,\sigma\in\mathbb{C}^d} u_{\rho,\sigma}$ where $u_{\rho,\sigma} \in U_{\rho,\sigma}$. We fix $i \in \{1,\dots,d\}$. Since $(\omega^{[i]}_1 - \zeta^{[i]})u \in W$, we have

$$(\omega^{[i]}_1 - \zeta^{[i]})u_{\rho,\sigma} = (\omega^{[i]}_1 - \zeta^{[i]})f^{\rho,\sigma}u = f^{\rho,\sigma}(\omega^{[i]}_1 - \zeta^{[i]})u \in W \cap U_{\rho,\sigma}. \tag{4.7}$$

Since the action of $\omega^{[i]}_1$ on W is semisimple,

$$(\omega^{[i]}_1 - \rho^{[i]})(\omega^{[i]}_1 - \zeta^{[i]})u_{\rho,\sigma} = 0. \tag{4.8}$$

Since $u_{\rho,\sigma} \in U_{\rho,\sigma}$, there exists $k \in \mathbb{Z}_{>0}$ such that $(\omega^{[i]}_1 - \rho^{[i]})^k u_{\rho,\sigma} = 0$. When $\rho^{[i]} \neq \zeta^{[i]}$, regarding $(\omega^{[i]}_1 - \rho^{[i]})^k$ and the left-hand side of (4.8) as polynomials in $\omega^{[i]}_1 - \rho^{[i]}$, and dividing the former by the latter, we get $(\omega^{[i]}_1 - \rho^{[i]})u_{\rho,\sigma} = 0$ and hence

$$u_{\rho,\sigma} = \frac{1}{\rho^{[i]} - \zeta^{[i]}}(\omega^{[i]}_1 - \zeta^{[i]})u_{\rho,\sigma} \in W \tag{4.9}$$

by (4.7). The same argument shows that

$$(H^{[i]}_3 - \sigma^{[i]})(H^{[i]}_3 - \xi^{[i]})u_{\rho,\sigma} = 0 \tag{4.10}$$

and if $\sigma^{[i]} \neq \xi^{[i]}$, then $u_{\rho,\sigma} \in W$. Thus if $(\rho,\sigma) \neq (\zeta,\xi)$, then $u_{\rho,\sigma} \in W$ and hence we can take $u = u_{\zeta,\xi} \in U_{\zeta,\xi}$. In this case, (4.4) and (4.5) hold by (4.8), and (4.10). □

Let $u \in N$ that satisfies (4.4) and (4.5). If $(W,B) \not\cong (M(1)^+, M(1)^-(0))$, then it follows from [2, Lemma 4.8] that

$$\epsilon(\omega^{[i]}, u) \leq 1 \text{ and } \epsilon(H^{[i]}, u) \leq 3 \tag{4.11}$$

for all $i = 1,\dots,d$, where $\epsilon = \epsilon_{Y_N}$.

For a pair of distinct elements $i, j \in \{1, \ldots, d\}$, a direct computation shows that

$$\begin{aligned}0 &= 6\omega^{[i]}_{-2}S_{ij}(1,1) + 2\omega^{[j]}_{-2}S_{ij}(1,1)\\&\quad - 4\omega_0\omega^{[i]}_{-1}S_{ij}(1,1) + \omega_0\omega_0\omega_0 S_{ij}(1,1)\\&\quad + 4\omega^{[i]}_{-1}S_{ij}(1,2) - 4\omega^{[j]}_{-1}S_{ij}(1,2)\\&\quad - 3\omega_0\omega_0 S_{ij}(1,2) + 6\omega_0 S_{ij}(1,3), \end{aligned} \tag{4.12}$$

$$\begin{aligned}0 &= 32\omega^{[i]}_{-3}S_{ij}(1,1) - 24H^{[i]}_{-1}S_{ij}(1,1)\\&\quad - 8\omega^{[j]}_{-3}S_{ij}(1,1) + 24H^{[j]}_{-1}S_{ij}(1,1)\\&\quad - 120\omega_0\omega^{[i]}_{-2}S_{ij}(1,1) + 36\omega_0\omega^{[j]}_{-2}S_{ij}(1,1)\\&\quad + 72\omega_0\omega_0\omega^{[i]}_{-1}S_{ij}(1,1) - 9\omega_0\omega_0\omega_0\omega_0 S_{ij}(1,1)\\&\quad + 12\omega^{[i]}_{-2}S_{ij}(1,2) + 12\omega^{[j]}_{-2}S_{ij}(1,2)\\&\quad - 72\omega_0\omega^{[i]}_{-1}S_{ij}(1,2) - 72\omega_0\omega^{[j]}_{-1}S_{ij}(1,2)\\&\quad + 18\omega_0\omega_0\omega_0 S_{ij}(1,2), \end{aligned} \tag{4.13}$$

$$\begin{aligned}0 &= 8\omega^{[j]}_{-3}S_{ij}(1,1) - 24H^{[j]}_{-1}S_{ij}(1,1)\\&\quad + 54\omega_0\omega^{[i]}_{-2}S_{ij}(1,1) - 36\omega_0\omega^{[j]}_{-2}S_{ij}(1,1)\\&\quad - 36\omega_0\omega_0\omega^{[i]}_{-1}S_{ij}(1,1) + 9\omega_0\omega_0\omega_0\omega_0 S_{ij}(1,1)\\&\quad + 54\omega^{[i]}_{-2}S_{ij}(1,2) - 12\omega^{[j]}_{-2}S_{ij}(1,2)\\&\quad + 72\omega_0\omega^{[j]}_{-1}S_{ij}(1,2) - 18\omega_0\omega_0\omega_0 S_{ij}(1,2)\\&\quad + 72\omega^{[i]}_{-1}S_{ij}(1,3), \end{aligned} \tag{4.14}$$

$$0 = 14\omega^{[j]}_{-3}S_{ij}(1,1) + 12H^{[j]}_{-1}S_{ij}(1,1) - 3\omega^{[j]}_{-2}S_{ij}(1,2) - 36\omega^{[j]}_{-1}S_{ij}(1,3). \tag{4.15}$$

The following result is a direct consequence of (2.59), Lemma 2.4, and (2.75):

Lemma 4.2. *Let K be an $M(1)^+$-module such that $K = M(1)^+ \cdot K(0)$. Then, K is spanned by $a^{(1)}_{i_1} \cdots a^{(n)}_{i_n} b$ where $n \in \mathbb{Z}_{\geq 0}$, $b \in K(0)$, $a^{(j)} \in \{\omega^{[k]}, J^{[k]} \mid k = 1, \ldots, d\} \cup \{S_{lm}(1,r) \mid 1 \leq m < l \leq d, r = 1, 2, 3\}$ and $i_j \in \mathbb{Z}_{\leq \mathrm{wt}\, a^j - 2}$ for $j = 1, \ldots, n$.*

Lemma 4.3. *Let U be a subspace of a weak $M(1)^+$-module which is stable under the actions of $\{\omega^{[i]}_1, H^{[i]}_3\}_{i=1}^d$. Assume $\epsilon(\omega^{[i]}, u) \leq 1$ and $\epsilon(H^{[i]}, u) \leq 3$ for all $u \in U$ and $i = 1, \ldots, d$. Let*

$i, j \in \{1, \ldots, d\}$ with $i \neq j$ and $\epsilon(S) \in \mathbb{Z}$ such that $\epsilon(S) \geq \epsilon(S_{ij}, u)$ for all $u \in U$. Then, for $u \in U$

$$\begin{aligned}
0 = &-\epsilon(S)(\epsilon(S)+1)^2 S_{ij}(1,1)_{\epsilon(S)}u - (\epsilon(S)+2)(3\epsilon(S)+1)S_{ij}(1,2)_{\epsilon(S)+1}u \\
&+ 4\epsilon(S)S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[i]}u - 4S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[j]}u \\
&- 2(3\epsilon(S)+1)S_{ij}(1,3)_{\epsilon(S)+2}u + 4S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[i]}u - 4S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[j]}u, \qquad (4.16)\\
0 = &-\epsilon(S)(\epsilon(S)+1)(3\epsilon(S)^2 + 27\epsilon(S) + 22)S_{ij}(1,1)_{\epsilon(S)}u \\
&- 2(3\epsilon(S)^3 + 39\epsilon(S)^2 + 82\epsilon(S) + 24)S_{ij}(1,2)_{\epsilon(S)+1}u \\
&+ 8\epsilon(S)(3\epsilon(S)+11)S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[i]}u + 8(3\epsilon(S)-13)S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[j]}u \\
&- 48(3\epsilon(S)+1)S_{ij}(1,3)_{\epsilon(S)+2}u + 8(3\epsilon(S)-13)S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[j]}u \\
&- 8S_{ij}(1,1)_{\epsilon(S)}H_3^{[i]}u + 8S_{ij}(1,1)_{\epsilon(S)}H_3^{[j]}u \\
&+ 8(3\epsilon(S)+11)S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[i]}u, \qquad (4.17)\\
0 = &\, 2(3\epsilon(S)^3 + 21\epsilon(S)^2 + 42\epsilon(S) + 14)S_{ij}(1,2)_{\epsilon(S)+1}u - 8(3\epsilon(S)-7)S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[j]}u \\
&+ 4(18\epsilon(S)+7)S_{ij}(1,3)_{\epsilon(S)+2}u - 8(3\epsilon(S)-7)S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[j]}u \\
&- 8S_{ij}(1,1)_{\epsilon(S)}H_3^{[j]}u + 3\epsilon(S)(\epsilon(S)+1)^2(\epsilon(S)+4)S_{ij}(1,1)_{\epsilon(S)}u \\
&- 12\epsilon(S)(\epsilon(S)+4)S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[i]}u - 36S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[i]}u \\
&+ 24S_{ij}(1,3)_{\epsilon(S)+2}\omega_1^{[i]}u, \qquad (4.18)\\
0 = &-S_{ij}(1,2)_{\epsilon(S)+1}u - 5S_{ij}(1,1)_{\epsilon(S)}\omega_1^{[j]}u - S_{ij}(1,3)_{\epsilon(S)+2}u - 11S_{ij}(1,2)_{\epsilon(S)+1}\omega_1^{[j]}u \\
&+ 2S_{ij}(1,1)_{\epsilon(S)}H_3^{[j]}u - 6S_{ij}(1,3)_{\epsilon(S)+2}\omega_1^{[j]}u. \qquad (4.19)
\end{aligned}$$

If u is a simultaneous eigenvector of $\{\omega_1^{[i]}, \omega_1^{[j]}, H_3^{[i]}, H_3^{[j]}\}$ with eigenvalues $\{\zeta^{[i]}, \zeta^{[j]}, \xi^{[i]}, \xi^{[j]}\}$:

$$(\omega_1^{[i]} - \zeta^{[i]})u = (\omega_1^{[j]} - \zeta^{[j]})u = (H_3^{[i]} - \xi^{[i]})u = (H_3^{[j]} - \xi^{[j]})u = 0, \qquad (4.20)$$

then

$$
\begin{aligned}
0 = &-(\epsilon(S)-1)\big((18\zeta^{[i]}+3)\epsilon(S)^5+(-54\zeta^{[i]}+6)\epsilon(S)^4\\
&\quad+(1-36\zeta^{[j]}-78\zeta^{[i]}-216\zeta^{[j]}\zeta^{[i]}+216(\zeta^{[i]})^2)\epsilon(S)^3\\
&\quad+(-2+4\zeta^{[j]}+22\zeta^{[i]}+744\zeta^{[j]}\zeta^{[i]}+24(\zeta^{[i]})^2)\epsilon(S)^2\\
&\quad+(12\zeta^{[i]}-192\zeta^{[j]}\zeta^{[i]}-48(\zeta^{[i]})^2-1152\zeta^{[j]}(\zeta^{[i]})^2)\epsilon(S)\\
&\quad+384\zeta^{[j]}(\zeta^{[i]})^2-16\zeta^{[j]}\zeta^{[i]}\big)\\
&+8\big((9\epsilon(S)^4+12\epsilon(S)^3+(-18\zeta^{[i]}-36\zeta^{[j]})\epsilon(S)^2+(-24\zeta^{[i]}-1)\epsilon(S)\\
&\quad-4\zeta^{[j]}-6\zeta^{[i]}-24\zeta^{[j]}\zeta^{[i]}+24(\zeta^{[i]})^2\big)\xi^{[i]}\\
&-8\big(((18\zeta^{[i]}+3)\epsilon(S)^2+(-24\zeta^{[i]}+1)\epsilon(S)+24(\zeta^{[i]})^2+(-24\zeta^{[j]}-6)\zeta^{[i]}-4\zeta^{[j]})\big)\xi^{[j]}, \qquad (4.21)\\
0 = &-(\epsilon(S)-1)\big((18\zeta^{[j]}+3)\epsilon(S)^5+(-54\zeta^{[j]}+6)\epsilon(S)^4\\
&\quad+(1-36\zeta^{[i]}-78\zeta^{[j]}-216\zeta^{[i]}\zeta^{[j]}+216(\zeta^{[j]})^2)\epsilon(S)^3\\
&\quad+(-2+4\zeta^{[i]}+22\zeta^{[j]}+744\zeta^{[i]}\zeta^{[j]}+24(\zeta^{[j]})^2)\epsilon(S)^2\\
&\quad+(12\zeta^{[j]}-192\zeta^{[i]}\zeta^{[j]}-48(\zeta^{[j]})^2-1152\zeta^{[i]}(\zeta^{[j]})^2)\epsilon(S)\\
&\quad+384\zeta^{[i]}(\zeta^{[j]})^2-16\zeta^{[i]}\zeta^{[j]}\big)\\
&+8\big((9\epsilon(S)^4+12\epsilon(S)^3+(-18\zeta^{[j]}-36\zeta^{[i]})\epsilon(S)^2+(-24\zeta^{[j]}-1)\epsilon(S)\\
&\quad-4\zeta^{[i]}-6\zeta^{[j]}-24\zeta^{[i]}\zeta^{[j]}+24(\zeta^{[j]})^2\big)\xi^{[j]}\\
&-8\big(((18\zeta^{[j]}+3)\epsilon(S)^2+(-24\zeta^{[j]}+1)\epsilon(S)+24(\zeta^{[j]})^2+(-24\zeta^{[i]}-6)\zeta^{[j]}-4\zeta^{[i]})\big)\xi^{[i]}. \qquad (4.22)
\end{aligned}
$$

Proof. We first note that interchanging the positions of $\zeta^{[i]}$ and $\zeta^{[j]}$, and $\xi^{[i]}$ and $\xi^{[j]}$ in (4.21), we get (4.22). Thus, these two equations (4.21) and (4.22) are essentially the same, however, we put them here because they are convenient for later use. By Lemma 3.2, $\epsilon(S_{ij}(1,r),u)\leq\epsilon(S)+r-1$ for all $r=1,2,\dots$ and $u\in U$. We shall apply Lemma 2.3 (2) to (4.12) with $A:=\{\omega^{[i]},\omega^{[j]},H^{[i]},H^{[j]}\}$, $B:=\{S_{ij}(1,r)\mid r=1,2,3\}$, $\epsilon(\omega^{[k]}):=\mathrm{wt}(\omega^{[k]})-1=1$, $\epsilon(H^{[k]}):=\mathrm{wt}(H^{[k]})-1=3$ for $k=i,j$, and $\epsilon(S_{ij}(1,r))=\epsilon(S)+r-1$ for $r=1,2,3$. By Lemma 2.4, we have $A\cdot\langle A_-\rangle\mathbf{1}\subset\langle A_-\rangle\mathbf{1}, A\cdot(\langle A_-\rangle B_-\mathbf{1})\subset\langle A_-\rangle B_-\mathbf{1}, B\cdot\langle A_-\rangle\mathbf{1}\subset B_-\langle A_-\rangle\mathbf{1}$, and $A\cdot(B_-\langle A_-\rangle\mathbf{1})\subset B_-\langle A_-\rangle\mathbf{1}$, where the symbols $\langle A_-\rangle\mathbf{1}, \langle A_-\rangle B_-\mathbf{1}$, and $B_-\langle A_-\rangle\mathbf{1}$ are defined in (2.10) and (2.11). The weight of each term in (4.12) is 5. The same argument as in (3.8) shows $\delta(5)=\epsilon(S)+3$, where δ is defined in (2.12). By Lemma 2.3 (2), the $(\epsilon(S)+3)$-th action of (4.12) on u is a linear combination of elements of the form

$$q_{\epsilon(q)}p^{(1)}_{\epsilon(p^{(1)})}\cdots p^{(m)}_{\epsilon(p^{(m)})}u \qquad (4.23)$$

where $m\in\mathbb{Z}_{\geq0}$, $p^{(1)},\dots,p^{(m)}\in A$, and $q\in B$. To obtain the explicit expression of the result (4.16) we use computer algebra system Risa/Asir[16]. By taking the $(\epsilon(S)+4)$-th actions of (4.13)–(4.15) on u, the same argument shows (4.17)–(4.19). Deleting the terms including $S_{ij}(1,3)_{\epsilon(S)+2}u$ and $S_{ij}(1,2)_{\epsilon(S)+2}u$ from (4.16)–(4.19), we have (4.21) and (4.22). □

Lemma 4.4. *Let*

$$0\to W\to N\xrightarrow{\pi}M\to 0 \qquad (4.24)$$

be an exact sequence of weak $M(1)^+$-modules where W is an irreducible $M(1)^+$-module, N is a weak $M(1)^+$-module and $M=\oplus_{i\in\gamma+\mathbb{Z}_{\geq0}}M_i$ is an $M(1)^+$-module. Let B be an irreducible

$A(M(1)^+)$-submodule of M_γ that is not isomorphic to $W(0)$ and v a simultaneous eigenvector in B of $\{\omega_1^{[i]}, H_3^{[i]}\}_{i=1}^d$ with eigenvalues $\{\zeta^{[i]}, \xi^{[i]}\}_{i=1}^d$:

$$(\omega_1^{[i]} - \zeta^{[i]})v = (H_3^{[i]} - \xi^{[i]})v = 0. \tag{4.25}$$

If $(W, B) \not\cong (M(1)^+, M(1)^-(0))$, then there exists a preimage $u \in N_\gamma$ of v under the canonical projection $N_\gamma \to M_\gamma$ such that

$$(\omega_1^{[i]} - \zeta^{[i]})u = (H_3^{[i]} - \xi^{[i]})u = 0 \tag{4.26}$$

for all $i = 1, \dots, d$.

Proof. Using [2, Proposition 4.3] and eigenvalues of $\omega_1^{[i]}$ and $H_3^{[i]}$ for $i = 1, \dots, d$ on irreducible $M(1)^+$-modules in [3, Table 1], we see that the result holds if $W = M(1)(\theta)^\pm$ or $B = M(1)(\theta)^\pm(0)$. We discuss the other cases. For $v \in B$ with (4.2) , we take $u \in N$ that satisfies (4.4) and (4.5).

Let $B = \mathbb{C}e^\lambda$ for some $\lambda \in \mathfrak{h} \setminus \{0\}$. In this case $\zeta^{[i]} = \langle\lambda, h^{[i]}\rangle^2/2$ and $\xi^{[i]} = 0$ for $i = 1, \dots, d$. We note that at least one of $\zeta^{[1]}, \dots, \zeta^{[d]}$ is not zero. Let $W = M(1, \mu)$. Since $B \not\cong W(0)$ as $A(M(1)^+)$-modules, we have $\mu \in \mathfrak{h} \setminus \{0, \pm\lambda\}$. Since $\cap_{j=1}^d \operatorname{Ker} H_3^{[j]} \cap M(1, \mu) = \mathbb{C}e^\mu$ by [2, Proposition 4.3],

$$M(1, \mu)_{\zeta,(0,\dots,0)} = \bigcap_{j=1}^d \operatorname{Ker}(\omega_1^{[j]} - \zeta^{[j]}) \cap \mathbb{C}e^\mu. \tag{4.27}$$

Assume

$$(\omega_1^{[i]} - \zeta^{[i]})u \neq 0 \text{ or } H_3^{[i]}u \neq 0 \text{ for some } i \in \{1, \dots, d\}. \tag{4.28}$$

It follows from (4.4) that $M(1, \mu)_{\zeta,(0,\dots,0)} \neq 0$ and hence $\langle\lambda, h^{[j]}\rangle = \pm\langle\mu, h^{[j]}\rangle$ for all $j = 1, \dots, d$ by (4.27). Thus, $\gamma = \langle\lambda, \lambda\rangle/2 = \langle\mu, \mu\rangle/2$. By this and $\lambda \neq \pm\mu$, we see that there exists an $A(M(1)^+)$-submodule of $N(0)$ which is isomorphic to $B \oplus M(1, \mu)(0) \cong M(1, \lambda)(0) \oplus M(1, \mu)(0)$. Thus, we have the result. If $W = M(1)^\pm$, then the result follows from the fact that $M(1)^\pm_{\zeta,(0,\dots,0)} = 0$.

If $B = \mathbb{C}\mathbf{1} = M(1)^+(0)$, then the same argument as above shows the result.

Let $B = M(1)^-(0)$, $W = M(1, \lambda)$ such that $\lambda \in \mathfrak{h} \setminus \{0\}$, and $v = h^{[j]}(-1)\mathbf{1}$ for some $j \in \{1, \dots, d\}$. Since $\xi^{[i]} = \delta_{ij}$ for all $i = 1, \dots, d$, it follows from [2, Proposition 4.3] that

$$M(1, \lambda)_{\zeta,\xi} \subset \mathbb{C}h^{[j]}(-1)e^\lambda. \tag{4.29}$$

Suppose there exists $i \in \{1, \dots, d\}$ such that $(\omega^{[i]} - \delta_{ij})u \neq 0$ or $(H^{[i]} - \delta_{ij})u \neq 0$. Then, $M(1, \lambda)_{\zeta,\xi} \neq 0$ and hence $\delta_{jk} = \langle\lambda, h^{[k]}\rangle^2/2 + \delta_{jk}$ for all $k = 1, \dots, d$, which contradicts that $\lambda \neq 0$. The proof is complete. □

We will prepare the following symbol for Lemmas 4.6 and 4.8:

Definition 4.5. Let $R[x]$ be a polynomial ring over a commutative ring R. For two polynomials $A_1 = \sum_{i=0}^{\deg A_1} A_{1,i}x^i, A_2 = \sum_{i=0}^{\deg A_2} A_{2,i}x^i \in R[x]$ with $A_{ki} \in R$, we define a polynomial $G(A_1, A_2) \in R[x]$ as follows. We first prepare indeterminates $\hat{A}_{1,0}, \dots, \hat{A}_{1,\deg A_1}, \hat{A}_{2,0}, \dots, \hat{A}_{2,\deg A_2}$ over $\mathbb{C}$. We define $\hat{R} := \mathbb{C}[\hat{A}_{1,0}, \dots, \hat{A}_{1,\deg A_1}, \hat{A}_{2,0}, \dots, \hat{A}_{2,\deg A_2}]$ and two polynomials $\hat{A}_1 := \sum_{i=0}^{\deg A_1} \hat{A}_{1,i}x^i, \hat{A}_2 := \sum_{i=0}^{\deg A_2} \hat{A}_{2,i}x^i \in \hat{R}[x]$. In the following, $\deg \hat{P}$ is the degree of $\hat{P} \in \hat{R}[x]$ with respect to x. If

$\deg \hat{A}_1 \geq \deg \hat{A}_2$, then we define $\hat{A}_3 \in \hat{R}[x]$ by the remainder after dividing $\hat{A}_{2,\deg \hat{A}_2}^{\deg \hat{A}_1-\deg \hat{A}_2+1}\hat{A}_1$ by $\hat{A}_2$:

$$\hat{A}_{2,\deg A_2}^{\deg \hat{A}_1-\deg \hat{A}_2+1}\hat{A}_1 = \hat{B}_2\hat{A}_2 + \hat{A}_3, \quad \hat{B}_2, \hat{A}_3 \in \hat{R}[x], \deg \hat{A}_3 < \deg \hat{A}_2. \tag{4.30}$$

Here we note that such an $\hat{A}_3$ exists uniquely since we took $\hat{A}_{2,\deg \hat{A}_2}^{\deg \hat{A}_1-\deg \hat{A}_2+1}\hat{A}_1$ instead of $\hat{A}_1$. If $\deg \hat{A}_2 > \deg \hat{A}_1$, then we define $\hat{A}_3$ by $\hat{A}_1$. Defining $A_3 := \hat{A}_3|_{\hat{A}_{1,0}=A_{1,0},\hat{A}_{1,1}=A_{1,1},\ldots,\hat{A}_{2,0}=A_{2,0},\ldots}$, $B_2 := \hat{B}_2|_{\hat{A}_{1,0}=A_{1,0},\hat{A}_{1,1}=A_{1,1},\ldots,\hat{A}_{2,0}=A_{2,0},\ldots} \in R[x]$, we have

$$A_{2,\deg A_2}^{\deg A_1-\deg A_2+1}A_1 = B_2A_2 + A_3, \quad B_2, A_3 \in R[x], \deg A_3 < \deg A_2. \tag{4.31}$$

We replace A_1 by A_2 and A_2 by A_3 and repeat this operation as many times as possible, which is the essentially Euclidean algorithm:

$$\vdots$$

$$A_{k+1,\deg A_{k+1}}^{\deg A_k-\deg A_{k+1}+1}A_k = B_{k+1}A_{k+1} + A_{k+2}, \quad B_{k+1}, A_{k+2} \in R[x], \deg A_{k+2} < \deg A_{k+1}\ (k = 0, 1, \ldots)$$

$$\vdots$$

$$A_{d+1,\deg A_{d+1}}^{\deg A_d-\deg A_{d+1}+1}A_d = B_{d+1}A_{d+1}, \quad B_{d+1} \in R[x].$$

Then, we define

$$G(A_1, A_2) := A_{d+1} \in R[x]. \tag{4.32}$$

Lemma 4.6. *Let*

$$0 \to W \to N \stackrel{\pi}{\to} M \to 0 \tag{4.33}$$

be an exact sequence of weak $M(1)^+$-modules where W is an irreducible $M(1)^+$-module, N is a weak $M(1)^+$-module and $M = \oplus_{i\in\gamma+\mathbb{Z}_{\geq 0}}M_i$ is an $M(1)^+$-module. Let B be an irreducible $A(M(1)^+)$-submodule of M_γ and v a simultaneous eigenvector in B for $\{\omega_1^{[i]}, H_3^{[i]}\}_{i=1}^d$ with eigenvalues $\{\zeta^{[i]}, \xi^{[i]}\}_{i=1}^d$:

$$(\omega_1^{[i]} - \zeta^{[i]})v = (H_3^{[i]} - \xi^{[i]})v = 0. \tag{4.34}$$

Let $w \in N_\gamma$ such that $(\omega_1^{[i]} - \zeta^{[i]})w = (H_3^{[i]} - \xi^{[i]})w = 0$ for all $i = 1, \ldots, d$. If $(W, B) \not\cong (M(1)^+, M(1)^-(0))$, then $w \in \Omega_{M(1)^+}(N_\gamma)$.

Proof. Assume $(W, B) \not\cong (M(1)^+, M(1)^-(0))$. By Lemma 3.2 and (4.11), it is enough to show that $\epsilon(S_{ij}(1,1), w) \leq 1$ for any pair of distinct elements $i, j \in \{1, \ldots, d\}$. For such a pair i, j, we write $\epsilon(S_{ij}) = \epsilon(S_{ij}(1,1), w)$ for simplicity.

(1) Let $B \cong M(1,\lambda)(0)$ for some $\lambda \in \mathfrak{h} \setminus \{0\}$. In this case $\xi^{[i]} = 0$ for all $i = 1, \ldots, d$. Assume $\langle\lambda, \lambda\rangle \neq 0$. Then, we may assume $\lambda \in \mathbb{C}h^{[1]}$ and hence $\langle\lambda, h^{[i]}\rangle = \zeta^{[i]} = 0$ for all $i = 2, \ldots, d$. For $i = 2, \ldots, d$, substituting $\zeta^{[i]} = 0$ and $\xi^{[1]} = \xi^{[i]} = 0$ into (4.21) and (4.22) with $j = 1$, we have

$$0 = \epsilon(S_{i1})^2(\epsilon(S_{i1}) - 1)(4(-9\epsilon(S_{i1}) + 1)\zeta^{[1]} + (\epsilon(S_{i1}) + 1)(3\epsilon(S_{i1})^2 + 3\epsilon(S_{i1}) - 2)) \quad \text{and} \tag{4.35}$$

$$\begin{aligned}0 = \epsilon(S_{i1})(\epsilon(S_{i1}) - 1)\Big(&(216\epsilon(S_{i1})^2 + 24\epsilon(S_{i1}) - 48)(\zeta^{[1]})^2 \\ &+ (18\epsilon(S_{i1})^4 - 54\epsilon(S_{i1})^3 - 78\epsilon(S_{i1})^2 + 22\epsilon(S_{i1}) + 12)\zeta^{[1]} \\ &+ 3\epsilon(S_{i1})^4 + 6\epsilon(S_{i1})^3 + \epsilon(S_{i1})^2 - 2\epsilon(S_{i1})\Big).\end{aligned} \tag{4.36}$$

If we take A_1 to the right-hand side of (4.36) and A_2 to the right-hand side of (4.35) and if we regard A_1 and A_2 as polynomials in the variable $\zeta^{[1]}$, then $G(A_1, A_2)$ in (4.32) is a non-zero scalar multiple of

$$\epsilon(S_{i1})^5\epsilon(S_{i1}-1)^4(\epsilon(S_{i1})+1)(2\epsilon(S_{i1})+1) \\ \times(3\epsilon(S_{i1})-2)(3\epsilon(S_{i1})+1)^2(3\epsilon(S_{i1})^2+3\epsilon(S_{i1})-2). \tag{4.37}$$

Since $G(A_1, A_2) = 0$, we have $\epsilon(S_{i1}) \le 1$.

Assume $\langle\lambda,\lambda\rangle = 0$. Then, we may assume $0 \ne \langle\lambda, h^{[1]}\rangle^2 = -\langle\lambda, h^{[2]}\rangle^2$ and $\langle\lambda, h^{[j]}\rangle = 0$ for all $j = 3, 4, \dots, d$. By substituting $\zeta^{[2]} = -\zeta^{[1]}$ into (4.21) and (4.22), the same argument as above shows that $\epsilon(S_{21}) \le 1$.

In both the cases of $\langle\lambda,\lambda\rangle \ne 0$ and $\langle\lambda,\lambda\rangle = 0$, for the other i, j, since one of $\zeta^{[i]}$ or $\zeta^{[j]}$ is 0, the same argument as above also shows that $\epsilon(S_{ij}) \le 1$.

(2) Let $B \cong M(1)^-(0)$. Assume $W \ncong M(1)^+$. If $d = 1$, then the result is shown in [2, Theorem 5.5]. Assume $d \ge 2$. Let i, j be a pair of distinct elements in $\{1, \dots, d\}$. If $(\zeta^{[i]}, \xi^{[i]}) = (\zeta^{[j]}, \xi^{[j]}) = (0, 0)$, then it follows from (4.11) and (4.21) that

$$0 = \epsilon(S_{ij})^2(\epsilon(S_{ij})-1)(\epsilon(S_{ij})+1)(3\epsilon(S_{ij})^2+3\epsilon(S_{ij})-2) \tag{4.38}$$

and hence $\epsilon(S_{ij}) \le 1$.

Assume $(\zeta^{[i]}, \xi^{[i]}) = (0, 0)$ and $(\zeta^{[j]}, \xi^{[j]}) = (1, 1)$. It follows from (4.11) and (4.21) that

$$0 = (\epsilon(S_{ij})-2)(\epsilon(S_{ij})-1)(3\epsilon(S_{ij})^4+12\epsilon(S_{ij})^3-11\epsilon(S_{ij})^2-20\epsilon(S_{ij})-16) \tag{4.39}$$

and hence $\epsilon(S_{ij}) = 1$ or 2. We further assume that $\epsilon(S_{ij}) = 2$. By (4.16)–(4.19),

$$S_{ij}(1,2)_3w = -2S_{ij}(1,1)_2w \text{ and } S_{ij}(1,3)_4w = 3S_{ij}(1,1)_2w. \tag{4.40}$$

We note that $S_{ij}(1,1)_2w \in W$. Using commutation relations (see Remark 2.5) and (4.40), we have

$$\omega_1^{[k]}S_{ij}(1,1)_2w = H_3^{[k]}S_{ij}(1,1)_2w = 0 \tag{4.41}$$

for all $k = 1, \dots, d$. It follows from [2, Proposition 4.3] that there is no non-zero element v in any irreducible $M(1)^+$-module except $\mathbf{1} \in M(1)^+$ that satisfies $\omega_1^{[k]}v = H_3^{[k]}v = 0$ for all $k = 1, \dots, d$. This is a contradiction.

The same argument as above shows the results for the case that $B \cong M(1)^+(0)$ or $M(1)(\theta)^\pm(0)$.

□

Lemma 4.7. *For any pair of non-isomorphic irreducible $M(1)^+$-modules M, W such that $(M, W) \ncong (M(1)^+, M(1)^-)$ and $(M(1)^-, M(1)^+)$, $\mathrm{Ext}^1_{M(1)^+}(M, W) = 0$.*

Proof. Let N be a weak $M(1)^+$-module and

$$0 \to W \to N \xrightarrow{\pi} M \to 0 \tag{4.42}$$

an exact sequence of weak $M(1)^+$-modules. We write $M = \oplus_{i\in\gamma+\mathbb{Z}_{\ge 0}}M_i$ with $M_\gamma \ne 0$ and $W = \oplus_{i\in\delta+\mathbb{Z}_{\ge 0}}W_i$ with $W_\delta \ne 0$. By Lemmas 4.4 and 4.6, there exists $u \in \Omega_{M(1)^+}(N)$ such that $0 \ne \pi(u) \in M_\gamma$.

Assume $W \cap (M(1)^+ \cdot u) \neq 0$. Since $W_\delta \subset W \cap (M(1)^+ \cdot u)$, $\delta \in \gamma + \mathbb{Z}_{\geq 0}$. Since $W \cap (M(1)^+ \cdot u) \neq 0$, we have $\mathrm{Ext}^1_{M(1)^+}(M, W) \neq 0$ and hence $\mathrm{Ext}^1_{M(1)^+}(W, M) \neq 0$ by [2, Proposition 2.5] and [4, Proposition 3.5]. Thus, there exists a non-split exact sequence $0 \to M \to N \to W \to 0$ of weak $M(1)^+$-modules. The same argument as above shows that $\gamma \in \delta + \mathbb{Z}_{\geq 0}$ and hence $\gamma = \delta$. Since $M \not\cong W$, $N(0) = N_\gamma \cong M_\gamma \oplus W_\delta$ as $A(M(1)^+)$-modules. Thus, the sequence (4.42) splits, a contradiction.

□

Lemma 4.8. *For $M = M(1)^+, M(1)^-, M(1)(\theta)^+$, and $M(1)(\theta)^-$, $\mathrm{Ext}^1_{M(1)^+}(M, M) = 0$.*

Proof. Let W be an $M(1)^+$-module such that $W \cong M$, N a weak $M(1)^+$-module, and

$$0 \to W \to N \xrightarrow{\pi} M \to 0 \tag{4.43}$$

an exact sequence of weak $M(1)^+$-modules. We take $v \in M(0)$ and $u \in N$ as in (4.2), (4.4), and (4.5). In the case of $M = M(1)^+$, the same argument as in the proof of [2, Proposition 5.1] shows that $\mathrm{Ext}^1_{M(1)^+}(M(1)^+, M(1)^+) = 0$.

For $M = M(1)^-$ or $M(1)(\theta)^\pm$, it is enough to show that $N(0) \cong W(0) \oplus M(0) \cong M(0) \oplus M(0)$ as $A(M(1)^+)$-modules. In the Zhu algebra $A(M(1)^+)$, we have

$$\omega^{[i]} * H^{[i]} \equiv H^{[i]} * \omega^{[i]} \tag{4.44}$$

and recall that the following relations from [10, (6.1.11) and (6.1.10)]:

$$(\omega^{[i]} - \mathbf{1}) * (\omega^{[i]} - \frac{1}{16}\mathbf{1}) * (\omega^{[i]} - \frac{9}{16}\mathbf{1}) * H^{[i]} \equiv 0, \tag{4.45}$$

$$(132(\omega^{[i]})^2 - 65\omega^{[i]} - 70H^{[i]} + 3) * H^{[i]} \equiv 0 \tag{4.46}$$

for $i = 1, \ldots, d$. Here, we note that H_a in [10, Section 6] is equal to the image of $-9H^{[a]}$ under the projection $M(1)^+ \to A(M(1)^+)$ for $a = 1, \ldots, d$. Let A_1 be the quotient of the right-hand side of (4.45) by $H^{[i]}$ and A_2 the quotient of the right-hand side of (4.46) by $H^{[i]}$: $A_1 := (\omega^{[i]} - \mathbf{1}) * (\omega^{[i]} - (1/16)\mathbf{1}) * (\omega^{[i]} - (9/16)\mathbf{1})$ and $A_2 := 132(\omega^{[i]})^2 - 65\omega^{[i]} - 70H^{[i]} + 3$. If we regard A_1 and A_2 as polynomials in $\omega^{[i]}$, then $G(A_1, A_2)$ in (4.32) is a non-zero scalar multiple of

$$(H^{[i]} - 1) * (H^{[i]} - \frac{-1}{128}) * (H^{[i]} - \frac{15}{128}) \tag{4.47}$$

and hence

$$H^{[i]} * (H^{[i]} - 1) * (H^{[i]} - \frac{-1}{128}) * (H^{[i]} - \frac{15}{128}) \equiv 0 \tag{4.48}$$

for all $i = 1, \ldots, d$.

(1) Let $M = M(1)(\theta)^+$. Since $S_{ij}(1,1)_1 \mathbf{1}_{\mathrm{tw}} = 0$ for any pair of distinct elements $i, j \in \{1, \ldots, d\}$, $S_{ij}(1,1)_1 u \in \mathbb{C}v$ in W. We note that $\omega^{[i]}_1 \mathbf{1}_{\mathrm{tw}} = (1/16)\mathbf{1}_{\mathrm{tw}}$ and $H^{[i]}_3 \mathbf{1}_{\mathrm{tw}} = (-1/128)\mathbf{1}_{\mathrm{tw}}$. By (4.45) and (4.48), $\omega^{[i]}_1 w = (1/16)w$ and $H^{[i]}_3 w = (-1/128)w$ for all $w \in N(0)$. We denote $\epsilon(S_{ij}(1,1), u)$ by $\epsilon(S)$ for simplicity. By (4.21),

$$0 = \epsilon(S)(11\epsilon(S)^2 - 15\epsilon(S) + 6)(6\epsilon(S)^3 + 6\epsilon(S)^2 - 7\epsilon(S) + 1) \tag{4.49}$$

and hence $\epsilon(S) = 0$. Thus, $S_{ij}(1,k)_k u = 0$ for all $k \in \mathbb{Z}_{\geq 1}$ by Lemma 3.2 (1) and hence $N(0) \cong M(1)(\theta)^+(0) \oplus M(1)(\theta)^+(0)$ as $A(M(1)^+)$-modules.

(2) Let $M = M(1)^-$. We consider $N(0)$. Since $N(0)/W(0) \cong M(1)^-(0)$, $A^u \cdot N(0) \neq 0$. Since $A^t \cdot N(0) \subset W(0)$, $A^u * A^t = 0$, and $A^u \cdot w \neq 0$ for any non-zero $w \in W(0)$, we have $A^t \cdot N(0) = 0$. For any pair of distinct elements $i, j \in \{1, \dots, d\}$, since $A^u * \Lambda_{ij} = 0$ by [10, Proposition 5.3.12], the same argument shows that $\Lambda_{ij} \cdot N(0) = 0$. We note that the eigenvalues for $\omega^{[i]}|_{N(0)}$ are 0 or 1, and those for $H^{[i]}|_{N(0)}$ are also 0 or 1. We take a non-zero $v \in M(0)$ and $u \in N$ as in (4.2), (4.4), and (4.5). We fix $i = 1, \dots, d$. By (4.48),

$$H^{[i]}_3 u = u \text{ or } H^{[i]}_3 u = 0. \tag{4.50}$$

We study the following three cases:

(2-1) If $H^{[i]}_3 u = u$, then it follows from (4.45) that $\omega^{[i]}_1 u = u$.

(2-2) The case that $H^{[i]}_3 u = 0$ and $(\omega^{[i]}_1 - 1)^2 u = 0$. Since $(\omega^{[i]}_1 - 1)u \in W(0)$ and there is no non-zero vector $w \in M(1)^-(0)$ such that $\omega^{[i]}_1 w = w$ and $H^{[i]}_3 w = 0$, we have $(\omega^{[i]}_1 - 1)u = 0$.

(2-3) The case that $H^{[i]}_3 u = 0$ and $(\omega^{[i]}_1)^2 u = 0$. Since $0 \neq u \in M(0) \cong M(1)^-(0)$, there exists k such that $H^{[k]}_3 u \neq 0$. The argument (2-1) above shows that $H^{[k]}_3 u = u$ and $\omega^{[k]}_1 u = u$. Since $\omega^{[k]}_1 \omega^{[i]}_1 u = \omega^{[i]}_1 u$, we have $E^u_{kk} \omega^{[i]}_1 u = \omega^{[i]}_1 u$ in $W(0)$. By [10, Lemma 5.2.2], $\omega^{[i]}_1 u = 0$.

Thus $A(M(1)^+) \cdot u = A^u \cdot u$. Since A^u is isomorphic to the matrix algebra, $A^u \cdot u$ is an irreducible $A(M(1)^+)$-module. Thus $N(0) \cong M(1)^-(0) \oplus M(1)^-(0)$.

(3) In the case of $M = M(1)(\theta)^-$, the same argument as in (2) above shows that $N(0) \cong M(1)(\theta)^-(0) \oplus M(1)(\theta)^-(0)$.

□

By Lemmas 4.7, 4.8, [2, Proposition 2.5], and [4, Proposition 3.5], we have the following result:

Proposition 4.9. *If a pair (M, W) of irreducible $M(1)^+$-modules satisfies one of the following conditions, then* $\mathrm{Ext}^1_{M(1)^+}(M, W) = \mathrm{Ext}^1_{M(1)^+}(W, M) = 0$.

(1) $M \cong M(1, \lambda)$ *with* $\lambda \in \mathfrak{h} \setminus \{0\}$ *and* $W \not\cong M(1, \lambda)$.

(2) $M \cong M(1)(\theta)^{\pm}$.

(3) $M \cong M(1)^+$ *and* $W \not\cong M(1)^-$.

(4) $M \cong M(1)^-$ *and* $W \not\cong M(1)^+$.

The following result is a direct consequence of Lemmas 4.4 and 4.6. Here we call the $\mathbb{N}$-graded module $\bar{M}(U)$ in [8, Theorem 6.2] the generalized Verma module associated with a module U for the Zhu algebra.

Corollary 4.10. *Let Ω be an irreducible $A(M(1)^+)$-module such that $\Omega \not\cong M(1)^+_0 = \mathbb{C}\mathbf{1}$. Then the generalized Verma module for $M(1)^+$ associated with Ω is irreducible.*

Proof. Let $N = \oplus_{i \in \delta + \mathbb{Z}_{\geq 0}} N_i$ with $N_\delta = \Omega$ be the generalized Verma module for $M(1)^+$ associated with Ω and $W = \oplus_{i \in \gamma + \mathbb{Z}_{\geq 0}} W_i$ the maximal submodule of M such that $\Omega \cap W = 0$. We take γ so that $W_\gamma \neq 0$ if $W \neq 0$. We note that $\gamma - \delta \in \mathbb{Z}_{>0}$. Taking the restricted dual of the exact sequence $0 \to W \to N \to N/W \to 0$, we have the following exact sequence

$$0 \to (N/W)' \to N' \to W' \to 0. \tag{4.51}$$

We note that $(N/W)' \ncong M(1)^+$ by [4, Proposition 3.5]. Assume $W_\gamma \neq 0$ and let B be an irreducible $A(M(1)^+)$-submodule of W'_γ. By Lemmas 4.4 and 4.6, there exists a non-zero $u' \in \Omega_{M(1)^+}(N'_\gamma)$. For any homogeneous element $a \in M(1)^+$ such that $\omega_2 a = 0$ and $i \in \mathbb{Z}_{\geq \mathrm{wt}\, a}$, it follows from [11, 5.2.4] that

$$0 = \langle a_i u', w\rangle = (-1)^{\mathrm{wt}\, a}\langle u', a_{2\,\mathrm{wt}\, a - i - 2} w\rangle \tag{4.52}$$

for all $w \in N$. Since $\omega_2 \omega^{[i]} = \omega_2 J^{[i]} = 0$ for all $i = 1, \dots, d$, it follows from Lemma 4.2 and (4.52) that $u' = 0$, a contradiction. □

Lemma 4.11. *Let W be the generalized Verma module associated to the $A(M(1)^+)$-module $\mathbb{C}\mathbf{1}$ and $\pi : W \to M(1)^+$ the canonical projection. Then,* $\mathrm{Ker}\,\pi \cong (M(1)^-)^{\oplus k}$ *as $M(1)^+$-modules for some $k \in \{1, \dots, d\}$.*

Proof. The same argument as in [2, (6.1)] shows that there is a non-split exact sequence

$$0 \to M(1)^- \to N \to M(1)^+ \to 0 \tag{4.53}$$

of $M(1)^+$-modules. Thus, $\mathrm{Ker}\,\pi \neq 0$. Let $u \in W$ such that $\pi(u) = \mathbf{1}$. Note that $u \in \Omega_{M(1)^+}(N)$. Since $M(1)^-_0 = 0$, we have $\omega^{[k]}_1 u = H^{[k]}_3 u = S_{ij}(1, r)_r u = 0$ for all $k = 1, \dots, d$, pairs of distinct elements $i, j \in \{1, \dots, d\}$, and $r = 1, 2, 3$. For $i = 1, \dots, d$, $P^{(8),H,i}$ denotes the element obtained by replacing ω by $\omega^{[i]}$ and H by $H^{[i]}$ in $P^{(8),H}$ in [19, (3.27)]. We have shown in [19, Lemma 3.5] that $P^{(8),H} = 0$. A direct computation shows that

$$0 = P^{(8),H,i}_6 u = 144(\omega^{[i]}_0 - 3H^{[i]}_2)u \tag{4.54}$$

for all $i = 1, \dots, d$. Taking the 3rd action of (4.12) on u, we have

$$0 = S_{ij}(1, 2)_1 u + S_{ij}(1, 3)_2 u \tag{4.55}$$

for any pair of distinct elements $i, j \in \{1, \dots, d\}$. By (4.54), (4.55), and

$$S_{ij}(1, 1)_0 u = -(\omega_0 S_{ij}(1, 1))_1 u = -S_{ji}(1, 2)_1 u - S_{ij}(1, 2)_1 u \tag{4.56}$$

for any pair of distinct elements $i, j \in \{1, \dots, d\}$, N_1 is spanned by $\{\omega^{[j]}_0 u, S_{ij}(1, 2)_1 u \mid i, j = 1, \dots, d, i \neq j\}$. For distinct $i, j, k \in \{1, \dots, d\}$, by (4.54), (4.55), and commutation relations (see

Remark 2.5), a direct computation shows that

$$\begin{aligned}
\omega^{[j]}_1\omega^{[j]}_0 u &= \omega^{[j]}_0 u,\\
\omega^{[i]}_1\omega^{[j]}_0 u &= 0,\\
H^{[j]}_3\omega^{[j]}_0 u &= \omega^{[j]}_0 u,\\
H^{[i]}_3\omega^{[j]}_0 u &= 0,\\
S_{ij}(1,1)_1\omega^{[j]}_0 u &= -S_{ij}(1,2)_1 u,\\
S_{ij}(1,2)_2\omega^{[j]}_0 u &= 2S_{ij}(1,2)_1 u,\\
S_{ij}(1,3)_3\omega^{[j]}_0 u &= -3S_{ij}(1,2)_1 u,\\
\omega^{[i]}_1 S_{ij}(1,2)_1 u &= S_{ij}(1,2)_1 u,\\
\omega^{[j]}_1 S_{ij}(1,2)_1 u &= 0,\\
H^{[i]}_3 S_{ij}(1,2)_1 u &= S_{ij}(1,2)_1 u,\\
H^{[j]}_3 S_{ij}(1,2)_1 u &= 0,\\
S_{ij}(1,1)_1 S_{ij}(1,2)_1 u &= -\omega^{[j]}_0 u,\\
S_{ij}(1,2)_2 S_{ij}(1,2)_1 u &= 0,\\
S_{ij}(1,3)_3 S_{ij}(1,2)_1 u &= 0,\\
S_{kj}(1,1)_1 S_{ij}(1,2)_1 u &= 0,\\
S_{kj}(1,2)_2 S_{ij}(1,2)_1 u &= 0,\\
S_{kj}(1,3)_3 S_{ij}(1,2)_1 u &= 0,\\
S_{ki}(1,1)_1 S_{ij}(1,2)_1 u &= S_{kj}(1,2)_1 u,\\
S_{ki}(1,2)_2 S_{ij}(1,2)_1 u &= -2S_{kj}(1,2)_1 u,\\
S_{ki}(1,3)_3 S_{ij}(1,2)_1 u &= 3S_{kj}(1,2)_1 u.
\end{aligned} \tag{4.57}$$

Thus, by (2.63), for each $j=1,\dots,d$, the linear subjective map $M(1)^-(0)\to U^{\langle j\rangle}:=\mathrm{Span}_{\mathbb{C}}\{\omega^{[j]}_0 u, S_{ij}(1,2)_1 u\mid i\neq j\}$ sending $h^{[j]}(-1)\mathbf{1}$ to $\omega^{[j]}_0 u$ and $h^{[i]}(-1)\mathbf{1}$ to $-S_{ij}(1,2)_1 u$ for $i\neq j$ is an $A(M(1)^+)$-homomorphism. Since $M(1)^-(0)$ is an irreducible $A(M(1)^+)$-module, if $U^{\langle j\rangle}\neq 0$, then $U^{\langle j\rangle}\cong M(1)^-(0)$ as $A(M(1)^+)$-modules. Since $\sum_{j=1}^d U^{\langle j\rangle}=N_1$, $(W/(M(1)^+\cdot(\sum_{j=1}^d U^{\langle j\rangle}))_1=0$. Thus $(W/(M(1)^+\cdot(\sum_{j=1}^d U^{\langle j\rangle}))\cong M(1)^+$ and hence $\mathrm{Ker}\,\pi = M(1)^+\cdot(\sum_{j=1}^d U^{\langle j\rangle})$. Now the result follows from Corollary 4.10. □

By Lemma 3.4, Proposition 3.5, Corollary 4.10, and Lemma 4.11, we have the following result.

Corollary 4.12. *Let L be a non-degenerate even lattice of finite rank and M a non-zero weak V_L^+-module. Then, there exists an irreducible $M(1)^+$-submodule of M.*

Proof. By Lemma 3.4, there exists an irreducible $A(M(1)^+)$-submodule Ω of $\Omega_{M(1)^+}(M)$. Let N be the generalized Verma module for $M(1)^+$ associated with Ω (Proposition 3.5) and $f:N\to M$ the associated $M(1)^+$-homomorphism. If $\Omega\ncong\mathbb{C}\mathbf{1}$, then by Corollary 4.10, N is irreducible and hence so is $f(N)$. If $\Omega\cong\mathbb{C}\mathbf{1}$, then by Lemma 4.11, $M(1)^-\subset f(N)$ or $f(N)\cong M(1)^+$. This completes the proof. □

Appendix A2

In this appendix, for some $a, b \in M(1)^+$, we put the computations of $a_k b$ for $k \in \mathbb{Z}_{\geq 0}$. For $k \in \mathbb{Z}_{\geq 0}$ not listed below, $a_k b = 0$. Using these results, we can compute the commutation relation $[a_i, b_j] = \sum_{k=0}^{\infty} \binom{i}{k} (a_k b)_{i+j-k}$. Let $h^{[1]}, \ldots, h^{[d]}$ be an orthonormal basis of $\mathfrak{h}$. The rest of this appendix, i, j, k, l are distinct elements of $\{1, \ldots, d\}$.

$$\omega_0^{[j]} S_{ij}(1,1) = S_{ij}(1,2)_{-1}\mathbf{1}, \qquad \omega_1^{[j]} S_{ij}(1,1) = S_{ij}(1,1)_{-1}\mathbf{1}, \tag{A2.1}$$

$$\omega_0^{[j]} S_{ij}(1,2) = 2S_{ij}(1,3)_{-1}\mathbf{1}, \quad \omega_1^{[j]} S_{ij}(1,2) = 2S_{ij}(1,2)_{-1}\mathbf{1}, \quad \omega_2^{[j]} S_{ij}(1,2) = 2S_{ij}(1,1)_{-1}\mathbf{1}, \tag{A2.2}$$

$$\begin{aligned}
\omega_0^{[j]} S_{ij}(1,3) &= -\omega_{-2}^{[j]} S_{ij}(1,1)_{-1}\mathbf{1} + 2\omega_{-1}^{[j]} S_{ij}(1,2)_{-1}\mathbf{1}, \\
\omega_1^{[j]} S_{ij}(1,3) &= 3S_{ij}(1,3)_{-1}\mathbf{1}, \\
\omega_2^{[j]} S_{ij}(1,3) &= 3S_{ij}(1,2)_{-1}\mathbf{1}, \\
\omega_3^{[j]} S_{ij}(1,3) &= 3S_{ij}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.3}$$

$$\begin{aligned}
H_0^{[j]} S_{ij}(1,1) &= -2\omega_{-2}^{[j]} S_{ij}(1,1)_{-1}\mathbf{1} + 4\omega_{-1}^{[j]} S_{ij}(1,2)_{-1}\mathbf{1}, \\
H_1^{[j]} S_{ij}(1,1) &= 4S_{ij}(1,3)_{-1}\mathbf{1}, \\
H_2^{[j]} S_{ij}(1,1) &= \frac{7}{3} S_{ij}(1,2)_{-1}\mathbf{1}, \\
H_3^{[j]} S_{ij}(1,1) &= S_{ij}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.4}$$

$$\begin{aligned}
H_0^{[j]} S_{ij}(1,2) &= -6\omega_0\omega_{-2}^{[j]} S_{ij}(1,1)_{-1}\mathbf{1} + 6\omega_{-2}^{[i]} S_{ij}(1,2)_{-1}\mathbf{1} \\
&\quad - 4\omega_0\omega_{-1}^{[i]} S_{ij}(1,2)_{-1}\mathbf{1} + 12\omega_0\omega_{-1}^{[j]} S_{ij}(1,2)_{-1}\mathbf{1} \\
&\quad + \omega_0\omega_0\omega_0 S_{ij}(1,2)_{-1}\mathbf{1} + 8\omega_{-1}^{[i]} S_{ij}(1,3)_{-1}\mathbf{1} \\
&\quad - 6\omega_0\omega_0 S_{ij}(1,3)_{-1}\mathbf{1}, \\
H_1^{[j]} S_{ij}(1,2) &= -6\omega_{-2}^{[j]} S_{ij}(1,1)_{-1}\mathbf{1} + 12\omega_{-1}^{[j]} S_{ij}(1,2)_{-1}\mathbf{1}, \\
H_2^{[j]} S_{ij}(1,2) &= \frac{38}{3} S_{ij}(1,3)_{-1}\mathbf{1}, \\
H_3^{[j]} S_{ij}(1,2) &= 8S_{ij}(1,2)_{-1}\mathbf{1}, \\
H_4^{[j]} S_{ij}(1,2) &= 4S_{ij}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.5}$$

$$\begin{aligned}
H_0^{[j]}S_{ij}(1,3) &= \frac{40}{29}\omega_{-3}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1} + \frac{60}{29}H_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + \frac{-60}{29}\omega_{-2}^{[j]}S_{ij}(1,3)_{-1}\mathbf{1},\\
H_1^{[j]}S_{ij}(1,3) &= -12\omega_0\omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} + 12\omega_{-2}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad - 8\omega_0\omega_{-1}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1} + 24\omega_0\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + 2\omega_0\omega_0\omega_0S_{ij}(1,2)_{-1}\mathbf{1} + 16\omega_{-1}^{[i]}S_{ij}(1,3)_{-1}\mathbf{1}\\
&\quad - 12\omega_0\omega_0S_{ij}(1,3)_{-1}\mathbf{1},\\
H_2^{[j]}S_{ij}(1,3) &= \frac{-37}{3}\omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} + \frac{74}{3}\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1},\\
H_3^{[j]}S_{ij}(1,3) &= 27S_{ij}(1,3)_{-1}\mathbf{1},\\
H_4^{[j]}S_{ij}(1,3) &= 18S_{ij}(1,2)_{-1}\mathbf{1},\\
H_5^{[j]}S_{ij}(1,3) &= 10S_{ij}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.6}$$

$$\begin{aligned}
\omega_0^{[i]}S_{ij}(1,1) &= \omega_0S_{ij}(1,1)_{-1}\mathbf{1} - S_{ij}(1,2)_{-1}\mathbf{1},\\
\omega_1^{[i]}S_{ij}(1,1) &= S_{ij}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.7}$$

$$\begin{aligned}
\omega_0^{[i]}S_{ij}(1,2) &= \omega_0S_{ij}(1,2)_{-1}\mathbf{1} - 2S_{ij}(1,3)_{-1}\mathbf{1},\\
\omega_1^{[i]}S_{ij}(1,2) &= S_{ij}(1,2)_{-1}\mathbf{1},
\end{aligned} \tag{A2.8}$$

$$\begin{aligned}
\omega_0^{[i]}S_{ij}(1,3) &= \omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} - 2\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1} + \omega_0S_{ij}(1,3)_{-1}\mathbf{1},\\
\omega_1^{[i]}S_{ij}(1,3) &= S_{ij}(1,3)_{-1}\mathbf{1},
\end{aligned} \tag{A2.9}$$

$$\begin{aligned}
H_0^{[i]}S_{ij}(1,1) &= 2\omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} + \omega_0\omega_0\omega_0S_{ij}(1,1)_{-1}\mathbf{1}\\
&\quad - 4\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1} - 3\omega_0\omega_0S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + 6\omega_0S_{ij}(1,3)_{-1}\mathbf{1},\\
H_1^{[i]}S_{ij}(1,1) &= 2\omega_0\omega_0S_{ij}(1,1)_{-1}\mathbf{1} - 4\omega_0S_{ij}(1,2)_{-1}\mathbf{1} + 4S_{ij}(1,3)_{-1}\mathbf{1},\\
H_2^{[i]}S_{ij}(1,1) &= \frac{7}{3}\omega_0S_{ij}(1,1)_{-1}\mathbf{1} + \frac{-7}{3}S_{ij}(1,2)_{-1}\mathbf{1},\\
H_3^{[i]}S_{ij}(1,1) &= S_{ij}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.10}$$

$$\begin{aligned}
H_0^{[i]}S_{ij}(1,2) &= -6\omega_{-2}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1} + 4\omega_0\omega_{-1}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1} - 8\omega_{-1}^{[i]}S_{ij}(1,3)_{-1}\mathbf{1},\\
H_1^{[i]}S_{ij}(1,2) &= -4\omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} + 8\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + 2\omega_0\omega_0S_{ij}(1,2)_{-1}\mathbf{1} - 8\omega_0S_{ij}(1,3)_{-1}\mathbf{1},\\
H_2^{[i]}S_{ij}(1,2) &= \frac{7}{3}\omega_0S_{ij}(1,2)_{-1}\mathbf{1} + \frac{-14}{3}S_{ij}(1,3)_{-1}\mathbf{1},\\
H_3^{[i]}S_{ij}(1,2) &= S_{ij}(1,2)_{-1}\mathbf{1},
\end{aligned} \tag{A2.11}$$

$$\begin{aligned}
H_0^{[i]}S_{ij}(1,3) &= \frac{-4}{3}H_{-1}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1} + \frac{16}{9}\omega_{-3}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + \frac{-11}{3}\omega_0\omega_{-2}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1} + 2\omega_0\omega_0\omega_{-1}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + \frac{4}{3}\omega_{-2}^{[i]}S_{ij}(1,3)_{-1}\mathbf{1} - 4\omega_0\omega_{-1}^{[i]}S_{ij}(1,3)_{-1}\mathbf{1},\\
H_1^{[i]}S_{ij}(1,3) &= -2\omega_0\omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} + 6\omega_{-2}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad - 4\omega_0\omega_{-1}^{[i]}S_{ij}(1,2)_{-1}\mathbf{1} + 4\omega_0\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1}\\
&\quad + \omega_0\omega_0\omega_0S_{ij}(1,2)_{-1}\mathbf{1} + 8\omega_{-1}^{[i]}S_{ij}(1,3)_{-1}\mathbf{1}\\
&\quad - 4\omega_0\omega_0S_{ij}(1,3)_{-1}\mathbf{1},\\
H_2^{[i]}S_{ij}(1,3) &= \frac{7}{3}\omega_{-2}^{[j]}S_{ij}(1,1)_{-1}\mathbf{1} + \frac{-14}{3}\omega_{-1}^{[j]}S_{ij}(1,2)_{-1}\mathbf{1} + \frac{7}{3}\omega_0S_{ij}(1,3)_{-1}\mathbf{1},\\
H_3^{[i]}S_{ij}(1,3) &= S_{ij}(1,3)_{-1}\mathbf{1},
\end{aligned} \tag{A2.12}$$

$$\begin{aligned}
S_{ij}(1,1)_0S_{ij}(1,1) &= \omega_0\omega_{-1}^{[i]}\mathbf{1} + \omega_0\omega_{-1}^{[j]}\mathbf{1},\\
S_{ij}(1,1)_1S_{ij}(1,1) &= 2\omega_{-1}^{[i]}\mathbf{1} + 2\omega_{-1}^{[j]}\mathbf{1},\\
S_{ij}(1,1)_2S_{ij}(1,1) &= 0,\\
S_{ij}(1,1)_3S_{ij}(1,1) &= \mathbf{1},
\end{aligned} \tag{A2.13}$$

$$\begin{aligned}
S_{ij}(1,1)_0S_{ij}(1,2) &= 2H_{-1}^{[i]}\mathbf{1} - 2H_{-1}^{[j]}\mathbf{1} + 2\omega_0^2\omega_{-1}^{[i]}\mathbf{1} + \omega_0^2\omega_{-1}^{[j]}\mathbf{1},\\
S_{ij}(1,1)_1S_{ij}(1,2) &= 2\omega_0\omega_{-1}^{[i]}\mathbf{1} + \omega_0\omega_{-1}^{[j]}\mathbf{1},\\
S_{ij}(1,1)_2S_{ij}(1,2) &= 4\omega_{-1}^{[i]}\mathbf{1},\\
S_{ij}(1,1)_3S_{ij}(1,2) &= 0,\\
S_{ij}(1,1)_4S_{ij}(1,2) &= 2\mathbf{1},
\end{aligned} \tag{A2.14}$$

$$\begin{aligned}
S_{ij}(1,1)_0 S_{ij}(1,3) &= 3\omega_0 H^{[i]}_{-1}\mathbf{1} - \omega_0 H^{[j]}_{-1}\mathbf{1} + \omega_0^3\omega^{[i]}_{-1}\mathbf{1} + \omega_0^3\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,1)_1 S_{ij}(1,3) &= 3H^{[i]}_{-1}\mathbf{1} + H^{[j]}_{-1}\mathbf{1} + \omega_0^2\omega^{[i]}_{-1}\mathbf{1} + \omega_0^2\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,1)_2 S_{ij}(1,3) &= 3\omega_0\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,1)_3 S_{ij}(1,3) &= 6\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,1)_4 S_{ij}(1,3) &= 0,\\
S_{ij}(1,1)_5 S_{ij}(1,3) &= 3\mathbf{1},
\end{aligned} \tag{A2.15}$$

$$\begin{aligned}
S_{ij}(1,1)_0 S_{kj}(1,1) &= S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,1)_1 S_{kj}(1,1) &= S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.16}$$

$$\begin{aligned}
S_{ij}(1,1)_0 S_{kj}(1,2) &= 2S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,1)_1 S_{kj}(1,2) &= 2S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,1)_2 S_{kj}(1,2) &= 2S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.17}$$

$$\begin{aligned}
S_{ij}(1,1)_0 S_{kj}(1,3) &= -3\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} + \omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + \omega_0^3 S_{ki}(1,1)_{-1}\mathbf{1} + 2\omega^{[k]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1}\\
&\quad - 3S_{ki}(1,2)_{-3}\mathbf{1} + 3\omega_0 S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,1)_1 S_{kj}(1,3) &= 3S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,1)_2 S_{kj}(1,3) &= 3S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,1)_3 S_{kj}(1,3) &= 3S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.18}$$

$$S_{ij}(1,1)_0 S_{ki}(1,1) = S_{kj}(1,2)_{-1}\mathbf{1}, \qquad S_{ij}(1,1)_1 S_{ki}(1,1) = S_{kj}(1,1)_{-1}\mathbf{1}, \tag{A2.19}$$

$$\begin{aligned}
S_{ij}(1,1)_0 S_{ki}(1,2) &= 2S_{kj}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,1)_1 S_{ki}(1,2) &= 2S_{kj}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,1)_2 S_{ki}(1,2) &= 2S_{kj}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.20}$$

$$\begin{aligned}
S_{ij}(1,1)_0 S_{ki}(1,3) &= 3\omega^{[k]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} - 2\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1}\\
&\quad + \omega_0^3 S_{kj}(1,1)_{-1}\mathbf{1} + 2\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1}\\
&\quad - 3\omega_0 S_{kj}(1,2)_{-2}\mathbf{1} + 3\omega_0 S_{kj}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,1)_1 S_{ki}(1,3) &= 3S_{kj}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,1)_2 S_{ki}(1,3) &= 3S_{kj}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,1)_3 S_{ki}(1,3) &= 3S_{kj}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.21}$$

$$\begin{aligned}
S_{ij}(1,2)_0 S_{ij}(1,2) &= -3\omega_0 H^{[i]}_{-1}\mathbf{1} - \omega_0 H^{[j]}_{-1}\mathbf{1} - \omega_0^3\omega^{[i]}_{-1}\mathbf{1} + \omega_0^3\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_1 S_{ij}(1,2) &= -6H^{[i]}_{-1}\mathbf{1} - 2H^{[j]}_{-1}\mathbf{1} - 2\omega_0^2\omega^{[i]}_{-1}\mathbf{1} + \omega_0^2\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_2 S_{ij}(1,2) &= -6\omega_0\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_3 S_{ij}(1,2) &= -12\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_4 S_{ij}(1,2) &= 0,\\
S_{ij}(1,2)_5 S_{ij}(1,2) &= -6\mathbf{1},
\end{aligned} \tag{A2.22}$$

$$\begin{aligned}
S_{ij}(1,2)_0 S_{ij}(1,3) &= -8\omega^{[i]}_{-1}H^{[i]}_{-1}\mathbf{1} - 2\omega^{[i]}_{-2}\omega^{[i]}_{-2}\mathbf{1} + 2\omega^{[j]}_{-2}\omega^{[j]}_{-2}\mathbf{1} + 8\omega^{[j]}_{-1}H^{[j]}_{-1}\mathbf{1}\\
&\quad - 7\omega_0^2 H^{[i]}_{-1}\mathbf{1} + 2\omega_0^2\omega^{[i]}_{-1}\omega^{[i]}_{-1}\mathbf{1} - 2\omega_0^2\omega^{[j]}_{-1}\omega^{[j]}_{-1}\mathbf{1} - 3\omega_0^2 H^{[j]}_{-1}\mathbf{1}\\
&\quad - 19\omega_0^4\omega^{[i]}_{-1}\mathbf{1} + 2\omega_0^4\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_1 S_{ij}(1,3) &= -6\omega_0 H^{[i]}_{-1}\mathbf{1} - \omega_0 H^{[j]}_{-1}\mathbf{1} - \omega_0^3\omega^{[i]}_{-1}\mathbf{1} + \omega_0^3\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_2 S_{ij}(1,3) &= -12H^{[i]}_{-1}\mathbf{1} - 4\omega_0^2\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_3 S_{ij}(1,3) &= -12\omega_0\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_4 S_{ij}(1,3) &= -24\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,2)_5 S_{ij}(1,3) &= 0,\\
S_{ij}(1,2)_6 S_{ij}(1,3) &= -12\mathbf{1},
\end{aligned} \tag{A2.23}$$

$$\begin{aligned}
S_{ij}(1,2)_0 S_{kj}(1,1) &= -2S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,2)_1 S_{kj}(1,1) &= -2S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,2)_2 S_{kj}(1,1) &= -2S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.24}$$

$$\begin{aligned}
S_{ij}(1,2)_0 S_{kj}(1,2) &= 6\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} - 2\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad - \omega_0^3 S_{ki}(1,1)_{-1}\mathbf{1} - 4\omega^{[k]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1}\\
&\quad + 6S_{ki}(1,2)_{-3}\mathbf{1} - 6\omega_0 S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,2)_1 S_{kj}(1,2) &= -6S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,2)_2 S_{kj}(1,2) &= -6S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,2)_3 S_{kj}(1,2) &= -6S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.25}$$

$$\begin{aligned}
S_{ij}(1,2)_0S_{kj}(1,3) &= 16\omega^{[k]}_{-1}S_{ki}(1,1)_{-3}\mathbf{1} - 4H^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1} \\
&\quad + 44\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} + 2\omega_0\omega^{[i]}_{-2}S_{ki}(1,1)_{-1}\mathbf{1} \\
&\quad - 16\omega_0^2\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1} - \omega_0^4S_{ki}(1,1)_{-1}\mathbf{1} \\
&\quad - 10\omega^{[k]}_{-2}S_{ki}(1,2)_{-1}\mathbf{1} - 4\omega^{[k]}_{-1}S_{ki}(1,2)_{-2}\mathbf{1} \\
&\quad + 2\omega_0S_{ki}(1,2)_{-3}\mathbf{1} - 4\omega_0\omega^{[i]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_1S_{kj}(1,3) &= 12\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} - 4\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1} \\
&\quad - 2\omega_0^3S_{ki}(1,1)_{-1}\mathbf{1} - 8\omega^{[k]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1} \\
&\quad + 12S_{ki}(1,2)_{-3}\mathbf{1} - 12\omega_0S_{ki}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_2S_{kj}(1,3) &= -12S_{ki}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_3S_{kj}(1,3) &= -12S_{ki}(1,2)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_4S_{kj}(1,3) &= -12S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.26}$$

$$S_{ij}(1,2)_0S_{ki}(1,1) = 2S_{kj}(1,3)_{-1}\mathbf{1}, \qquad S_{ij}(1,2)_1S_{ki}(1,1) = S_{kj}(1,2)_{-1}\mathbf{1}, \tag{A2.27}$$

$$\begin{aligned}
S_{ij}(1,2)_0S_{ki}(1,2) &= 6\omega^{[k]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} - 4\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&\quad + \omega_0^3S_{kj}(1,1)_{-1}\mathbf{1} + 4\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} \\
&\quad - 3\omega_0S_{kj}(1,2)_{-2}\mathbf{1} + 6\omega_0S_{kj}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_1S_{ki}(1,2) &= 4S_{kj}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_2S_{ki}(1,2) &= 2S_{kj}(1,2)_{-1}\mathbf{1},
\end{aligned} \tag{A2.28}$$

$$\begin{aligned}
S_{ij}(1,2)_0S_{ki}(1,3) &= -16\omega^{[k]}_{-1}S_{kj}(1,1)_{-3}\mathbf{1} + 4H^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&\quad - 98\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-2}\mathbf{1} + 34\omega_0^2\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&\quad + 3\omega_0^4S_{kj}(1,1)_{-1}\mathbf{1} + 2\omega^{[k]}_{-1}S_{kj}(1,2)_{-2}\mathbf{1} \\
&\quad + 22\omega_0\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} - 4\omega_0^3S_{kj}(1,2)_{-1}\mathbf{1} \\
&\quad + 6\omega_0S_{kj}(1,3)_{-2}\mathbf{1}, \\
S_{ij}(1,2)_1S_{ki}(1,3) &= 9\omega^{[k]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} - 6\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&\quad + 3\omega_0^3S_{kj}(1,1)_{-1}\mathbf{1} + 6\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} \\
&\quad - 9\omega_0S_{kj}(1,2)_{-2}\mathbf{1} + 9\omega_0S_{kj}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_2S_{ki}(1,3) &= 6S_{kj}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,2)_3S_{ki}(1,3) &= 3S_{kj}(1,2)_{-1}\mathbf{1},
\end{aligned} \tag{A2.29}$$

$$\begin{aligned}
S_{ij}(1,3)_0 S_{ij}(1,3) &= 2\omega_0\omega^{[i]}_{-1}H^{[i]}_{-1}\mathbf{1} + 5\omega_0\omega^{[i]}_{-2}\omega^{[i]}_{-2}\mathbf{1} + \omega_0\omega^{[j]}_{-2}\omega^{[j]}_{-2}\mathbf{1} + 2\omega_0\omega^{[j]}_{-1}H^{[j]}_{-1}\mathbf{1}\\
&\quad + \omega_0^3 H^{[i]}_{-1}\mathbf{1} - \omega_0^3\omega^{[i]}_{-1}\omega^{[i]}_{-1}\mathbf{1} - \omega_0^3\omega^{[j]}_{-1}\omega^{[j]}_{-1}\mathbf{1} - 3\omega_0^3 H^{[j]}_{-1}\mathbf{1}\\
&\quad + 7\omega_0^5\omega^{[i]}_{-1}\mathbf{1} + \omega_0^5\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,3)_1 S_{ij}(1,3) &= 4\omega^{[i]}_{-1}H^{[i]}_{-1}\mathbf{1} + 5\omega^{[i]}_{-2}\omega^{[i]}_{-2}\mathbf{1} + \omega^{[j]}_{-2}\omega^{[j]}_{-2}\mathbf{1} + 4\omega^{[j]}_{-1}H^{[j]}_{-1}\mathbf{1}\\
&\quad + 7\omega_0^2 H^{[i]}_{-1}\mathbf{1} - \omega_0^2\omega^{[i]}_{-1}\omega^{[i]}_{-1}\mathbf{1} - \omega_0^2\omega^{[j]}_{-1}\omega^{[j]}_{-1}\mathbf{1} - 3\omega_0^2 H^{[j]}_{-1}\mathbf{1}\\
&\quad + 19\omega_0^4\omega^{[i]}_{-1}\mathbf{1} + \omega_0^4\omega^{[j]}_{-1}\mathbf{1},\\
S_{ij}(1,3)_2 S_{ij}(1,3) &= 15\omega_0 H^{[i]}_{-1}\mathbf{1} + 5\omega_0^3\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,3)_3 S_{ij}(1,3) &= 30H^{[i]}_{-1}\mathbf{1} + 10\omega_0^2\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,3)_4 S_{ij}(1,3) &= 30\omega_0\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,3)_5 S_{ij}(1,3) &= 60\omega^{[i]}_{-1}\mathbf{1},\\
S_{ij}(1,3)_6 S_{ij}(1,3) &= 0,\\
S_{ij}(1,3)_7 S_{ij}(1,3) &= 30\mathbf{1},
\end{aligned} \tag{A2.30}$$

$$\begin{aligned}
S_{ij}(1,3)_0 S_{kj}(1,1) &= -3\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} + \omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + \omega_0^3 S_{ki}(1,1)_{-1}\mathbf{1} + 2\omega^{[k]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1}\\
&\quad - 3S_{ki}(1,2)_{-3}\mathbf{1} + 3\omega_0 S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_1 S_{kj}(1,1) &= 3S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_2 S_{kj}(1,1) &= 3S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,3)_3 S_{kj}(1,1) &= 3S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.31}$$

$$\begin{aligned}
S_{ij}(1,3)_0 S_{kj}(1,2) &= -16\omega^{[k]}_{-1}S_{ki}(1,1)_{-3}\mathbf{1} + 4H^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad - 44\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} - 2\omega_0\omega^{[i]}_{-2}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + 16\omega_0^2\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1} + \omega_0^4 S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + 10\omega^{[k]}_{-2}S_{ki}(1,2)_{-1}\mathbf{1} + 4\omega^{[k]}_{-1}S_{ki}(1,2)_{-2}\mathbf{1}\\
&\quad - 2\omega_0 S_{ki}(1,2)_{-3}\mathbf{1} + 4\omega_0\omega^{[i]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,3)_1 S_{kj}(1,2) &= -12\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} + 4\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + 2\omega_0^3 S_{ki}(1,1)_{-1}\mathbf{1} + 8\omega^{[k]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1}\\
&\quad - 12S_{ki}(1,2)_{-3}\mathbf{1} + 12\omega_0 S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_2 S_{kj}(1,2) &= 12S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_3 S_{kj}(1,2) &= 12S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,3)_4 S_{kj}(1,2) &= 12S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned} \tag{A2.32}$$

$$
\begin{aligned}
S_{ij}(1,3)_0 S_{kj}(1,3) &= 30H^{[i]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1} + 20\omega^{[i]}_{-3}S_{ki}(1,2)_{-1}\mathbf{1} - 30\omega^{[i]}_{-2}S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_1 S_{kj}(1,3) &= -40\omega^{[k]}_{-1}S_{ki}(1,1)_{-3}\mathbf{1} + 10H^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad - 110\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} - 5\omega_0\omega^{[i]}_{-2}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + 40\omega_0^2\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1} + 5\omega_0^4 S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + 25\omega^{[k]}_{-2}S_{ki}(1,2)_{-1}\mathbf{1} + 10\omega^{[k]}_{-1}S_{ki}(1,2)_{-2}\mathbf{1}\\
&\quad - 5\omega_0 S_{ki}(1,2)_{-3}\mathbf{1} + 10\omega_0\omega^{[i]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,3)_2 S_{kj}(1,3) &= -30\omega^{[k]}_{-1}S_{ki}(1,1)_{-2}\mathbf{1} + 10\omega_0\omega^{[k]}_{-1}S_{ki}(1,1)_{-1}\mathbf{1}\\
&\quad + 5\omega_0^3 S_{ki}(1,1)_{-1}\mathbf{1} + 20\omega^{[k]}_{-1}S_{ki}(1,2)_{-1}\mathbf{1}\\
&\quad - 30 S_{ki}(1,2)_{-3}\mathbf{1} + 30\omega_0 S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_3 S_{kj}(1,3) &= 30 S_{ki}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_4 S_{kj}(1,3) &= 30 S_{ki}(1,2)_{-1}\mathbf{1},\\
S_{ij}(1,3)_5 S_{kj}(1,3) &= 30 S_{ki}(1,1)_{-1}\mathbf{1},
\end{aligned}
\tag{A2.33}
$$

$$
\begin{aligned}
S_{ij}(1,3)_0 S_{ki}(1,1) &= 3\omega^{[k]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} - 2\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1}\\
&\quad + \omega_0^3 S_{kj}(1,1)_{-1}\mathbf{1} + 2\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1}\\
&\quad - 3\omega_0 S_{kj}(1,2)_{-2}\mathbf{1} + 3\omega_0 S_{kj}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_1 S_{ki}(1,1) &= S_{kj}(1,3)_{-1}\mathbf{1},
\end{aligned}
\tag{A2.34}
$$

$$
\begin{aligned}
S_{ij}(1,3)_0 S_{ki}(1,2) &= -16\omega^{[k]}_{-1}S_{kj}(1,1)_{-3}\mathbf{1} + 4H^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1}\\
&\quad - 98\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-2}\mathbf{1} + 34\omega_0^2\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1}\\
&\quad + 3\omega_0^4 S_{kj}(1,1)_{-1}\mathbf{1} + 2\omega^{[k]}_{-1}S_{kj}(1,2)_{-2}\mathbf{1}\\
&\quad + 22\omega_0\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} - 4\omega_0^3 S_{kj}(1,2)_{-1}\mathbf{1}\\
&\quad + 6\omega_0 S_{kj}(1,3)_{-2}\mathbf{1},\\
S_{ij}(1,3)_1 S_{ki}(1,2) &= 6\omega^{[k]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} - 4\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1}\\
&\quad + \omega_0^3 S_{kj}(1,1)_{-1}\mathbf{1} + 4\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1}\\
&\quad - 3\omega_0 S_{kj}(1,2)_{-2}\mathbf{1} + 6\omega_0 S_{kj}(1,3)_{-1}\mathbf{1},\\
S_{ij}(1,3)_2 S_{ki}(1,2) &= 2 S_{kj}(1,3)_{-1}\mathbf{1},
\end{aligned}
\tag{A2.35}
$$

$$
\begin{aligned}
S_{ij}(1,3)_0 S_{ki}(1,3) &= -120\omega^{[k]}_{-1}\omega^{[j]}_{-1}S_{kj}(1,1)_{-2}\mathbf{1} - 300\omega^{[j]}_{-4}S_{kj}(1,1)_{-1}\mathbf{1} \\
&- 180H^{[j]}_{-1}S_{kj}(1,1)_{-2}\mathbf{1} - 60H^{[j]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} \\
&+ 40\omega_0\omega^{[k]}_{-1}\omega^{[j]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} + 240\omega_0\omega^{[j]}_{-3}S_{kj}(1,1)_{-1}\mathbf{1} \\
&- 90\omega_0^2\omega^{[j]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} + 20\omega_0^3\omega^{[j]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&+ 330\omega^{[k]}_{-2}S_{kj}(1,2)_{-2}\mathbf{1} + 780\omega^{[k]}_{-1}S_{kj}(1,2)_{-3}\mathbf{1} \\
&- 90H^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} + 180H^{[j]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} \\
&- 120\omega_0^2\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} - 135\omega_0^4 S_{kj}(1,2)_{-1}\mathbf{1} \\
&- 470\omega^{[k]}_{-2}S_{kj}(1,3)_{-1}\mathbf{1} - 500\omega^{[k]}_{-1}S_{kj}(1,3)_{-2}\mathbf{1} \\
&+ 750S_{kj}(1,3)_{-4}\mathbf{1}, \\
S_{ij}(1,3)_1 S_{ki}(1,3) &= -8\omega^{[k]}_{-1}S_{kj}(1,1)_{-3}\mathbf{1} + 2H^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&- 49\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-2}\mathbf{1} + 17\omega_0^2\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&+ 9\omega_0^4 S_{kj}(1,1)_{-1}\mathbf{1} + \omega^{[k]}_{-1}S_{kj}(1,2)_{-2}\mathbf{1} \\
&+ 11\omega_0\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} - 6\omega_0^3 S_{kj}(1,2)_{-1}\mathbf{1} \\
&+ 9\omega_0 S_{kj}(1,3)_{-2}\mathbf{1}, \\
S_{ij}(1,3)_2 S_{ki}(1,3) &= 9\omega^{[k]}_{-2}S_{kj}(1,1)_{-1}\mathbf{1} - 6\omega_0\omega^{[k]}_{-1}S_{kj}(1,1)_{-1}\mathbf{1} \\
&+ 3\omega_0^3 S_{kj}(1,1)_{-1}\mathbf{1} + 6\omega^{[k]}_{-1}S_{kj}(1,2)_{-1}\mathbf{1} \\
&- 9\omega_0 S_{kj}(1,2)_{-2}\mathbf{1} + 9\omega_0 S_{kj}(1,3)_{-1}\mathbf{1}, \\
S_{ij}(1,3)_3 S_{ki}(1,3) &= 3S_{kj}(1,3)_{-1}\mathbf{1},
\end{aligned} \tag{A2.36}
$$

Notation

V	a vertex algebra.
U	a subspace of a weak V-module.
$\Omega_V(U)$	$= \{u \in U \mid a_i u = 0$ for all homogeneous $a \in V$ and $i > \mathrm{wt}\, a - 1\}$.
$\mathfrak{h}$	a finite dimensional vector space equipped with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$.
$h^{[1]}, \ldots, h^{[d]}$	an orthonormal basis of $\mathfrak{h}$.
$M(1)$	the vertex operator algebra associated to the Heisenberg algebra.
L	a non-degenerate even lattice of finite rank.
d	the rank of L.
V_L	the vertex algebra associated to L.
θ	the automorphism of V_L induced from the -1 symmetry of L.
$M(1)^+$	the fixed point subalgbra of $M(1)$ under the action of θ.
V_L^+	the fixed point subalgbra of V_L under the action of θ.
$I(\ ,x)$	an intertwining operator for $M(1)^+$.
$\epsilon(u,v)$	$u_{\epsilon(u,v)}v \neq 0$ and $u_i v = 0$ for all $i > \epsilon(u,v)$ if $I(u,x)v \neq 0$ and $\epsilon(u,v) = -\infty$ if $I(u,x)v = 0$, where $I : M \times W \to N((x))$ is an intertwining operator and $u \in M$, $v \in W$ (see (2.9)).
$A(V)$	the Zhu algebra of a vertex operator algebra V.
$A_{-}B$	$:= \mathrm{Span}_{\mathbb{C}}\{a_{-i}b \mid a \in A, b \in B,$ and $i \in \mathbb{Z}_{>0}\}$ (see (2.10)).
$\langle A_{-}\rangle B$	see (2.11).
ω	$= (1/2)\sum_{i=1}^{d} h^{[i]}(-1)^2\mathbf{1}$.
$E(\alpha)$	$= e^{\alpha} + \theta(e^{\alpha})$ where $\alpha \in \mathfrak{h}$.
$\omega^{[i]}$	$= (1/2)h^{[i]}(-1)^2$.
$H^{[i]}$	$= (1/3)(h^{[i]}(-3)h^{[i]}(-1)\mathbf{1} - h^{[i]}(-2)^2\mathbf{1})$.

Conflict of Interest

The author declares that he has no conflict of interest.

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