NESTED PRODUCTS AND A STRONGLY CENTERED FILTER

MASAHIRO SHIOYA

Abstract. Assume an almost huge cardinal with Mahlo target exists. We construct a model of ZFC in which a small cardinal carries a strongly centered filter by forcing with an iteration of two nested products of Levy collapses.

1. Introduction

One of the main themes in set theory is to construct models of ZFC (Zermelo–Fraenkel set theory with the axiom of choice) in which small cardinals have strong combinatorial properties, assuming the existence of large cardinals. In [8] Kunen devised a powerful method to construct a model in which ω_1 carries a saturated filter. To sketch his method, let $j : V \to M$ witness that κ is a huge cardinal. Kunen constructed the model by forcing with an iteration of the form $P * \dot{S}(\kappa, j(\kappa))$, where $\dot{S}(\kappa, j(\kappa))$ is a P-name for the Silver collapse. The "universal collapse" P, which forms the core of the method, is defined by recursion so that $P * \dot{S}(\kappa, j(\kappa))$ can be completely embedded into $j(P)$ among other conditions. We refer the reader to [3] for a comprehensive survey of Kunen's method.

Laver [9] and Foreman–Laver [4] used Kunen's method to construct models in which ω_1 carries a strongly saturated filter and a centered filter respectively. In doing so, they replaced Silver collapses in the original construction by Laver collapses and nested products of Silver collapses respectively.

In [11] the Laver construction was simplified and extended to the case of $\mathcal{P}_\kappa\lambda$. As it turned out, there is no need of the universal collapse: An iteration of two Easton collapses works. In this paper we do the corresponding task for the Foreman–Laver construction. More specifically, we prove

Theorem 1. *Suppose κ is almost huge, $j : V \to M$ is a witness and $j(\kappa)$ is Mahlo. Let $\mu < \lambda$ be both strongly regular with $\mu < \kappa \leq \lambda < j(\kappa)$. Then $P(\mu, \kappa) * \dot{P}(\lambda, j(\kappa))$ forces that $\kappa = \mu^+$, $j(\kappa) = \lambda^+$ and $\mathcal{P}_\kappa\lambda$ carries a strongly centered normal filter.*

Here Mahloness is a large cardinal property much weaker than almost hugeness. $P(\mu, \kappa)$ denotes the nested product of Levy collapses. Regular cardinals are strongly regular under GCH (the generalized continuum hypothesis). A strongly centered normal filter on a small cardinal can be viewed as an analogue of a normal ultrafilter on a large cardinal. As in [11], the key to the proof is the existence of a projection from $j(P(\mu, \kappa))$ to $P(\mu, \kappa) * \dot{P}(\lambda, j(\kappa))$ with a suitable quotient (see Lemma 6).

While there is no implication between strong saturation and strong centeredness, they are both stronger than saturation and weaker than density. In [12] a poset for a model with a dense filter is defined explicitly, but it has only the Baire property.

1991 *Mathematics Subject Classification.* 03E05, 03E35, 03E55.

Key words and phrases. Strongly centered filter, nested product, almost huge cardinal.

This work was supported by JSPS Grant-in-Aid 21K03334.

In contrast, the poset of Theorem 1 is μ-directed closed, so that we can make μ supercompact in the extension by incorporating the Laver preparation. This in turn allows one to do Prikry forcing at μ, which we leave for future work.

2. Preliminaries

Our notation is standard. We refer the reader to [7] for the background material. Unless otherwise stated, μ, κ, λ and ν denote regular cardinals.

A cardinal κ is almost huge if there is an elementary embedding $j : V \to M$ such that κ is the critical point of j and $M^{<j(\kappa)} \subset M$. We say that κ is strongly regular if $|\kappa^{<\kappa}| = \kappa$. Note that if a poset P is κ-cc and of size $\leq \kappa$, then the class of strongly regular cardinals $\geq \kappa$ remains the same after forcing with P. In what follows, SR denotes the class of strongly regular cardinals (in V).

We identify a poset P with its separative quotient. Thus for $p, p' \in P$ we have $p' \leq p \Leftrightarrow \forall p'' \leq p'(p'' \parallel p) \Leftrightarrow p' \Vdash p \in \dot{G}$, where $\dot{G}$ is the canonical P-name for a generic filter. By our convention a complete embedding between posets is injective.

Suppose $(P_i, \leq_i)$ is a poset for $i \in I$. The $< \kappa$-support product $\prod^{<\kappa}_{i\in I} P_i$ is the set $\bigcup_{d\in[I]^{<\kappa}} \prod_{i\in d} P_i$ ordered by: $p' \leq p$ iff $\operatorname{dom} p' \supset \operatorname{dom} p$ and $p'(i) \leq_i p(i)$ for every $i \in \operatorname{dom} p$. The full support product $\prod_{i\in I} P_i$ is ordered similarly.

For a set A of ordinals, the Levy collapse $C(\kappa, A)$ is defined as $\prod^{<\kappa}_{\gamma\in\mathsf{SR}\cap A-\kappa} \gamma^{<\kappa}$. Here $\gamma^{<\kappa}$ is ordered by reverse inclusion. Thus $C(\kappa, A)$ forces $|\gamma| \leq \kappa$ for every $\gamma \in \mathsf{SR} \cap A$. Note that $X \subset C(\kappa, A)$ has a lower bound iff the coordinatewise union of X is in $C(\kappa, A)$, in which case the coordinatewise union of X is $\inf X$. In particular, $C(\kappa, A)$ is κ-linked closed, i.e. every linked (pairwise compatible) subset of size $< \kappa$ has a lower bound. If $\nu > \kappa$ is inaccessible and $\sup(\mathsf{SR} \cap \nu) = \nu$, then $C(\kappa, \nu)$ is ν-cc and forces $\nu = \kappa^+$.

The following definition is due to Shelah. Let $S \subset \nu$ be stationary. A poset P is S-layered if $P = \bigcup_{\xi<\nu} P_\xi$ for some increasing sequence $\langle P_\xi : \xi < \nu\rangle$ of complete suborders of P of size $< \nu$ such that $S \cap C \subset \{\xi < \nu : P_\xi = \bigcup_{\zeta<\xi} P_\zeta\}$ for some club $C \subset \nu$. A poset is ν-cc if it is S-layered for some stationary $S \subset \nu$ (see [5]).

Lemma 2. *Let $\nu > \kappa$ be inaccessible. Suppose S is a stationary subset of ν such that $S \cap C \subset \{\xi < \nu : \operatorname{cf} \xi \geq \kappa\}$ for some club $C \subset \nu$. Assume P_γ is S-layered for every $\gamma < \nu$. Then $\prod^{<\kappa}_{\gamma<\nu} P_\gamma$ is S-layered.*

Proof. For each $\gamma < \nu$ let $\langle P_{\gamma,\xi} : \xi < \nu\rangle$ and $C_\gamma \subset \nu$ club witness that P_γ is S-layered. Then $\langle \prod^{<\kappa}_{\gamma<\xi} P_{\gamma,\xi} : \xi < \nu\rangle$ and $C \cap \triangle_{\gamma<\nu} C_\gamma$ witness the desired result. □

A poset P is $(\lambda, < \mu)$-centered if there is $f : P \to \lambda$ such that every $X \in [P]^{<\mu}$ on which f is constant has a lower bound. The $(\lambda, < \mu)$-centeredness implies the $(\lambda^+, \lambda^+, < \mu)$-cc in the sense of [11], which in turn implies the λ^+-cc. By our convention a complete suborder of a $(\lambda, < \mu)$-centered poset is $(\lambda, < \mu)$-centered. The following modification of [4, Lemma 4] simplifies the proof of centeredness:

Lemma 3. *Let $\mu \leq \kappa \leq \lambda$. Suppose P_i is a poset for $i \in I$, R is μ-Baire and forces λ to be strongly regular. Assume one of the following:*

1. *$P = \prod^{<\kappa}_{i\in I} P_i$ and $R \Vdash |I| \leq 2^\lambda$.*
2. *$P = \prod_{i\in I} P_i$ and $|I| < \lambda$.*

Then R forces that P is $(\lambda, < \mu)$-centered if P_i is $(\lambda, < \mu)$-centered for every $i \in I$.

Proof. Let $H \subset R$ be V-generic. Work in $V[H]$. For $i \in I$ let $f_i : P_i \to \lambda$ witness that P_i is $(\lambda, < \mu)$-centered.

(1) Fix an injection $t : I \to \{0,1\}^\lambda$. To give a witness for P, let $p \in P$. Note that $|\operatorname{dom} p| \leq |\operatorname{dom} p|^V < \kappa \leq \lambda$. So there is $\delta < \lambda$ such that $\langle t(i) \mid \delta : i \in \operatorname{dom} p\rangle$ is injective. Define $f(p) : \{t(i) \mid \delta : i \in \operatorname{dom} p\} \to \lambda$ by $f(p)(t(i) \mid \delta) = f_i(p(i))$. Note that $f : P \to \bigcup\{\lambda^d : d \in \bigcup_{\delta<\lambda}[\{0,1\}^\delta]^{<\kappa}\}$. So we have $|\operatorname{ran} f| \leq |\lambda^{<\lambda}| = \lambda$. Suppose f is constant on $X \in [P]^{<\mu}$. Note that $X \in V$ because R is μ-Baire in V. Let $D = \bigcup_{p \in X} \operatorname{dom} p \in V$. Note that $|D|^V < \kappa$. For every $i \in D$, f_i is constant on $X_i = \{p(i) : p \in X,\ i \in \operatorname{dom} p\} \in [P_i]^{<\mu}$, so that X_i has a lower bound in P_i. Since $\langle X_i : i \in D\rangle \in V$, we get a lower bound of X (in V), as desired.

(2) For $p \in P$ define $f(p) : I \to \lambda$ by $f(p)(i) = f_i(p(i))$. Then $|\operatorname{ran} f| \leq |\lambda^I| \leq \lambda$. Suppose f is constant on $X \in [P]^{<\mu}$. Then $X \in V$ as in (1). For every $i \in I$, f_i is constant on $X_i = \{p(i) : p \in X\} \in [P_i]^{<\mu}$, so that X_i has a lower bound in P_i. Since $\langle X_i : i \in I\rangle \in V$, we get a lower bound of X (in V), as desired. □

Suppose F is a filter on $\mathcal{P}_\kappa\lambda$, where $\omega < \kappa \leq \lambda$. Then F^+ denotes the set of F-positive subsets ordered by inclusion. By our convention $X, Y \in F^+$ are equivalent iff $X \triangle Y \notin F^+$. When $\kappa = \mu^+$, we say that F is strongly centered if F^+ is $(\lambda, < \mu)$-centered.

Let P and R be posets. Suppose $\pi : R \to P$ is a projection, i.e. an order-preserving map such that $\pi(1_R) = 1_P$, and $p \leq_P \pi(r)$ implies $\pi(r') \leq_P p$ for some $r' \leq_R r$. If $D \subset P$ is dense open, then $\pi^{-1}[D] \subset R$ is dense. So if $H \subset R$ is V-generic, then $\pi[H]$ generates a V-generic filter over P in $V[H]$. If $G \subset P$ is V-generic, then in $V[G]$ we can define the quotient of R by π as the suborder $\pi^{-1}[G]$ of R. Let $\dot{Q}$ be a P-name for the quotient. Then the map $i : r \mapsto (\pi(r), \hat{r})$ is a dense embedding of R into $P * \dot{Q}$, where $\hat{r}$ is a P-name such that $\pi(r) \Vdash \hat{r} = r$ and $p \Vdash \hat{r} = 1_R$ for every $p \perp \pi(r)$.

Let $\dot{Q}$ be a P-name for a poset. Then $T(P, \dot{Q})$ denotes the term forcing: $T(P, \dot{Q})$ is the set of P-names for elements of $\dot{Q}$ ordered by: $\dot{q}' \leq \dot{q} \Leftrightarrow P \Vdash \dot{q}' \leq_{\dot{Q}} \dot{q}$. It is easy to see that $\mathrm{id} : P \times T(P, \dot{Q}) \to P * \dot{Q}$ is a projection. Lemma 4 is essentially proved in [1] (see also [11]).

Lemma 4. *Suppose P is κ-cc and of size $\leq \kappa$, and $|\gamma^{<\kappa}| = \gamma$. Then $T(P, \dot{\gamma^{<\kappa}})$ is equivalent to $\gamma^{<\kappa}$.*

Note that the isomorphisms of Lemma 5 are defined coordinatewise.

Lemma 5. *Suppose $\dot{Q}_i$ is a P-name for a poset for $i \in I$. Then the following hold.*

(1) *If P is κ-cc, then $T(P, \prod^{<\kappa}_{i\in I} \dot{Q}_i) \simeq \prod^{<\kappa}_{i \in I} T(P, \dot{Q}_i)$.*

(2) *$T(P, \prod_{i\in I} \dot{Q}_i) \simeq \prod_{i\in I} T(P, \dot{Q}_i)$.*

Proof. (1) By the κ-cc of P, a subset of I of size $< \kappa$ in the extension can be covered by a set of size $< \kappa$ in the ground model. Thus

$$D = \{\dot{q} \in T(P, \textstyle\prod^{<\kappa}_{i\in I} \dot{Q}_i) : \exists d \in [I]^{<\kappa}(P \Vdash \operatorname{dom} \dot{q} = d)\}$$

is dense in $T(P, \prod^{<\kappa}_{i\in I} \dot{Q}_i)$. Define $e : \langle \dot{q}_i : i \in d\rangle \mapsto \dot{q}$, where $P \Vdash \dot{q} = \langle \dot{q}_i : i \in d\rangle$. It is easy to see that $e : \prod^{<\kappa}_{i\in I} T(P, \dot{Q}_i) \to D$ is an isomorphism.

(2) is proved in a similar (even simpler) way. □

Suppose P is κ-cc and of size $\leq \kappa$, and $\nu > \kappa$ is Mahlo. Then by Lemmas 4 and 5 there is a projection $\pi : P \times C(\kappa, \nu) \to P * \dot{C}(\kappa, \nu)$ that is the identity on the first coordinate. Moreover, the following holds by the proof of Lemma 5: Suppose $X \subset C(\kappa,\nu)$ is linked and of size $< \kappa$, and q^* is the coordinatewise union of X. Let $\pi(1_P, q) = (1_P, \dot{q})$ for $q \in X$, and $\pi(1_P, q^*) = (1_P, \dot{q}^*)$. Then P forces that $\{\dot{q} : q \in X\}$ is linked with the coordinatewise union $\dot{q}^*$.

3. The Nested Product of Levy Collapses

In this section we define our poset and prove the key lemma.

Let $\kappa < \nu$ be both strongly regular. The nested product $P(\kappa,\nu)$ of Levy collapses is $\prod_{n<\omega} P_n(\kappa,\nu)$, where $P_n(\gamma,\nu)$ is defined by recursion for each $\gamma \in \mathsf{SR} \cap \nu$ so that $P_0(\gamma,\nu) = C(\gamma,\nu)$ and $P_{n+1}(\gamma,\nu) = \prod^{<\gamma}_{\delta \in \mathsf{SR}\cap[\gamma,\nu)} P_n(\delta,\nu)$.

Assume in addition ν is Mahlo. Then the set S of inaccessible cardinals $< \nu$ is stationary in ν. Using Lemma 2, we have by induction on $n < \omega$ that $P_n(\gamma,\nu)$ is S-layered for every $\gamma \in \mathsf{SR} \cap \nu$. By Lemma 2 again $P(\kappa,\nu) = \prod^{<\omega_1}_{n<\omega} P_n(\kappa,\nu)$ is S-layered, and thus is ν-cc. Having $C(\kappa,\nu)$ as a factor, $P(\kappa,\nu)$ forces $\nu = \kappa^+$.

By induction on $n < \omega$ we have $P_n(\gamma,\nu) \subset V_\nu$ for every $\gamma \in \mathsf{SR} \cap \nu$, so that $P(\kappa,\nu) \subset V_\nu$. A similar induction shows that $P(\kappa,\nu)$ is κ-linked closed. Moreover, if $X \subset P(\kappa,\nu)$ is linked and of size $< \kappa$, then $\inf X$ is given by the "coordinatewise union" of X.

If $\alpha < \gamma < \nu$ are both strongly regular, then $P(\alpha,\nu)$ is the product of $P(\gamma,\nu)$ and an α-closed poset because $P_0(\alpha,\nu)$ is α-closed, and $P_{n+1}(\alpha,\nu)$ is the product of $P_n(\gamma,\nu)$ and an α-closed poset. Claim 1 below can be regarded as a refinement of this observation.

Lemma 6. *Suppose $\mu < \kappa \leq \lambda < \nu$ are all strongly regular with $\kappa < \nu$ both Mahlo. Then there is a projection $\pi : P(\mu,\nu) \to P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ such that the quotient of $P(\mu,\nu)$ by π is forced to be $(\lambda, <\mu)$-centered. Moreover, the first coordinate of $\pi(p)$ is the canonical restriction of $p \in P(\mu,\nu)$.*

Proof. We define π as the composition of two projections

$$P(\mu,\nu) \xrightarrow{\sigma} P(\mu,\kappa) \times P(\lambda,\nu) \xrightarrow{\tau} P(\mu,\kappa) * \dot{P}(\lambda,\nu).$$

Here the projection σ is the canonical restriction of maps induced from Claim 1. The projection τ is the identity on the first coordinate, so that the first coordinate of $\pi(p)$ is the canonical restriction of $p \in P(\mu,\nu)$.

Claim 1. *$P(\mu,\nu) \simeq P(\mu,\kappa) \times P(\lambda,\nu) \times R$ for some μ-closed R.*

Proof. By induction on $n < \omega$ we prove that for every $\gamma \in \mathsf{SR} \cap \kappa$ there is a γ-closed $R_n(\gamma)$ such that

$$P_{n+1}(\gamma,\nu) \simeq P_{n+1}(\gamma,\kappa) \times P_n(\lambda,\nu) \times R_n(\gamma).$$

Note that this completes the proof: Clearly, $R = C(\mu,[\kappa,\nu)) \times \prod_{n<\omega} R_n(\mu)$ works.

For $n = 0$, we have

$$\begin{aligned} P_1(\gamma,\nu) &= \prod^{<\gamma}_{\delta \in \mathsf{SR}\cap[\gamma,\nu)} P_0(\delta,\nu) \\ &\simeq \prod^{<\gamma}_{\delta \in \mathsf{SR}\cap[\gamma,\kappa)} P_0(\delta,\nu) \times \prod^{<\gamma}_{\delta \in \mathsf{SR}\cap[\kappa,\lambda)} P_0(\delta,\nu) \times P_0(\lambda,\nu) \times \prod^{<\gamma}_{\delta \in \mathsf{SR}\cap(\lambda,\nu)} P_0(\delta,\nu) \\ &\simeq P_1(\gamma,\kappa) \times P_0(\lambda,\nu) \times R_0(\gamma), \end{aligned}$$

where

$$R_0(\gamma) = \prod_{\delta \in \mathsf{SR} \cap [\gamma,\kappa)}^{<\gamma} C(\delta, [\kappa, \nu)) \times \prod_{\delta \in \mathsf{SR} \cap [\kappa,\lambda)}^{<\gamma} P_0(\delta, \nu) \times \prod_{\delta \in \mathsf{SR} \cap (\lambda,\nu)}^{<\gamma} P_0(\delta, \nu).$$

Assuming the claim for n, we have

$$\begin{aligned} P_{n+2}(\gamma,\nu) &= \prod_{\delta \in \mathsf{SR} \cap [\gamma,\nu)}^{<\gamma} P_{n+1}(\delta,\nu) \\ &\simeq \prod_{\delta \in \mathsf{SR} \cap [\gamma,\kappa)}^{<\gamma} P_{n+1}(\delta,\nu) \times \prod_{\delta \in \mathsf{SR} \cap [\kappa,\lambda)}^{<\gamma} P_{n+1}(\delta,\nu) \times P_{n+1}(\lambda,\nu) \times \prod_{\delta \in \mathsf{SR} \cap (\lambda,\nu)}^{<\gamma} P_{n+1}(\delta,\nu) \\ &\simeq P_{n+2}(\gamma,\kappa) \times P_{n+1}(\lambda,\nu) \times R_{n+1}(\gamma), \end{aligned}$$

where

$$R_{n+1}(\gamma) = \prod_{\delta \in \mathsf{SR} \cap [\gamma,\kappa)}^{<\gamma} (P_n(\lambda,\nu) \times R_n(\delta)) \times \prod_{\delta \in \mathsf{SR} \cap [\kappa,\lambda)}^{<\gamma} P_{n+1}(\delta,\nu) \times \prod_{\delta \in \mathsf{SR} \cap (\lambda,\nu)}^{<\gamma} P_{n+1}(\delta,\nu),$$

as desired. □

Claim 2. *$P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ forces that $P(\mu,\nu)$ is $(\lambda, <\mu)$-centered.*

Proof. By induction on $n < \omega$ we prove that $P(\mu,\kappa) * \dot{P}(\alpha,\nu)$ forces $P_n(\gamma,\nu)$ to be $(\alpha, <\mu)$-centered for every $\alpha \in \mathsf{SR} \cap [\lambda,\nu)$ and $\gamma \in \mathsf{SR} \cap [\mu,\nu)$. Note that this completes the proof: $P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ forces that $P(\mu,\nu) = \prod_{n<\omega} P_n(\mu,\nu)$ is $(\lambda, <\mu)$-centered by Lemma 3, as desired.

For $n = 0$, assume $\alpha \geq \gamma$ first. Note that $P(\mu,\kappa) * \dot{P}(\alpha,\nu)$ forces $|\delta^{<\gamma}| \leq \alpha$ for every $\delta \in \mathsf{SR} \cap [\gamma,\nu)$. Then $P(\mu,\kappa) * \dot{P}(\alpha,\nu)$ forces $P_0(\gamma,\nu) = \prod_{\delta \in \mathsf{SR} \cap [\gamma,\nu)}^{<\gamma} \delta^{<\gamma}$ to be $(\alpha, <\mu)$-centered by Lemma 3. Next assume $\alpha < \gamma$. Then $P(\mu,\kappa)$ forces that $\dot{P}(\alpha,\nu)$ is the product of $\dot{P}(\gamma,\nu)$ and an α-closed poset. Note that $P(\mu,\kappa) * \dot{P}(\gamma,\nu)$ forces $P_0(\gamma,\nu)$ to be $(\gamma, <\mu)$-centered. Thus $P(\mu,\kappa) * \dot{P}(\alpha,\nu)$ forces that $P_0(\gamma,\nu)$ is $(|\gamma|, <\mu)$-centered and $|\gamma| = \alpha$, as desired.

Assume the claim for n. Then by Lemma 3, $P(\mu,\kappa) * \dot{P}(\alpha,\nu)$ forces that $P_{n+1}(\gamma,\nu) = \prod_{\delta \in \mathsf{SR} \cap [\gamma,\nu)}^{<\gamma} P_n(\delta,\nu)$ is $(\alpha, <\mu)$-centered if $\alpha \geq \gamma$. The rest of the proof is the same as before. □

From Claims 1 and 2 it follows that $P(\mu,\kappa) \times P(\lambda,\nu)$ and R are forced to be $(\lambda, <\mu)$-centered as well.

Claim 3. *There is a projection $\tau : P(\mu,\kappa) \times P(\lambda,\nu) \to P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ that is the identity on the first coordinate. Moreover, $\tau(\inf X) = \inf \tau[X]$ holds for every linked $X \subset P(\mu,\kappa) \times P(\lambda,\nu)$ of size $< \mu$.*

Proof. Using Lemmas 4 and 5, we have $T(P(\mu,\kappa), \dot{P}_n(\gamma,\nu)) \simeq P_n(\gamma,\nu)$ for every $\gamma \in \mathsf{SR} \cap [\lambda,\nu)$ by induction on $n < \omega$. Thus $T(P(\mu,\kappa), \dot{P}(\lambda,\nu)) \simeq P(\lambda,\nu)$ by Lemma 5. Therefore we get a projection $\tau : P(\mu,\kappa) \times P(\lambda,\nu) \to P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ that is the identity on the first coordinate.

Suppose $X = \{(p_\xi, q_\xi) : \xi < \gamma\} \subset P(\mu,\kappa) \times P(\lambda,\nu)$ is linked and $\gamma < \mu$. Then $\inf X = (p^*, q^*)$, where p^* and q^* are the "coordinatewise unions" of $\{p_\xi : \xi < \gamma\}$ and $\{q_\xi : \xi < \gamma\}$ respectively. Let $\tau(p_\xi, q_\xi) = (p_\xi, \dot{q}_\xi)$ for $\xi < \gamma$. Since $\tau[X]$ is linked in $P(\mu,\kappa) \times T(P(\mu,\kappa), \dot{P}(\lambda,\nu))$, $P(\mu,\kappa)$ forces that $\{\dot{q}_\xi : \xi < \gamma\}$ is linked in $\dot{P}(\lambda,\nu)$. Let $\tau(p^*, q^*) = (p^*, \dot{q}^*)$. By the proof of Lemma 5, $P(\mu,\kappa)$ forces that $\dot{q}^*$

is the "coordinatewise union" of $\{\dot{q}_\xi : \xi < \gamma\}$. Thus $(p^*, \dot{q}^*) = \inf\{(p_\xi, \dot{q}_\xi) : \xi < \gamma\}$ holds in $P(\mu,\kappa) * \dot{P}(\lambda,\nu)$, as desired. □

Now let $G * H \subset P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ be V-generic. Work in $V[G*H]$. Note that the quotient of $P(\mu,\nu)$ by π is $\tau^{-1}[G*H] \times R$, where R is $(\lambda, <\mu)$-centered by Claims 1 and 2. Thus it remains to prove that $\tau^{-1}[G*H]$ is $(\lambda,<\mu)$-centered. Let f witness that $P(\mu,\kappa) \times P(\lambda,\nu)$ is $(\lambda,<\mu)$-centered. We claim that $f \mid \tau^{-1}[G*H]$ is the desired witness. Suppose f is constant on $X \subset \tau^{-1}[G*H]$ of size $<\mu$. Then $X \in V$ because $P(\mu,\kappa) * \dot{P}(\lambda,\nu)$ is μ-closed in V. Thus by Claim 3 we have $\tau(\inf X) = \inf \tau[X] \in G * H$, i.e. $\inf X \in \tau^{-1}[G*H]$, as desired. □

4. Proof of Theorem 1

The rest of the proof proceeds as in [11]:

Proof of Theorem 1. Let $\pi : P(\mu, j(\kappa)) \to P(\mu,\kappa) * \dot{P}(\lambda, j(\kappa))$ be a projection as in Lemma 6 with $\nu = j(\kappa)$. Note that $P(\mu, j(\kappa)) = P(\mu, j(\kappa))^M = j(P(\mu,\kappa))$ because $M^{<j(\kappa)} \subset M$ holds.

Let $G * H \subset P(\mu,\kappa) * \dot{P}(\lambda, j(\kappa))$ be V-generic. Work in $V[G*H]$. Let Q be the quotient of $P(\mu, j(\kappa))^V$ by π, which is $(\lambda,<\mu)$-centered. We claim that Q forces the existence of a $V[G*H]$-normal ultrafilter over $\mathcal{P}(\mathcal{P}_\kappa\lambda)^{V[G*H]}$. Note that this completes the proof: Let $\dot{U}$ be a Q-name for a witness. The standard arguments show that $F = \{X \subset \mathcal{P}_\kappa\lambda : Q \Vdash X \in \dot{U}\}$ is a normal filter on $\mathcal{P}_\kappa\lambda$, and the map $e : X \mapsto \|X \in \dot{U}\|$ is a complete embedding of F^+ into $B(Q)$. Thus F^+ is $(\lambda,<\mu)$-centered, as desired.

Let $K \subset Q$ be $V[G*H]$-generic. Then there is a V-generic $\bar{G} \simeq G * H * K$ over $P(\mu, j(\kappa))^V$ such that $\pi[\bar{G}]$ generates $G*H$. Work in $V[\bar{G}]$. Note that $j[G] = G \subset \bar{G}$ by the choice of π. So we can extend $j : V \to M$ to $j : V[G] \to M[\bar{G}]$. Note that $P(\mu, j(\kappa))$ is $j(\kappa)$-cc in V. Since $M^{<j(\kappa)} \subset M$ in V, we have $M[\bar{G}]^{<j(\kappa)} \subset M[\bar{G}]$.

Work in $V[G]$. Then $j(\kappa)$ remains Mahlo, so that $P(\lambda, j(\kappa))$ is $j(\kappa)$-cc. Thus we can list with cofinal repetition the set of $P(\lambda, j(\kappa))$-names for subsets of $\mathcal{P}_\kappa\lambda$ as $\{\dot{X}_\zeta : \zeta \in \mathsf{SR} \cap (\lambda, j(\kappa))\}$. Note that each $\dot{X}_\zeta$ can be viewed as a $P(\lambda,\xi)$-name for some $\xi \in \mathsf{SR} \cap (\lambda, j(\kappa))$.

Now work in $V[\bar{G}]$. Let $\xi \in \mathsf{SR} \cap (\lambda, j(\kappa))$. Note that $j[H \cap P(\lambda,\xi)^{V[G]}]$ is a directed subset of $P(j(\lambda), j(\xi))^{M[\bar{G}]}$. Since $M[\bar{G}]^{<j(\kappa)} \subset M[\bar{G}]$,

$$r_\xi = \text{the "coordinatewise union" of } j[H \cap P(\lambda,\xi)^{V[G]}]$$

is in $P(j(\lambda), j(\xi))^{M[\bar{G}]}$. Note that if $\zeta < \xi$, then r_ζ is the canonical restriction of r_ξ. Thus we can define by recursion a descending sequence $\langle r^*_\xi : \xi \in \mathsf{SR} \cap (\lambda, j(\kappa))\rangle$ in $P(j(\lambda), j^2(\kappa))^{M[\bar{G}]}$ so that

- $r^*_\xi \leq r_\xi$ in $P(j(\lambda), j(\xi))^{M[\bar{G}]}$ and
- if $\dot{X}_\xi$ is a $P(\lambda,\xi)^{V[G]}$-name, then r^*_ξ decides $j[\lambda] \in j(\dot{X}_\xi)$ in $M[\bar{G}]$.

The standard argument shows that

$$U = \{(\dot{X}_\xi)^H : \xi \in \mathsf{SR} \cap (\lambda, j(\kappa)),\ M[\bar{G}] \models r^*_\xi \Vdash j[\lambda] \in j(\dot{X}_\xi)\}$$

is a $V[G*H]$-normal ultrafilter over $\mathcal{P}((\mathcal{P}_\kappa\lambda)^{V[G]})^{V[G*H]} = \mathcal{P}(\mathcal{P}_\kappa\lambda)^{V[G*H]}$, as desired. □

Remark. Suppose κ is huge, $j : V \to M$ is a witness, $\mu < \kappa < \lambda < j(\kappa)$ are all regular and GCH holds. Then the standard argument shows that $(\lambda^+, \kappa) \twoheadrightarrow (\kappa, \mu)$ holds in the model of Theorem 1. Moreover, $[\lambda^+]^\kappa$ carries a κ-complete filter F such that F^+ is $(\lambda, < \mu)$-centered by an argument of Magidor [10]. See [6] for a related result.

5. Application of Theorem 1

As an application of Theorem 1, we give a quick proof of

Theorem 7. *Suppose κ_n is almost huge, $j_n : V \to M_n$ is a witness and $j_n(\kappa_n) = \kappa_{n+1}$ for every $n < \omega$. Then there is a generic extension in which $\kappa_n = \omega_{n+1}$ carries a strongly centered normal filter for every $n < \omega$.*

If there is a huge cardinal, then we get a sequence $\langle \kappa_n : n < \omega \rangle$ as in Theorem 7 by the standard argument. Compare the proof of [2, Theorem 2] and the following

Proof. Let P be the inverse limit of $\langle P_n : n < \omega \rangle$, where P_n is defined by recursion so that $P_0 = P(\omega, \kappa_0)$ and $P_{n+1} = P_n * \dot{P}(\kappa_n, \kappa_{n+1})$. We claim that P works.

By induction we show that P_{n+1} forces $\kappa_n = \omega_{n+1}$ to carry a strongly centered normal filter. The case $n = 0$ follows from Theorem 1. Let $n > 0$. Then $P_{n-1} \subset V_{\kappa_{n-1}}$ forces κ_n to be almost huge, as witnessed by an extension of j_n. Applying Theorem 1 in the generic extension by P_{n-1}, we have that $P_{n+1} = P_{n-1} * \dot{P}(\kappa_{n-1}, \kappa_n) * \dot{P}(\kappa_n, \kappa_{n+1})$ forces $\kappa_n = \kappa_{n-1}^+$ to carry a strongly centered normal filter, as desired.

By the standard argument we have $P \simeq P_{n+1} * \dot{Q}_{n+1}$, where $\dot{Q}_{n+1}$ is forced to be κ_{n+1}-closed. Thus a strongly centered normal filter on $\kappa_n = \omega_{n+1}$ in the generic extension by P_{n+1} remains so after forcing with the interpretation of $\dot{Q}_{n+1}$, as desired. □

References

[1] J. Cummings, *A model in which GCH holds at successors but fails at limits*, Trans. Amer. Math. Soc. 329 (1992) 1–39.

[2] M. Foreman, *More saturated ideals*, Cabal Seminar 79–81, pp. 1–27, Lecture Notes in Math. 1019, Springer, Berlin, 1983.

[3] M. Foreman, *Ideals and generic elementary embeddings*, Handbook of Set Theory, pp. 885–1147, Springer, Berlin, 2010.

[4] M. Foreman and R. Laver, *Some downwards transfer properties for $\aleph_2$*, Adv. Math. 67 (1988) 230–238.

[5] M. Foreman, M. Magidor and S. Shelah, *Martin's Maximum, saturated ideals, and non-regular ultrafilters. Part II*, Ann. Math. 127 (1988) 521–545.

[6] M. Huberich, *Large ideals on small cardinals*, Ann. Pure Appl. Logic 64 (1993) 241–271.

[7] A. Kanamori, The Higher Infinite, Springer Monogr. Math., Springer, Berlin, 2003.

[8] K. Kunen, *Saturated ideals*, J. Symbolic Logic 43 (1978) 65–76.

[9] R. Laver, *An $(\aleph_2, \aleph_2, \aleph_0)$-saturated ideal on ω_1*, Logic Colloquium '80, pp. 173–180, Stud. Logic Foundations Math. 108, North-Holland, Amsterdam, 1982.

[10] M. Magidor, *On the existence of nonregular ultrafilters and the cardinality of ultrapowers*, Trans. Amer. Math. Soc. 249 (1979) 97–111.

[11] M. Shioya, *Easton collapses and a strongly saturated filter*, Arch. Math. Logic 59 (2020) 1027–1036.

[12] M. Shioya, *An explicit collapse for a dense filter*, Israel J. Math., to appear.

Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571 Japan.
E-mail address: shioya@math.tsukuba.ac.jp