MAXIMAL L^1-REGULARITY AND FREE BOUNDARY PROBLEMS FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS IN CRITICAL SPACES

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Abstract. Time-dependent free surface problem for the incompressible Navier–Stokes equations which describes the motion of viscous incompressible fluid nearly half-space are considered. We obtain global well-posedness of the problem for a small initial data in scale invariant critical Besov spaces. Our proof is based on maximal L^1-regularity of the corresponding Stokes problem in the half-space and special structures of the quasi-linear term appearing from the Lagrangian transform of the coordinate.

1. Introduction and Main Results

We consider a time-dependent free surface problem for the Navier–Stokes equations which describes the motion of viscous incompressible fluid. The domain $\Omega_t \subset \mathbb{R}^n$ $(n \geq 2)$ is occupied by the fluid and the velocity of fluid $\bar{u}(t, y)$ and the pressure $\bar{p}(t, y)$ for $y \in \Omega_t$ satisfy the incompressible Navier–Stokes equations:

$$\begin{cases} \partial_t \bar{u} + (\bar{u} \cdot \nabla)\bar{u} - \operatorname{div} T(\bar{u}, \bar{p}) = 0, & t > 0, \quad y \in \Omega_t, \\ \operatorname{div} \bar{u} = 0, & t > 0, \quad y \in \Omega_t, \\ T(\bar{u}, \bar{p})\nu_t = 0, & t > 0, \quad y \in \partial\Omega_t, \\ \bar{u}(0, y) = u_0(y), & \qquad\quad y \in \Omega_0. \end{cases} \tag{1.1}$$

Here, $\partial\Omega_t$ denotes the boundary of Ω_t, $\nu_t = \nu_t(y)$ is the unit outward normal at a point $y \in \partial\Omega_t$, $T(\bar{u}, \bar{p})$ is the stress tensor defined by $T(\bar{u}, \bar{p}) = (\nabla\bar{u} + (\nabla\bar{u})^{\mathsf{T}}) - \bar{p}I$, where I is the $n \times n$ identity matrix, $(\nabla_y \bar{u})_{ij} = \frac{\partial \bar{u}_j}{\partial y_i}$, and $(\nabla\bar{u})^{\mathsf{T}}$ denotes the transposed matrix of $\nabla\bar{u}$. u_0 is the given initial velocity. In our setting (1.1), we do not take into account the effect from the gravity force or the surface tension.

Free boundary problems for the incompressible fluids are considered by many authors. The pioneer work was done by Solonnikov [64], he established local well-posedness of (1.1) whose initial state Ω is a bounded domain in the frame work of Hölder spaces $C^{2+\alpha,1+\alpha/2}$ with $\alpha \in (\frac{1}{2}, 1)$. Solonnikov also proved global well-posedness of (1.1) in the class of Sobolev space $W_p^{2,1}$ with $n < p < \infty$ when $n = 2$, 3, where surface tension is excluded. When initial state is bounded and the surface tension is excluded, Mucha–Zajaczkowski considered the case where the self-gravitational force exists, they proved in [39], [40] the local in time unique solvability in $W_p^{2,1}$ with $n = 3$ and $3 < p < \infty$ for arbitrary initial data. Shibata–Shimizu [61], [62] developed the L^p-theory for the problem and showed global well-posedness of (1.1) in the class of Sobolev space $W_{q,p}^{2,1}$ with $n < q < \infty$ and $2 < p < \infty$ when $n \geq 2$.

In the case when initial state is bounded and the surface tension is included, Solonnikov proved the global in time solvability in $W_2^{2+\alpha,1+\alpha/2}$ with $1/2 < \alpha < 1$ provided that initial data are sufficiently small and the

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initial domain is sufficiently close to a ball. There are many other contributions in the case when the effect of surface tension is included, for instance [33], [37], [49], [63], [65]-[69] and references therein, we do not get involved with the case because in this paper we consider without surface tension case.

Another typical free boundary problem describes the motion of a fluid which occupies a semi-infinite domain between the moving upper surface and a fixed bottom. Beale [7], [8] considered the free surface problem in a three dimensional region with a bottom, in the L^2-based Bessel potential spaces $H_2^{r/2,r}$ where $3 < r < 7/2$. His problem (called as the ocean problem) has a similar setting of the following Lagrange coordinate equations and showed that the global in time solvability for small initial data in L_2 Bessel-potential space setting. Since the ocean problem has a finite depth, however, the spectral property for the linearized problem is different from the case for a domain close to the half-space. Prüss–Simonett [53], [54] proved local well-posedness of (1.1) whose initial state Ω_0 is close to the half-space $\mathbb{R}^n_+$ in the class of Sobolev space $W_p^{2,1}$ with $p > n+2$. There are many other contributions on this direction, for instance, [1], [9], [10], [19]-[21], [29], [30], [39]-[42], [53]-[55], [61], [70]-[72] and reference therein.

Recently, Shibata [57], [58] considered local and global well-posedness on general unbounded domain in the space $W_{q,p}^{2,1}$ with $n < q < \infty$ and $2 < p < \infty$.

The incompressible Navier–Stokes equations are invariant under the following scaling: For all $\lambda > 0$,

$$\begin{cases} \bar{u}(t,y) \to \bar{u}_\lambda(t,y) \equiv \lambda \bar{u}(\lambda^2 t, \lambda y), \\ \bar{p}(t,y) \to \bar{p}_\lambda(t,y) \equiv \lambda^2 \bar{p}(\lambda^2 t, \lambda y). \end{cases}$$

Subsequently, it is well-known that the Cauchy problem of the Navier–Stokes equations can be solved globally in time in the invariant Bochner–Sobolev space $L^\rho\big(\mathbb{R}_+; \dot{H}^s_p(\mathbb{R}^n;\mathbb{R}^n)\big)$

$$\frac{2}{\rho} + \frac{n}{p} = 1 + s, \tag{1.2}$$

which is observed in the celebrated result by Fujita–Kato [26] (see also Prodi [52] and Serrin [56] for the relation between regularity of solutions and the scaling invariance). When we choose $\rho = \infty$, we obtain $s = -1 + n/p$ by (1.2), and the critical class at $s = 0$ is given, in particular, by $L^\infty(0,T;L^n(\mathbb{R}^n))$, where Kato [32] considered global well-posedness of the Cauchy problem. Such a critical setting for the Cauchy problem is considered by several authors in the framework of the scaling critical Besov spaces $\dot{B}^{-1+n/p}_{p,\sigma}(\mathbb{R}^n)$, where $1 \le p < \infty$ and $1 \le \sigma \le \infty$ ([4], [13], [14], [15], [34]). Meanwhile, it is proved ill-posedness of the problem in [12], [77], [80], namely the continuous dependence on the initial data in the classes $u_0 \in \dot{B}^{-1}_{\infty,\sigma}(\mathbb{R}^n)$, $1 \le \sigma \le \infty$ breaks down. In view of those of well-posedness results to the Cauchy problem, it is natural to ask if the free surface problem can be solvable in such a scaling critical function class. Our main motivation is to consider the free surface problem (1.1) near the half-space $\mathbb{R}^n_+$ in the scaling critical function space.

In this paper, we show global in time well-posedness of the Lagrangian transformed problem for (1.1) under small data in the scaling critical Besov space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ for all $n \le p < 2n-1$, via maximal L^1-regularity of the linearized problem associated with (1.1). As far as the authors know, there is almost no result of global well-posedness to (1.1) in the scale critical space whose initial state Ω_0 is an unbounded domain except the recent result due to Danchin–Hieber–Mucha–Tolksdorf [18]. They consider the analogous problem in the scaling critical Besov spaces for $n-1 < p < n$.

Let the half Euclidean space and its boundary be denoted by

$$\begin{aligned} \mathbb{R}^n_+ \equiv& \{(x', x_n);\ x' \in \mathbb{R}^{n-1},\ x_n > 0\}, \\ \partial\mathbb{R}^n_+ \equiv& \mathbb{R}^{n-1} \times \{0\} = \{(x', x_n);\ x' \in \mathbb{R}^{n-1},\ x_n = 0\}. \end{aligned}$$

We also set $\mathbb{R}^n_-$ as the negative part of $\mathbb{R}^n$, i.e., $\mathbb{R}^n = \mathbb{R}^n_+ \cup \partial\mathbb{R}^n_+ \cup \mathbb{R}^n_-$. Aside from the dynamical boundary condition, a further kinematic condition for the free surface is satisfied which gives $\partial\Omega_t$ as a set of points $y = y(t,x)$, $x \in \partial\Omega_0 = \partial\mathbb{R}^n_+$, where $y(t,x)$ is the solution of the Cauchy problem:

$$\frac{dy}{dt} = \bar{u}\big(t, y(t)\big), \quad t > 0, \qquad y(0) = x. \tag{1.3}$$

Let the Euler coordinates $y \in \Omega_t$ be transformed into the Lagrangian coordinates $x \in \mathbb{R}^n_+$ connected by (1.3). If $\bar{u}(t,y)$ is Lipschitz continuous with respect to y, then (1.3) can be solved uniquely by

$$y(t,x) = x + \int_0^t \bar{u}\big(s, y(s,x)\big) ds. \tag{1.4}$$

By the kinematic condition of the original boundary Ω_t, it is described by the map $Y_{\bar{u}} : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n_+ \to (t,y) \in \Omega_t$, where Ω_t is given by

$$\Omega_t = Y_{\bar{u}}(t, \mathbb{R}^n_+) \equiv \big\{(t, y(t,x));\ t > 0,\ \ y(t,x) \text{ satisfies (1.4) and } x \in \mathbb{R}^n_+\big\}.$$

Setting

$$\begin{cases} u(t,x) \equiv \bar{u}(t, y(t,x)), \\ p(t,x) \equiv \bar{p}(t, y(t,x)), \end{cases}$$

and applying the Lagrangian coordinate to the original problem (1.1) yields that the system is transformed into the following form:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = F_u(u) + F_p(u,p), & t > 0, \quad x \in \mathbb{R}^n_+, \\ \qquad \operatorname{div} u = G_{\operatorname{div}}(u), & t > 0, \quad x \in \mathbb{R}^n_+, \\ \big(\nabla u + (\nabla u)^\mathsf{T} - pI)\big)\nu_n = H_u(u) + H_p(u,p), & t > 0, \quad x \in \partial\mathbb{R}^n_+, \\ \qquad u(0,x) = u_0(x), & \qquad\qquad x \in \mathbb{R}^n_+, \end{cases} \tag{1.5}$$

where $\nu_n = (0, \cdots, 0, -1)^\mathsf{T}$ denotes the outward normal [1] and the nonlinear terms of (1.5) are given by

$$F_u(u) \equiv \operatorname{div}\Big(J(Du)^{-1}\big(J(Du)^{-1}\big)^\mathsf{T}\nabla u - \nabla u\Big) = \Pi_u^{2n-2}\left(\int_0^t Du\, ds\right) D^2 u, \tag{1.6}$$

$$F_p(u,p) \equiv -\big(J(Du)^{-1} - I\big)^\mathsf{T}\nabla p = \Pi_p^{n-1}\left(\int_0^t Du\, ds\right)\nabla p, \tag{1.7}$$

$$\begin{aligned} G_{\operatorname{div}}(u) \equiv& -\operatorname{tr}\Big(\big(J(Du)^{-1} - I\big)^\mathsf{T}\nabla u\Big) = \operatorname{tr}\left(\Pi_{\operatorname{div}}^{n-1}\left(\int_0^t Du\, ds\right) Du\right) \\ =& \operatorname{div}\left(\Pi_{\operatorname{div}}^{n-1}\left(\int_0^t Du\, ds\right) u\right), \end{aligned} \tag{1.8}$$

$$\begin{aligned} H_u(u) \equiv& -\Big(\big(J\big(Du\big)^{-1}\big)^\mathsf{T}\nabla u + (\nabla u)^\mathsf{T} J\big(Du\big)^{-1}\Big)\big(J(Du)^{-1} - I\big)^\mathsf{T}\nu_n \\ & - \Big(\big(J\big(Du\big)^{-1} - I\big)^\mathsf{T}\nabla u + (\nabla u)^\mathsf{T}\big(J\big(Du\big)^{-1} - I\big)\Big)\nu_n \\ =& \Pi_{bu}^{2n-2}\left(\int_0^t Du\, ds\right) Du\,\nu_n, \end{aligned} \tag{1.9}$$

$$H_p(u,p) \equiv pI\big(J(Du)^{-1} - I\big)^\mathsf{T}\nu_n = \Pi_{bp}^{n-1}\left(\int_0^t Du\, ds\right) p\,\nu_n. \tag{1.10}$$

Here $J(Du)^{-1}$ denotes the inverse of the Jacobian matrix, I denotes the identity matrix, $(Du)_{ij} = \frac{\partial u_i}{\partial x_j}$ and $\Pi_*^m(d)$ denote m-th order polynomials of $d = (d_{jk})_{1 \le j,k \le n}$ with

$$d_{jk} = \Big(\int_0^t Du(s) ds\Big)_{jk} \equiv \int_0^t \partial_{x_k} u_j(s)\, ds$$

(here the notation $*$ stands for either u, p, div or bu, bp). Those polynomials are indeed given by the inverse matrix of the Jacobi matrix, $J(Du)^{-1}$ as follows:

$$\Pi_*^m\left(\int_0^t Du\, ds\right) = \sum_{\ell=1}^m \prod_{1 \le j_\ell, k_\ell \le n}^{\ell} \sigma_{k_\ell j_\ell}\left(\int_0^t \partial_{x_{k_\ell}} u_{j_\ell}(s,x) ds\right)$$

[1] Practically natural setting is $\Omega_0 = \mathbb{R}^n_-$ under the gravity circumstance.

with $\sigma_{k_\ell j_\ell}$ is either 1 or -1.

By using the Lagrangian transformation, the free surface problem (1.1) can be transformed into the initial-boundary value problem in $\mathbb{R}^n_+$ with the fixed boundary $\partial\mathbb{R}^n_+$ and the system is transformed into the quasi-linear parabolic equation (1.5) (see e.g., [67]).

Before stating our results, we define the Besov spaces and Lizorkin–Triebel spaces in the half-space. Since the global estimate requires the base space for spatial variable x in the homogeneous Besov space, we introduce the homogeneous Besov space over $\mathbb{R}^n_+$ (see for details, Bergh–Löfström [11], Lizorkin [36], Peetre [50], [51], Triebel [74]-[76]).

Definition (The Besov spaces). Let $s \in \mathbb{R}$, $1 \le p, \sigma \le \infty$. Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity for $x \in \mathbb{R}^n$, namely $\widehat{\phi}$ is the Fourier transform of a smooth radial function ϕ with $\widehat{\phi}(\xi) \ge 0$ and $\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n \mid 2^{-1} < |\xi| < 2\}$, and

$$\begin{aligned} &\widehat{\phi}_j(\xi) = \widehat{\phi}(2^{-j}\xi), \quad \sum_{j\in\mathbb{Z}} \widehat{\phi}_j(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}, \quad j \in \mathbb{Z} \\ \text{and} \quad &\widehat{\phi}_{\widehat{0}}(\xi) + \sum_{j\ge 1} \widehat{\phi}_j(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n, \end{aligned} \tag{1.11}$$

where $\widehat{\phi}_{\widehat{0}}(\xi) \equiv \widehat{\zeta}(|\xi|)$ with a low frequency cut-off

$$\widehat{\zeta}(r) = \begin{cases} 1, & 0 \le r < 1, \\ \text{decreasing in} & 1 \le r < 2, \\ 0, & 2 \le r. \end{cases} \tag{1.12}$$

For $s \in \mathbb{R}$ and $1 \le p, \sigma \le \infty$, let $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ be the homogeneous Besov space with norm

$$\|\tilde{f}\|_{\dot{B}^s_{p,\sigma}} \equiv \begin{cases} \Big(\displaystyle\sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * \tilde{f}\|_p^\sigma\Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \displaystyle\sup_{j\in\mathbb{Z}} 2^{sj}\|\phi_j * \tilde{f}\|_p, & \sigma = \infty, \end{cases}$$

where $\phi_j * f$ stands for the convolution operation with a constant correction $c_n = (2\pi)^{-n/2}$ given by

$$\phi_j * f = c_n \int_{\mathbb{R}^n} \phi_j(x-y) f(y)dy \tag{1.13}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ and its standard extension to $f \in \mathcal{S}'(\mathbb{R}^n)$. In what follows, we always regard this correction of the constant against the convolution operations for all kinds of the Littlewood–Paley decompositions.

Also let $B^s_{p,\sigma}(\mathbb{R}^n)$ be the inhomogeneous Besov space with norm

$$\|\tilde{f}\|_{B^s_{p,\sigma}} \equiv \begin{cases} \Big(\|\phi_{\hat{0}} * \tilde{f}\|_p + \displaystyle\sum_{j\in\mathbb{Z}} 2^{s\sigma j}\|\phi_j * \tilde{f}\|_p^\sigma\Big)^{1/\sigma}, & 1 \le \sigma < \infty, \\ \|\phi_{\hat{0}} * \tilde{f}\|_p + \displaystyle\sup_{j\in\mathbb{Z}} 2^{sj}\|\phi_j * \tilde{f}\|_p, & \sigma = \infty. \end{cases}$$

We define the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ as the set of all the restriction f of the distribution $\tilde{f} \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$, i.e., $f = \tilde{f}\big|_{\mathbb{R}^n_+}$ with

$$\|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)} \equiv \inf\Big\{\|\tilde{f}\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} < \infty; \quad \tilde{f} = \sum_{j\in\mathbb{Z}} \phi_j * \tilde{f} \text{ in } \mathcal{S}', \quad f = \tilde{f}\big|_{\mathbb{R}^n_+}\Big\}. \tag{1.14}$$

Analogously we define the inhomogeneous Besov space $B^s_{p,\sigma}(\mathbb{R}^n_+)$ in a similar manner.

Definition (The Bochner–Lizorkin–Triebel spaces). Let $s \ge 0$, $1 \le p, \sigma \le \infty$ and X be a Banach space with the norm $\|\cdot\|_X$. Let $\{\psi_k\}_{k\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity for $t \in \mathbb{R}$. For $s \in \mathbb{R}$

and $1 \le p < \infty$, $\dot{F}^s_{p,\sigma}(\mathbb{R};X)$ be the Bochner–Lizorkin–Triebel space with norm

$$\|\tilde{f}\|_{\dot{F}^s_{p,\sigma}(\mathbb{R};X)} \equiv \begin{cases} \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{s\sigma k} \|\psi_k * \tilde{f}(t,\cdot)\|_X^\sigma \Big)^{1/\sigma} \Big\|_{L^p(\mathbb{R})}, & 1 \le \sigma < \infty, \\ \Big\| \sup_{k\in\mathbb{Z}} 2^{sk} \|\psi_k * \tilde{f}(t,\cdot)\|_X \Big\|_{L^p(\mathbb{R})}, & \sigma = \infty. \end{cases}$$

Analogously above, we define the Bochner–Lizorkin–Triebel spaces $\dot{F}^s_{p,\sigma}(I;X)$ as the set of all the restriction f of a distribution $\tilde{f} \in \dot{F}^s_{p,\sigma}(\mathbb{R};X)$ i.e., $f = \tilde{f}\big|_I$ on X with

$$\|f\|_{\dot{F}^s_{p,\sigma}(I;X)} \equiv \inf \Big\{ \|\tilde{f}\|_{\dot{F}^s_{p,\sigma}(\mathbb{R};X)} < \infty; \quad f = \tilde{f}\big|_I \Big\},$$

where $I = (0,T)$ denotes the time interval. We denote $\mathbb{R}_+ = (0,\infty)$ as the half real line and $\overline{\mathbb{R}_+} = [0,\infty)$ as its closure. We note that all those homogeneous spaces are understood as the Banach spaces by introducing the quotient spaces identifying any difference of polynomials.

Let $C_b(I;X)$ be a set of all bounded continuous functions from an interval I to a Banach space X. We also use the notation $C_v(\mathbb{R}^n_+)$ for a set of all continuous functions vanishing at $|x| \to \infty$. Obviously $C_v(\mathbb{R}^n_+) \subset C_b(\mathbb{R}^n_+)$.

Theorem 1.1 (Global well-posedness under the Lagrangian coordinates)**.** *Let $n \le p < 2n-1$. There exists small $\varepsilon_0 > 0$ such that if the initial data $u_0 \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with $\operatorname{div} u_0 = 0$ in the sense of distribution satisfying*

$$\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \le \varepsilon_0, \tag{1.15}$$

then (1.5) *admits a unique global solution*

$$\begin{aligned} &u \in C_b(\overline{\mathbb{R}_+}; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \\ &\Delta u,\ \nabla p \in L^1(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)), \\ &p|_{x_n=0} \in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})) \end{aligned}$$

with the estimate

$$\begin{aligned} \|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} &+ \|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\ &+ \big\|p|_{x_n=0}\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \le \varepsilon_1, \end{aligned} \tag{1.16}$$

where D^2u denotes all the second order derivatives of u by x and $\varepsilon_1 = \varepsilon_1(n,p,\varepsilon_0)$ is a constant.

Corollary 1.2 (Global well-posedness)**.** *Let $n \le p < 2n-1$. For the same ε_0 in Theorem 1.1 and $u_0 \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with $\operatorname{div} u_0 = 0$ in the sense of distribution satisfying* (1.15)*, let (u,p) be the global solution of* (1.5) *obtained in Theorem 1.1. Then the pull-back $(\bar{u},\bar{p})$ of (u,p) with the estimate* (1.16) *satisfies* (1.1)*.*

Concerning the half-space problem, Danchin–Mucha [19] proved well-posedness of the Cauchy–Dirichlet problem of the density-dependent incompressible Navier–Stokes equations with the 0-Dirichlet boundary data. The result there is also applicable for the incompressible Navier–Stokes equations in the scaling invariant Besov spaces $p = n$, namely in $\dot{B}^0_{n,1}(\mathbb{R}^n_+)$.

Let us mention on the two results between Danchin–Hieber–Mucha–Tolksdorf [18] and ours. Their result based on the abstract interpolation spaces based on the original idea that goes back to Da Prato–Grisvard [22] and based on the result due to Danchin–Mucha [19] and [20], where the authors considered the 0-Neumann boundary condition for the linearized system of the Stokes equation. Our approach is very much different from theirs. We handle the boundary potential for non-stress boundary condition directly in the homogeneous Besov spaces and as a result, our result covers the initial data as a class of distributions (negative indices of regularity in the scaling critical homogeneous Besov space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ with $n \le p < 2n-1$), while they treats the function case $n-1 < p < n$ for $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ case in [18]. Theorem 1.1 (and hence Corollary 1.2) is the first result for treating the scaling invariant distribution as an initial data for the free boundary value problem, as far as the authors can find.

Our proof of Theorem 1.1 is heavily depending on the end-point estimate of maximal regularity for the initial-boundary value problem of the Stokes system in the half-space $\mathbb{R}^n_+$. Many of the existence results are related to the spectral analysis for the linearized equation and derive the decay property for the linearized Stokes equations. On the other hand, maximal regularity for the parabolic equation gives a suitable estimate for treating the quasi-linear terms effectively ([3], [5]). In contrast with those results, our method is a direct application of maximal L^1-regularity for the half time line $\mathbb{R}_+$ to treat the system under the Lagrange transformations. This method enables us to handle main terms appearing the quasi-linear perturbations (1.6)–(1.10) directly and we may treat them globally in the transformed problem (1.5). Namely, to obtain global well-posedness of (1.5), it is required to treat the terms with

$$\tilde{d}_{kj} \equiv \delta_{k,j} + \lim_{t\to\infty} \int_0^t \partial_{x_k} u_j(s,x)ds, \qquad k,j = 1,2,\cdots,n.$$

We then establish maximal L^1-regularity for the transformed Stokes system via maximal regularity estimate for the initial-boundary value problem of the heat equations obtained in the previous work of authors [48] (see for its announce [45]). Such argument was developed by Danchin [16], [17] for treating global well-posedness for the Cauchy problem of the compressible or incompressible density dependent Navier–Stokes equations. The main difference here is to treat the boundary inhomogeneous terms appearing in $H_b(u)$ by maximal L^1-regularity and usage of the sharp trace estimate of the boundary terms. Such an estimate is available for analyzing the potential expression of the pressure term p for the Stokes system with the free surface boundary condition obtained in [61]. Maximal regularity and its sharpness is obtained by establishing the almost orthogonal estimates for the pressure potential and the Littlewood–Paley space-time decompositions of unity that defines our sharp function class of the well-posedness.

In order to enlarge the solution class into the critical Besov spaces, the divergence free condition is crucial. In particular to enlarge the class for the bilinear estimate remains valid, the multiple divergence-rotation-free structure is another crucial point (cf. [46]). This nonlinear structure was partially observed by Solonnikov [64] and Shibata–Shimizu [59] for treating the terms in the Sobolev spaces. However in order to apply the bilinear estimate in the critical Besov space, we need to ensure such a special structure for each decomposition steps of sub-matrix expansion of the inverse of Jacobi matrix. In this stage, we show that *a divergence-curl free structure* (div-curl structure, in short) holds for each step of sub-cofactor of expansion involving the *null-Lagrangian structure* (cf. Evans [25]). This was shown in [46] for the initial value problem for the Lagrangian coordinate case. We develop the analogous estimate and establish the multiple Besov estimate in the half-spaces. It is well-known that the convection term $\bar{u}\cdot\nabla\bar{u}$ maintains the div-curl structure and it helps to enlarge the solution class. Although the convection term vanishes after the transformation into the Lagrangian coordinate, all the nonlinear terms inherit the div-curl structure from the divergence free condition and then the solution class can be reach the critical homogeneous Besov space.

We should like to notice that regularity for the solution obtained in Theorem 1.1 is weaker than known results, we do not assume the compatibility conditions on the initial and boundary data. The regularity of solution ensures us that the velocity fields has a sufficient regularity $\nabla u \in L^1((0,\infty); \dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+))$ so that the Lagrange transformation (1.4) is uniquely determined and the inverse of the transformation has meaningful by $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+) \subset C_v(\mathbb{R}^n_+)$. Thus the original problem (1.1) is solvable.

The rest of this paper is organized as follows. We present a solution formula of the linear problem of (1.1) in the next section. Maximal L^1-regularity of the Stokes system (Theorem 2.1 stated in Section 2) is a key estimate for our argument. Section 3 is devoted to prove almost orthogonality between the pressure potential and the space-time Littlewood–Paley dyadic decomposition, which is crucial to prove maximal L^1-regularity of the Stokes system. Using the almost orthogonal estimates, we show maximal regularity for the Stokes system in Section 4. The bilinear estimates as well as the div-curl lemma are discussed in Section 5, both of them are necessary to treat nonlinear equations. Finally we devote to the proof of Theorem 1.1 in Section 6. Some supplementary estimates are described in the Appendix.

Throughout this paper we use the following notations. For $x \in \mathbb{R}^n$, $\langle x\rangle \equiv (1+|x|^2)^{1/2}$. The transpose of a matrix A is denoted by A^{T}. The Fourier and the inverse Fourier transforms are defined with $c_n = (2\pi)^{-n/2}$

by

$$\widehat{f}(\xi) = \mathcal{F}[f](\xi) \equiv c_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \mathcal{F}^{-1}[f](x) \equiv c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)d\xi.$$

For any functions $f = f(t, x', x_n)$ and $g = g(t, x', x_n)$, $f \underset{(t)}{*} g$, $f \underset{(t,x')}{*} g$ and $f \underset{(x_n)}{*} g$ stand for the convolution between f and g with respect to the variable indicated under $*$, respectively. If both f and g are vector field functions, $f \underset{(t,x')}{\cdot *} g$ denotes the convolution in x' as well as the inner-product of f and g, i.e.,

$$f \underset{(t,x')}{\cdot *} g = \sum_{\ell=1}^{n-1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f_\ell(t-s, x'-y') g_\ell(s, y') dy' ds. \tag{1.17}$$

In the summation $\sum_{k\in\mathbb{Z}}$, the parameter k runs for all integers $k \in \mathbb{Z}$ and for $\sum_{k\le j}$, k runs for all integers less than or equal to $j \in \mathbb{Z}$. We denote $\mathcal{D}'(\mathbb{R}^n_+)$ the distribution over $\mathbb{R}^n_+$ and the norm of the Lebesgue space $L^p(\mathbb{R}^{n-1})$ with $x' \in \mathbb{R}^{n-1}$ variable by $\|\cdot\|_{L^p_{x'}}$. In the norm for the Bochner spaces on $\dot{F}^s_{p,\rho}\big(I; X(\mathbb{R}^{n-1})\big)$ we use

$$\|f\|_{\dot{F}^s_{p,\rho}(I;X)} = \|f\|_{\dot{F}^s_{p,\rho}(I;X(\mathbb{R}^{n-1}))}$$

unless it may cause any confusion. For the Besov spaces, we abbreviate $\mathbb{R}^n$ for $\dot{B}^s_{p,\sigma} = \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ and its norm $\|\cdot\|_{\dot{B}^s_{p,\sigma}}$. For $a \in \mathbb{R}^n$, we denote $B_R(a)$ as the open ball centered at a with its radius $R > 0$. We also denote the compliment of $B_R(0)$ by B^c_R. $\Gamma(\cdot)$ denotes the Gamma function. Various constants are simply denoted by C unless otherwise stated.

2. Maximal L^1-regularity for the Stokes equation in the half-space

2.1. Maximal L^1-regularity for the Stokes flow. Maximal L^1-regularity in the half-space is considered in [45], [48] (see also Danchin–Mucha [19] for 0-boundary data). Here we develop maximal L^1-regularity for the Stokes system corresponding (1.1) and (1.5) with inhomogeneous free stress boundary condition:

$$\begin{cases} \partial_t u - \Delta u + \nabla p = f, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \quad \operatorname{div} u = g, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \big(\nabla u + (\nabla u)^\mathsf{T} - pI\big)\, \nu_n = h, & t > 0, \quad x \in \partial\mathbb{R}^n_+, \\ \quad u(0, x) = u_0(x), & \qquad x \in \mathbb{R}^n_+, \end{cases} \tag{2.1}$$

where u_0, f, g and h are given initial, external and boundary data, respectively and $\nu_n = (0, 0 \cdots, 0, -1)^\mathsf{T}$ denotes the outer normal on $\partial\mathbb{R}^n_+$. The following theorem is the main result of this section.

Theorem 2.1 (Maximal L^1-regularity)**.** *Let $1 < p < \infty$ and $-1 + 1/p < s \le 0$. The problem* (2.1) *admits a unique solution (u, p) with*

$$\begin{aligned} & u \in C_b(\overline{\mathbb{R}_+}; \dot{B}^s_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ & \Delta u,\ \nabla p \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ & p\big|_{x_n=0} \in \dot{F}^{\frac12 - \frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})) \end{aligned}$$

if and only if the data in (2.1) *satisfy*

$$\begin{aligned} & u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n_+),\ \operatorname{div} u_0 = g\big|_{t=0} \text{in } \mathcal{D}'(\mathbb{R}^n_+), \quad f \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ & \nabla g \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \nabla(-\Delta)^{-1} g \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ & h \in \dot{F}^{\frac12 - \frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})), \end{aligned}$$

where $(-\Delta)^{-1}g$ *is given by* $G * \tilde g|_{\mathbb{R}^n_+}$ *with* $\tilde g$ *as the even extension of* g *(see* (2.4) *blow). Besides the solution* (u,p) *satisfies the following estimate for some constant* $C_M > 0$ *depending only on* p, s *and* n

$$
\begin{aligned}
&\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\quad + \|p|_{x_n=0}\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))} + \|p|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C_M\Big(\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)} + \|f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\qquad + \|\nabla g\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_t \nabla(-\Delta)^{-1} g\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\qquad\qquad + \|h\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))} + \|h\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}
\tag{2.2}
$$

The general theory of maximal regularity for the parabolic type partial differential equation is extensively developed in the UMD Banach space (see, for instance, [23], [24], [27], [28], [31], [35], [43], [54], [60], [78], [79], [81]). However the end-point exponent is normally excluded in the general theory. If the space is restricted the homogeneous Besov space or Fourier transformed measures, one can see the end-point estimate holds as is seen in [6], [16], [20], [27], [44], [45], [48].

To establish maximal regularity of the half-space problem (2.1), we reduce the problem (2.1) into the several partial components of the data and reduce the problem into the inhomogeneous problem with only boundary data. At first we remove the divergence data. Introducing the even extension of divergence data g with respect to x_n;

$$
\widetilde{g}(t,x) = \begin{cases} g(t,x',x_n), & x_n > 0,\\ g(t,x',-x_n), & x_n < 0 \end{cases}
$$

for $x' = (x_1, x_2, \cdots, x_{n-1})$, we consider the problem

$$
\begin{cases} -\Delta\phi = \widetilde{g}, & t>0,\ \ x\in\mathbb{R}^n,\\ \phi\Big|_{x_n=0} = 0, & t>0,\ \ x'\in\mathbb{R}^{n-1}. \end{cases}
\tag{2.3}
$$

One of the solution of (2.3) is given by the Newtonian potential $\phi = (-\Delta)^{-1}\widetilde{g} \equiv G * \widetilde{g}$ with the Newtonian kernel G in $\mathbb{R}^n$;

$$
G(x) = \begin{cases} \frac{1}{2\pi}\log|x|^{-1}, & n=2,\\ ((n-2)\omega_n)^{-1}|x|^{-(n-2)}, & n\ge 3, \end{cases} \qquad \omega_n = \frac{2\pi^{\frac n2}}{\Gamma\big(\frac n2\big)}.
\tag{2.4}
$$

Then the gradient of potential $\nabla\phi$ satisfies the estimate for $1<p<\infty$ and $-1+1/p<s<1/p$

$$
\begin{cases} \|\nabla^3\phi\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} \le C\|\nabla g\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))},\\ \|\partial_t\nabla\phi\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))} \le C\|\partial_t\nabla(-\Delta)^{-1}g\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}. \end{cases}
\tag{2.5}
$$

Indeed the corresponding estimate to (2.5) in $\mathbb{R}^n$ follows directly from the elliptic estimate for the Poisson equation (or the Bernstein type estimate) and hence the estimate (2.5) in the half-space naturally follows from the definition of the Besov space in $\mathbb{R}^n_+$. Setting $w = u + \nabla\phi|_{\mathbb{R}^n_+}$, the pair of functions (w,p) satisfy the equations

$$
\begin{cases} \partial_t w - \Delta w + \nabla p = f + \big(\partial_t\nabla\phi - \Delta\nabla\phi\big)\big|_{x_n>0}, & t>0,\ \ x\in\mathbb{R}^n_+,\\ \qquad \operatorname{div} w = 0, & t>0,\ \ x\in\mathbb{R}^n_+,\\ \big(\nabla w + (\nabla w)^{\mathsf T} - pI\big)\nu_n = h + \big(\nabla^2\phi + (\nabla^2\phi)^{\mathsf T}\big)\nu_n, & t>0,\ \ x\in\partial\mathbb{R}^n_+,\\ \quad w(0,x) = u_0(x) + \nabla\phi(0,x)\big|_{x_n>0}, & x\in\mathbb{R}^n_+, \end{cases}
\tag{2.6}
$$

where ν_n denotes the outer normal to $\partial\mathbb{R}^n_+$.

In order to exclude the external and initial data, we extend them into $\mathbb{R}^n$, more precisely, we extend f_j $(1\le j\le n-1)$ by odd functions and for the n-th component f_n, we employ the even extension (we write them f_j^o and f_n^e, respectively) and set $\overline{f} = (f_1^o, \cdots f_{n-1}^o, f_n^e)^\mathsf{T}$. For the initial data u_0, we also employ the same extension with respect to x_n and write it $\overline{u_0}$ and set

$$\begin{cases} \qquad \widetilde{f} = \overline{f} + (\partial_t \nabla\phi - \Delta\nabla\phi), & t > 0, x \in \mathbb{R}^n, \\ \widetilde{u_0}(x) = \overline{u_0}(x) + \nabla\phi(0,x), & x \in \mathbb{R}^n, \end{cases} \tag{2.7}$$

and we consider the Cauchy problem:

$$\begin{cases} \partial_t \widetilde{u} - \Delta\widetilde{u} + \nabla\widetilde{p} = \widetilde{f} & t > 0, \quad x \in \mathbb{R}^n, \\ \qquad \operatorname{div} \widetilde{u} = 0, & t > 0, \quad x \in \mathbb{R}^n, \\ \qquad \widetilde{u}(0,x) = \widetilde{u_0}(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.8}$$

Then it is known that the solution $(\widetilde{u}, \widetilde{p})$ of the equation (2.8) satisfies maximal L^1-regularity

$$\begin{aligned} \|\partial_t \widetilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))} &+ \|\nabla^2 \widetilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))} + \|\nabla\widetilde{p}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))} \\ &\le C_M \Big(\|\widetilde{u_0}\|_{\dot{B}^s_{p,1}(\mathbb{R}^n)} + \|\widetilde{f}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))} \Big) \end{aligned} \tag{2.9}$$

for any $-1/p' < s < 1/p$ and $1 < p < \infty$ (see Danchin–Mucha [19] and Ogawa–Shimizu [44], see also [81]). By restricting the solution $(\widetilde{u}, \widetilde{p})$ over the half-space $\mathbb{R}^n_+$ (and we denote them in the same notation) we directly obtain from (2.5) and (2.9) that

$$\begin{aligned} \|\partial_t \widetilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} &+ \|\nabla^2 \widetilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla\widetilde{p}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\le C_M \Big(\|\widetilde{u_0}\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|\widetilde{f}\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \Big) \\ &\le C_M \Big(\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\ &\qquad + \|\nabla g\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\partial_t \nabla(-\Delta)^{-1} g\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \Big), \end{aligned} \tag{2.10}$$

where the inverse operator $(-\Delta)^{-1}$ is given by the solution operator to the elliptic problem (2.3) and it is realized by the Green's function (2.4).

Finally we consider the difference between the solutions (w, p) to (2.6) and $(\widetilde{u}, \widetilde{p})$ to (2.8) restricted in $\mathbb{R}^n_+$. Letting $v = w - \widetilde{u}|_{x_n>0} \equiv u + \nabla\phi|_{x_n>0} - \widetilde{u}|_{x_n>0}$ and $q = p - \widetilde{p}|_{x_n>0}$ and we reduce the original problem into the following initial boundary value problem for (v, q):

$$\begin{cases} \partial_t v - \Delta v + \nabla q = 0, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \qquad \operatorname{div} v = 0, & t > 0, \quad x \in \mathbb{R}^n_+, \\ \quad \big(\nabla v + (\nabla v)^\mathsf{T} - qI\big)\, \nu_n = H, & t > 0, \quad x \in \partial\mathbb{R}^n_+, \\ \qquad v(0,x) = 0, & x \in \mathbb{R}^n_+, \end{cases} \tag{2.11}$$

where we set

$$\begin{aligned} H \equiv& \widetilde{h} - \Big(\nabla\widetilde{u} + (\nabla\widetilde{u})^\mathsf{T} - \widetilde{p}I \Big) \nu_n \\ =& h - \big(\nabla^2\phi + (\nabla^2\phi)^\mathsf{T}\big)\nu_n - \Big(\nabla\widetilde{u} + (\nabla\widetilde{u})^\mathsf{T} - \widetilde{p}I \Big) \nu_n. \end{aligned} \tag{2.12}$$

In order to prove Theorem 2.1, it is essential to show maximal L^1-regularity for (2.11).

Theorem 2.2. *Let $-1 + 1/p < s \le 0$ and $1 < p < \infty$. The problem (2.11) admits a unique solution*

$$\begin{aligned} & v \in C_b(\overline{\mathbb{R}_+}; \dot{B}^s_{p,1}(\mathbb{R}^n_+)) \cap \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ & \Delta v,\ \nabla q \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \\ & q|_{x_n=0} \in \dot{F}^{\frac12 - \frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}, (\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})) \end{aligned} \tag{2.13}$$

if and only if the data in (2.11) satisfy

$$H \in \dot{F}^{\frac12 - \frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})). \tag{2.14}$$

Besides the solution (v,q) satisfies the following estimate for some constant $C_M>0$ depending only on p, s and n

$$
\begin{aligned}
&\|\partial_t v\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\|D^2 v\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\
&\quad+\|q|_{x_n=0}\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}+\|q|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\Big(\|H\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}+\|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}
\tag{2.15}
$$

Once we obtain corresponding maximal regularity Theorem 2.2 to the solution (v,q) for (2.11), we may prove maximal L^1-regularity for the original Stokes system (2.1) combining with those estimates (2.5) and (2.10) and the relation

$$
\begin{cases}
u(t,x)=\widetilde{u}(t,x)+v(t,x)+\nabla\phi(t,x), & t>0,\quad x\in\mathbb{R}^n_+,\\
p(t,x)=\widetilde{p}(t,x)+q(t,x), & t>0,\quad x\in\mathbb{R}^n_+,
\end{cases}
\tag{2.16}
$$

as well as the following trace estimate (see Appendix below, cf. [45], [48]):

$$
\begin{aligned}
&\Big\|\Big(\nabla\widetilde{u}+(\nabla\widetilde{u})^{\mathsf T}\Big)\nu_n\Big\|_{\dot{F}^{\frac12-\frac1{2p}}_{1,1}(I;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}+\Big\|\Big(\nabla\widetilde{u}+(\nabla\widetilde{u})^{\mathsf T}\Big)\nu_n\Big\|_{L^1(I;\dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\Big(\|\partial_t\widetilde{u}\|_{L^1(I;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\|\Delta\widetilde{u}\|_{L^1(I;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\Big).
\end{aligned}
\tag{2.17}
$$

2.2. **Solution formula of the Stokes equation.** We construct the solution formula of (2.1) according to the method by Shibata–Shimizu [59] and [62].

Let $H=H(t,x')\equiv(H'(t,x'),H_n(t,x'))$ be the boundary data extended into $t<0$ by the zero extension. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and the inverse Fourier transform with respect to $x'\in\mathbb{R}^{n-1}$, and $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and the inverse Laplace transform for t and τ, respectively. Namely

$$
\begin{aligned}
\mathcal{L}\widehat{f}(\lambda,\xi',x_n)&=(2\pi)^{-\frac n2}\int_{\mathbb{R}_+}\int_{\mathbb{R}^{n-1}}e^{-\lambda t-ix'\cdot\xi'}f(t,x',x_n)\,dx'dt,\\
\mathcal{L}^{-1}\mathcal{F}^{-1}f(t,x',x_n)&=(2\pi)^{-\frac{n-1}{2}}\frac{1}{2\pi i}\int_\Gamma\int_{\mathbb{R}^{n-1}}e^{\lambda t+ix'\cdot\xi'}f(\lambda,\xi',x_n)\,d\xi'd\lambda,
\end{aligned}
$$

where Γ denotes an integral path given for some $\gamma>0$ by $\Gamma=\{\lambda=\gamma+i\tau;\,\tau\in\mathbb{R}\}$. Applying the Fourier-Laplace transform with respect to (x',t) to (2.11), we have the solution formula for the n-th component of the velocity and the pressure as follows:

$$
\begin{aligned}
\widehat{v_n}(\tau,\xi',x_n)=&\frac{|\xi'|}{(B(\tau,\xi')-|\xi'|)D(\tau,\xi')}\Big(2i|\xi'|B(\tau,\xi')\big(\xi'\cdot\widehat{H'}\big)-(|\xi'|^2+B(\tau,\xi')^2)\widehat{H_n}\Big)e^{-|\xi'|x_n}\\
&+\frac{|\xi'|}{(B(\tau,\xi')-|\xi'|)D(\tau,\xi')}\Big(-(|\xi'|^2+B(\tau,\xi')^2)i\xi'\cdot\widehat{H'}+2|\xi'|B(\tau,\xi')\widehat{H_n}\Big)e^{-B(\tau,\xi')x_n},
\end{aligned}
\tag{2.18}
$$

$$
\widehat{q}(\tau,\xi',x_n)=\frac{|\xi'|+B(\tau,\xi')}{D(\tau,\xi')}\Big(2B(\tau,\xi')\big(i\xi'\cdot\widehat{H'}\big)-(|\xi'|^2+B(\tau,\xi')^2)\widehat{H_n}\Big)e^{-|\xi'|x_n},
\tag{2.19}
$$

where we have set $\widehat{H}=(\widehat{H'}(\tau,\xi'),\widehat{H_n}(\tau,\xi'))$ as the Fourier-Laplace transform of the given boundary data and

$$
B(\tau,\xi')=\sqrt{i\tau+|\xi'|^2},\quad \operatorname{Re}B(\tau,\xi')\ge 0,
\tag{2.20}
$$

$$
D(\tau,\xi')=B(\tau,\xi')^3+|\xi'|B(\tau,\xi')^2+3|\xi'|^2B(\tau,\xi')-|\xi'|^3.
\tag{2.21}
$$

Hence we see for any smooth rapidly decreasing boundary data $(\widehat{H'}, \widehat{H_n})$ in both (τ, ξ') variables, we see by passing $\gamma \to 0$ to obtain

$$
\begin{aligned}
&v_n(t,x',x_n)\\
&=c_{n+1}\text{p.v.} \iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}\Big\{\frac{|\xi'|}{(B-|\xi'|)D(\tau,\xi')}\Big(2B(i\xi'\cdot\widehat{H}') - (|\xi'|^2+B^2)\widehat{H_n}\Big)e^{-|\xi'|x_n}\\
&\qquad\qquad + \frac{|\xi'|}{(B-|\xi'|)D(\tau,\xi')}\Big(-(|\xi'|^2+B^2)i\xi'\cdot\widehat{H}') + 2|\xi'|B\widehat{H_n}\Big)e^{-Bx_n}\Big\}d\tau d\xi', \qquad (2.22)\\
&q(t,x',x_n)\\
&=c_{n+1}\text{p.v.} \iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}\Big\{\frac{|\xi'|+B}{D(\tau,\xi')}\Big(2B(i\xi'\cdot\widehat{H}') - (|\xi'|^2+B^2)\widehat{H_n}\Big)\Big\}e^{-|\xi'|x_n}d\tau d\xi', \qquad (2.23)
\end{aligned}
$$

where we take a limit of the integral pass avoiding the singularity at $(\tau, \xi') = (0, 0)$. All the other components of the velocity fields $v_\ell(t,x)$ $(\ell = 1, 2, \cdots, n-1)$ are given by the above two components (v_n, q) and the boundary data $H = (H', H_n)$ from the equation (2.11) (see for the details [62]).

Our main task is to prove maximal L^1-regularity of the pressure term q in (2.11) which is directly obtained from the inhomogeneous boundary data. Then the maximal L^1-regularity estimate for the velocity term v of (2.11) follows from the estimate for q. Applying the gradient to the solution formula (2.23), we obtain the explicit expression of ∇q as

$$
\begin{aligned}
&\nabla q(t,x',x_n)\\
&=c_{n+1}\iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}(i\xi', -|\xi'|)^{\mathsf{T}}\Big\{\frac{|\xi'|+B}{D(\tau,\xi')}\Big(2B(i\xi'\cdot\widehat{H}') - (|\xi'|^2+B^2)\widehat{H_n}\Big)\Big\}e^{-|\xi'|x_n}d\tau d\xi', \qquad (2.24)
\end{aligned}
$$

where $B = B(\tau, \xi')$ and $D(\tau, \xi')$ are defined by (2.20) and (2.21), respectively. We also set the following Fourier multiplier $m(\tau, \xi') : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ as

$$
\begin{aligned}
m(\tau,\xi') =&\big(m'(\tau,\xi'), m_n(\tau,\xi')\big)\\
\equiv&\Big(\frac{2B(|\xi'|+B)}{D(\tau,\xi')}i\xi',\ -\frac{\big(|\xi'|+B\big)(|\xi'|^2+B^2)}{D(\tau,\xi')}\Big).
\end{aligned} \tag{2.25}
$$

2.3. **The homogeneous Besov spaces on the half-space.** First we recall the summary for the homogeneous Besov spaces over the half Euclidean space $\mathbb{R}^n_+$. We recall the retraction and the coretraction defined in the way of Triebel [75] as follows:

Definition ([75]). Let A and B be Banach spaces and let R and E be linear operators as

$$
\begin{aligned}
&R : A \to B \text{ bounded},\\
&E : B \to A \text{ bounded},\\
&RE = Id : B \to B \text{ bounded},
\end{aligned} \tag{2.26}
$$

where Id is the identity operator from B to B. Then R is called as *retraction* and E is called as *coretraction.*

Definition. Let $1 \le p < \infty$ and $1 \le \sigma < \infty$ with $s \in \mathbb{R}$. Let

$$
\overset{\circ}{B}{}^s_{p,\sigma}(\mathbb{R}^n_+) \equiv \overline{C_0^\infty(\mathbb{R}^n_+)}^{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)}, \tag{2.27}
$$

$$
\overset{\odot}{B}{}^s_{p,\sigma}(\mathbb{R}^n_+) \equiv \overline{\{f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n);\ \text{supp}\, f \subset \mathbb{R}^n_+\}}^{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)}. \tag{2.28}
$$

It is shown that the above defined space coincides with the space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ defined by the restriction in (1.14). First we observe that the duality is well-defined in certain range of the exponent.

Proposition 2.3 (cf. [75])**.** *Let $1 \le p < \infty$ and $1 \le \sigma < \infty$. For $0 \le s < 1/p$, it holds*

$$
\big(\overset{\circ}{B}{}^s_{p,\sigma}(\mathbb{R}^n_+)\big)' \simeq \dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n_+).
$$

where $\simeq$ stands for the both spaces being equivalent as the normed space.

The part of the following proposition is shown by Danchin–Mucha [19].

Proposition 2.4 ([19]). *Let $1 \le p < \infty$ and $1 \le \sigma < \infty$. For $-1+1/p < s < 1/p$,*

$$
\begin{aligned}
\mathring{\dot{B}}^s_{p,\sigma}(\mathbb{R}^n_+) &\simeq \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),\\
\overset{\odot}{\dot{B}}{}^s_{p,\sigma}(\mathbb{R}^n_+) &\simeq \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),
\end{aligned}
$$

where $\simeq$ stands for the both spaces being equivalent as the normed space.

We consider the restriction operator R_0 by

$$
R_0 f(x) = f(x)\Big|_{x\in\mathbb{R}^n_+} \tag{2.29}
$$

for all $f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ with $s > 0$ and it is understood in the sense of distribution for $s \le 0$.

Let χ_+ be a cut-off operation defined by multiplying a cut-off function

$$
\chi_{\mathbb{R}^n_+}(x) = \begin{cases} 1, & \text{in } \mathbb{R}^n_+,\\ 0, & \text{in } \overline{\mathbb{R}^n_-}. \end{cases}
$$

Let the extension operator E_0 from $\overset{\odot}{\dot{B}}{}^s_{p,\sigma}(\mathbb{R}^n_+)$ given by the zero-extension, i.e., for any $f \in \overset{\odot}{\dot{B}}{}^s_{p,\sigma}(\mathbb{R}^n_+)$, set

$$
E_0 f = \begin{cases} f(x), & \text{in } \mathbb{R}^n_+,\\ 0, & \text{in } \overline{\mathbb{R}^n_-}. \end{cases} \tag{2.30}
$$

One can find that those operators are basic tool to recognize the homogeneous Besov spaces. Using Proposition 2.4, the following statement is a variant introduced by Triebel [75, p.228] .

Proposition 2.5. *Let $1 \le p < \infty$, $1 \le \sigma < \infty$ and $-1+1/p < s < 1/p$, and let R_0 and E_0 be operators defined in* (2.29) *and* (2.30) *It holds that*

$$
R_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+), \tag{2.31}
$$

$$
E_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n), \tag{2.32}
$$

are linear bounded operators. Besides it holds that

$$
R_0 E_0 = Id : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+), \tag{2.33}
$$

where Id denotes the identity operator. Namely R_0 and E_0 are a retraction and a coretraction, respectively.

The proof of Proposition 2.5 is along the same line of the proof in [75]. Note that the spaces are homogeneous Besov spaces and then the arrangement appears in Proposition 3 in Danchin–Mucha [19] is required.

Proof of Proposition 2.5. To see the first operator (2.31) is bounded, let $f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ and we show that

$$
\|R_0 f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)} = \inf \|\widetilde{R_0 f}\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} \le \|\chi_{\mathbb{R}^n_+} f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} \le \|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)}
$$

is valid under the restriction $s > 0$. Let $-1/p' < s < 0$ and $f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$. For any test $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$
\begin{aligned}
\big|\langle \chi_{\mathbb{R}^n_+} f, \phi\rangle\big| =& \big|\langle f, \chi_{\mathbb{R}^n_+}\phi\rangle\big| \le \|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} \|\chi_{\mathbb{R}^n_+}\phi\|_{\dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n)}\\
\le& \|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} \|\phi\|_{\dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n)},
\end{aligned} \tag{2.34}
$$

since $0 < -s < 1/p'$ and the last inequality follows from the pointwise sense. Thus from the definition of the norm in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$, it holds similarly to the above that

$$
\|R_0 f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)} \le \|\chi_{\mathbb{R}^n_+} f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)} = \sup_{\phi\in \dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n)\setminus\{0\}} \frac{\big|\langle \chi_{\mathbb{R}^n_+} f, \phi\rangle\big|}{\|\phi\|_{\dot{B}^{-s}_{p',\sigma'}(\mathbb{R}^n)}} \le \|f\|_{\dot{B}^s_{p,\sigma}(\mathbb{R}^n)}.
$$

For the second bound (2.32), see [19, Proposition 3]. Since the both operators are bounded, we see that

$$
R_0 E_0 = Id : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),
$$

holds by the density argument in Proposition 2.4. ❑

Proposition 2.6 (cf. [18], [75]). *Let $1 < p < \infty$ and $-1 + 1/p < s \le n/p - 1$. Then for any $f \in \dot{B}^s_{p,1}(\mathbb{R}^n_+)$,*

$$\|\nabla f\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} \simeq \|f\|_{\dot{B}^{s+1}_{p,1}(\mathbb{R}^n_+)},$$

where $\simeq$ stands for that the both side of the norm is equivalent.

For the proof, see [18, Proposition 3.19, Corollary 3.20].

Remark. In what follows, we restrict ourselves to the regularity range of the Besov spaces $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ in $-1 + 1/p < s < 1/p$ for $1 < p < \infty$ unless otherwise stated. According to Proposition 2.5, we can regard that any distribution in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ under such restriction on s and p can be extended into a distribution over whole space $\mathbb{R}^n$ and conversely any distribution in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ is restricted into a distribution over the half-space $\mathbb{R}^n_+$. We frequently use those facts without noticing for every case below.

2.4. **The L-P decomposition with separation of variables.** In order to split the variables $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}_+$, we introduce an x'-parallel decomposition and an x_n-parallel decomposition by Littlewood–Paley type. In what follows $\eta \in \mathbb{R}_+$ denotes a parameter for x_n-axis in $\mathbb{R}^n_+$. We introduce $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$ as a Littlewood–Paley dyadic frequency decomposition of unity in separated variables (ξ', ξ_n).

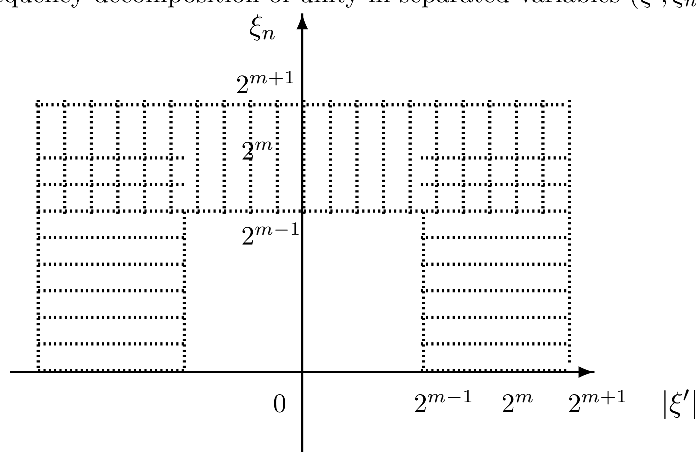

Fig 1: The support of Littlewood–Paley decomposition $\{\overline{\Phi_m}\}_{m\in\mathbb{Z}}$

Definition (The Littlewood–Paley decomposition of separated variables). For $m \in \mathbb{Z}$, let

$$\begin{aligned} \widehat{\zeta_m}(\xi_n) &= \begin{cases} 1, & 0 \le |\xi_n| \le 2^m, \\ \text{smooth}, & 2^m < |\xi_n| < 2^{m+1}, \\ 0, & 2^{m+1} \le |\xi_n|, \end{cases} \\ \widehat{\zeta_m}(\xi_n) &= \widehat{\zeta_{m-1}}(\xi_n) + \widehat{\phi_m}(\xi_n) \end{aligned} \tag{2.35}$$

(one can choose $\widehat{\zeta_m}(r) = \sum_{\ell \le m-1} \widehat{\phi_\ell}(r) + \widehat{\phi_{-\infty}}(r)$ with a correction distribution $\widehat{\phi_{-\infty}}(r)$ supported at $r = 0$) and set

$$\widehat{\overline{\Phi_m}}(\xi) \equiv \widehat{\phi_m}(|\xi'|) \otimes \widehat{\zeta_{m-1}}(\xi_n) + \widehat{\zeta_m}(|\xi'|) \otimes \widehat{\phi_m}(\xi_n). \tag{2.36}$$

Then it is obvious from Fig. 2 (restricted on the upper half region in $\mathbb{R}^n$) that

$$\sum_{m\in\mathbb{Z}} \widehat{\overline{\Phi_m}}(\xi) \equiv 1, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{2.37}$$

Indeed, from (2.35) and (2.36),

$$\begin{aligned} &\sum_{m\in\mathbb{Z}} \widehat{\overline{\Phi_m}}(\xi) \\ &= \sum_{m\in\mathbb{Z}} \widehat{\phi_m}(|\xi'|) \otimes \sum_{-\infty \le \ell \le m-1} \widehat{\phi_\ell}(\xi_n) + \sum_{m\in\mathbb{Z}} \sum_{\ell \le m} \widehat{\phi_\ell}(|\xi'|) \otimes \widehat{\phi_m}(\xi_n) + \sum_{m\in\mathbb{Z}} \widehat{\phi_{-\infty}}(|\xi'|) \otimes \widehat{\phi_m}(\xi_n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m\in\mathbb{Z}} \widehat{\phi_m}(|\xi'|) \otimes \Big(\sum_{-\infty\le \ell\le m-1} \widehat{\phi_\ell}(\xi_n) + \sum_{\ell\ge m} \widehat{\phi_\ell}(\xi_n)\Big) + \widehat{\phi_{-\infty}}(|\xi'|) \otimes \sum_{m\in\mathbb{Z}} \widehat{\phi_m}(\xi_n) \\
&= \sum_{m\in\mathbb{Z}} \widehat{\phi_m}(|\xi'|) \otimes \sum_{\ell\in\mathbb{Z}\cup\{-\infty\}} \widehat{\phi_\ell}(\xi_n) + \widehat{\phi_{-\infty}}(|\xi'|) \otimes \sum_{\ell\in\mathbb{Z}\cup\{-\infty\}} \widehat{\phi_m}(\xi_n) - \widehat{\phi_{-\infty}}(|\xi'|) \otimes \widehat{\phi_{-\infty}}(\xi_n) \\
&=1 - \widehat{\phi_{-\infty}}(|\xi'|) \otimes \widehat{\phi_{-\infty}}(\xi_n).
\end{aligned}
$$

Definition (Varieties of the Littlewood–Paley dyadic decompositions). Let $(\tau, \xi', \xi_n) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$ be Fourier adjoint variables corresponding to $(t, x', \eta) \in \mathbb{R}\times\mathbb{R}^{n-1}\times\mathbb{R}$.

- $\{\Phi_m(x)\}_{m\in\mathbb{Z}}$: the standard (annulus type) Littlewood–Paley dyadic decomposition by $x = (x', \eta) \in \mathbb{R}^n$.
- $\{\overline{\Phi_m}(x)\}_{m\in\mathbb{Z}}$: the Littlewood–Paley dyadic decomposition over $x = (x', \eta) \in \mathbb{R}^n$ given by (2.36).
- $\{\psi_k(t)\}_{k\in\mathbb{Z}}$: the Littlewood–Paley dyadic decompositions in $t \in \mathbb{R}$.
- $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ and $\{\phi_j(\eta)\}_{j\in\mathbb{Z}}$: the standard (annulus type) Littlewood–Paley dyadic decompositions in $x' \in \mathbb{R}^{n-1}$ and $\eta \in \mathbb{R}$, respectively.
- $\{\zeta_m(x')\}_{m\in\mathbb{Z}}$ and $\{\zeta_m(\eta)\}_{m\in\mathbb{Z}}$: the lower frequency smooth cut-off given by (2.35), respectively.
- Let $\widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}$ be the Littlewood–Paley dyadic decompositions with its j-neighborhood to ϕ_j.
- All the above defined decompositions are even functions.

Then in view of Proposition 2.5 and the remark at the end of the previous sub-section, we see that the norm of the Besov spaces on $\mathbb{R}^n$ defined by $\{\Phi_m\}_m$ is equivalent to the one from the Littlewood–Paley decomposition of direct sum type, $\{\overline{\Phi_m}\}_m$ over $\mathbb{R}^n$ and hence one can identify those norms as it appears the homogeneous Besov space over $\mathbb{R}^n_+$ as follows. Indeed, for any $1 < p < \infty$ and $-1 + 1/p < s < 1/p$,

$$
\begin{aligned}
\|\nabla q(t)\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)} \le& C\|\nabla q(t)\|_{\overset{\odot}{B}{}^s_{p,1}(\mathbb{R}^n_+)} = C\|E_0\big[\nabla q(t)\big]\|_{\dot B^s_{p,1}(\mathbb{R}^n)} \\
=& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\Phi_m \underset{(x)}{*} \sum_{|m-k|\le 1} \overline{\Phi_k} \underset{(x)}{*} E_0\big[\nabla q(t)\big]\Big\|_{L^p(\mathbb{R}^n)} \\
\le& 3C\sum_{m\in\mathbb{Z}} 2^{sm}\big\|\overline{\Phi_m} \underset{(x)}{*} E_0\big[\nabla q(t)\big]\big\|_{L^p(\mathbb{R}^n)} \\
\le& 3C\sum_{m\in\mathbb{Z}} 2^{sm}\big\|\overline{\Phi_m} \underset{(x)}{*} \sum_{|m-k|\le 1} \Phi_k \underset{(x)}{*} E_0\big[\nabla q(t)\big]\big\|_{L^p(\mathbb{R}^n)} \\
\le& 3^2C\sum_{m\in\mathbb{Z}} 2^{sm}\big\|\Phi_k \underset{(x)}{*} E_0\big[\nabla q(t)\big]\big\|_{L^p(\mathbb{R}^n)} \\
\le& 3^2C\big\|\nabla q(t)\big\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}.
\end{aligned} \tag{2.38}
$$

In what follows, we freely use the retraction and coretraction operators as observed above and for simplicity we avoid reprised usage of them.

3. Almost orthogonality of the pressure potential

Almost orthogonality is the key lemma to obtain the maximal L^1-regularity estimate. In this section, we derive almost orthogonality concerning the pressure.

Defintion (The pressure potentials). For $j, k \in \mathbb{Z}$, let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$, $\{\phi_j(x')\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley decompositions for $t \in \mathbb{R}$ and $x' \in \mathbb{R}^{n-1}$ valuables, respectively. We set for $\eta = x_n > 0$,

$$
\begin{cases}
\pi(t, x', \eta) \equiv c_{n+1} \displaystyle\iint_{\mathbb{R}\times\mathbb{R}^{n-1}} e^{it\tau + ix'\cdot\xi'} (i\xi', -|\xi'|)^{\mathsf{T}} m(\tau, \xi') e^{-|\xi'|\eta} d\tau d\xi', \\
\pi_{k,j}(t, x', \eta) \equiv \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \pi(t, x', \eta) \\
\qquad\qquad\quad\; = \big(\pi'_{k,j}(t, x', \eta), \pi_{n,k,j}(t, x, \eta)\big),
\end{cases} \tag{3.1}
$$

where $m : \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ is defined in (2.25). We extend the potential $\pi(t, x', \eta)$ into all $\eta \in \mathbb{R}$ by the even extension (i.e. exchange η into $|\eta|$).

Setting $\widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}$, $\widetilde{\psi_k} = \psi_{k-1} + \psi_k + \psi_{k+1}$ and noting that

$$\sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widehat{\psi}_k(\tau)\widehat{\phi_j}(\xi) \equiv 1, \quad \tau, \xi \neq 0,$$

we have for $x_n > 0$ that

$$\begin{aligned}
&\nabla q(t, x', x_n) \\
&= c_{n+1} \iint_{\mathbb{R}^n} e^{it\tau + ix'\cdot\xi'} (i\xi', -|\xi'|)^{\mathsf{T}} \Big(m'(\tau, \xi') \cdot \widehat{H'} + m_n(\tau, \xi') \widehat{H_n} \Big) e^{-|\xi'|x_n} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \widehat{\widetilde{\psi_k}}(\tau)\widehat{\widetilde{\phi_j}}(\xi) d\tau d\xi' \\
&\equiv \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \Big({\pi'}_{k,j} \underset{(t,x')}{\cdot *} \Big(\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_j} \underset{(x')}{*} H' \Big) + \pi_{n\,k,j} \underset{(t,x')}{*} \Big(\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_j} \underset{(x')}{*} H_n \Big) \Big),
\end{aligned}$$

where we use the notion of the inner product-convolution (1.17) and the data is extended by the zero extension for $t \le 0$. We show the almost orthogonality and its variation in the following.

3.1. **The almost orthogonality.** For the symbol of the gradient of the pressure, we introduce the useful notation for a part of the symbol defined by (2.20); $B(\tau, \xi') = \sqrt{i\tau + |\xi'|^2}$.

Definition. Let $\sigma \in \mathbb{R}$ and $\zeta' \in \mathbb{R}^{n-1}$ with $1/2 < |\sigma|, |\zeta'| < 2$. For $a > 0$, we set

$$\begin{cases} b(\sigma, \zeta', a) = \sqrt{i\sigma + a^{-2}|\zeta'|^2}, \\ d(\sigma, \zeta', a) = \sqrt{a^{-2}i\sigma + |\zeta'|^2}. \end{cases} \tag{3.2}$$

Lemma 3.1. *Let $\sigma \in \mathbb{R}$ and $\zeta' \in \mathbb{R}^{n-1}$.*
(1) *For the time dominated region $k \ge 2j$,*

$$\frac{1}{\sqrt{2}} \le |b(\sigma, \zeta', 2^{\frac{k}{2}-j})| \le 20^{1/4}, \tag{3.3}$$

in particular, there exist constants $0 < c < C$ independent of j and k such that

$$c2^{\frac{k}{2}} \le |B(2^k\sigma, 2^j\zeta')| = 2^{\frac{k}{2}} |b(\sigma, \zeta', 2^{\frac{k}{2}-j})| \le C2^{\frac{k}{2}}. \tag{3.4}$$

(2) *For the space dominated region $k < 2j$,*

$$\frac{1}{2} \le |d(\sigma, \eta', 2^{j-\frac{k}{2}})| \le 20^{1/4},$$

in particular, there exist constants $0 < c < C$ independent of j and k such that

$$c2^j \le |B(2^k\sigma, 2^j\zeta')| = 2^j |d(\sigma, \zeta', 2^{j-\frac{k}{2}})| \le C2^{j+3}. \tag{3.5}$$

Proof of Lemma 3.1. (1) In the case when $k \ge 2j$, by using $2^{-1} < |\sigma| < 2$, $2^{-1} < |\zeta'| < 2$, it holds that

$$\begin{aligned}
B(2^k\sigma, 2^j\zeta') =& 2^{\frac{k}{2}} b(\sigma, \zeta', a)\big|_{a=2^{\frac{k}{2}-j}} = 2^{\frac{k}{2}} \sqrt{i\sigma + (2^{j-\frac{k}{2}})^2 |\zeta'|^2} \\
=& 2^{\frac{k}{2}} \cdot \sqrt[4]{\sigma^2 + (2^{j-\frac{k}{2}}|\zeta'|)^4} \exp\Big(\frac{i}{2} \tan^{-1} \frac{2^k\sigma}{2^{2j}|\zeta'|^2} \Big),
\end{aligned}$$

and (3.4) follows from

$$2^{-\frac{1}{2}} 2^{\frac{k}{2}} \le 2^{\frac{k}{2}} \cdot \sqrt[4]{\sigma^2} \le |B(2^k\sigma, 2^j\zeta')| = 2^{\frac{k}{2}} \cdot \sqrt[4]{\sigma^2 + 2^{4j-2k}|\zeta'|^4} \le 20^{1/4} \cdot 2^{\frac{k}{2}}.$$

(2) In the case when $k < 2j$, it holds that

$$2^{-1}2^j \le 2^j \cdot \sqrt[4]{|\zeta'|^4} \le |B(2^k\sigma, 2^j\zeta')| = 2^j \cdot \sqrt[4]{2^{2k-4j}\sigma^2 + |\eta'|^4} \le 20^{1/4} \cdot 2^j.$$

The constants c and C can be taken as $c = 1/\sqrt{2}$ and $C = \sqrt{2\sqrt{5}}$. ❑

Lemma 3.2 (Almost orthogonality I)**.** *For $k, j \in \mathbb{Z}$, let $\pi_{k,j}(t, x', \eta)$ be the pressure potentials defined by* (3.1) *and let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x)\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley decompositions for time and space, respectively.*

(1) *For the time-dominated region $k \ge 2j$, there exists $C_n > 0$ such that for any $\eta \in \mathbb{R}_+$ and $t \in \mathbb{R}$,*

$$\big\|\pi_{k,j}(t,\cdot,\eta)\big\|_{L^1_{x'}} \le C_n 2^j\big(1+(2^j\eta)^{n+2}\big)e^{-2^{(j-1)}\eta}\frac{2^k}{\langle 2^k t\rangle^2}, \tag{3.6}$$

where $\|\cdot\|_{L^1_{x'}}$ denotes the $L^1(\mathbb{R}^{n-1})$ norm in x'-variable.

(2) *For the space-dominated region $k < 2j$, there exists $C_n > 0$ such that for any $\eta \in \mathbb{R}_+$ and $t \in \mathbb{R}$,*

$$\Big\|\sum_{k<2j}\pi_{k,j}(t,\cdot,\eta)\Big\|_{L^1_{x'}} \le C_n 2^j\big(1+(2^j\eta)^{n+2}\big)e^{-2^{(j-1)}\eta}\frac{2^{2j}}{\langle 2^{2j} t\rangle^2}. \tag{3.7}$$

The estimates are extended to $\eta \in \mathbb{R}$ by the even extensions.

Proof of Lemma 3.2. (1) In the time-dominated region $k \ge 2j$, by using the expression of the fundamental solution and using change of variables $\tau = 2^k\sigma$, $\xi' = 2^j\zeta'$ and then $x' = 2^{-j}y'$, we first observe that

$$\begin{aligned}
&\|\pi_{k,j}(t,\cdot,\eta)\|_{L^1_{x'}}\\
&=\Big\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}}m(\tau,\xi')e^{-|\xi'|\eta}\widehat{\psi}(2^{-k}\tau)\widehat{\phi}(2^{-j}\xi')d\xi'd\tau\Big\|_{L^1_{x'}}\\
&=\Big\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{i2^kt\sigma+i2^jx'\cdot\zeta'}(2^ji\zeta',-2^j|\zeta'|)^{\mathsf{T}}m(2^k\sigma,2^j\zeta')e^{-2^j|\zeta'|\eta}\widehat{\psi}(\sigma)\widehat{\phi}(\zeta')2^{(n-1)j}d\zeta'\,2^kd\sigma\Big\|_{L^1_{x'}}\\
&=2^{j+k}e^{-(2^{j-1}\eta)}\Big\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{i2^kt\sigma+i2^jx'\cdot\zeta'}(i\zeta',-|\zeta'|)^{\mathsf{T}}m(2^k\sigma,2^j\zeta')\\
&\qquad\qquad\times\exp\big(-2^j\eta(|\zeta'|-\tfrac{1}{2})\big)\widehat{\psi}(\sigma)\widehat{\phi}(\zeta')2^{(n-1)j}d\zeta'd\sigma\Big\|_{L^1_{x'}}\\
&=2^{k+j}e^{-(2^{j-1}\eta)}\Big\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{i2^kt\sigma+iy'\cdot\zeta'}(i\zeta',-|\zeta'|)^{\mathsf{T}}\\
&\qquad\qquad\times m(2^k\sigma,2^j\zeta')\exp\big(-2^j\eta(|\zeta'|-\tfrac{1}{2})\big)\widehat{\psi}(\sigma)\widehat{\phi}(\zeta')d\zeta'\,d\sigma\Big\|_{L^1_{y'}}.
\end{aligned}\tag{3.8}$$

Since the Fourier inverse transform of the most right term of the above equation contains the Littlewood–Paley cut-off for σ and ζ', it is integrable absolutely with respect to σ and ζ'. If the symbol $m(2^k\tau, 2^j\zeta')$ is bounded, then $\pi_{k,j}(t, 2^{-j}y', \eta)$ are integrable with respect to y' when $|y'| < 1$. Therefore we check the boundedness of the symbol $m(2^k\sigma, 2^j\zeta')$. Recalling the definition of $m'(\tau, \xi')$ in (2.25) with using $b(\sigma, \zeta, 2^{\frac{k}{2}-j})$ in (3.2) and its bound (3.4), it holds that

$$\begin{aligned}
&m'(2^k\sigma,2^j\zeta')\\
&=2i\frac{\xi'}{|\xi'|}\;\frac{|\xi'|^2B+|\xi'|B^2}{B^3+|\xi'|B^2+3|\xi'|^2B-|\xi'|^3}\bigg|_{\tau=2^k\sigma,\xi'=2^j\zeta'}\\
&=2i\frac{\zeta'}{|\zeta'|}\frac{2^{\frac{k}{2}+2j}|\zeta'|^2b(\sigma,\zeta,2^{\frac{k}{2}-j})+2^{k+j}|\zeta'|b(\sigma,\zeta,2^{\frac{k}{2}-j})^2}{2^{\frac{3}{2}k}b(\sigma,\zeta,2^{\frac{k}{2}-j})^3+2^{k+j}|\zeta'|b(\sigma,\zeta,2^{\frac{k}{2}-j})^2+3\cdot2^{\frac{k}{2}+2j}|\zeta'|^2b(\sigma,\zeta,2^{\frac{k}{2}-j})-2^{3j}|\zeta'|^3}\\
&=2i\frac{\zeta'}{|\zeta'|}\frac{2^{-2(\frac{k}{2}-j)}|\zeta'|^2b(\sigma,\zeta',2^{2j-k})+2^{-(\frac{k}{2}-j)}|\zeta'|b(\sigma,\zeta,2^{\frac{k}{2}-j})^2}{(b(\sigma,\zeta,2^{\frac{k}{2}-j})^3+2^{-(\frac{k}{2}-j)}|\zeta'|\,b(\sigma,\zeta,2^{\frac{k}{2}-j})^2+3\cdot2^{-2(\frac{k}{2}-j)}|\zeta'|^2\,b(\sigma,\zeta,2^{\frac{k}{2}-j})-2^{-3(\frac{k}{2}-j)}|\zeta'|^3}
\end{aligned}$$

and thus for the case $k - 2j > 3$, it holds $2^{-m(\frac{k}{2}-j)} \le 2^{-\frac{3}{2}m}$ with $m = 1, 2, 3$ and from (3.3), we have

$$|m'(2^k\sigma,2^j\zeta')| \le C\frac{2^{-(\frac{k}{2}-j)}|\zeta'|}{b(\sigma,\zeta,2^{\frac{k}{2}-j})} \le C, \tag{3.9}$$

and otherwise $0 \le k-2j \le 3$ it is obviously bounded from above and below since the denominator never vanishes. Furthermore,

$$
\begin{aligned}
&m_n(2^k\sigma, 2^j\zeta')\\
&= \frac{(|\xi'|+B)(|\xi'|^2+B^2)}{B^3+|\xi'|B^2+3|\xi'|^2B-|\xi'|^3}\bigg|_{\tau=2^k\sigma,\xi'=2^j\zeta'}\\
&=\frac{\big(2^j|\zeta'|+2^{\frac{k}{2}}b(\sigma,\zeta,2^{\frac{k}{2}-j})\big)\big(2^{2j}|\zeta'|^2+2^k b(\sigma,\zeta,2^{\frac{k}{2}-j})^2\big)}{(2^{\frac{3k}{2}}b(\sigma,\zeta,2^{\frac{k}{2}-j})^3+2^{k+j}|\zeta'|b(\sigma,\zeta,2^{\frac{k}{2}-j})^2+3\cdot 2^{\frac{k}{2}+2j}|\zeta'|^2 b(\sigma,\zeta,2^{\frac{k}{2}-j})-2^{3j}|\zeta'|^3}\\
&=\frac{\big(2^{-(\frac{k}{2}-j)}|\zeta'|+b(\sigma,\zeta,2^{\frac{k}{2}-j})\big)\big(2^{-2(\frac{k}{2}-j)}|\zeta'|^2+b(\sigma,\zeta,2^{\frac{k}{2}-j})^2\big)}{(b(\sigma,\zeta,2^{\frac{k}{2}-j})^3+2^{-(\frac{k}{2}-j)}|\zeta'|\,b(\sigma,\zeta,2^{\frac{k}{2}-j})^2+3\cdot 2^{-2(\frac{k}{2}-j)}|\zeta'|^2\,b(\sigma,\zeta,2^{\frac{k}{2}-j})-2^{-3(\frac{k}{2}-j)}|\zeta'|^3}
\end{aligned}
$$

and similarly

$$
|m_n(2^k\sigma,2^j\zeta')| \lesssim \frac{b(\sigma,\zeta,2^{\frac{k}{2}-j})^3}{b(\sigma,\zeta,2^{\frac{k}{2}-j})^3} \le C. \tag{3.10}
$$

Note that the common denominator $B^3+|\xi'|B^2+3|\xi'|^2B-|\xi'|^3$ has no zero point except $(\tau,\xi')=(0,0)$ (cf. Lemma 4.4 in [59]).

For $t<1$, we obtain from (3.9) and (3.10) that

$$
\begin{aligned}
&\|\pi_{k,j}(t,\cdot)\|_{L^1_{x'}(B_{2^{-j}})} \qquad (3.11)\\
=&C_n 2^{k+j}e^{-(2^{j-1}\eta)}\bigg\|\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{i(2^k t\sigma+y'\cdot\zeta')}\\
&\qquad\times(i\zeta',-|\zeta'|)^{\mathsf{T}} m(2^k\sigma,2^j\zeta')\exp\big(-2^j\eta(|\zeta'|-\frac{1}{2})\big)\widehat{\psi}(\sigma)\widehat{\phi}(\zeta')d\zeta' d\sigma\bigg\|_{L^1_{y'}(B_1)}\\
\le& C_n 2^{k+j}e^{-(2^{j-1}\eta)}.
\end{aligned}
$$

Next we consider the case when $t>1$. It is important that we gain decay of time for $t>1$ by integration by parts. Noting that

$$
e^{i(2^k t\sigma+y'\cdot\zeta')} = \left(\frac{1}{2^k it}\right)^2 \partial_\sigma^2 e^{i(2^k t\sigma+y'\cdot\zeta')},
$$

and integrating by parts with respect to σ twice, we obtain

$$
\begin{aligned}
&\|\pi_{k,j}(t,\cdot)\|_{L^1_{x'}(B_{2^{-j}})} \qquad (3.12)\\
=&2^{k+j}e^{-(2^{j-1}\eta)}\bigg\| c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\frac{1}{(2^k it)^2}e^{i(2^k t\sigma+y'\cdot\zeta')}\\
&\qquad\times(i\zeta',-|\zeta'|)^{\mathsf{T}}\partial_\sigma^2\Big(m(2^k\sigma,2^j\zeta')\widehat{\psi}(\sigma)\Big)\exp\big(-2^j\eta(|\zeta'|-\frac{1}{2})\big)\widehat{\phi}(\zeta')d\zeta' d\sigma\bigg\|_{L^1_{y'}(B_1)}.
\end{aligned}
$$

Again we separate the region $k\ge 2j$ and $k<2j$. For m', from (3.9) we use $b=b(\sigma,\zeta',2^{\frac{k}{2}-j})$ defined in (3.2) and $a=2^{\frac{k}{2}-j}$ to see

$$
\begin{aligned}
&\partial_\sigma m'(2^k\sigma,2^j\zeta') \qquad (3.13)\\
=&\frac{\partial}{\partial\sigma}\left(2i\frac{\xi'}{|\xi'|}\frac{|\xi'|^2B+|\xi'|B^2}{B^3+|\xi'|B^2+3|\xi'|^2B-|\xi'|^3}\bigg|_{\tau=2^k\sigma,\xi'=2^j\zeta'}\right)\\
=&2i\frac{\zeta'}{|\zeta'|}\frac{\partial b}{\partial\sigma}\frac{\partial}{\partial b}\frac{|\zeta'|^2a^{-2}b+|\zeta'|a^{-1}b^2}{b^3+a^{-1}|\zeta'|\,b^2+3a^{-2}|\zeta'|^2\,b-a^{-3}|\zeta'|^3}\\
=&\frac{\zeta'}{|\zeta'|b}\cdot\left\{\frac{a^{-1}|\zeta'|b^4+2a^{-2}|\zeta'|^2\,b^3-2a^{-3}|\zeta'|^3\,b^2+2a^{-4}|\zeta'|^4b+a^{-5}|\zeta'|^5}{\big(b^3+a^{-1}|\zeta'|\,b^2+3a^{-2}|\zeta'|^2\,b-a^{-3}|\zeta'|^3\big)^2}\right\}.
\end{aligned}
$$

Analogously

$$
\begin{aligned}
&\partial_\sigma m_n(2^k\sigma, 2^j\zeta') \qquad\qquad (3.14)\\
=&\frac{\partial}{\partial\sigma}\left(\frac{(|\xi'|+B)(|\xi'|^2+B^2)}{B^3+|\xi'|B^2+3|\xi'|^2B-|\xi'|^3}\bigg|_{\tau=2^k\sigma,\xi'=2^j\zeta'}\right)\\
=&\frac{\partial b}{\partial\sigma}\frac{\partial}{\partial b}\frac{(a^{-1}|\zeta'|+b)(a^{-2}|\zeta'|^2+b^2)}{\left(b^3+a^{-1}|\zeta'|b^2+3a^{-2}|\zeta'|^2b-a^{-3}|\zeta'|^3\right)}\\
=&\frac{2i}{b}\frac{a^{-2}|\zeta'|^2b^3-a^{-3}|\zeta'|^3b^2-a^{-4}|\zeta'|^4b-a^{-5}|\zeta'|^5}{\left(b^3+a^{-1}|\zeta'|b^2+3a^{-2}|\zeta'|^2b-a^{-3}|\zeta'|^3\right)^2}.
\end{aligned}
$$

By Lemma 3.1, $\partial_\sigma m'(2^k\sigma, 2^j\zeta')$ and $\partial_\sigma m_n(2^k\sigma, 2^j\zeta')$ are bounded from above for all (k,j) when $k \ge 2j$. Since the denominator does not vanish because it is smooth on the support of σ and η such that $|\sigma|, |\zeta'| \in (1/2, 2)$, there is no diverging coefficient from σ-derivative of m, $\partial_\sigma^2 m$ is bounded on the support of $\widehat{\psi}(\sigma)$. The situation is same for the second derivative with respect to σ. Therefore for $t > 1$, it holds from (3.12)-(3.14) that

$$
\begin{aligned}
&\|\pi_{k,j}(t,\cdot)\|_{L^1_{x'}(B_{2^{-j}})}\\
=&2^{k+j}e^{-(2^{j-1}\eta)}\frac{1}{(2it)^2}\bigg\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}e^{i(2^kt\sigma+y'\cdot\zeta')}\\
&\qquad\times(i\zeta', -|\zeta'|)^{\mathsf{T}}\partial_\sigma^2\Big(m(2^k\sigma, 2^j\zeta')\widehat{\psi}(\sigma)\Big)\exp\big(-2^j\eta(|\zeta'|-\frac{1}{2})\big)\widehat{\phi}(\zeta')d\zeta'd\sigma\bigg\|_{L^1_{y'}(B_1)}\\
\le&C2^je^{-(2^{j-1}\eta)}\frac{2^k}{(2^kt)^2}. \qquad\qquad (3.15)
\end{aligned}
$$

On the other hand, in the case $y' \in B_1^c(0)$ we differentiate the symbol

$$
m(2^k\sigma, 2^j\zeta')(i\zeta', -|\zeta'|)e^{-2^j\eta(|\zeta'|-1/2)}\widehat{\phi}(\zeta')
$$

n times with respect to ζ'. If we obtain boundedness of the symbol uniformly k and j, then we obtain (t, y')-decay estimate by using

$$
e^{i(2^kt\sigma+y'\cdot\zeta')} = \left(\frac{1}{|y'|^n}\frac{1}{(2^kit)^2}\right)\partial_\sigma^2(-\Delta_{\zeta'})^{\frac{n}{2}}e^{i(2^kt\sigma+y'\cdot\zeta')}.
$$

In this way it holds that

$$
\begin{aligned}
&\|\pi_{k,j}(t,\cdot)\|_{L^1_{x'}(B^c_{2^{-j}})}\\
=&2^{k+j}e^{-(2^{j-1}\eta)}\frac{1}{(2^kt)^2}\bigg\|\frac{c_{n+1}}{\langle y'\rangle^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}e^{i(2^kt\sigma+y'\cdot\zeta')}\\
&\times(1-\partial_\sigma^2)\cdot(1-\Delta_{\zeta'})^{\frac{n}{2}}\Big((i\zeta', -|\zeta'|)^{\mathsf{T}}m(2^k\sigma, 2^j\zeta')\exp\big(-2^j\eta(|\zeta'|-\frac{1}{2})\big)\widehat{\psi}(\sigma)\widehat{\phi}(\zeta')\Big)d\zeta'd\sigma\bigg\|_{L^1_{y'}(B_1^c)}\\
\le&C2^j\big(1+(2^j\eta)^{n+2}\big)e^{-(2^{j-1}\eta)}\frac{2^k}{(2^kt)^2}. \qquad\qquad (3.16)
\end{aligned}
$$

Combining with (3.11), (3.15) and (3.16), we obtain the estimate (3.6).

(2) In the space-dominated region $k < 2j$, the proof is almost the same as in (1). Using the notation $\widehat{\zeta}(2^{-2j}\tau) = \sum_{k<2j}\widehat{\psi}(2^{-k}\tau)$ for the Littlewood–Paley decomposition (see (1.12) and (2.35)), and applying the change of variables $\xi' = 2^j\zeta'$, $\tau = 2^{2j}\sigma$ and then $x' = 2^{-j}y'$, we have for $\widehat{\zeta}(2^{-2j}\tau) = \sum_{k\le 2j}\widehat{\psi}(2^{-k}\tau)$

that

$$
\begin{aligned}
&\Big\|\sum_{k<2j}\pi_{k,j}(t,\cdot,\eta)\Big\|_{L^1_{x'}}\\
=&\left\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}}m(\tau,\xi')e^{-|\xi'|\eta}\sum_{k<2j}\widehat{\psi}(2^{-k}\tau)\widehat{\phi}(2^{-j}\xi')d\xi'd\tau\right\|_{L^1_{x'}}\\
=&\left\|c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}e^{i2^{2j}t\sigma+i2^jx'\cdot\zeta'}(2^ji\zeta',-2^j|\zeta'|)^{\mathsf{T}}m(2^{2j}\sigma,2^j\zeta')e^{-2^j|\zeta'|\eta}\widehat{\zeta}(2^{-2j}\tau)\widehat{\phi}(\zeta')2^{(n-1)j}d\zeta'\cdot 2^{2j}d\sigma\right\|_{L^1_{x'}}\\
=&C_n2^{3j}e^{-(2^{j-1}\eta)}\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}e^{2^{2j}it\sigma+iy'\cdot\zeta'}(i\zeta',-|\zeta'|)^{\mathsf{T}}m(2^{2j}\sigma,2^j\zeta')\\
&\qquad\qquad\times\exp\big(-2^j\eta(|\zeta'|-1/2)\big)\widehat{\zeta}(\sigma)\widehat{\phi}(\zeta')d\zeta'\,d\sigma\Big\|_{L^1_{y'}}.
\end{aligned}
$$

Using $d=d(\sigma,\zeta,1)$, we have from (3.5) that

$$
\begin{aligned}
m'(2^{2j}\sigma,2^j\zeta')=&2i\frac{\zeta'}{|\zeta'|}\frac{2^{3j}|\zeta'|^2\sqrt{i\sigma+|\zeta'|^2}+2^{3j}|\zeta'|(i\sigma+|\zeta'|^2)}{2^{3j}\sqrt{i\sigma+|\zeta'|^2}^3+2^{3j}|\zeta'|(i\sigma+|\zeta'|^2)+3\cdot2^{3j}|\zeta'|^2\sqrt{i\sigma+|\zeta'|^2}-2^{3j}|\zeta'|^3}\\
=&2i\frac{\zeta'}{|\zeta'|}\frac{|\zeta'|^2\sqrt{i\sigma+|\zeta'|^2}+|\zeta'|(i\sigma+|\zeta'|^2)}{\sqrt{i\sigma+|\zeta'|^2}^3+|\zeta'|(i\sigma+|\zeta'|^2)+3\cdot|\zeta'|^2\sqrt{i\sigma+|\zeta'|^2}-|\zeta'|^3}\\
=&2i\frac{\zeta'}{|\zeta'|}\frac{|\zeta'|^2d+|\zeta'|d^2}{d^3+|\zeta'|d^2+3|\zeta'|^2d-|\zeta'|^3}\simeq i\frac{|\zeta'|^2\zeta'}{|\zeta'|^3},\\
m_n(2^{2j}\sigma,2^j\zeta')=&\frac{\big(|\zeta'|+\sqrt{i\sigma+|\zeta'|^2}\big)\big(|\zeta'|^2+(i\sigma+|\zeta'|^2)\big)}{\sqrt{i\sigma+|\zeta'|^2}^3+|\zeta'|(i\tau+2^{2j}|\zeta'|^2)+3|\zeta'|^2\sqrt{i\tau+2^{2j}|\zeta'|^2}-|\zeta'|^3}\\
=&\frac{\big(|\zeta'|+d\big)\big(|\zeta'|^2+d^2\big)}{d^3+|\zeta'|d^2+3|\zeta'|^2d-|\zeta'|^3}\qquad\simeq C,
\end{aligned}
\tag{3.17}
$$

where $\simeq$ stands for the equivalence with a constant. Besides by $d\equiv d(\sigma,\zeta',1)$ for simplicity, and noting $\frac{\partial}{\partial\sigma}d(\sigma,\zeta',1)=i(2d)^{-1}$, we see

$$
\begin{aligned}
\partial_\sigma m'(2^{2j}\sigma,2^j\zeta')=&2i\frac{\zeta'}{|\zeta'|}\frac{\partial d}{\partial\sigma}\frac{\partial}{\partial d}\frac{|\zeta'|^2d+|\zeta'|d^2}{d^3+|\zeta'|\,d^2+3|\zeta'|^2\,d-|\zeta'|^3}\simeq\frac{C}{d^3(\sigma,\zeta',1)}\simeq\frac{C}{\langle\sigma\rangle^{\frac32}},\\
\partial_\sigma m_n(2^{2j}\sigma,2^j\zeta')=&\frac{\partial d}{\partial\sigma}\frac{\partial}{\partial d}\frac{|\zeta'|^3+|\zeta'|^2d+|\zeta'|d^2+d^3}{\big(d^3+|\zeta'|d^2+3|\zeta'|^2d-|\zeta'|^3\big)}\simeq\frac{C}{d^2}\simeq\frac{C}{\langle\sigma\rangle}.
\end{aligned}
\tag{3.18}
$$

Similarly one can estimate the second derivative of the symbol $m(2^{2j}\sigma,2^j\zeta')$ and it is now clear that the second derivatives are also bounded over the support of $\widehat{\psi}(\sigma)$ and $\widehat{\phi}(\zeta')$.

Hence by the boundedness obtained from (3.17) and (3.18), the rest of computation go through along the same line to (3.11), (3.15) and (3.16), and we conclude for the ball $B_{2^{-j}}$ with radius 2^{-j} around the origin that

$$
\begin{aligned}
&\Big\|\sum_{k<2j}\pi_{k,j}(t,\cdot)\Big\|_{L^1_{x'}(B_{2^{-j}})}\\
\le&C_n2^{3j}e^{-(2^{j-1}\eta)}\frac{1}{\langle2^{2j}t\rangle^2}\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}e^{i(2^{2j}t\sigma+y'\cdot\zeta')}(i\zeta',-|\zeta'|)^{\mathsf{T}}(1+\partial_\sigma^2)\Big(m(2^{2j}\sigma,2^j\zeta')\widehat{\zeta}(\sigma)\Big)\\
&\qquad\qquad\times\exp\big(-2^j\eta(|\zeta'|-\frac12)\big)\widehat{\phi}(\zeta')d\zeta'd\sigma\Big\|_{L^1_{y'}(B_1)}\\
\le&C2^je^{-(2^{j-1}\eta)}\frac{2^{2j}}{\langle2^{2j}t\rangle^2}.
\end{aligned}
\tag{3.19}
$$

Very much similar way to the case except the cut off function $\widehat{\zeta}(\sigma)$ instead of $\widehat{\psi}(\sigma)$, we proceed as before that

$$
\begin{aligned}
&\Big\| \sum_{k<2j} \pi_{k,j}(t,\cdot) \Big\|_{L^1_{x'}(B^c_{2^{-j}})} \\
&\quad = 2^{3j} e^{-(2^{j-1}\eta)} \frac{1}{(2^{2j}t)^2} \Bigg\| \frac{c_{n+1}}{\langle y'\rangle^n} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i(2^k t\sigma + y'\cdot\zeta')} \\
&\qquad \times (1-\partial_\sigma^2)\cdot(1-\Delta_{\zeta'})^{\frac{n}{2}} \Big((i\zeta', -|\zeta'|)^{\mathsf{T}} m(2^k\sigma, 2^j\zeta') \exp\big(-2^j\eta(|\zeta'| - \frac{1}{2})\big) \widehat{\zeta}(\sigma)\widehat{\phi}(\zeta') \Big) d\zeta' d\sigma \Bigg\|_{L^1_{y'}(B^c_1)} \\
&\quad \le C 2^j \big(1 + (2^j\eta)^{n+2}\big) e^{-(2^{j-1}\eta)} \frac{2^{2j}}{\langle 2^{2j}t\rangle^2}.
\end{aligned} \tag{3.20}
$$

From (3.19) and (3.20), we conclude the desired estimate. ❑

3.2. **The second almost orthogonality.** We consider the almost orthogonality estimate of second type which will be used for the triumphal arch type Littlewood–Paley dyadic decomposition.

Lemma 3.3 (Almost orthogonality II). *Let $k, j, m \in \mathbb{Z}$ and $\pi_{k,j}(\tau, \xi', \eta)$ be the pressure potential given by (3.1) and let $\{\psi_k(t)\}_{k\in\mathbb{Z}}$ and $\{\phi_j(x)\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley decompositions for time and space, respectively. Assume that $j \le m$. Then for any $N \in \mathbb{N}$, there exists a constant $C_{n,N} > 0$ depending on n and N such that the following estimates hold:*

(1) *For the time-dominated region $k \ge 2j$,*

$$
\big\| \phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta) \big\|_{L^1_{x'}} \le C_{n,N} \frac{2^j 2^{-(m-j)}}{\langle 2^j\eta\rangle^N} \frac{2^k}{\langle 2^k t\rangle^2}. \tag{3.21}
$$

(2) *For the space-dominated region $k < 2j$, it holds that*

$$
\Big\| \sum_{k<2j} \phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta) \Big\|_{L^1_{x'}} \le C_{n,N} \frac{2^j 2^{-(m-j)}}{\langle 2^j\eta\rangle^N} \frac{2^{2j}}{\langle 2^{2j} t\rangle^2}. \tag{3.22}
$$

Proof of Lemma 3.3. (1) In the time-dominated region $k \ge 2j$, by using the expression of the fundamental solution and using change of variables $\tau = 2^k\sigma$, $\xi' = 2^j\zeta'$ and then $x' = 2^{-j}y'$, we have

$$
\begin{aligned}
&\|\phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta)\|_{L^1_{x'}} \\
&\quad = \Bigg\| c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{it\tau + ix'\cdot\xi'} (i\xi', -|\xi'|)^{\mathsf{T}} m(\tau,\xi') \big(\phi_m \underset{(\eta)}{*} e^{-|\xi'|\eta}\big) \widehat{\psi}(2^{-k}\tau)\widehat{\phi}(2^{-j}\xi') d\xi' d\tau \Bigg\|_{L^1_{x'}} \\
&\quad = 2^k \Bigg\| c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i2^k t\sigma + iy'\cdot\zeta'} 2^j (i\zeta', -|\zeta'|)^{\mathsf{T}} m(2^k\sigma, 2^j\zeta') \big(\phi_m \underset{(\eta)}{*} e^{-2^j|\zeta'|\eta}\big) \widehat{\psi}(\sigma)\widehat{\phi}(\zeta') d\zeta'\, d\sigma \Bigg\|_{L^1_{y'}}.
\end{aligned} \tag{3.23}
$$

Applying

$$
e^{i(2^k t\sigma + y'\cdot\zeta')} = \frac{1}{(2^k it)^2} \frac{1}{(iy')^n} \partial_\sigma^2 \partial_{\zeta'}^n e^{i(2^k t\sigma + y'\cdot\zeta')}, \tag{3.24}
$$

and integration by parts in the right hand side of (3.23), we see that

$$
\begin{aligned}
&\|\phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta)\|_{L^1_{x'}} \\
&\simeq \frac{2^k}{\langle 2^k t\rangle^2} \Bigg\| \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{1}{\langle y'\rangle^n} e^{2^k it\sigma + iy'\cdot\zeta'} (1-\partial_\sigma^2) \Big(\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|\le n} C_n \partial_{\zeta'}^{\alpha_1} \big(2^j (i\zeta', -|\zeta'|)^{\mathsf{T}} (\phi_m \underset{(\eta)}{*} e^{-2^j|\zeta'|\eta})\big) \\
&\qquad\qquad \times \partial_{\zeta'}^{\alpha_2} \big(m(2^k\sigma, 2^j\zeta')\big) \partial_{\zeta'}^{\alpha_3} \widehat{\phi}(\zeta') \Big) \widehat{\psi}(\sigma) d\xi' d\tau \Bigg\|_{L^1_{y'}}
\end{aligned}
$$

$$\simeq \frac{2^k}{\langle 2^k t\rangle^2}\Bigg\| \frac{1}{\langle y'\rangle^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{2^k it\sigma + iy'\cdot\zeta'} 2^j(1-\partial_\sigma^2)\Bigg(\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|\le n} C_n\Big(\phi_m \underset{(\eta)}{*} \big(1+|\zeta'|+(2^j\eta)^{\alpha_1}\big)e^{-2^j|\zeta'|\eta}\Big)$$

$$\times\, \partial_{\zeta'}^{\alpha_2}\big(m(2^k\sigma, 2^j\zeta')\big)\partial_{\zeta'}^{\alpha_3}\widehat{\phi}(\zeta')\Bigg)\widehat{\psi}(\sigma)d\zeta' d\sigma\Bigg\|_{L^1_{y'}}. \tag{3.25}$$

Note that the above estimates are also valid even for $\alpha = \alpha_1+\alpha_2+\alpha_3 = 0$. Since all the functions in the integrand involving $\xi'(=2^j\zeta')$ are spherically symmetric, we employ integration by parts with respect to $|\zeta'|$, then $\bar{\eta} = 2^j|\zeta'|\eta$ appears. Namely, if we take integration by parts $|\alpha|$ times, the same estimate holds. There is no influence from integration by parts with respect to σ.

To consider the effect from the convolution $\phi_m \underset{(\eta)}{*}$, we restrict $j \le m$ (cf. (4.11)). Applying the change of variable $\tilde{\theta} = 2^m\theta$, $\tilde{\eta} = 2^m\eta$ and setting

$$p_\alpha(\zeta',\eta,\theta,\nu) = \Big(|\zeta'|\big(1+|\zeta'|+(2^j(\eta-\nu\theta))^\alpha\big) + \alpha\big(2^j(\eta-\nu\theta)\big)^{\alpha-1}\Big)$$

and noting $\int_{\mathbb{R}}\phi_m(\theta)d\theta = 0$, it follows that

$$\begin{aligned}
&\Big|\phi_m \underset{(\eta)}{*} \big(1+|\zeta'|+(2^j\eta)^\alpha\big)e^{-2^j|\zeta'|\eta}\Big| \\
&\quad= \Big|\int_{\mathbb{R}}\phi_m(\theta)2^j\Big(\big(1+|\zeta'|+(2^j(\eta-\theta))^\alpha\big)\exp\big(-2^j|\zeta'||\eta-\theta|\big) \\
&\qquad\qquad -\big(1+|\zeta'|+(2^j\eta)^\alpha\big)\exp\big(-2^j|\zeta'||\eta|\big)\Big)d\theta\Big| \\
&\quad\le \int_{\mathbb{R}}\Big|\phi_m(\theta)\Big(2^j\int_0^1 \frac{d}{d\nu}\big(1+|\zeta'|+(2^j(\eta-\nu\theta))^\alpha\big)\exp\big(-2^j|\zeta'||\eta-\nu\theta|\big)d\nu\Big)\Big|\,d\theta \\
&\quad\le \int_0^1\int_{\mathbb{R}}\big|\phi_m(\theta)\big|2^j\Big(2^j|\theta||\zeta'|\big(1+|\zeta'|+(2^j(\eta-\nu\theta))^\alpha\big)+2^j|\theta|\alpha\big(2^j(\eta-\nu\theta)\big)^{\alpha-1}\Big) \\
&\qquad\qquad \times \exp\big(-2^j|\zeta'||\eta-\nu\theta|\big)d\theta d\nu \\
&\quad= \int_0^1\int_{\mathbb{R}} 2^{2j}|2^m\phi(2^m\theta)||\theta|p_\alpha(\zeta',\eta,\theta,\nu)\exp\big(-2^j|\zeta'||\eta-\nu\theta|\big)d\theta d\nu \\
&\quad= 2^j|\zeta'|^{-1}\int_0^1\int_{|\bar{\theta}|>\frac{1}{2}|\bar{\eta}|}\big|\phi(2^m 2^{-j}|\zeta'|^{-1}\bar{\theta})\big|2^m 2^{-j}|\zeta'|^{-1}|\bar{\theta}|\bar{p}_\alpha(\bar{\eta},\bar{\theta})\exp\big(-|\bar{\eta}-\nu\bar{\theta}|\big)d\bar{\theta}d\nu \\
&\qquad + 2^j|\zeta'|^{-1}\int_0^1\int_{|\bar{\theta}|\le\frac{1}{2}|\bar{\eta}|}\big|\phi(2^m 2^{-j}|\zeta'|^{-1}\bar{\theta})\big|2^m 2^{-j}|\zeta'|^{-1}|\bar{\theta}|\bar{p}_\alpha(\bar{\eta},\bar{\theta})\exp\big(-|\bar{\eta}-\nu\bar{\theta}|\big)d\bar{\theta}d\nu \\
&\quad\equiv I + II,
\end{aligned} \tag{3.26}$$

where we set

$$\bar{\eta} = 2^j|\zeta'|\eta, \quad \bar{\theta} = 2^j|\zeta'|\theta, \tag{3.27}$$

$$\bar{p}_\alpha(\bar{\eta},\bar{\theta}) = p_\alpha(\zeta',\bar{\eta},\bar{\theta},\nu). \tag{3.28}$$

By $1/2 \le |\zeta'| \le 2$ we note that

$$\big|\bar{p}_\alpha(\bar{\eta},\bar{\theta})\big|\exp\big(-2^{-1}|\bar{\eta}-\nu\bar{\theta}|\big) \le C\Big(1+|\bar{\eta}-\nu\bar{\theta}|^\alpha\Big)\exp\big(-2^{-1}|\bar{\eta}-\nu\bar{\theta}|\big) \le C. \tag{3.29}$$

For the first term I of (3.26), by using the decay property of $\phi \in \mathcal{S}$, we know for sufficiently large $N \in \mathbb{N}$ there exists $C_N > 0$ such that by (3.29) that

$$
\begin{aligned}
I \le & 2^j|\zeta'|^{-1}\int_0^1\int_{|\bar\theta|>\frac{1}{2}|\bar\eta|}\frac{C_N 2^m 2^{-j}|\zeta'|^{-1}|\bar\theta|}{\langle 2^m 2^{-j}|\zeta'|^{-1}\bar\theta\rangle^{2N}}\big|\bar p_\alpha(\bar\eta,\bar\theta)\big|\exp\big(-|\bar\eta-\nu\bar\theta|\big)d\bar\theta d\nu \\
= & 2^j|\zeta'|^{-1}\frac{C_N(2^{m-j}|\zeta'|^{-1})^{-1}}{\langle 2^{m-1-j}|\zeta'|^{-1}\bar\eta\rangle^N}\int_0^1\int_{|\bar\theta|>\frac{1}{2}|\bar\eta|}\frac{(2^{m-j}|\zeta'|^{-1})^2|\bar\theta|}{\langle 2^{m-j}|\zeta'|^{-1}\bar\theta\rangle^N}\big|\bar p_\alpha(\bar\eta,\bar\theta)\big|\exp\big(-|\bar\eta-\nu\bar\theta|\big)d\bar\theta d\nu \\
\le & 2^j\frac{C_N 2^{-m+j}}{\langle 2^{m-1}\eta\rangle^N}\int_{\mathbb{R}}\frac{(2^{m-j}|\zeta'|^{-1})^2|\bar\theta|}{\langle 2^{m-j}|\zeta'|^{-1}\bar\theta\rangle^N}d\bar\theta \\
\le & \frac{C_N 2^{-m+j}2^j|\zeta'|^{-2}}{\langle 2^m\eta\rangle^N}\le C_N 2^j\frac{2^{-m+j}}{\langle 2^j\eta\rangle^N},
\end{aligned}
\tag{3.30}
$$

where we used $j \le m$. For an estimate of the second term II of (3.26), we use the following lemma.

Lemma 3.4. *For $N \in \mathbb{N}\setminus\{1\}$ and $a > 0$, it holds that*

$$
\int_{-a\le x\le a}\frac{dx}{(1+|x|^2)^{N/2}} \le \frac{4a}{(1+a^2)^{1/2}}. \tag{3.31}
$$

Proof of Lemma 3.4. For any $N \ge 1$ and $a \le 1$

$$
\int_{-a\le x\le a}\frac{dx}{(1+|x|^2)^{N/2}} \le \int_{|x|\le a}dx = 2a \le \frac{4a}{(1+a^2)^{1/2}},
$$

while for any $N \ge 2$ and $a \ge 1$,

$$
\int_{-a\le x\le a}\frac{dx}{(1+|x|^2)^{N/2}} \le \int_{|x|\le a}\frac{dx}{1+|x|^2} = 2\tan^{-1}a \le \frac{4a}{(1+a^2)^{1/2}}.
$$

This shows (3.31). ❑

Proof of Lemma 3.3, continued. Under the condition $|\bar\theta| \le |\bar\eta|/2$, it holds that $|\bar\eta-\nu\bar\theta| \ge |\bar\eta|-|\bar\theta| \ge |\bar\eta|-|\bar\eta|/2 = |\bar\eta|/2$. By using the above estimate, (3.29) and (3.31), by changing the integral variables

$$
\bar{\bar\eta} = 2^m 2^{-j}|\zeta'|^{-1}\bar\eta, \qquad \bar{\bar\theta} = 2^m 2^{-j}|\zeta'|^{-1}\bar\theta,
$$

the second term II of (3.26) is estimated as follows:

$$
\begin{aligned}
II = & 2^j|\zeta'|^{-1}\int_0^1\int_{|\bar\theta|\le\frac{1}{2}|\bar\eta|}\frac{C_N 2^m 2^{-j}|\zeta'|^{-1}|\bar\theta|}{\langle 2^m 2^{-j}|\zeta'|^{-1}\bar\theta\rangle^{N+1}}\big|\bar p_\alpha(\bar\eta,\bar\theta)\big|\exp\big(-|\bar\eta-\nu\bar\theta|\big)d\bar\theta d\nu \\
= & C_N 2^j 2^{-m+j}\int_0^1\int_{|\bar\theta|\le\frac{1}{2}|\bar\eta|}\frac{(2^m 2^{-j}|\zeta'|^{-1})^2|\bar\theta|}{\langle 2^m 2^{-j}|\zeta'|^{-1}\bar\theta\rangle^{N+1}}\big|\bar p_\alpha(\bar\eta,\bar\theta)\big|\exp\big(-|\bar\eta-\nu\bar\theta|\big)d\bar\theta d\nu \\
\le & C_N 2^j 2^{-m+j}\int_0^1\int_{|\bar\theta|\le\frac{1}{2}|\bar\eta|}\frac{(2^m 2^{-j}|\zeta'|^{-1})^2|\bar\theta|}{\langle 2^{m-j}|\zeta'|^{-1}\bar\theta\rangle^{N+1}}\exp\big(-2^{-1}|\bar\eta-\nu\bar\theta|\big)d\bar\theta d\nu \\
\le & C_N 2^j 2^{-m+j}\exp\big(-\frac{1}{4}|\bar\eta|\big)\int_{|\bar{\bar\theta}|\le\frac{1}{2}|\bar{\bar\eta}|}\frac{\bar{\bar\theta}}{\langle\bar{\bar\theta}\rangle^{N+1}}d\bar{\bar\theta} \\
\le & C_N 2^j 2^{-m+j}\exp\big(-\frac{1}{4}|\bar\eta|\big)\frac{8\cdot 2^m 2^{-j}|\zeta|^{-1}\bar\eta}{\langle 2^m 2^{-j}|\zeta|^{-1}\bar\eta\rangle} \\
\le & C_N\frac{2^j 2^{-m+j}}{\langle 2^j\eta\rangle^{N-1}},
\end{aligned}
\tag{3.32}
$$

where we used (3.27) and C_N is a constant depending on sufficiently large $N \in \mathbb{N}$. Hence the similar estimate as in Lemma 3.2 holds and we gain the decay for variables y' and t. Applying (3.26), (3.30), (3.32) into

(3.25) and recalling (3.18), we see that

$$
\begin{aligned}
&\|\phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta)\|_{L^1_{x'}} \\
&\quad\simeq \frac{2^k}{\langle 2^k t\rangle^2}\Big\| \frac{1}{\langle y'\rangle^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{2^k it\sigma+iy'\cdot\zeta'}2^j(1-\partial_\sigma^2)\Big(\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|\le n} C_n\partial_{\zeta'}^{\alpha_2}\big(m(2^k\sigma,2^j\zeta')\big) \\
&\qquad\qquad \times\big(\phi_m \underset{(\eta)}{*} \big(1+|\zeta'|+(2^j\eta)^{\alpha_1}\big)e^{-2^j|\zeta'|\eta}\big)\widehat{\psi}(\sigma)\partial_{\zeta'}^{\alpha_3}\widehat{\phi}(\zeta')\Big)d\zeta' d\sigma\Big\|_{L^1_{y'}} \\
&\quad\le \frac{2^k}{\langle 2^k t\rangle^2}\Big\| \frac{1}{\langle y'\rangle^n}\int_{2^{-1}<|\sigma|<2}\int_{2^{-1}<|\zeta'|<2} 2^j\big|\phi_m \underset{(\eta)}{*} \big(1+|\zeta'|+(2^j\eta)^n\big)e^{-2^j|\zeta'|\eta}\big|d\zeta' d\sigma\Big\|_{L^1_{y'}} \\
&\quad\le \frac{C_N 2^j 2^{-(m-j)}}{\langle 2^j\eta\rangle^N}\frac{2^k}{\langle 2^k t\rangle^2}. \qquad (3.33)
\end{aligned}
$$

From the estimate (3.33), we conclude that (3.21) holds.

(2) To see the estimate (3.22), we recall the low frequency restriction ζ given by (1.12) and it follows

$$
\begin{aligned}
&\Big\|\sum_{k<2j}\phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta)\Big\|_{L^1_{x'}} \\
&=\Big\| c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}} m(\tau,\xi')\big(\phi_m \underset{(\eta)}{*} e^{-|\xi'|\eta}\big)\sum_{k<2j}\widehat{\psi}(2^{-k}\tau)\widehat{\phi}(2^{-j}\xi')d\xi' d\tau\Big\|_{L^1_{x'}} \\
&=\Big\| c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}} m(\tau,\xi')\big(\phi_m \underset{(\eta)}{*} e^{-|\xi'|\eta}\big)\widehat{\zeta}(2^{-2j}\tau)\widehat{\phi}(2^{-j}\xi')d\xi' d\tau\Big\|_{L^1_{x'}} \\
&=C_n 2^{2j}\Big\|\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{i2^{2j}t\sigma+iy'\cdot\zeta'}2^j(i\zeta',-|\zeta'|)^{\mathsf{T}} m(2^{2j}\sigma,2^j\zeta')\big(\phi_m \underset{(\eta)}{*} e^{-2^j|\zeta'|\eta}\big)\widehat{\zeta}(\sigma)\widehat{\phi}(\zeta')d\zeta' d\sigma\Big\|_{L^1_{y'}}. \qquad (3.34)
\end{aligned}
$$

Applying (3.24) and integration by parts in the right hand side of (3.34), we see by using (3.26), (3.30) and (3.32) again that

$$
\begin{aligned}
&\Big\|\sum_{k<2j}\phi_m \underset{(\eta)}{*} \pi_{k,j}(t,\cdot,\eta)\Big\|_{L^1_{x'}} \\
&\quad\simeq \frac{2^{2j}}{\langle 2^{2j} t\rangle^2}\Big\| \frac{1}{\langle y'\rangle^n}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{2^{2j} it\sigma+iy'\cdot\zeta'}2^j(1-\partial_\sigma^2)\Big(\sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|\le n} C_n\partial_{\zeta'}^{\alpha_2}\big(m(2^{2j}\sigma,2^j\zeta')\big) \\
&\qquad\qquad \times\big(\phi_m \underset{(\eta)}{*} \big(1+|\zeta'|+(2^j\eta)^{\alpha_1}\big)e^{-2^j|\zeta'|\eta}\big)\widehat{\zeta}(\sigma)\partial_{\zeta'}^{\alpha_3}\widehat{\phi}(\zeta')\Big)d\zeta' d\sigma\Big\|_{L^1_{y'}} \\
&\quad\le \frac{C_N 2^j 2^{-(m-j)}}{\langle 2^j\eta\rangle^N}\frac{2^{2j}}{\langle 2^{2j} t\rangle^2}. \qquad (3.35)
\end{aligned}
$$

The estimate (3.35) shows (3.22). This completes the proof of Lemma 3.3. ❑

4. Proof of maximal L^1-regularity

In this section, we prove maximal L^1-regularity Theorem 2.2. The key estimate is the bound for the derivative of the pressure term $\nabla q(t,x)$. Indeed, once we obtain the required estimate for the pressure, then the estimate for the velocity directly follows from the estimate for the heat equation. Note that the velocity term can be also expressed by the potential as is shown in (2.18).

4.1. **Maximal regularity for the pressure.**

To show Theorem 2.2, we show maximal L^1-regularity for the pressure term. We recall the notations for the potential (3.1) for the pressure ∇q.

Proposition 4.1. *Let $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. For given data*

$$H \in \dot{F}_{1,1}^{\frac{1}{2}-\frac{1}{2p}}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1})),$$

there exists $C > 0$ independent of H such that the pressure part q of the problem (2.11) satisfies the estimate

$$\big\|\nabla q\big\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \le C\Big(\|H\|_{\dot{F}_{1,1}^{\frac{1}{2}-\frac{1}{2p}}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big). \tag{4.1}$$

To show the pressure estimate (4.1), we use the potential expression $\pi(t, x', \eta)$ in (2.24) and the Littlewood–Paley decomposition of unity (2.36);

$$\begin{aligned}
&\overline{\Phi_m} \underset{(x',\eta)}{*} \big(\pi(t,x',\eta)\big) \\
&= c_{n+1}\overline{\Phi_m}(x',\eta) \underset{(x',\eta)}{*} \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}} e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}} m(\tau,\xi')e^{-|\xi'|\eta}d\xi' d\tau \\
&= \zeta_{m-1}(\eta) \underset{(\eta)}{*} c_{n+1}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} e^{it\tau+ix'\cdot\xi'}\widehat{\phi_m}(|\xi'|)(i\xi',-|\xi'|)^{\mathsf{T}} m(\tau,\xi')e^{-|\xi'\eta}d\xi' d\tau \\
&\quad + \phi_m(\eta) \underset{(\eta)}{*} c_{n+1}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} e^{it\tau+ix'\cdot\xi'}\widehat{\zeta_m}(|\xi'|)(i\xi',-|\xi'|)^{\mathsf{T}} m(\tau,\xi')e^{-|\xi'|\eta}d\xi' d\tau \\
&\equiv \zeta_{m-1}(\eta) \underset{(\eta)}{*} \phi_m(x') \underset{(x')}{*} \pi(t,x',\eta) + \phi_m(\eta) \underset{(\eta)}{*} \zeta_m(x') \underset{(x')}{*} \pi(t,x',\eta).
\end{aligned} \tag{4.2}$$

Concerning the first term of the right-hand side of (4.2), we estimate that the convolution with $\zeta_{m-1}(\eta)$ can be treated by the Hausdorff–Young inequality in η-variable. Note that the potential $\pi(t, x', \eta)$ has the even extension in $\eta \in \mathbb{R}$ and hence the $L^p(\mathbb{R}^n_+)$ norm of the term is estimated as follows:

$$\begin{aligned}
&\Big\|\zeta_{m-1} \underset{(\eta)}{*} \Big(\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta)\Big)\Big\|_{L^p(\mathbb{R}_{+,\eta};L^p(\mathbb{R}^{n-1}))} \\
&\le \Big\|\zeta_{m-1} \underset{(\eta)}{*} \Big(\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta)\Big)\Big\|_{L^p(\mathbb{R}_{+,\eta};L^p(\mathbb{R}^{n-1}))} \\
&\le \|\zeta_{m-1}\|_{L^1(\mathbb{R}_{+,\eta})}\big\|\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta)\big\|_{L^p(\mathbb{R}_{+,\eta};L^p(\mathbb{R}^{n-1}))} \\
&\le C\big\|\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta)\big\|_{L^p(\mathbb{R}_{+,\eta};L^p(\mathbb{R}^{n-1}))}
\end{aligned} \tag{4.3}$$

and we apply Lemma 3.2. Concerning the second term of the right-hand side of (4.2), the number of overlapping supports of the kernel $\zeta_m(x') \underset{(x')}{*} \phi_j(x')$ is infinite, i.e., m and j run independently. We apply the almost orthogonality of the second type stated in Lemma 3.3.

Proof of Proposition 4.1. Let us recall that the boundary data $H(t,x') = (H'(t,x'), H_n(t.x'))$ is extended into $t < 0$ by the zero extension. By (4.2), we divide the term into two terms.

$$\begin{aligned}
&\nabla q(t,x',x_n) \\
&= c_{n+1}\iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}(i\xi',-|\xi'|)^{\mathsf{T}}\Big(m'(\tau,\xi')\cdot\widehat{H'} + m_n(\tau,\xi')\widehat{H_n}\Big)e^{-|\xi'|x_n}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\widehat{\phi_k}(\tau)\widehat{\phi_j}(\xi)d\tau d\xi' \\
&\equiv \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\Big({\pi'}_{k,j} \underset{(t,x')}{\cdot *} \Big(\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_j} \underset{(x')}{*} H'\Big) + {\pi_n}_{k,j} \underset{(t,x')}{*} \Big(\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_j} \underset{(x')}{*} H_n\Big)\Big).
\end{aligned}$$

Then observing the estimate (4.3), we see

$$\begin{aligned}
&\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} \\
&\le C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big\|\Big(\int_{\mathbb{R}}\Big|\zeta_{m-1}(\tilde\eta) \underset{(\tilde\eta)}{*} \phi_m(x') \underset{(x')}{*} \pi(t,x',\eta) \underset{(t,x')}{\cdot *} H\Big|^p d\tilde\eta\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)} \\
&\quad + C\Big\|\sum_{m\in\mathbb{Z}}2^{sm}\Big\|\Big(\int_{\mathbb{R}}\Big|\phi_m(\tilde\eta) \underset{(\tilde\eta)}{*} \zeta_m(x') \underset{(x')}{*} \pi(t,x',\eta) \underset{(t,x')}{\cdot *} H\Big|^p d\tilde\eta\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}
\end{aligned}$$

$$
\begin{aligned}
&\le C\Big\| \sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}} \Big\|\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta) \underset{(t,x')}{\cdot *} H(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1})} d\tilde\eta\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\quad + C\Big\| \sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}} \Big\|\Big(\phi_m(\tilde\eta) \underset{(\tilde\eta)}{*} \zeta_m(x') \underset{(x')}{*} \pi(t,x',\eta)\Big) \underset{(t,x')}{\cdot *} H(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} d\tilde\eta\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\equiv \|P_1(t)\|_{L^1_t(\mathbb{R}_+)} + \|P_2(t)\|_{L^1_t(\mathbb{R}_+)},
\end{aligned}
\tag{4.4}
$$

where we denote the inner product-convolution by (1.17). Noting that the data H is divided into the the time-dominated region $k \ge 2j$ and the space-dominated region $k < 2j$, respectively, as

$$
H(t,x') = \sum_{k\in\mathbb{Z}} \sum_{2j\le k} H_{k,j}(t,x') + \sum_{k\in\mathbb{Z}} \sum_{2j>k} H_{k,j}(t,x'),
\tag{4.5}
$$

where we set

$$
\begin{aligned}
H_{k,j}(t,x') &= \widetilde{\psi_k}(t) \underset{(t)}{*} \widetilde{\phi_j}(x') \underset{(x')}{*} H(t,x'),\\
H_j(t,x') &= \widetilde{\phi_j}(x') \underset{(x')}{*} H(t,x'),
\end{aligned}
\tag{4.6}
$$

where we use $\widetilde{\phi_j} = \phi_{j-1} + \phi_j + \phi_{j+1}$ and $\widetilde{\psi_k}$ similar arrangement. Then applying $\widetilde{\phi_j} \underset{(x')}{*} \phi_j = \phi_j$ and $\widetilde{\psi_k} \underset{(t)}{*} \psi_k = \psi_k$, and Proposition 2.5, we divide $P_1(t)$ into $L_1(t)$ and $L_2(t)$ to have the following:

$$
\begin{aligned}
P_1(t) \le& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\Big\|\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta) \underset{(t,x')}{\cdot *} \sum_{j\in\mathbb{Z}}\sum_{k\ge 2j} H_{k,j}(t,x')\Big\|_{L^p(\mathbb{R}^{n-1}_{x'})}\Big\|_{L^p(\mathbb{R}_{+,\eta})}\\
&+ C\sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\Big\|\phi_m(x') \underset{(x')}{*} \pi(t,x',\eta) \underset{(t,x')}{\cdot *} \sum_{j\in\mathbb{Z}}\sum_{k< 2j} H_{k,j}(t,x')\Big\|_{L^p(\mathbb{R}^{n-1}_{x'})}\Big\|_{L^p(\mathbb{R}_{+,\eta})}\\
=& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+} \Big\|\sum_{k\ge 2m} \pi_{k,m} \underset{(t,x')}{\cdot *} H_{k,m}(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} d\eta\Big)^{1/p}\\
&+ C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+} \Big\|\sum_{k< 2m} \pi_{k,m} \underset{(t,x')}{\cdot *} H_m(t,x'))\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})} d\eta\Big)^{1/p}\\
\equiv& L_1(t) + L_2(t),
\end{aligned}
\tag{4.7}
$$

where $\{\pi_{k,m}\}_{k,m}$ are defined in (3.1). For the time dominated part L_1, since $k \ge 2m$, we apply the almost orthogonality estimate (3.6) in Lemma 3.2, by using the change of valuable $2^m\eta = \bar\eta$ it holds that

$$
\begin{aligned}
\|L_1\|_{L^1(\mathbb{R}_+)} \le& C\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k\ge 2m}\int_{\mathbb{R}} \Big\|\pi_{k,m}(t-s,x',\eta)\Big\|_{L^1_{x'}} \Big\|H_{k,m}(s,x')\Big\|_{L^p_{x'}} ds\Big\}^p d\eta\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k\ge 2m} \big(2^m(1+(2^m\eta)^{n+2}\big)e^{-(2^{m-1}\eta)}\\
&\qquad\times \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\widetilde{\psi_k} \underset{(t)}{*} \widetilde{\phi_m} \underset{(x')}{*} H(s,\cdot)\Big\|_{L^p_{x'}} ds\Big\}^p d\eta\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
=& C\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\Big\{\sum_{k\ge 2m} 2^m \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot)\Big\|_{L^p_{x'}} ds\Big\}^p\\
&\qquad\times 2^{-m}\Big(\int_{\mathbb{R}_+} \Big((1+\bar\eta^{n+2})e^{-\bar\eta/2}\Big)^p d\bar\eta\Big)\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}2^{(1-\frac{1}{p})m}\sum_{k\ge 2m}\int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2}\Big\|\widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot)\Big\|_{L^p_{x'}} ds\Big\|_{L^1_t(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}(1-\frac{1}{p})}\Big\| \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2}\sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\widetilde{\psi_k} \underset{(t)}{*} H_m(s,\cdot)\Big\|_{L^p_{x'}} ds\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}(1-\frac{1}{p})}\Big\| \sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\widetilde{\psi_k} \underset{(t)}{*} H_m(t,\cdot)\Big\|_{L^p_{x'}}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{k\in\mathbb{Z}} 2^{\frac{k}{2}(1-\frac{1}{p})}\Big\|\psi_k \underset{(t)}{*} H_m(t,\cdot)\Big\|_{\dot{B}^s_{p,1}(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\big\|H\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned}\tag{4.8}
$$

On the other hand, when $k<2m$, For the space dominated part L_2, applying the almost orthogonality estimate (3.7) in Lemma 3.2 with using the Minkowski inequality, the Hausdorff–Young inequality, we obtain

$$
\begin{aligned}
\|L_2\|_{L^1_t} \le&\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\int_{\mathbb{R}}\Big\|\sum_{k<2m}\pi_{k,m}(t-s,x',\eta)\Big\|_{L^1_{x'}}\|H_m(s,\cdot)\|_{L^p_{x'}}ds\Big\}^p d\eta\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\big(2^m(1+(2^m\eta)^{n+2})e^{-(2^{m-1}\eta)}\big)\\
&\qquad\qquad\times\int_{\mathbb{R}}\frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2}\|H_m(s,\cdot)\|_{L^p_{x'}}ds\Big\}^p d\eta\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(2^{mp(1-\frac{1}{p})}\int_{\mathbb{R}_+}\Big((1+\bar\eta^{n+2})e^{-(2^{-1}\bar\eta)}\Big)^p d\bar\eta\\
&\qquad\qquad\times\Big\{\int_{\mathbb{R}}\frac{2^{2m}}{\langle 2^{2m}(t-s)\rangle^2}\|H_m(s,\cdot)\|_{L^p_{x'}}ds\Big\}^p\Big)^{1/p}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\sum_{m\in\mathbb{Z}} 2^{(s+1-\frac{1}{p})m}\Big\|\frac{2^{2m}}{\langle 2^{2m}t\rangle^2}\Big\|_{L^1_t(\mathbb{R})}\Big\|\|H_m(t,\cdot)\|_{L^p_{x'}}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\big\|H\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned}\tag{4.9}
$$

In the same way for $P_1(t)$, we decompose $P_2(t)$ as a space-dominated region and a time-dominated region.

$$
\begin{aligned}
P_2(t) \le& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\Big\|\phi_m(\eta)\underset{(\eta)}{*}\zeta_m(x')\underset{(x')}{*}\pi(t,x',\eta)\underset{(t,x')}{\cdot *}\sum_{k\in\mathbb{Z}}\sum_{2j\le k}H_{k,j}(t,x')\Big\|_{L^p(\mathbb{R}^{n-1}_{x'})}\Big\|_{L^p(\mathbb{R}_{+,\eta})}\\
&+C\sum_{m\in\mathbb{Z}} 2^{sm}\Big\|\Big\|\phi_m(\eta)\underset{(\eta)}{*}\zeta_m(x')\underset{(x')}{*}\pi(t,x',\eta)\underset{(t,x')}{\cdot *}\sum_{k\in\mathbb{Z}}\sum_{2j> k}H_{k,j}(t,x')\Big\|_{L^p(\mathbb{R}^{n-1}_{x'})}\Big\|_{L^p(\mathbb{R}_{+,\eta})}\\
\le& C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\|\sum_{k\in\mathbb{Z}}\sum_{2j\le\min(2m,k)}\phi_m(\eta)\underset{(\eta)}{*}\pi_{k,j}\underset{(t,x')}{\cdot *}H_{k,j}(t,x')\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})}d\eta\Big)^{1/p}\\
&+C\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\|\sum_{k\in\mathbb{Z}}\sum_{k<2j\le 2m}\phi_m(\eta)\underset{(\eta)}{*}\pi_{k,j}\underset{(t,x')}{\cdot *}H_j(t,x'))\Big\|^p_{L^p(\mathbb{R}^{n-1}_{x'})}d\eta\Big)^{1/p}\\
\equiv& M_1(t)+M_2(t).
\end{aligned}\tag{4.10}
$$

For the time dominated part M_1, Setting $h_j\equiv\widetilde{\phi_j}*h$, using the Minkowski inequality and the Hausdorff–Young inequality, and also using (3.21) in Lemma 3.3 (1) (the second almost orthogonality), we have

$$
\begin{aligned}
&\big\|M_1\big\|_{L^1_t(\mathbb{R}_+)}\\
&\le\Big\|\sum_{m\in\mathbb{Z}} 2^{sm}\Big(\int_{\mathbb{R}_+}\Big\{\sum_{k\in\mathbb{Z}}\sum_{2j\le\min(2m,k)}\int_{\mathbb{R}}\Big\|\phi_m(\eta)\underset{(\eta)}{*}\pi_{k,j}(t-s,x',\eta)\Big\|_{L^1_{x'}}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \Big\| \psi_k \underset{(s)}{*} H_j(s,x') \Big\|_{L^p_{x'}} ds \Big\}^p d\eta \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\{ \sum_{k\in\mathbb{Z}} \sum_{2j\le \min(2m,k)} \frac{2^j 2^{-(m-j)}}{\langle 2^j \eta\rangle^N} \\
&\qquad\qquad\qquad\qquad \times \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\}^p d\eta \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
=& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \sum_{k\in\mathbb{Z}} \sum_{2j\le\min(2m,k)} \\
&\qquad\qquad \times 2^j 2^{-(m-j)} \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} ds \Big(\int_{\mathbb{R}_+} \frac{1}{\langle \bar\eta\rangle^{pN}} 2^{-j} d\bar\eta \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
=& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \sum_{k\in\mathbb{Z}} \sum_{2j\le\min(2m,k)} 2^j 2^{-(m-j)} 2^{-\frac{j}{p}} \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \Big\| \sum_{k\in\mathbb{Z}} \sum_{2j\le k} \sum_{m\ge j} 2^{sm} 2^{-(m-j)} 2^{(1-\frac{1}{p})j} \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\
&\text{(setting } m = m'+j \text{ and changing } m\to m')\\
\le& C \Big\| \sum_{k\in\mathbb{Z}} \sum_{2j\le k} \sum_{m'\ge 0} 2^{s(m'+j)} 2^{-m'} 2^{(1-\frac{1}{p})j} \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \sum_{k\in\mathbb{Z}} \sum_{2j\le k} 2^{(1-\frac{1}{p})j} 2^{sj} \Big\| \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \Big\| \sum_{k\in\mathbb{Z}} 2^{(1-\frac{1}{p})\frac{k}{2}} \sum_{2j\le k} 2^{sj} \Big\| \psi_k \underset{(s)}{*} H_j(s,\cdot) \Big\|_{L^p_{x'}} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \| H \|_{\dot F^{\frac12 - \frac{1}{2p}}_{1,1}(\mathbb{R}_+; \dot B^s_{p,1}(\mathbb{R}^{n-1}))},
\end{aligned}
\tag{4.11}
$$

where we use $s<1$ for the convergence of the 4th line from the bottom.

The space dominated part M_2 is estimated in the similar way as M_1. We apply the Minkowski inequality and the Hausdorff–Young inequality

$$
M_2(t) \le \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\{ \sum_{j\le m} \int_{\mathbb{R}} \Big\| \phi_m(\eta) \underset{(\eta)}{*} \sum_{k<2j} \pi_{k,j}(t-s,x',\eta) \Big\|_{L^1_{x'}} \| H_j(s,x') \|_{L^p_{x'}} ds \Big\}^p d\eta \Big)^{1/p}.
$$

Using the almost orthogonality (3.22) in Lemma 3.3 (2) for $k<2j$ we have

$$
\begin{aligned}
\| M_2 \|_{L^1(\mathbb{R}_+)} \le& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \Big(\int_{\mathbb{R}_+} \Big\{ \sum_{j\le m} \frac{C_N 2^j 2^{-(m-j)}}{\langle 2^j\eta\rangle^N} \int_{\mathbb{R}} \frac{2^{2j}}{\langle 2^{2j}(t-s)\rangle^2} \Big\| H_j(s) \Big\|_{L^p_{x'}} ds \Big\}^p d\eta \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \sum_{j\le m} 2^j 2^{-(m-j)} \int_{\mathbb{R}} \frac{2^{2j}}{\langle 2^{2j}(t-s)\rangle^2} \Big\| H_j(s) \Big\|_{L^p_{x'}} ds \Big(\int_{\mathbb{R}_+} \frac{1}{\langle \widetilde\eta\rangle^{pN}} 2^{-j} d\widetilde\eta \Big)^{1/p} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \Big\| \sum_{m\in\mathbb{Z}} 2^{sm} \sum_{j\le m} 2^j 2^{-\frac{j}{p}} 2^{-(m-j)} \Big(\int_{\mathbb{R}} \frac{2^{2j}}{\langle 2^{2j}(t-s)\rangle^2} \Big\| H_j(s) \Big\|_{L^p_{x'}} ds \Big) \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le& C \sum_{m\in\mathbb{Z}} \sum_{j\le m} 2^{-(m-j)} 2^{sm} 2^{(1-\frac{1}{p})j} \Big\| \int_{\mathbb{R}} \frac{2^{2j}}{\langle 2^{2j}(t-s)\rangle^2} \Big\| H_j(s) \Big\|_{L^p_{x'}} ds \Big\|_{L^1_t(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
\le & C\Big\| \sum_{j\in\mathbb{Z}} 2^{(s+1-\frac{1}{p})j} \sum_{m'\ge 0} 2^{-m'+sm'} \Big\| \phi_j \underset{(x')}{*} H(t) \Big\|_{L^p_{x'}} \Big\|_{L^1_t(\mathbb{R}_+)} \\
= & C\big\| H \big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))},
\end{aligned} \tag{4.12}
$$

where we use $m-j=m'\ge 0$, $m\to m'$ in the third line from the bottom, and $s<1$ to obtain the convergence of $\sum_{m'\ge 0} 2^{-m'+sm'}$. In the last line, we enter the t-integral in the sum of of ℓ', and delete the convolution with respect to t by using the the sum of of ℓ'. Combining (4.4), (4.7)–(4.12), we obtain the estimate (4.1). The restriction on the regularity exponent s stems from the structure of the homogeneous Besov space stated in Propositions 2.3–2.6.

This complete the proof. ❑

The following estimate is required for showing maximal regularity for the velocity part of the Stokes equation.

Proposition 4.2. *Let $1\le p<\infty$ and $s\in\mathbb{R}$. Given boundary data*

$$
H\in \dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1})),
$$

let q be the pressure term defined by (2.23)*. Then there exists a constant $C>0$ such that the following estimates hold:*

$$
\big\| q|_{x_n=0} \big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} \le C\big\| H \big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}. \tag{4.13}
$$

Proof of Proposition 4.2. Let $\{\psi_k\}_{k\in\mathbb{Z}}$ and $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of the unity in $t\in\mathbb{R}$ and $x'\in\mathbb{R}^{n-1}$ variables, respectively. For simplicity, we assume that $q\in\mathcal{S}_0(\mathbb{R}^{n-1})$ and show the estimates (4.13). The results follows by the density $\mathcal{S}_0(\mathbb{R}^{n-1})\subset \dot{B}^s_{p,1}(\mathbb{R}^{n-1})$, where $\mathcal{S}_0(\mathbb{R}^{n-1})$ denotes the rapidly decreasing functions with vanishing at the origin of their Fourier images. Then the resulting estimates follows from the following bounds.

$$
\Big\| \| \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} q\big|_{x_n=0} \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \le C \Big\| \| \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)}. \tag{4.14}
$$

Indeed, admitting the above estimate (4.14), the Minkowski inequality yields

$$
\begin{aligned}
\big\| q\big|_{x_n=0} \big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} \le & C\sum_{k\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})k} \sum_{j\in\mathbb{Z}} 2^{sj} \Big\| \| \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} q\big|_{x_n=0} \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le & C\sum_{k\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})k} \sum_{j\in\mathbb{Z}} 2^{sj} \Big\| \| \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\le & C\Bigg\| \sum_{k\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})k} \sum_{j\in\mathbb{Z}} 2^{sj} \| \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} H \|_{L^p(\mathbb{R}^{n-1})} \Bigg\|_{L^1_t(\mathbb{R}_+)} \\
\le & C\big\| H \big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))},
\end{aligned}
$$

which implies (4.13).

To see (4.14), from (2.23), it follows

$$
\begin{aligned}
& \psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} q(t,x',x_n)\big|_{x_n=0} \\
& = c_{n+1} \iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'} \Big\{ \frac{|\xi'|+B}{D(\tau,\xi')} \Big(2B(i\xi'\cdot \widehat{H}') - (|\xi'|^2+B^2)\widehat{H}_n \Big) \Big\} \widehat{\psi_k}(\tau)\widehat{\phi_j}(\xi') d\tau d\xi'
\end{aligned} \tag{4.15}
$$

and the support of the symbol on the right hand side is in an annulus domain and hence there is no singular point in both τ, $|\xi'|$- variables and it gives a smooth symbol. Besides, for the time-like region $k\ge 2j$, by

Lemma 3.1, (3.4), (3.9) and (3.10) implies that

$$\left|\frac{2^j|\zeta'|+\sqrt{2^ki\sigma+2^{2j}|\zeta'|^2}}{D(2^k\sigma,2^j\zeta')}\Big(-2\cdot 2^j|\zeta'|\sqrt{2^ki\sigma+2^{2j}|\zeta'|^2}i\frac{\zeta'}{|\zeta'|}\Big)\right| = O(1).$$

Analogously for the space-like region, we see from (3.5) that

$$\left|\frac{2^j|\zeta'|+\sqrt{2^ki\sigma+2^{2j}|\zeta'|^2}}{D(2^k\sigma,2^j\zeta')}\Big((i2^k\sigma+2\cdot 2^{2j}|\zeta'|^2)\Big)\right| = O(1).$$

Those bounds enable us to treat the operator given by (4.15) is $L^p(\mathbb{R}^{n-1})$ bounded in x' and L^1 bound in t-variable. Thus the estimate (4.14) holds for all $1\le p\le\infty$. ❑

Proposition 4.3. *Let $1<p<\infty$ and $-1+1/p<s<1/p$. There exists $C>0$ such that for any $\nabla(-\Delta)^{-1}f\in C(\overline{\mathbb{R}_+};\dot B^s_{p,1}(\mathbb{R}^n_+))\cap \dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$, $\nabla f\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$, it holds that*

$$\begin{aligned}\sup_{x_n\in\mathbb{R}_+}&\Big(\big\|f(\cdot,\cdot,x_n)\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\big\|f(\cdot,\cdot,x_n)\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big)\\ &\le C\Big(\big\|\nabla f\big\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))},+\big\|\partial_t\nabla(-\Delta)^{-1}f\big\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\Big),\end{aligned}\tag{4.16}$$

In particular, for $1\le p<\infty$,

$$\sup_{x_n\in\mathbb{R}_+}\big\|f(\cdot,\cdot,x_n)\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\le C\big\|\nabla f\big\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}.\tag{4.17}$$

The proof of the trace estimate (4.16) is along the line of proof for the trace estimate (7.1) in Theorem 7.1 shown in Appendix and we show it in subsection 7.1 of the Appendix below.

4.2. **Estimate for the velocity.** Once we obtain the estimates for the pressure ∇q to (2.11), the required estimates for the velocities v of the solution to (2.11) can be obtained by applying maximal regularity for the initial boundary value of the heat equations:

$$\begin{cases}\partial_t u-\Delta u=f, & t>0,\ \ x\in\mathbb{R}^n_+,\\ \quad\partial_n u(t,x',x_n)\big|_{x_n=0}=h(t,x'), & t>0,\ \ x'\in\mathbb{R}^{n-1},\\ \quad u(t,x)\big|_{t=0}=u_0(x), & \qquad x\in\mathbb{R}^n_+,\end{cases}\tag{4.18}$$

where $x=(x',x_n)\in\mathbb{R}^n_+$ and ∂_n denotes the normal derivative $\partial/\partial x_n$ at any boundary point of $\mathbb{R}^n_+$.

Proposition 4.4 (Maximal L^1-regularity [45], [48])**.** *Let $1<p<\infty$ and $-1+1/p<s\le 0$. The problem (4.18) admits a unique solution*

$$\begin{aligned}&u\in C_b(\overline{\mathbb{R}_+};\dot B^s_{p,1}(\mathbb{R}^n_+))\cap\dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+)),\\ &\Delta u\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))\end{aligned}$$

if and only if the external, the initial and the boundary data in (4.18) satisfy

$$\begin{aligned}&u_0\in\dot B^s_{p,1}(\mathbb{R}^n_+),\quad f\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+)),\\ &h\in\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})),\end{aligned}$$

respectively. Moreover following the maximal L^1-regularity estimate holds:

$$\begin{aligned}&\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\ \le& C\Big(\|u_0\|_{\dot B^s_{p,1}(\mathbb{R}^n_+)}+\|f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|h\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\|h\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big),\end{aligned}\tag{4.19}$$

where C is depending only on p, s and n.

For the proof of Proposition 4.4, see [45] and [48].

Proof of Theorem 2.2. Let the boundary data satisfy the regularity assumption (2.14). First we consider the n-th component of the unknown velocity that satisfies the the corresponding system (cf. Shibata-Shimizu [59, (4.24)], and [62, (5.19)]). Namely (v_n, q) is given by the expressions (2.22) and (2.23). In particular from (2.22), we see that

$$\begin{aligned}
&\partial_t v_n(t,x',x_n)\\
=&c_{n+1}\text{p.v.} \iint_{\mathbb{R}^n} e^{it\tau+ix'\cdot\xi'}\Big\{\frac{(B+|\xi'|)|\xi'|}{D(\tau,\xi')}\Big(2B(i\xi'\cdot\widehat{H}')-(|\xi'|^2+B^2)\widehat{H_n}\Big)e^{-|\xi'|x_n}\\
&\qquad+\frac{(B+|\xi'|)|\xi'|}{D(\tau,\xi')}\Big(-(|\xi'|^2+B^2)(i\xi'/|\xi'|\cdot\widehat{H}')+2|\xi'|^2\widehat{H_n}\Big)e^{-Bx_n}\Big\}d\tau d\xi'. \qquad (4.20)
\end{aligned}$$

Via a very much similar argument for the pressure estimate in Proposition 4.1, we may derive the estimates for $\partial_t v_n$, Δv_n. Namely the n-component of the velocity fulfills the estimate:

$$\begin{aligned}
&\|\partial_t v_n\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\|D^2 v_n\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\
&\le C\big(\|H\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}+\|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\big).
\end{aligned} \qquad (4.21)$$

Note that the first term of the Fourier image of $\partial_t v_n$ in (4.20) is indeed expressed by the pressure and the rest of the symbol which is the parabolic part involving the symbol

$$\widetilde{m}(\tau,\xi')=\frac{(B(\tau,\xi')+|\xi'|)}{D(\tau,\xi')}\Big(-(|\xi'|^2+B(\tau,\xi')^2)\frac{i\xi'}{|\xi'|},2|\xi'|^2\Big)$$

and the above symbol denotes the singular integral part and it is analogous to $m(\tau,\xi)$ in (2.25) so that the estimate (4.21) follows from the estimate of the pressure term and maximal regularity for the parabolic part with quite similar argument found in the previous work, in particular using the [48, Lemma 6.5] with a modification involving $\widetilde{m}$ as is shown in (3.9), (3.10), (3.13) and (3.14). Hence the maximal regularity estimate for the n-th component of the velocity as well as the pressure follows from the estimate (4.21), Proposition 4.1 and Proposition 4.2 and we obtain that

$$\begin{aligned}
&\|\partial_t v_n\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\|D^2 v_n\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}\\
&\qquad+\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\big\|q|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\big(\|H\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1}))}+\|H\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\big).
\end{aligned} \qquad (4.22)$$

The trace estimate (4.17) in Proposition 4.3 enable us to control the term $\big\|q|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}$.

The other components of the velocity fields $v'=(v_1(t,x),v_2(t,x),\cdots,v_{n-1}(t,x))$ satisfy the initial boundary value problem of the heat equations as the pressure and the n-th component velocity as the external force and boundary condition as follows: For $\ell=1,2\cdots,n-1$,

$$\begin{cases}
\partial_t v_\ell-\Delta v_\ell=-\partial_\ell q, & t>0,\ x\in\mathbb{R}^n_+,\\
\partial_n v_\ell=-H_\ell-\partial_\ell v_n, & t>0,\ x\in\partial\mathbb{R}^n_+,\\
v_\ell(0,x)=0, & x\in\mathbb{R}^n_+.
\end{cases} \qquad (4.23)$$

Similarly to the above estimate, we have from Proposition 4.1, Proposition 4.3, Proposition 4.4 and the estimate (4.21) that the solution $v_\ell(t,x)$ to the problem (4.23) has the estimate

$$\begin{aligned}
&\|\partial_t v_\ell\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|D^2 v_\ell\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\le C\big(\|\partial_\ell q\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|H_\ell\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\|H_\ell\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\qquad+\big\|\partial_\ell v_n\big|_{x_n=0}\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\big\|\partial_\ell v_n\big|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\big)\\
&\le C\big(\|\nabla q\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|H_\ell\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\|H_\ell\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\qquad+\|\partial_t v_n\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla^2 v_n\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\big)\\
&\le C\big(\|H\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\big).
\end{aligned}\tag{4.24}$$

Combining the estimates (4.21) and (4.24) for all $\ell=1,2,\cdots,n-1$ as well as the pressure estimate (4.1) in Proposition 4.1, we conclude that the desired estimate (2.15) holds.

Conversely, if the solution (v,q) to the problem (2.11) exists, then it holds by letting f by v in the trace estimate (7.1) of Theorem 7.1 in Appendix and Proposition 4.3 that

$$\begin{aligned}
&\|H\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})))}+\|H\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le 2\|\nabla v\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})))}+2\|\nabla v\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\qquad+\big\|q\,|_{x_n=0}\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})))}+\big\|q\,|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\Big(\|\partial_t v\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|\nabla^2 v\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\\
&\qquad+\|\nabla q\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\big\|q\,|_{x_n=0}\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})))}+\big\|q\,|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}\tag{4.25}$$

This shows regularity for the boundary data is necessary.

Concerning the uniqueness, we invoke the argument employed in [62, Theorem 4.3 and 5.7] for the half space. Let (v,q) be a solution of the Stokes system (2.1) with vanishing data and satisfying the regularity Theorem 2.1. For any $\phi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n_+)$ and for any $T>0$ with $\phi=0$ for $t\in(-\infty,-1)\cup(T/2,\infty)$, set $\phi_T(t,x)=\phi(T-t,x)$. Let (Φ,θ) be the solution of (2.1) with vanishing data except the external force $f=\phi_T$ and set $v_*(t,x)=\Phi(T-t,x)$ and $q_*(t,x)=\theta(T-t,x)$ and arrange its support into the time interval $(-1,T/2)$. Then (v_*,q_*) solves the adjoint Stokes system except the pressure sign in the subset of the dual space $L^\infty(I;\dot H^{-s,p'}(\mathbb{R}^n_+))\subset L^\infty(I;\dot B^{-s}_{p',\infty}(\mathbb{R}^n_+))$. If we choose $\rho>2p/(p+1)$, then $v_*\in W^{1,\rho}(I;L^{p'}(\mathbb{R}^n_+))\cap L^\rho(I;H^{2,p'}(\mathbb{R}^n_+))\subset L^\infty(I;\dot H^{-s,p'}(\mathbb{R}^n_+))$. Here we note that $\dot H^{-s,p'}(\mathbb{R}^n_+)\subset\dot B^{-s}_{p',\infty}(\mathbb{R}^n_+)=(\dot B^s_{p,1}(\mathbb{R}^n_+))^*$, where $0\le -s<1/p'$. Let $\tilde\chi(r)$ be a smooth cut-off function of $r>0$ over the annulus $B_2(0)\setminus\bar B_1(0)$ with $\chi(x)\equiv\tilde\chi(|x|)$ and set $\chi_R(x)\equiv R^{-1}\chi(R^{-1}x)$ for any $R>0$ and $D_R=\operatorname{supp}\chi_R(x)\equiv\{x\in\mathbb{R}^n_+;\ R\le|x|\le 2R\}$. By the Poincaré–Wirtinger inequality for $I=(-2,T)$, there exists $\theta\in(1,\infty)$ such that

$$\begin{aligned}
&\Big|\int_I\int_{\mathbb{R}^n_+}q(t,x)\chi_R(x)v_*(t,x)\,dx\,dt\Big|\\
&\le C\Big(\|q|_{x_n=0}\|_{L^1(I;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}+\|\nabla q\|_{L^1(I;\dot B^s_{p,1}(\mathbb{R}^n_+))}\Big)\|\chi_R\,v_*\|_{C_b(I;\dot H^{-s,p'}(D_R))},
\end{aligned}\tag{4.26}$$

$$\begin{aligned}
&\Big|\int_I\int_{\mathbb{R}^n_+}q_*(t,x)\chi_R(x)v(t,x)\,dx\,dt\Big|\\
&\le C\Big(\|q_*|_{x_n=0}\|_{L^\theta(I;\dot W^{-s+1-\frac1{p'}}_{p'}(\mathbb{R}^{n-1}))}+\|\nabla q_*\|_{L^\theta(I;\dot H^{-s,p'}(\mathbb{R}^n_+))}\Big)\|\chi_R\,v\|_{C_b(I;\dot B^s_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}\tag{4.27}$$

By passing $R\to\infty$ the both terms in the right hand side of (4.26) and (4.27) are vanishing (cf. by the bilinear estimate (7.26) and $\|\chi_R\|_{\dot B^{n/p}_{p,1}(\mathbb{R}^n_+)}=O(R^{-1})$), which justify the integration by parts (cf. [59] and

[62] for the details). Since the external force is smooth, the exceptional regularity can be avoided. The analogous estimate above also justify the dual couplings at the boundary. Those observations ensure that the following argument remains valid;

$$\begin{aligned}\langle v,\phi\rangle_{\mathbb{R}\times\mathbb{R}^n_+} &= \langle v, -\partial_t v_* - \Delta v_* + \nabla q_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+}\\ &= \langle \partial_t v, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} + \langle \nabla v, \nabla v_* + (\nabla v_*)^{\mathsf{T}} - q_* I\rangle_{\mathbb{R}\times\mathbb{R}^n_+} - (v|_{x_n=0}, T(v_*,q_*)\nu_n|_{x_n=0}\rangle_{\mathbb{R}\times\mathbb{R}^{n-1}}\\ &= \langle \partial_t v, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} + \langle \nabla v + (\nabla v)^{\mathsf{T}} - qI, \nabla v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+}\\ &= \langle \partial_t v, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} + \langle -\Delta v + \nabla q, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} + \langle T(v,q)\nu_n|_{x_n=0}, v_*\rangle_{\mathbb{R}\times\mathbb{R}^{n-1}}\\ &= \langle \partial_t v - \Delta v + \nabla q, v_*\rangle_{\mathbb{R}\times\mathbb{R}^n_+} = 0,\end{aligned}$$

from which we conclude $v = 0$ by the arbitrariness of ϕ, and hence $q = 0$ by $\nabla q = 0$ in $\mathbb{R}^n_+$ and $q(\cdot, 0) = 0$ by (2.11). This proves Theorem 2.2. ❑

Proof of Theorem 2.1. Applying the maximal L^1-regularity result to the initial-boundary value problem of the Stokes equations with the boundary condition, we obtain end-point maximal L^1-maximal regularity from (4.21), (4.24) and Since by (2.16) $u = \widetilde{u} + v - \nabla\phi|_{x_n>0}$ and $p = \widetilde{p} + q$ is the solution to (2.1). Hence by combining (2.3)-(2.5), (2.7), (2.10), (2.12), (2.15) in Theorem 2.2, we obtain (2.2).

Conversely, by using (2.17), (4.16) in Proposition 4.3 as well as (7.1) in Theorem 7.1, we conclude that regularity for data is necessary for the existence of the solution (u,p) for the Stokes system (2.1). This completes the proof of Theorem 2.1. ❑

5. Multiple div-curl structure and critical multi-linear estimates

In this section, we show *the multiple divergence-free-curl-free structure* related to Jacobi matrix of transformation from the Euler coordinates to the Lagrange coordinate, which is essential to obtain global well-posedness in the critical Besov space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$. The single divergence structure was firstly pointed out by Solonnikov [67], and it is applied by Shibata–Shimizu [61] for the free boundary value problem. Our case is a multiple extension from those divergence structure. Namely in order to apply the bilinear estimate in the critical Besov spaces, we need to ensure that the divergence-free, rotation-free structure for every step when we apply the bilinear estimate. Namely for multi-linear case, we need to make it clear that the nonlinear terms in the equation maintains the multiple div-curl free structures. This was shown in [46] for the initial value problem for the Lagrangian coordinate case. We develop the analogous estimate and establish the multiple Besov estimate in the half-spaces.

5.1. **Multiple div-curl structure.** We show that the inverse matrix of Jacobian for the Lagrangian transform and consequently the perturbation terms F_u, F_p and G_{div} have a special divergence structure. We call a inner product of two vector fields $f\cdot g$ for $f,g\in\mathcal{D}^*$ maintains the divergence free rotation-free structure (in short div-curl structure) if $\operatorname{rot} f = 0$ and $\operatorname{div} g = 0$ and the multiple-div-curl structure for $f\cdot\Pi(g_1\otimes g_2\otimes\cdots\otimes g_\ell)$ if $\operatorname{rot} f = 0$ and $\operatorname{div}\Pi = 0$ and Π can be decomposed into lower order component consisting of div-curl structure. Such kind of structure easily yields us the original term can be expressed in the divergence form and the bilinear estimate can be enlarged in the critical Besov framework.

We show such a structure holds for each of the semi-linear terms of the system (1.5).

Proposition 5.1 (Multiple divergence structure)**.** *Let $I = (0,T)$ with $T \le \infty$ and suppose that (u,p) has the following regularities;*

$$\nabla u \in L^1\big(I; \dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)\big), \quad \nabla p \in L^1\big(I; \dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)\big).$$

Let $F_p(u,p)$, $G_{\mathrm{div}}(u)$ be the polynomials of d_{jk} of order $n-1$ defined in (1.7), (1.8). Then the terms are subject to the multiple div-curl structure. Namely every component of those polynomial consist of the inner products of divergence free vector and rotation free vectors.

Before going into details of the proof, we introduce several notations. Let

$$\tilde{d}_{ij} = \delta_{ij} + d_{ij} = \delta_{ij} + \int_0^t \partial_j u_i(s)\,ds, \tag{5.1}$$

then the Jacobi matrix is written as

$$J(Du) = \begin{pmatrix} 1+d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & 1+d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & 1+d_{nn} \end{pmatrix} = \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \cdots & \tilde{d}_{1n} \\ \tilde{d}_{21} & \tilde{d}_{22} & \cdots & \tilde{d}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{d}_{n1} & \tilde{d}_{n2} & \cdots & \tilde{d}_{nn} \end{pmatrix}$$

and set

$$J(Du)^{-1} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

For any $1 \le \ell \le n-1$, let us denote an $\ell \times \ell$-submatrix of $J(Du)$ as

$$J(Du)^{[\ell]} \equiv \begin{pmatrix} \tilde{d}_{\sigma_1\tau_1} & \cdots & \tilde{d}_{\sigma_1\tau_\ell} \\ \vdots & \ddots & \vdots \\ \tilde{d}_{\sigma_\ell\tau_1} & \cdots & \tilde{d}_{\sigma_\ell\tau_\ell} \end{pmatrix}, \tag{5.2}$$

where $(\sigma_1, \sigma_2, \cdots, \sigma_\ell)$ and $(\tau_1, \tau_2, \cdots, \tau_\ell)$ are any combination of ordered sub-factor from $(1, 2, \cdots, n)$, namely $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_\ell \le n$ and $1 \le \tau_1 < \tau_2 < \cdots < \tau_\ell \le n$. We notice that

$$b_{kj} = (-1)^{k+j} \det J(Du)^{[n-1]}_{jk}.$$

We prove *the multiple div-curl structure* for those (1.7)–(1.8) by induction. The proof for the case of $F_p(u,p)$ in (1.7), $G_{\rm div}(u)$ in (1.8) can be shown along a similar way. Hence we mainly show (1.7) for the case $F_p(u,p)$. In the case of $n=2$ and $n=3$, such a structure is shown in an explicit way for $\mathbb{R}^n$ case (see [46]). It is easy to show (1.7) in the case of $n=2$.

According to Evans [25] (section 8.1), we recall the null Lagrangian structure for the Jacobian of a Lipschitz continuous function u. Let A be a $n \times n$ matrix and consider its $\ell \times \ell$ sub-matrix $A^{[\ell]}$.

Lemma 5.2 (Divergence free for sub-cofator)**.** *Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function and for any $1 \le \ell \le n$, let $J(Du)^{[\ell]}$ be an $\ell \times \ell$-submatrix of the Jacobi matrix $J(Du)$ and $\mathrm{cof}\,(J(Du)^{[\ell]})_{kj} = (-1)^{\sigma_k} \det J(Du)^{[\ell-1]}_{kj}$ be the cofactor matrix of the sub-matrix $J(Du)^{[\ell]}$. Then for (k,j) component of $\mathrm{cof}\,(J(Du)^{[\ell]})$, it holds*

$$\mathrm{div}_j\, \mathrm{cof}\,\big(J(Du)^{[\ell]}\big)_{kj} = 0$$

for any point $x \in \mathbb{R}^\ell$ with $\det(J(Du)^{[\ell]})(x) \ne 0$.

We show the proof of Lemma 5.2 in the Appendix (Lemma 7.2) below.

Proof of Proposition 5.1. Assume that $\nabla u \in L^1\big(I; \dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)\big)$. Since $\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+) \subset C_v(\mathbb{R}^n_+)$ for almost every $t \in I = (0,T)$, we regard that u is the Lipschitz continuous function in $x \in \mathbb{R}^n_+$ for almost all $t \in I$.

Each component of $J(Du)^{-1}$ can be realized by the cofactor expansion by $J(Du)$, namely

$$b_{kj} = (\det J(Du))^{-1} \sum_{j=1}^n (-1)^{k+j} \tilde{d}_{kj} \det(J(Du)^{[n-1]})_{kj} = \sum_{j=1}^n \tilde{d}_{kj} \mathrm{cof}\,\big(J(Du)\big)_{kj},$$

where we recall that $\mathrm{div}\,\bar{u} = 0$ implies $\det(J(D\bar{u})) = \det(J(Du)) = 1$. Then it is easy to see that from Lemma 5.2 and for each k,

$$\tilde{d}_{kj} = \delta_{kj} + \int_0^t \partial_j u_k(s) ds$$

is a rotation-free vector and hence the each component of cofactor of $J(Du)$, namely $J(Du)^{-1}$ has the div-curl structure and this structure can be decomposed into any order of its sub-factor by expanding the determinant of cofactor matrices $J(Du)^{[n-1]}_{kj}$. It can be realized by the form

$$\det\big(J(Du)^{[\ell]}\big)=\sum_{j=1}^{\ell}(-1)^{k+j}\tilde{d}_{kj}\,\det\big(J(Du)^{[\ell-1]}\big)_{kj}=\sum_{j=1}^{\ell}\tilde{d}_{kj}\,\mathrm{cof}\,\big(J(Du)^{[\ell]}\big)$$

for all $\ell=1,2,\cdots,n-1$. It is clear from Lemma 5.2 that the above expression also maintains div-curl structure unless $\det\big(J(Du)^{[\ell]}\big)=0$, since $\tilde{d}_k\equiv(\tilde{d}_{k1},\tilde{d}_{k2},\cdots,\tilde{d}_{kn})$ is a rotation free vector for each $k=1,2,\cdots,n$.

Now we finalized the proof to see that

$$\big(F_p(u,p)\big)_k=\Big(\big(J(Du)^{-1}-I\big)^{\mathsf{T}}\nabla p\Big)_k=\sum_{j=1}^{n}\mathrm{cof}\,\big(J(Du)^{[n-1]}\big)_{kj}\partial_j p-\partial_k p$$

with observing that the first term is an inner product of the rotation free vector ∇p and divergence free element $\mathrm{cof}\,\big(J(Du)^{[n-1]}_{kj}\big)$ and the second is also with a trivial curl-free element δ_{kj}.

The proof for $G_{\mathrm{div}}(u)$ goes almost the same way since each component of trace has the div-curl structure as the above. The boundary term $H_u(u)$ is also decomposed into the div-curl free structure before taking the inner product with ν_n. This completes the proof. $\square$

5.2. **Bilinear estimate for div-curl structure.** In general, the following bilinear estimate does not hold in the Besov space over $\mathbb{R}^n$:

$$\|fg\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\le C\|f\|_\infty\|g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}.$$

However it is possible to change the norm of L^∞ into slightly stronger norm of $\dot{B}^{n/q}_{q,1}$, which have the same scaling invariance with L^∞. The following bilinear estimate is essentially obtained by Abidi-Paicu [2] (cf. [44], [46]) in $\mathbb{R}^n$.

Proposition 5.3. *Let $1\le p,p_1,p_2\le\infty$ and $1/p=1/p_1+1/p_2$. Under the assumption $s+s'>1$, in particular, for $1\le p<2n$, for all $f\in\dot{B}^{s-1}_{p_1,1}(\mathbb{R}^n_+)$ and $g\in L^{p_2}(\mathbb{R}^n_+)\cap\dot{B}^{s'}_{p_2,\infty}(\mathbb{R}^n_+)$, there exists $C>0$ independent of f, g such that the following estimate holds:*

$$\|fg\|_{\dot{B}^{-1+s}_{p,1}(\mathbb{R}^n_+)}\le C\|f\|_{\dot{B}^{-1+s}_{p_1,1}(\mathbb{R}^n_+)}\big(\|g\|_{L^{p_2}(\mathbb{R}^n_+)}+\|g\|_{\dot{B}^{s'}_{p_2,\infty}(\mathbb{R}^n_+)}\big).\tag{5.3}$$

In particular for $g\in\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$

$$\|fg\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}.\tag{5.4}$$

Proof of Proposition 5.3. Since the bilinear estimates in (5.3) and (5.4) are established in the whole space, we merely show the case for the half-space for (5.4). Let $f\in\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ and $g\in\dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$ then from the definition of the Besov space in the half-space, for any $\varepsilon>0$ there exists $\tilde{f}\in\dot{B}^{-1+n/p}_{p,\sigma}(\mathbb{R}^n)$ and $\tilde{g}\in\dot{B}^{n/p}_{p,1}(\mathbb{R}^n)$ such that $f=\tilde{f}$ in $\mathcal{D}'(\mathbb{R}^n_+)$ and $g=\tilde{g}$ over $\mathbb{R}^n_+$ and

$$\begin{aligned}\|\widetilde{f}\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}&\le\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}+\varepsilon,\\ \|\widetilde{g}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}&\le\|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}+\varepsilon.\end{aligned}$$

Then the corresponding estimate (5.3) in $\mathbb{R}^n$ now implies

$$\begin{aligned}\|fg\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\le&\Big\|\widetilde{fg}|_{\mathbb{R}^n_+}\Big\|_{\dot{B}^{s-1}_{p,1}}\le\Big\|\widetilde{f}\widetilde{g}\Big\|_{\dot{B}^{s-1}_{p,1}}\le C\|\widetilde{f}\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}}\|\widetilde{g}\|_{\dot{B}^{\frac{n}{p}}_{p,1}}\\ \le&C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitraly, this proves the estimate (5.4) in $\mathbb{R}^n_+$. The estimate (5.3) follows in a similar way. $\square$

Proposition 5.4 (Bilinear estimate under div-curl structure)**.** *Let $1 \le p < \infty$. For any vector valued functions $f \in \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+)$ and $g \in \dot{B}^{n/p}_{p,1}(\mathbb{R}^n_+)$ with $\operatorname{div} f = 0$ and $\operatorname{rot} g = 0$ in the distribution sense, there exists a constant $C > 0$ independent of f, g such that*

$$\|f \cdot g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}. \tag{5.5}$$

If f satisfies $\operatorname{rot} f = 0$ and g satisfies $\operatorname{div} g = 0$ in $\mathcal{D}'$, then (5.5) also holds.

Proof of Proposition 5.4. The corresponding estimate to (5.5) in the whole space is shown in Proposition 7.5 in Appendix below (cf. [46]). To show the half-space case, the argument of proof of Proposition 5.3 works as well and this shows the proof. ❑

5.3. **Multi-linear estimate under the div-curl structure.** The perturbation terms for the Navier–Stokes equations in the Lagrangian coordinate exhibit the multiple-div-structure. In this case, we use the improved bilinear estimate in Proposition 5.4 in the critical Besov space for the nonlinear term(cf. for the whole space case [46], Proposition 4.5–4.6).

Proposition 5.5 (Multiple div-curl estimates 1)**.** *Let $n \ge 2$, $1 \le p < \infty$. For $\partial_t u$, $D^2 u$ and $\nabla p \in L^1(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, let $F_p(u,p)$ and $G_{\mathrm{div}}(u)$ be the terms defined in (1.7) and (1.8), respectively. Then the following estimates hold:*

$$\|F_p(u,p)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}, \tag{5.6}$$

$$\|\nabla G_{\mathrm{div}}(u)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C\sum_{k=1}^{n-1}\|D^2u\|^{k+1}_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}. \tag{5.7}$$

In particular for $1 \le p < 2n$, it holds that

$$\|\partial_t(-\Delta)^{-1}\nabla G_{\mathrm{div}}(u)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \le C\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}. \tag{5.8}$$

Proof of Proposition 5.5. First we show the estimate for F_p:

$$F_p(u,p) = -\big(J(Du)^{-1} - I\big)^{\mathsf{T}}\,\nabla p \equiv \Pi_p^{n-1}\left(\int_0^t Du\,ds\right)\nabla p.$$

Here $\Pi_p^{n-1}(\cdot)$ is the $n-1$-th order polynomial of the component of inverse of Jacobi matrix $J(Du)^{-1}$ without a constant term:

$$\Pi_p^{n-1}\Big(\int_0^t Du\,ds\Big) = \sum_{k=1}^{n-1} c_k \prod_{m,\ell\le n}^{k}\Big(\int_0^t \partial_\ell u_m\,ds\Big).$$

By using (5.5) with $1 \le p < \infty$ repeatedly, we have by inductively that

$$\begin{aligned}
&\|F_p(u,p)\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\le C\sup_{t>0}\Big\|\int_0^t Du\,ds\Big\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\Big(\|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\qquad\qquad + \Big\|\sum_{k=2}^{n-2} c_k \prod_{m,\ell\le n}^{k}\Big(\int_0^t \partial_\ell u_m\,ds\Big)\nabla p\Big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)\\
&\le C\Big\|\int_0^t Du\,ds\Big\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\qquad + \Big(\sup_{t>0}\Big\|\int_0^t Du\,ds\Big\|_{\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\Big)^2\Big\|\sum_{k=0}^{n-3} c_k \prod_{m,\ell\le n}^{k}\Big(\int_0^t \partial_\ell u_m\,ds\Big)\nabla p\Big\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}
\end{aligned}$$

$$\le C\sum_{k=1}^{n-1}\Big\|\int_0^t Du\,ds\Big\|^k_{L^\infty(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\nabla p\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}$$
$$\le C\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\nabla p\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.$$

This proves the estimate (5.6).

Next we show (5.7). The estimate of ∇G_{div} :

$$\nabla G_{\mathrm{div}}(u)\equiv \nabla \mathrm{tr}\left(\Pi^{n-1}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)Du\right).$$

Here $\Pi^{n-1}_{\mathrm{div}}(\cdot)$ is the $(n-1)$-th order polynomial without a constant term:

$$\Pi^{n-1}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)=\sum_{k=1}^{n-1}c_k\prod_{m,\ell\le n}^{k}\Big(\int_0^t\partial_\ell u_m ds\Big).$$

By using (5.5), we have

$$\begin{aligned}
&\|\nabla G_{\mathrm{div}}(u)\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le&\Big\|\Pi^{n-1}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)D^2u\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\Big\|\Big(D\,\Pi^{n-1}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\Big)\mathrm{div}\;u\Big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le&C\sup_{t>0}\Big\|\int_0^t Du\,ds\Big\|_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\Big\|\Pi^{n-2}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)D^2\,u\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&+C\sup_{t>0}\Big\|\int_0^t Du\,ds\Big\|_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\Big\|D\,\Pi^{n-2}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\mathrm{div}\;u\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le&C\,\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\times\Bigg(\Big\|\Pi^{n-2}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)D^2\,u\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\qquad+\Big\|\Pi^{n-3}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)D\Big(\int_0^t Du\,ds\Big)\mathrm{div}\;u\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Bigg)\\
\le&C\sum_{k=1}^{n-2}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Bigg(\Big\|\Big(\int_0^t Du\,ds\Big)D\mathrm{div}\;u\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\qquad\qquad+\|\mathrm{div}\;u\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big\|\int_0^t D^2u\,ds\Big\|_{L^\infty(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Bigg)\\
\le&C\sum_{k=1}^{n-1}\|D^2u\|^{k+1}_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}\tag{5.9}$$

This shows (5.7).

Finally we show (5.8). $\Pi^{n-1}_{\mathrm{div}}(\cdot)$ is the $n-1$-th order polynomial of the component of inverse of Jacobi matrix $J(Du)^{-1}$ without a constant term. By the div-curl structure it holds that

$$G_{\mathrm{div}}(u)=\Pi^{n-1}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\mathrm{div}\,u=\mathrm{div}_{\,k}\Big(\sum_{k=1}^{n-1}c_k\prod_{m,\ell\le n}^{k}\Big(\int_0^t\partial_\ell u_m ds\Big)u_k\Big)=\mathrm{div}\,\overline{G_{\mathrm{div}}}(u).\tag{5.10}$$

By using (5.5) several times, we have

$$\|\partial_t(-\Delta)^{-1}\nabla G_{\mathrm{div}}(u)\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}$$

$$
\begin{aligned}
&\le \|\partial_t \overline{G_{\mathrm{div}}}(u)\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\le C\left\|\Pi^{n-1}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + C\left\|\Pi^{n-2}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\nabla u\otimes u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\le C\sup_{t>0}\left\|\int_0^t Du\,ds\right\|_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\left\|\Pi^{n-2}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\partial_t\, u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\quad + C\sup_{t>0}\left\|\int_0^t Du\,ds\right\|_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\left\|\Pi^{n-2}_{\mathrm{div}}\Big(\int_0^t Du\,ds\Big)\nabla u\otimes u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\le C\sum_{k=1}^{n-1}\left\|D^2u\right\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\left\|\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\quad + C\sum_{k=0}^{n-2}\left\|D^2u\right\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\left\|u\,Du\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\le C\sum_{k=1}^{n-1}\left\|D^2u\right\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\left\|\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\quad + C\sum_{k=0}^{n-2}\left\|D^2u\right\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\left\|\nabla u\right\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\sup_{t>0}\|u\|_{\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\\
&\le C\sum_{k=1}^{n-1}\left\|D^2u\right\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\left(\left\|\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \sup_{t>0}\left\|\int_t^\infty \partial_t u ds\right\|_{\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\right)\\
&\le C\sum_{k=1}^{n-1}\left\|D^2u\right\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\left\|\partial_t u\right\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}
\tag{5.11}
$$

$\square$

For the quasi-linear term $F_u(u)$ associated with the Laplace operator defined in (1.6), the structure is slightly different from the others since total *multiple-div-structure* fails in the following whole form:

$$
F_u(u) = \mathrm{div}\,\Big(\Big(J(Du)^{-1}\big(J(Du)^{-1}\big)^{\mathsf T} - I\Big)\nabla u\Big).
$$

Indeed, the *multiple-div-curl structure* remains valid for $\big(J(Du)^{-1} - I\big)^{\mathsf T}\nabla u$ partially, as well as for div $\Big(J(Du)^{-1}\ F\Big)$ with any vector field F. However since the coefficient function is $J(Du)^{-1}$ which is adjoint of $\big(J(Du)^{-1}\big)^{\mathsf T}$, the derivatives for the divergence 'div' outside does not commute with $J(Du)^{-1}$ and the whole *multiple div-curl structure* does not hold. To recover this difficulty, we use the bilinear estimate in Proposition 5.3.

Proposition 5.6 (Multiple div-curl estimate 2)**.** *Let $n \ge 2$, $1 \le p < \infty$. For $D^2u \in L^1(\mathbb{R}_+;\dot B^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$, let $F_u(u)$ be the terms defined by* (1.6)*. Then the following estimate holds:*

$$
\|F_u(u)\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1})} \le C\sum_{k=1}^{2n-2}\|D^2u\|^{k+1}_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\tag{5.12}
$$

Proof of Proposition 5.6. To show the estimate (5.12), we first decompose the terms as

$$
\begin{aligned}
F_u(u) &= \mathrm{div}\,\Big(J(Du)^{-1}\big(J(Du)^{-1}\big)^{\mathsf T}\nabla u - \nabla u\Big)\\
&= \mathrm{div}\,\Big(J(Du)^{-1}\big(J(Du)^{-1} - I\big)^{\mathsf T}\,\nabla u\Big) + \mathrm{div}\,\Big(\big(J(Du)^{-1} - I\big)\nabla u\Big)\\
&= \mathrm{div}\,\Big(\big(J(Du)^{-1} - I\big)\big(J(Du)^{-1} - I\big)^{\mathsf T}\,\nabla u\Big) + \mathrm{div}\,\Big(\big(J(Du)^{-1} - I\big)^{\mathsf T}\nabla u\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\operatorname{div}\Big(\big(J(Du)^{-1}-I\big)\nabla u\Big)\\
\equiv &F_u^1(u)+F_u^2(u)+F_u^3(u).
\end{aligned}
$$

Here $\operatorname{div} E$ stands for $[\nabla^{\mathsf{T}} E]^{\mathsf{T}}$, where E denotes the $n\times n$-matrix valued function. To show the estimate F_u^1, one can use the *div-structure* up to estimate for the terms for $\operatorname{div}\Big(\big(J(Du)^{-1}-I\big)F\Big)$ for the vector field $F\equiv \big(J(Du)^{-1}-I\big)^{\mathsf{T}}\nabla u$ since it maintains the *multiple-div-structure*. Namely since $\operatorname{div}\Big(\big(J(Du)^{-1}-I\big)F\Big)$ is the adjoint operator of $\big(J(Du)^{-1}-I\big)^{\mathsf{T}}\nabla F$, it maintains the structure and it follows that for any $1\le p<\infty$,

$$
\Big\|\operatorname{div}\Big(\big(J(Du)^{-1}-I\big)\ F\Big)\Big\|_{\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\le C\sum_{k=1}^{n-1}\|\int_0^t\nabla u ds\|^k_{\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\|\operatorname{div} F\|_{\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}
$$

and

$$
\begin{aligned}
&\|\operatorname{div}\Big(\big(J(Du)^{-1}-I\big)\ F\Big)\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le& C\sum_{k=1}^{n-1}c_k\|\nabla u\|^k_{L^1(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\operatorname{div} F\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le& C\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\|\operatorname{div} F\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}
\tag{5.13}
$$

Letting $F\equiv (J(Du)-1)^{\mathsf{T}}\nabla u$ and using Proposition 5.3 and the div-curl bilinear estimate (5.5) in Proposition 5.4 for $n-2$-times, we see for the last term of the right hand side of (5.13) that

$$
\begin{aligned}
\|\operatorname{div} F\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}=&\Big\|\operatorname{div}\Big(\big(J(Du)^{-1}-I\big)^{\mathsf{T}}\ \nabla u\Big)\Big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le& C\big\|\big(J(Du)^{-1}-I\big)^{\mathsf{T}}\nabla u\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le& C\sum_{k=1}^{n-1}c_k\Big\|\int_0^t\nabla u ds\Big\|^k_{L^\infty(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\big\|\nabla u\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
\le& C\sum_{k=1}^{n-1}\big\|D^2u\big\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\big\|D^2u\big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}
\tag{5.14}
$$

Combining (5.13), (5.14), the estimates for $F_u^1(u)$ and $F_u^2(u)$ are proven. It is then easy to see that a similar argument of (5.14) can be applicable for estimating the last term $F_u^3(u)$ and we conclude that

$$
\|F_u(u)\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\le C\sum_{k=1}^{2n-2}\|D^2u\|^{k+1}_{L^1(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
$$

□

We finally treat the boundary nonlinearity as follows.

Proposition 5.7 (Multiple estimates for boundary nonlinearity)**.** *Let $n\ge 2$, $1<p<2n-1$ and assume that functions u and p satisfy $\partial_t u$, D^2u, $\nabla p\in L^1(\mathbb{R}_+;\dot B^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$ with $p|_{x_n=0}\in \dot F^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot B^{-1+n/p}_{p,1}(\mathbb{R}^{n-1}))\cap$*

$L^1(\mathbb{R}_+;\dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$. *For the boundary terms $H_b(u)$ and $H_p(u,p)$ defined by* (1.9) *and* (1.10)*, respectively. Then the following estimates hold:*

$$\begin{aligned}
&\|H_p(u,p)\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
&\qquad\le C\Big(\big\|p|_{x_n=0}\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}+\big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\Big)\\
&\qquad\qquad\times\sum_{k=1}^{n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}\Big)^k,
\end{aligned}\tag{5.15}$$

$$\|H_p(u,p)\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\le C\big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))},\tag{5.16}$$

$$\|H_u(u)\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\le C\sum_{k=2}^{2n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}\Big)^k,\tag{5.17}$$

$$\|H_u(u)\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\le C\sum_{k=2}^{2n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}.\tag{5.18}$$

Proof of Proposition 5.7. From (1.9) and (1.10) and from the regularity assumptions; we notice that the sharp trace estimate implies

$$Du\big|_{x_n=0},\ p\big|_{x_n=0}\in\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})).\tag{5.19}$$

We first see the estimate (5.15). Since $J(Du)^{-1}-I$ consists of a polynomial of $\int_0^t Du\,ds$ with its order up to $n-1$, we show that

$$H_p(u,p)=\Pi^{n-1}_{bp}\left(\int_0^t Du\,ds\right)p\,\nu_n\in\dot F^{\frac12-\frac1{2p}}_{1,1}\big(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1})\big).\tag{5.20}$$

We introduce auxiliary norms of Chemin–Lerner type (cf. [15]) for the proof of Proposition 5.7.

Definition. For $1\le p,\rho\le\infty$ and $r,s\in\mathbb{R}$, the Bochner–Besov spaces of Chemin–Lerner type $\widetilde{\dot B^r_{\rho,1}\big(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})\big)}$ and $\widetilde{L^\rho\big(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})\big)}$ are defined by the following norms:

$$\begin{aligned}
\|f\|_{\widetilde{\dot B^r_{\rho,1}\left(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})\right)}}&\equiv\sum_{k\in\mathbb{Z}}2^{rk}\sum_{j\in\mathbb{Z}}2^{sj}\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}f(t,x')\big\|_{L^\rho_t(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))},\\
\|f\|_{\widetilde{L^\rho\left(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})\right)}}&\equiv\sum_{j\in\mathbb{Z}}2^{sj}\big\|\phi_j\underset{(x')}{*}f(t,x')\big\|_{L^\rho_t(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))}.
\end{aligned}\tag{5.21}$$

Lemma 5.8 (Multiple estimates for boundary nonlinearity)**.** *Let $n\ge2$, $1<p<2n-1$ and assume that functions F and G over $\mathbb{R}_+\times\mathbb{R}^{n-1}$ satisfy $F\in\dot F^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot B^{-1+n/p}_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ and $G\in\widetilde{\dot B^{1/2-1/2p}_{\infty,1}}(\mathbb{R}_+;\dot B^{-1+n/p}_{p,1}(\mathbb{R}^{n-1}))\cap\widetilde{L^\infty(\mathbb{R}_+}; \dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$. Then the following estimate holds:*

$$\begin{aligned}
\|F\,G\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\le&\,C\Big(\big\|F\big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}+\big\|F\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)\\
&\times\Big(\big\|G\big\|_{\widetilde{\dot B^{\frac12-\frac1{2p}}_{\infty,1}}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}+\big\|G\big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}\tag{5.22}$$

The proof of Lemma 5.8 directly follows from Proposition 7.6 with $\rho=1$ and (7.30)–(7.31) in Appendix below.

Now we set

$$F(t,x') \equiv p(t,x',x_n)|_{x_n=0},$$
$$G(t,x') \equiv \Pi_{bp}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds\Big)\Big|_{x_n=0}$$

in Lemma 5.8 with regarding (5.19) to find that

$$\begin{aligned}
&\|H_p(u,p)\|_{\dot F_{1,1}^{\frac12-\frac1{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))}\\
&\le C\Big(\big\|p|_{x_n=0}\big\|_{\dot F_{1,1}^{\frac12-\frac1{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))}+\big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{\frac{n-1}{p}}(\mathbb{R}^{n-1}))}\Big)\\
&\quad\times\Big(\Big\|\Pi_{bp}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds\Big)\Big|_{x_n=0}\Big\|_{\widetilde{\dot B_{\infty,1}^{\frac12-\frac1{2p}}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))}\\
&\qquad+\Big\|\Pi_{bp}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds\Big)\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B_{p,1}^{\frac{n-1}{p}}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}\tag{5.23}$$

To complete the estimate we use the following lemma.

Lemma 5.9. *For any* $1\le p<\infty$,

$$\Big\|\int_0^t Du(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{\dot B_{\infty,1}^{\frac12-\frac1{2p}}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))}\le C\big\|Du|_{x_n=0}\big\|_{\dot F_{1,1}^{\frac12-\frac1{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))},\tag{5.24}$$

$$\Big\|\int_0^t Du(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B_{p,1}^{\frac{n-1}{p}}(\mathbb{R}^{n-1}))}\le C\big\|Du|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{\frac{n-1}{p}}(\mathbb{R}^{n-1}))}.\tag{5.25}$$

Proof of Lemma 5.9. The first estimate (5.24) follows by using $\widetilde{\psi}_k(t)=\psi_{k-1}(t)+\psi_k(t)+\psi_{k+1}(t)$ and noticing $\|\partial_t^{-1}\psi_k\|_{L^\infty(\mathbb{R}_+)}\le\|\psi_k\|_{L^1(\mathbb{R}_+)}$, where $\partial_t^{-1}\psi_k$ is defined as in (7.6) below that

$$\begin{aligned}
&\Big\|\int_0^t Du(s)\,ds\,\Big|_{x_n=0}\Big\|_{\widetilde{\dot B_{\infty,1}^{\frac12-\frac1{2p}}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))}\\
&=\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac1{2p})k}\Big\|\Big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\Big(\int_0^t Du(s)\,ds\,\Big|_{x_n=0}\Big)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
&\le\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac1{2p})k}\Big\|\Big\|(\partial_t^{-1}\psi_k)\underset{(t)}{*}\widetilde{\psi_k}\underset{(t)}{*}\phi_j\underset{(x')}{*}\partial_t\Big(\int_0^t Du(s)\,|_{x_n=0}\,ds\Big)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
&\le\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac1{2p})k}\big\|\partial_t^{-1}\psi_k\big\|_{L^\infty_t(\mathbb{R}_+)}\Big\|\Big\|\widetilde{\psi_k}\underset{(t)}{*}\phi_j\underset{(x')}{*}Du\,|_{x_n=0}\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac1{2p})k}\big\|\psi_k\big\|_{L^1_t(\mathbb{R}_+)}\Big\|\Big\|\widetilde{\psi_k}\underset{(t)}{*}\phi_j\underset{(x')}{*}Du\,|_{x_n=0}\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\Big\|\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\in\mathbb{Z}}2^{(\frac12-\frac1{2p})k}\Big\|\widetilde{\psi_k}\underset{(t)}{*}\phi_j\underset{(x')}{*}Du\,|_{x_n=0}\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\le C\big\|Du\,|_{x_n=0}\big\|_{\dot F_{1,1}^{\frac12-\frac1{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))}.
\end{aligned}$$

The second inequality (5.25) follows from the following estimate:

$$
\begin{aligned}
&\Big\| \int_0^t Du(s)\, ds \,\Big|_{x_n=0} \Big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}^{n-1}))} \le \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \| \int_0^t \phi_j \underset{(x')}{*} Du(s)\,|_{x_n=0} ds \|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\le \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \|\phi_j \underset{(x')}{*} Du\,|_{x_n=0}\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \\
&= \Big\| \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \|\phi_j \underset{(x')}{*} Du\,|_{x_n=0}\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} = \big\| Du\,|_{x_n=0} \big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})}.
\end{aligned}
$$

❑

Lemma 5.10. *$\widetilde{L^\infty(\mathbb{R}_+};\dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ is the Banach algebra, namely for any $f, g \in \widetilde{L^\infty(\mathbb{R}_+};\dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ it holds*

$$
\big\| f\, g \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}^{n-1}))} \le C \big\| f \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)} \big\| g \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)}. \tag{5.26}
$$

In particular for $Du|_{x_n=0} \in L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))$,

$$
\Big\| \Pi^{n-1}_{bp} \Big(\int_0^t Du(s)\, ds \Big) \Big|_{x_n=0} \Big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}^{n-1}))} \le \sum_{k=1}^{n-1} \big\| Du\,|_{x_n=0} \big\|^k_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})}. \tag{5.27}
$$

Proof of Lemma 5.10. To see that (5.26) holds, we start from Bony's paraproduct decomposition: Setting $P_m \underset{(x')}{*} F \equiv \sum_{m'\le m} \phi_{m'} \underset{(x')}{*} F$ and $Q_\ell \underset{(t)}{*} F \equiv \sum_{\ell'\le \ell} \psi_{\ell'} \underset{(t)}{*} F$, it follows that

$$
\begin{aligned}
&\big\| f\, g \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}^{n-1}))} \\
&\le \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \|\phi_j\|_1 \big\| \widetilde{\phi}_j \underset{(x')}{*} f \big\|_{L^p(\mathbb{R}^{n-1}_{x'})} \big\| P_{j-2} \underset{(x')}{*} g \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\quad + \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \|\phi_j\|_1 \big\| P_{j-2} \underset{(x')}{*} f \big\|_{L^\infty(\mathbb{R}^{n-1})} \big\| \widetilde{\phi}_j \underset{(x')}{*} g \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\qquad + \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \|\phi_j\|_r \sum_{m\ge j-2} \big\| \phi_m \underset{(x')}{*} f \big\|_{L^p(\mathbb{R}^{n-1})} \big\| \widetilde{\phi}_m \underset{(x')}{*} g \big\|_{L^{r'}(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\le \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \big\| \widetilde{\phi}_j \underset{(x')}{*} f \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \Big\| \big\| P_{j-2} \big\|_{L^1(\mathbb{R}^{n-1})} \big\| g \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\quad + \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \big\| P_{j-2} \big\|_{L^1(\mathbb{R}^{n-1})} \big\| f \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \Big\| \big\| \widetilde{\phi}_j \underset{(x')}{*} g \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\qquad + \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \sum_{m\ge j-2} 2^{\frac{n-1}{r'}j} \\
&\qquad\qquad \times \Big\| \big\| \phi_m \underset{(x')}{*} f \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \Big\| \big\| \widetilde{\phi}_m \underset{(x')}{*} g \big\|_{L^{r'}(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\le \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \big\| \widetilde{\phi}_j \underset{(x')}{*} f \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \Big\| \big\| g \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\quad + \Big\| \big\| f \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \sum_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \big\| \widetilde{\phi}_j \underset{(x')}{*} g \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\qquad + \sum_{m\in\mathbb{Z}} \sum_{j\le m+2} 2^{\frac{2(n-1)}{p}j} \Big\| \big\| \phi_m \underset{(x')}{*} f \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \Big\| \big\| \widetilde{\phi}_m \underset{(x')}{*} g \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\le \big\| f \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)} \big\| g \big\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))} + \big\| f \big\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))} \big\| g \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{m\in\mathbb{Z}} 2^{\frac{(n-1)}{p}m}\Big\|\|\phi_m \underset{(x')}{*} f\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)} \sup_{m\in\mathbb{Z}} 2^{\frac{(n-1)}{p}m}\Big\|\|\widetilde{\phi}_m \underset{(x')}{*} g\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
\le& C\|f\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)}\|g\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))}+C\|f\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))}\|g\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)}\\
&+C\|f\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)}\|g\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,\infty}}(\mathbb{R}_+)}\\
\le& C\|f\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)}\|g\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}_+)},
\end{aligned}
$$

which shows (5.26). The estimate (5.27) follows from the estimate (5.25). ❑

The polynomial term can be estimated as the following way: From Proposition 7.6 in Appendix with $\rho=\infty$, Lemma 5.10 and (5.24) in Lemma 5.9 we see that

$$
\begin{aligned}
&\Big\|\Pi^{n-1}_{bp}\Big(\int_0^t Du(s)ds\Big)\Big|_{x_n=0}\Big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}}(\mathbb{R}^{n-1}))}\\
&\le C\sum_{k=1}^{n-1}\Big(\Big\|\int_0^t Du(s)ds\Big|_{x_n=0}\Big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}}(\mathbb{R}^{n-1}))}+\Big\|\int_0^t Du(s)ds\Big|_{x_n=0}\Big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}}(\mathbb{R}^{n-1}))}\Big)^k\\
&\le C\sum_{k=1}^{n-1}\Big(\big\|Du\,|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|Du\,|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)^k\\
&\le C\sum_{k=1}^{n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\|\Delta u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)^k
\end{aligned}\tag{5.28}
$$

by the sharp trace estimate Proposition 4.3. Combining the estimates (5.23)-(5.28), we obtain (5.15).

To show the estimate (5.16), we notice that $L^\infty(\mathbb{R}_+;\dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ is the Banach algebra and from the sharp trace estimate (4.17) , it follows from (4.17) that

$$
\begin{aligned}
&\|H_p(u,p)\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\|p\,|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big\|\Pi^{n-1}_{bp}\Big(\int_0^t Du\,ds\Big)\Big|_{x_n=0}\Big\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\|p\,|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big\|\int_0^t Du\,ds\Big|_{x_n=0}\Big\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big\|\Pi^{n-2}_{bp}\Big(\int_0^t Du\,ds\Big)\Big|_{x_n=0}\Big\|_{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\|p\,|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\sum_{k=1}^{n-1}\big\|Du\,|_{x_n=0}\big\|^k_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\le C\|p\,|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\sum_{k=1}^{n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}\tag{5.29}
$$

On the other hand, for the estimate (5.17) of the velocity boundary term, we split $H_u(u)$ into two parts as

$$
\begin{aligned}
H_u(u)=&\Big((J(Du)^{-1})^{\mathsf{T}}\nabla u+(\nabla u)^{\mathsf{T}}J(Du)^{-1}\Big)\big((J(Du)^{-1})^{\mathsf{T}}-I\big)\nu_n\\
&+\Big(\big(J(Du)^{-1}-I\big)^{\mathsf{T}}\nabla u+\big(\big(J(Du)^{-1}-I\big)^{\mathsf{T}}\nabla u\big)^{\mathsf{T}}\Big)\nu_n\\
\equiv& H^1_u(u)+H^2_u(u).
\end{aligned}\tag{5.30}
$$

By setting

$$F(t,x') \equiv \Big(\big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big) \Big|_{x_n=0},$$

$$G(t,x') \equiv \Pi_{bu}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds \Big)\Big|_{x_n=0}$$

in Lemma 5.8 with

$$\big((J(Du)^{-1})^{\mathsf{T}} - I \big)\Big|_{x_n=0} = \Pi_{bu}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds \Big)\Big|_{x_n=0},$$

we find that

$$\begin{aligned}
&\|H_u^1\|_{\dot F_{1,1}^{\frac12-\frac{1}{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))} \\
&\le C\Big(\Big\| \Big(\big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big) \Big|_{x_n=0} \Big\|_{\dot F_{1,1}^{\frac12-\frac{1}{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))} \\
&\qquad + \Big\| \Big(\big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big) \Big|_{x_n=0} \Big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{\frac{n-1}{p}}(\mathbb{R}^{n-1}))} \Big) \\
&\quad \times \Big(\Big\| \Pi_{bu}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds \Big)\Big|_{x_n=0} \Big\|_{\widetilde{\dot B_{\infty,1}^{\frac12-\frac{1}{2p}}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))} \\
&\qquad + \Big\| \Pi_{bu}^{n-1}\Big(\int_0^t Du(s,x',x_n)ds \Big)\Big|_{x_n=0} \Big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B_{p,1}^{\frac{n-1}{p}}(\mathbb{R}^{n-1}))} \Big).
\end{aligned} \tag{5.31}$$

The first term of the right hand side of (5.31) is estimated by applying the sharp trace estimate (4.16) as well as a similar way in (5.9), (5.11) to obtain

$$\begin{aligned}
&\Big\| \big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big|_{x_n=0} \Big\|_{\dot F_{1,1}^{\frac12-\frac{1}{2p}}(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^{n-1}))} \\
&\le C\Big(\Big\| \partial_t(-\Delta)^{-1}\nabla \Big(\big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big) \Big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \\
&\qquad + \Big\| \nabla \Big(\big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big) \Big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \Big) \\
&\le C\Big(\Big\| \Big(\big(J(Du)^{-1}\big)^{\mathsf{T}} \partial_t u + (\partial_t u)^{\mathsf{T}} J(Du)^{-1} \Big) \Big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \\
&\qquad + \Big\| \sum_{k=1}^{n-1} \sigma_k \Big(\int_0^t Du(s)\,ds \Big)^{k-1} Du \times u \Big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \\
&\qquad + \Big\| \big(J(Du)^{-1}\big)^{\mathsf{T}} \nabla u + (\nabla u)^{\mathsf{T}} J(Du)^{-1} \Big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{\frac np}(\mathbb{R}^n_+))} \Big) \\
&\le C\Big(\sum_{k=0}^{n-1} \Big\| \int_0^t Du(s)\,ds \Big\|_{L^\infty(\mathbb{R}_+;\dot B_{p,1}^{\frac np}(\mathbb{R}^n_+))}^k \big\|\partial_t u\big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \\
&\qquad + \sum_{k=1}^{n-1} \Big\| \int_0^t Du(s)\,ds \Big\|_{L^\infty(\mathbb{R}_+;\dot B_{p,1}^{\frac np}(\mathbb{R}^n_+))}^{k-1} \big\|Du\big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{\frac np}(\mathbb{R}^n_+))} \big\|u\big\|_{L^\infty(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \\
&\qquad + \sum_{k=0}^{n-1} \Big\| \int_0^t Du(s)ds \Big\|_{L^\infty(\mathbb{R}_+;\dot B_{p,1}^{\frac np}(\mathbb{R}^n_+))}^k \big\|\nabla u\big\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{\frac np}(\mathbb{R}^n_+))} \Big) \\
&\le C \sum_{k=0}^{n-1} \|D^2 u\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))}^k \Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(\mathbb{R}_+;\dot B_{p,1}^{-1+\frac np}(\mathbb{R}^n_+))} \Big),
\end{aligned} \tag{5.32}$$

while

$$\Big\|\big(J(Du)^{-1}\big)^{\mathsf T}\nabla u+(\nabla u)^{\mathsf T}J(Du)^{-1}\Big|_{x_n=0}\Big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}$$

is estimated by the right hand side of (5.32) in much simpler way as in (5.27), since $\dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1})$ is the Banach algebra. By (5.31), (5.32) and the estimates (5.27) and (5.28) with the sharp trace estimate imply

$$\begin{aligned}
&\Big\|H^1_u(u)\Big\|_{\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}\\
\le& C\sum_{k=2}^{2n-1}\Big(\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}\Big)^k.
\end{aligned}\tag{5.33}$$

The estimate for the term $H^2_u(u)$ can be shown in the same way as is shown in (5.32), which shows that (5.17) holds.

For the proof of (5.18), we notice that $\dot B^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1})$ is the Banach algebra, the same argument to (5.29) shows for $1\le p<\infty$ that

$$\begin{aligned}
&\|H_u(u)\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
\le& C\Big\|\Big(\big(J(Du)^{-1}\big)^{\mathsf T}\nabla u-(\nabla u)^{\mathsf T}J(Du)^{-1}\Big)\Big|_{x_n=0}\Big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\qquad\times\Big\|\Pi^{n-1}_{bu}\Big(\int_0^t Du\,ds\Big)\Big|_{x_n=0}\Big\|_{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&+C\Big\|\Big(\big(J(Du)^{-1}-I\big)^{\mathsf T}\nabla u-(\nabla u)^{\mathsf T}\big(J(Du)^{-1}-I\big)\Big)\Big|_{x_n=0}\Big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
\le& C\|Du\,|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\sum_{k=1}^{2n-2}\Big\|\int_0^t Du(s)\,ds\Big|_{x_n=0}\Big\|^k_{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
\le& C\sum_{k=2}^{2n-1}\big\|Du\,|_{x_n=0}\big\|^k_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\le C\sum_{k=2}^{2n-1}\|D^2u\|^k_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}.
\end{aligned}$$

This show the estimate (5.18). ❑

6. The proof of main theorem

Proof of Theorem 1.1. We define the complete metric space

$$X=\left\{(u,p):\begin{array}{l}u\in C(\overline{\mathbb{R}_+};\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))\cap\dot W^{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+)),\\ (\Delta u,\nabla p)\in L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+)),\\ p|_{x_n=0}\in\dot F^{\frac12-\frac1{2p}}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))\cap L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})),\end{array}\ \|(u,p)\|_X\le M\right\},$$

where

$$\begin{aligned}
\|(u,p)\|_X\equiv&\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\|D^2u\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}+\|\nabla p\|_{L^1(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^n_+))}\\
&+\big\|p|_{x_n=0}\big\|_{\dot F^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1}))}+\big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}.
\end{aligned}$$

The constant $M>0$ is chosen to be small enough depending on the norm of the initial data. Given $(\tilde u,\tilde p)\in X$, we consider the liner inhomogeneous initial boundary value problem:

$$\begin{cases}\partial_t u-\Delta u+\nabla p=F_u(\tilde u)+F_p(\tilde u,\tilde p), & t>0,\ x\in\mathbb{R}^n_+,\\ \qquad\operatorname{div}\,u=G_{\mathrm{div}}(\tilde u). & t>0,\ x\in\mathbb{R}^n_+,\\ \big(\nabla u+(\nabla u)^{\mathsf T}-pI\big)\cdot\nu_n=H_u(\tilde u)+H_p(\tilde u,\tilde p). & t>0,\ x\in\partial\mathbb{R}^n_+,\\ \qquad u(0,x)=u_0(x), & x\in\mathbb{R}^n_+,\end{cases}\tag{6.1}$$

where $\nu_n = (0, 0, \cdots, 0, -1)^\mathsf{T}$ denotes the outer normal and we set

$$F_u(\tilde{u}) = \mathrm{div}\left(J(D\tilde{u})^{-1}\big(J(D\tilde{u})^{-1}\big)^\mathsf{T}\nabla\tilde{u} - \nabla\tilde{u}\right) = \Pi_{\tilde{u}}^{2n-2}\left(\int_0^t D\tilde{u}\, ds\right) D^2\tilde{u}, \tag{6.2}$$

$$F_p(\tilde{u}, \tilde{p}) = -\big(J(D\tilde{u})^{-1} - I\big)\,\nabla\tilde{p} = \Pi_p^{n-1}\left(\int_0^t D\tilde{u}\, ds\right)\nabla\tilde{p}, \tag{6.3}$$

$$\begin{aligned} G_{\mathrm{div}}(\tilde{u}) &= -\,\mathrm{tr}\Big(\big(J(D\tilde{u})^{-1} - I\big)\,\nabla\tilde{u}\Big) = \mathrm{tr}\left(\Pi_{\mathrm{div}}^{n-1}\left(\int_0^t D\tilde{u}\, ds\right) D\tilde{u}\right) \\ &= \mathrm{div}\left(\Pi_{\mathrm{div}}^{n-1}\left(\int_0^t D\tilde{u}\, ds\right)\tilde{u}\right), \end{aligned} \tag{6.4}$$

$$\begin{aligned} H_b(\tilde{u}) &= -\Big(\big(J\big(D(\tilde{u})\big)^{-1}\big)^\mathsf{T}\,\nabla\tilde{u} + (\nabla\tilde{u})^\mathsf{T} J\big(D(\tilde{u})\big)^{-1}\Big)\,\big(J\big(D(\tilde{u})\big)^{-1} - I\big)^\mathsf{T}\nu_n \\ &\quad - \Big(\big(J\big(D(\tilde{u})\big)^{-1} - I\big)^\mathsf{T}\,\nabla\tilde{u} + (\nabla\tilde{u})^\mathsf{T}\big(J\big(D(\tilde{u})\big)^{-1} - I\big)\Big)\nu_n \\ &= \Pi_{bu}^{2n-2}\left(\int_0^t D\tilde{u}\, ds\right) D\tilde{u}\,\nu_n, \end{aligned} \tag{6.5}$$

$$H_p(\tilde{u}, \tilde{p}) = \tilde{p}\big(J(D\tilde{u})^{-1} - I\big)^\mathsf{T}\nu_n = \Pi_{bp}^{n-1}\left(\int_0^t D\tilde{u}\, ds\right) p\,\nu_n. \tag{6.6}$$

We define the map

$$\Phi : X \to X$$

by

$$(\tilde{u}, \tilde{p}) \to (u, p) \equiv \Phi\big[\tilde{u}, \tilde{p}\big]$$

and prove that Φ is contraction on X.

First we show that a priori estimate of $\Phi[u,p]$ in $L^1(\mathbb{R}_+; \dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n_+))$. Let (u,p) solve (6.1). Applying Theorem 2.1 to the equation (6.1), we have by (2.2), Propositions 5.5–5.7 to the nonlinear terms (6.2)–(6.6) that

$$\begin{aligned}
&\|\partial_t u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2 u\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\quad + \|\nabla p\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \big\|p|_{x_n=0}\big\|_{\dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \big\|p|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\le C_M\Big(\|u_0\|_{\dot{B}^0_{p,1}(\mathbb{R}^n_+)} + \|F_u(\tilde{u})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|F_p(\tilde{u},\tilde{p})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad + \|\nabla G_{\mathrm{div}}(\tilde{u})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\partial_t(-\Delta)^{-1}\nabla G_{\mathrm{div}}(\tilde{u})\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad + \|H_u(\tilde{u})\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \|H_u(\tilde{u})\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\
&\qquad + \|H_p(\tilde{u},\tilde{p})\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \|H_p(\tilde{u},\tilde{p})\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big) \\
&\le C\Bigg(\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)} + \sum_{k=1}^{2n-2}\|D^2\tilde{u}\|^{k+1}_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} \\
&\qquad + \sum_{k=1}^{2n-1}\|D^2\tilde{u}\|^k_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big(\|\partial_t\tilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2\tilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big) \\
&\qquad + \sum_{k=1}^{n-1}\Big(\|\partial_t\tilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|D^2\tilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k \\
&\qquad \times\Big(\|\nabla\tilde{p}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))} + \|\tilde{p}|_{x_n=0}\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} + \|\tilde{p}|_{x_n=0}\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)
\end{aligned}$$

$$+\sum_{k=2}^{2n-1}\Big(\|\partial_t\tilde{u}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2\tilde{p}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k\Big). \tag{6.7}$$

By (6.7), it holds that

$$\|\Phi[\tilde{u},\tilde{p}]\|_X\le C_1\Big(\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}+\sum_{k=1}^{2n-1}M^{k+1}\Big). \tag{6.8}$$

Therefore if we choose the initial data small enough

$$\|u_0\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+)}\le\frac{1}{2C_1}M<\frac{1}{4}$$

then we obtain from (6.8) that

$$\|\Phi[\tilde{u},\tilde{p}]\|_X\le M.$$

Moreover, for all (u_1,p_1), $(u_2,p_2)\in X$, we know that the difference

$$w=u_1-u_2,\qquad q=p_1-p_2$$

satisfy the same estimate (6.7):

$$\begin{aligned}
&\|\partial_t w\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2 w\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\quad+\big\|q|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|q|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
\le& C_2\max_{i=1,2}\sum_{k=1}^{2n-2}\Big(\|\partial_t u_i\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2 u_i\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k\\
&\quad\times\Big(\|\partial_t w\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2 w\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|\nabla q\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\\
&\quad+\big\|q|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|q|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big).
\end{aligned}$$

Therefore if we choose

$$C_2\max_{i=1,2}\sum_{k=1}^{2n-2}\Big(\|\partial_t u_i\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|D^2 u_i\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)^k\le C_2\sum_{k=1}^{2n-2}M^k\le\frac12,$$

then it holds that

$$\|\Phi[w,q]\|_X\le\frac12\|(w,q)\|_X,$$

which shows the map

$$\Phi:X\to X$$

is contraction. By the fixed point theorem of Banach-Cacciopolli, there exists a unique fixed point (u,p) of the map Φ in X.

We finally confirm that the boundary equation in (6.1) is fulfilled. Let the difference between the solution and the date as

$$\widetilde{w}=\widetilde{u}-u.$$

The sharp trace estimate Proposition 4.3 ensure that

$$\begin{aligned}
&\big\|\nabla\widetilde{w}|_{x_n=0}\big\|_{\dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|\nabla\widetilde{w}|_{x_n=0}\big\|_{L^1(\mathbb{R}_+;\dot{B}^{\frac{n}{p}-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
\le& C\Big(\|\partial_t\widetilde{w}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}+\|\Delta\widetilde{w}\|_{L^1(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^n_+))}\Big)
\end{aligned}\tag{6.9}$$

and the right hand side of (6.9) converges to 0 as the iterative process. Then the unique fixed point (u,p) satisfies (6.1) with the all right members changed into (u,p) and it is a time global strong solution of (1.5). This completes the proof of Theorem 1.1. ❑

7. Appendix

7.1. **The optimal boundary trace.** The proof of Proposition 4.3 is based on the following trace estimate (cf. [38]).

Theorem 7.1 (Sharp boundary derivative trace [48])**.** *For $1<p<\infty$ and $-1+1/p<s$, there exists a constant $C>0$ such that for any function $f=f(t,x',\eta)$ with $f\in C_b(\overline{\mathbb{R}_+};\dot B^s_{p,1}(\mathbb{R}^n_+))\cap \dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$, $D^2f\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$, it holds for all $\partial_\ell=(\partial_{x_\ell},\partial_\eta)$ with $1\le \ell\le n-1$ that*

$$\begin{aligned}\sup_{\eta\in\mathbb{R}_+}\Big(\big\|\partial_\ell f(\cdot,\cdot,\eta)\big\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1}))}+\big\|\partial_\ell f(\cdot,\cdot,\eta)\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\Big)&\\ \le C\Big(\|\partial_t f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\|D^2 f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}\Big).&\qquad(7.1)\end{aligned}$$

Proof of Theorem 7.1. For $1<p<\infty$ and $-1+1/p<s$, assume $f\in C_b(\overline{\mathbb{R}_+};\dot B^s_{p,1}(\mathbb{R}^n_+))\cap\dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$, $D^2f\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$. Then by the definition (1.14), for any $\varepsilon>0$, there exists $\widetilde f\in C_b(\overline{\mathbb{R}_+};\dot B^s_{p,1}(\mathbb{R}^n))\cap \dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n))\cap L^1(\mathbb{R}_+;\dot B^{s+2}_{p,1}(\mathbb{R}^n))$ such that

$$\begin{aligned}\|\widetilde f\|_{L^\infty(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n))\cap\dot W^{1,1}(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^n))}&\le\|f\|_{L^\infty(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))\cap\dot W^{1,1}(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\varepsilon,\\ \|D^2\widetilde f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n))}&\le\|D^2 f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+\varepsilon.\end{aligned}$$

We then extend $\widetilde f$ into $t<0$ as an even extension. For simplicity, we denote $\widetilde f$ as f in the following. It directly follows that

$$\partial_\ell f\in L^2(\mathbb{R};\dot B^{s+1}_{p,1}(\mathbb{R}^n)).\qquad(7.2)$$

Then

$$\begin{aligned}\big\|\partial_\ell f(\cdot,\cdot,\eta)\big\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}\left(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^{n-1})\right)}\le&\big\|\partial_\ell f(\cdot,\cdot,\eta)\big\|_{\dot F^{\frac12-\frac{1}{2p}}_{1,1}\left(\mathbb{R};\dot B^s_{p,1}(\mathbb{R}^{n-1})\right)}\\ \le&\Big\|\sum_{j\in\mathbb{Z}}2^{sj}\sum_{k\ge 2j}2^{(\frac12-\frac{1}{2p})k}\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\partial_\ell f(t,\cdot,\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R})}\\ &+\Big\|\sum_{j\in\mathbb{Z}}2^{sj}\sum_{k\le 2j}2^{(\frac12-\frac{1}{2p})k}\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\partial_\ell f(t,\cdot,\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R})}\\ \equiv& I+II.\end{aligned}\qquad(7.3)$$

From (7.2) and $1/p<s+1$, one can approximate $\partial_\ell f$ by a function satisfying

$$\lim_{t\to\pm\infty}\partial_\ell f(t,x',\eta)=0,\qquad \text{a.a. } (x',\eta)\in\mathbb{R}^{n-1}\times\mathbb{R},\qquad(7.4)$$

and using the assumption (7.4) and noting $\widehat{\psi_k}(0)=\int_{\mathbb{R}}\psi_k(t)dt=0$,

$$\begin{aligned}\psi_k(t)\underset{(t)}{*}\partial_\ell f(t,x',\eta)=&\int_{\mathbb{R}}\psi_k(s)\,\partial_\ell f(t-s,x',\eta)ds\\ =&-\int_{\mathbb{R}}\partial_s\Big(\int_s^\infty\psi_k(r)dr\Big)\partial_\ell f(t-s,x',\eta)ds\\ =&-\Big[\Big(\int_s^\infty\psi_k(r)dr\Big)\partial_\ell f(t-s,x',\eta)\Big]_{s=-\infty}^{\infty}+\int_{\mathbb{R}}\Big(\int_s^\infty\psi_k(r)dr\Big)\partial_s\partial_\ell f(t-s,x',\eta)ds\\ =&-\int_{\mathbb{R}}\partial_s^{-1}\psi_k(s)\partial_s\partial_\ell f(t-s,x',\eta)ds\\ =&\,\partial_t^{-1}\psi_k(t)\underset{(t)}{*}\partial_t\partial_\ell f(t,x',\eta),\end{aligned}\qquad(7.5)$$

where we set

$$\partial_t^{-1}\psi_k(s)\equiv-\int_s^\infty\psi_k(r)dr.\qquad(7.6)$$

Then $\partial_t^{-1}\psi_k(t) = 2^{-k}\big(\partial_t^{-1}\psi_0\big)_k(t)$. Hence from (7.4) and (7.5) and using the Hausdorff–Young inequality, it follows for $\partial_\ell = \partial_\eta$ (i.e., $\ell = n$) that

$$
\begin{aligned}
I \leq& C\Big\| \sum_{j\in\mathbb{Z}} 2^{sj} \sum_{k\geq 2j} 2^{-(\frac{1}{2}+\frac{1}{2p})k} \int_{\mathbb{R}} |(\partial_t^{-1}\psi_0)_k(t-s)| \big\|\phi_j \underset{(x')}{*} \partial_t\partial_\eta f(s,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} ds \Big\|_{L^1_t(\mathbb{R})} \\
\leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \sum_{k\geq 2j} 2^{-(\frac{1}{2}+\frac{1}{2p})k} \Big\| \int_{\mathbb{R}} \frac{2^k}{\langle 2^k(t-s)\rangle^2} \big\|\phi_j \underset{(x')}{*} \partial_t\partial_\eta f(s,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} ds \Big\|_{L^1_t(\mathbb{R})} \\
\leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \sum_{k\geq 2j} 2^{-(\frac{1}{2}+\frac{1}{2p})k} \Big\| \frac{2^k}{\langle 2^k t\rangle^2} \Big\|_{L^1_t(\mathbb{R})} \Big\| \big\| \sum_{m\in\mathbb{Z}} \overline{\Phi_m} \underset{(x',\eta)}{*} \phi_j \underset{(x')}{*} \partial_t\partial_\eta f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R})} \\
\leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \Big\| \sum_{k\geq 2j} 2^{-(\frac{1}{2}+\frac{1}{2p})k} \\
&\qquad \times \big\| \sum_{|m-j|\leq 1} \overline{\Phi_m} \underset{(x',\eta)}{*} \phi_j \underset{(x')}{*} \partial_\eta\zeta_{j+1}(\eta) \underset{(\eta)}{*} \zeta_{j+2}(\eta) \underset{(\eta)}{*} \partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \Big\| \sum_{k\geq 2j} 2^{-(\frac{1}{2}+\frac{1}{2p})k} \\
&\qquad \times \int_{\mathbb{R}} \big|\partial_\eta\zeta_{j+1} \underset{(\eta)}{*} \zeta_{j+2}(\eta-\xi)\big| \Big\| \sum_{|m-j|\leq 1} \overline{\Phi_m} \underset{(x',\xi)}{*} \phi_j \underset{(x')}{*} \partial_t f(t,x',\xi)\Big\|_{L^p(\mathbb{R}^{n-1})} d\xi \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \Big\| \sum_{k\geq 2j} 2^{-(\frac{1}{2}+\frac{1}{2p})k} \|\partial_\eta\zeta_{j+1}(\eta)\|_{L^{p'}(\mathbb{R}_\eta)} \\
&\qquad \times \Big\| \big\| \overline{\Phi_j} \underset{(x',\eta)}{*} \phi_j \underset{(x')}{*} \partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^p(\mathbb{R}_\eta)} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\Big\| \sum_{j\in\mathbb{Z}} 2^{sj} \big\|\overline{\Phi_j} \underset{(x',\eta)}{*} \partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^n_{(x',\eta)})} \Big\|_{L^1_t(\mathbb{R}_+)} \leq C\|\partial_t f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \tilde{\varepsilon}.
\end{aligned} \tag{7.7}
$$

For the second term of (7.3), we use the Minkowski inequality to see

$$
\begin{aligned}
II \leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \sum_{k\leq 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| |\psi_k| \underset{(t)}{*} \big\|\phi_j \underset{(x')}{*} \partial_\eta f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R})} \\
\leq& C\sum_{j\in\mathbb{Z}} 2^{sj} \sum_{k\leq 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
&\qquad \times \Big\| \big\| \sum_{m\in\mathbb{Z}} \overline{\Phi_m} \underset{(x',\eta)}{*} \zeta_{j+1}(\eta) \underset{(\eta)}{*} \zeta_{j+2}(\eta) \underset{(\eta)}{*} \phi_j \underset{(x')}{*} \partial_\eta f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\Big\| \sum_{j\in\mathbb{Z}} 2^{(s+1-\frac{1}{p})j} \big\| \sum_{|m-j|\leq 1} \overline{\Phi_m} \underset{(x',\eta)}{*} \partial_\eta\zeta_{j+1}(\eta) \underset{(\eta)}{*} \zeta_{j+2}(\eta) \underset{(\eta)}{*} \phi_j \underset{(x')}{*} f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\Big\| \sum_{j\in\mathbb{Z}} 2^{(s+1-\frac{1}{p})j} \|\partial_\eta\widetilde{\zeta}_j\|_{L^{p'}(\mathbb{R}_\eta)} \big\|\overline{\Phi_j} \underset{(x',\eta)}{*} f(t,x',\eta)\big\|_{L^p(\mathbb{R}^n)} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\Big\| \sum_{j\in\mathbb{Z}} 2^{(s+2)j} 2^{-2j} \big\|\overline{\Phi_j} \underset{(x',\eta)}{*} D^2 f(t,x',\eta)\big\|_{L^p(\mathbb{R}^n)} \Big\|_{L^1_t(\mathbb{R}_+)} \\
\leq& C\big\|D^2 f\big\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))} \leq C\big\|D^2 f\big\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \tilde{\varepsilon},
\end{aligned} \tag{7.8}
$$

where we used $1 < p$ and $\tilde{\varepsilon} = C\varepsilon$.

The estimate for the spatial trace term is shown along the following way:

$$
\begin{aligned}
&\big\|\partial_\eta f(\cdot,\cdot,\eta)\big\|_{L^1(\mathbb{R}_+;\dot{B}^{s+1-\frac{1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\\
&\qquad=\Big\|\sum_{j\in\mathbb{Z}}2^{sj}2^{(1-\frac{1}{p})j}\big\|\phi_j \underset{(x')}{*} \partial_\eta f(t,\cdot,\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\qquad\le C\Big\|\sum_{j\in\mathbb{Z}}2^{sj}2^{(1-\frac{1}{p})j}\big\|\sum_{m\in\mathbb{Z}}\overline{\Phi_m}\underset{(x',\eta)}{*}\phi_j\underset{(x')}{*}\partial_\eta f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\qquad\le C\Big\|\sum_{j\in\mathbb{Z}}2^{sj}2^{(1-\frac{1}{p})j}\big\|\sum_{m\in\mathbb{Z}}\overline{\Phi_m}\underset{(x',\eta)}{*}\partial_\eta\zeta_{j+1}(\eta)\underset{(\eta)}{*}\zeta_{j+2}(\eta)\underset{(\eta)}{*}\phi_j\underset{(x')}{*}f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\qquad\le C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+2)j}\,2^{-2j}\big\|\overline{\Phi_j}\underset{(x',\eta)}{*}D^2 f(t,x',\eta)\big\|_{L^p(\mathbb{R}^n)}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\qquad\le C\big\|D^2 f\big\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n))}\le C\big\|D^2 f\big\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))}+\tilde{\varepsilon}.
\end{aligned}\tag{7.9}
$$

Since $\tilde{\varepsilon}=C\varepsilon>0$ is arbitrary small, we conclude our result.

The other cases $1\le\ell\le n-1$ can be shown similar way by changing $\partial_\eta\zeta_j$ into ζ_j and ϕ_j into $\partial_\ell\widetilde{\phi_j}*\phi_j$. This completes the proof of Theorem 7.1.

❑

Proof of Proposition 4.3. The proof of the trace estimate (4.16) is almost the same line of (7.1) in Theorem 7.1 except the regularity. Hence we show an outlined proof. For $1<p<\infty$ and $-1+1/p<s<1/p$, assume $\nabla(-\Delta)^{-1}f\in C_b(\overline{\mathbb{R}^n_+};\dot{B}^s_{p,1}(\mathbb{R}^n_+))\cap\dot{W}^{1,1}(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$ and $\nabla f\in L^1(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^n_+))$. We employ the similar argument and use the extended element $\widetilde{f}$ associated with f as above. We regard f as the extended element $\widetilde{f}$. From $f\in L^2(\mathbb{R};\dot{B}^{s+1}_{p,1}(\mathbb{R}^n))$ like in (7.2), we may assume that

$$
\lim_{t\to\pm\infty}f(t,x',\eta)=0,\qquad \text{a.a. }(x',\eta)\in\mathbb{R}^{n-1}\times\mathbb{R},\tag{7.10}
$$

we have like (7.5) that

$$
\psi_k(t)\underset{(t)}{*}f(t,x',\eta)=\partial_t^{-1}\psi_k(t)\underset{(t)}{*}\partial_t f(t,x',\eta),\tag{7.11}
$$

with (7.6). Then it follows

$$
\begin{aligned}
\big\|f(\cdot,\cdot,\eta)\big\|_{\dot{F}^{\frac{1}{2}-\frac{1}{2p}}_{1,1}\left(\mathbb{R}_+;\dot{B}^s_{p,1}(\mathbb{R}^{n-1})\right)}
\le&\Big\|\sum_{j\in\mathbb{Z}}2^{sj}\sum_{k\ge 2j}2^{(\frac{1}{2}-\frac{1}{2p})k}\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}f(t,\cdot,\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R})}\\
&+\Big\|\sum_{j\in\mathbb{Z}}2^{sj}\sum_{k\le 2j}2^{(\frac{1}{2}-\frac{1}{2p})k}\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}f(t,\cdot,\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R})}\\
\equiv& I+II.
\end{aligned}\tag{7.12}
$$

Hence from (7.10) and (7.11) and using the Hausdorff–Young inequality

$$
\begin{aligned}
I=&\Big\|\sum_{j\in\mathbb{Z}}2^{sj}\sum_{k\ge 2j}2^{(\frac{1}{2}-\frac{1}{2p})k}\big\|\partial_t^{-1}\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R})}\\
\le& C\sum_{j\in\mathbb{Z}}2^{sj}\Big\|\sum_{k\ge 2j}2^{-(\frac{1}{2}+\frac{1}{2p})k}\\
&\qquad\times\big\|\sum_{|m-j|\le 1}\overline{\Phi_m}\underset{(x',\eta)}{*}\phi_j\underset{(x')}{*}\zeta_{j+1}(\eta)\underset{(\eta)}{*}\zeta_{j+2}(\eta)\underset{(\eta)}{*}\partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\Big\|\sum_{j\in\mathbb{Z}}2^{(s-1)j}\big\|\overline{\Phi_j}\underset{(x',\eta)}{*}\partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^n_{x',\eta})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\le& C\Big\|\sum_{j\in\mathbb{Z}}2^{sj}\big\|\nabla(-\Delta)^{-1}\overline{\Phi_j}\underset{(x',\eta)}{*}\partial_t f(t,x',\eta)\big\|_{L^p(\mathbb{R}^n_{x',\eta})}\Big\|_{L^1_t(\mathbb{R}_+)}
\end{aligned}
$$

$$\leq C\|\partial_t\nabla(-\Delta)^{-1}f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n))}\leq C\|\partial_t\nabla(-\Delta)^{-1}f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+C\varepsilon. \tag{7.13}$$

For the second term of (7.12), we use the Minkowski inequality to see

$$\begin{aligned}
II\leq &C\sum_{j\in\mathbb{Z}}2^{sj}\sum_{k\leq 2j}2^{(\frac12-\frac{1}{2p})k}\\
&\qquad\times\Big\|\Big\|\sum_{m\in\mathbb{Z}}\overline{\Phi_m}\underset{(x',\eta)}{*}\zeta_{j+1}(\eta)\underset{(\eta)}{*}\zeta_{j+2}(\eta)\underset{(\eta)}{*}\phi_j\underset{(x')}{*}f(t,x',\eta)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\leq &C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+1-\frac1p)j}\Big\|\sum_{|m-j|\leq 1}\overline{\Phi_m}\underset{(x',\eta)}{*}\zeta_{j+1}(\eta)\underset{(\eta)}{*}\zeta_{j+2}(\eta)\underset{(\eta)}{*}\phi_j\underset{(x')}{*}f(t,x',\eta)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\leq &C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+1)j}\Big\||\nabla|^{-1}\overline{\Phi_j}\underset{(x',\eta)}{*}\nabla f(t,x',\eta)\Big\|_{L^p(\mathbb{R}^n_{x',\eta})}\Big\|_{L^1_t(\mathbb{R}_+)}\leq C\big\|\nabla f\big\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+C\varepsilon.
\end{aligned}\tag{7.14}$$

Combining the estimates (7.12), (7.13), (7.14), we obtain the first part of the left hand side of (4.16). The estimate for the spatial direction (4.17) is slightly simpler. For $1\leq p<\infty$, $\nabla f\in L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))$ and noting (2.37), we obtain that

$$\begin{aligned}
&\big\|f(\cdot,\cdot,\eta)\big\|_{L^1(\mathbb{R}_+;\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1}))}\\
\leq &C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+1-1/p)j}\Big\|\phi_j\underset{(x')}{*}\sum_{m\in\mathbb{Z}}\overline{\Phi_m}(x',\eta)\underset{(x',\eta)}{*}f(t,x,\eta)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\leq &C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+1-\frac1p)j}\|\zeta_{j+1}\|_{L^{p'}(\mathbb{R}_\eta)}\Big\|\phi_j\underset{(x')}{*}\zeta_{j+2}(\eta)\underset{(\eta)}{*}\overline{\Phi_j}(x',\eta)\underset{(x',\eta)}{*}f(t,x,\eta)\Big\|_{L^p(\mathbb{R}^n_{x',\eta})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
\leq &C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+1)j}\Big\|\overline{\Phi_j}(x',\eta)\underset{(x',\eta)}{*}f(t,\cdot,\eta)\Big\|_{L^p(\mathbb{R}^n_{x',\eta})}\Big\|_{L^1_t(\mathbb{R}_+)}\leq C\|\nabla f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+C\varepsilon.
\end{aligned}$$

For almost every $t\in\mathbb{R}_+$ and $\eta\in\mathbb{R}$,

$$\begin{aligned}
&\big\|f(t,\cdot,\eta)\big\|_{\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})}\\
\leq &C\sum_{j\in\mathbb{Z}}2^{(s+1-1/p)j}\Big\|\phi_j\underset{(x')}{*}\sum_{m\in\mathbb{Z}}\overline{\Phi_m}(\cdot,\eta)\underset{(x',\eta)}{*}f(t,\cdot,\eta)\Big\|_{L^p(\mathbb{R}^{n-1})}\\
\leq &C\sum_{j\in\mathbb{Z}}2^{(s+1-\frac1p)j}\|\zeta_{j+1}\|_{L^{p'}(\mathbb{R}_\eta)}\Big\|\phi_j\underset{(x')}{*}\zeta_{j+2}(\eta)\underset{(\eta)}{*}\overline{\Phi_j}(\cdot,\eta)\underset{(x',\eta)}{*}f(t,\cdot,\eta)\Big\|_{L^p(\mathbb{R}^n_{x',\eta})}\\
\leq &C\sum_{j\in\mathbb{Z}}2^{(s+1)j}\Big\|\overline{\Phi_j}(x',\eta)\underset{(x',\eta)}{*}f(t,\cdot,\cdot)\Big\|_{L^p(\mathbb{R}^n_{x',\eta})}.
\end{aligned}$$

After taking supremum in $\eta>0$ and integrate it over $\mathbb{R}_+$, we obtain that

$$\begin{aligned}
&\Big\|\sup_{\eta>0}\big\|f(\cdot,\cdot,\eta)\big\|_{\dot B^{s+1-\frac1p}_{p,1}(\mathbb{R}^{n-1})}\Big\|_{L^1(\mathbb{R}_+)}\\
&\qquad\leq C\Big\|\sum_{j\in\mathbb{Z}}2^{(s+1)j}\Big\|\overline{\Phi_j}(x',\eta)\underset{(x',\eta)}{*}f(t,\cdot,\eta)\Big\|_{L^p(\mathbb{R}^n_{x',\eta})}\Big\|_{L^1_t(\mathbb{R}_+)}\\
&\qquad\leq C\|\nabla f\|_{L^1(\mathbb{R}_+;\dot B^s_{p,1}(\mathbb{R}^n_+))}+C\varepsilon.
\end{aligned}\tag{7.15}$$

The estimate (4.17) follows immediately from (7.15). This completes the proof of Proposition 4.3. ❑

7.2. **Null-Lagrangian structure.** According to Evans [25, section 8.1], we recall the null Lagrangian structure for the Jacobian of a Lipschitz continuous function u.

For $n \in \mathbb{N}$, let A be a $n \times n$ matrix whose components are denoted by $\{a_{kj}\}$ and consider its $\ell \times \ell$ sub-matrix $A^{[\ell]}$ given by

$$A^{[\ell]} = \begin{pmatrix} a_{\sigma_1\tau_1} & \cdots & a_{\sigma_1\tau_\ell} \\ \vdots & \ddots & \vdots \\ a_{\sigma_\ell\tau_1} & \cdots & a_{\sigma_\ell\tau_\ell} \end{pmatrix}, \tag{7.16}$$

where $\sigma_k, \tau_j \in \{1, 2, \cdots, n\}$ with $1 \le \sigma_1 < \sigma_2 < \cdots, < \sigma_\ell \le n$ and $1 \le \tau_1 < \tau_2 < \cdots, < \tau_\ell \le n$

Lemma 7.2 (Evans [25])**.** *Let $1 \le \ell \le n$ and let $u : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function and $J(Du)^{[\ell]}$ denotes the $\ell \times \ell$ sub-matrix of the Jacobi matrix defined by (7.16) (cf. (5.2)), $\mathrm{cof}\,(J(Du)^{[\ell]})_{kj}$ denotes the (k, j)- cofactor and $\mathrm{cof}\,(J(Du)^{[\ell]})$ be the cofactor matrix. Then for any $x \in \mathbb{R}^n$ with $\det(J(Du)(x) \neq 0$ it holds that*

$$\mathrm{div}_{\,j}\big(\mathrm{cof}\,(J(Du)^{[\ell]})\big)_{kj} = 0.$$

Proof of Lemma 7.2. Let $P = [p_{ij}]_{1\le i,j\le \ell}$ be a matrix whose (i, j)-components are p_{ij} and its cofactor matrix be $\mathrm{cof}\,(P)$. Then (k, j)-component of cofactor is given by

$$\mathrm{cof}\,(P)_{kj} = (-1)^{k+j} \det P^{[\ell-1]}_{kj}.$$

Let I be the $\ell \times \ell$ unit matrix and by $I = P^{\mathsf{T}}\,(P^{-1})^{\mathsf{T}}$, it follows

$$\det P \cdot \delta_{ij} = \big(P^{\mathsf{T}}\,(\mathrm{cof}\,(P))\big)_{ij} = \sum_{k=1}^{n} (P^{\mathsf{T}})_{ik}\mathrm{cof}\,(P)_{kj} = \sum_{k=1}^{n} p_{ki}\mathrm{cof}\,(P)_{kj}. \tag{7.17}$$

Taking the partial derivative of the both side of (7.17) by p_{km}, the component p_{kj} is missing in $\mathrm{cof}\,(P)_{kj}$

$$\frac{\partial}{\partial p_{km}} \det P = \mathrm{cof}\,(P)_{km}. \tag{7.18}$$

Choose P as the sub-matrix of the Jacobian $J(Du)^{[\ell]}$, the relation (7.17) is now reduced into

$$\det J(Du)^{[\ell]} \cdot \delta_{ij} = \sum_{k=1}^{\ell} \widetilde{d}_{ki}\mathrm{cof}\,(J(Du)^{[\ell]})_{kj}, \tag{7.19}$$

where $\widetilde{d}$ denotes the component of the Jacobi matrix $J(Du)$ defined by (5.1). Taking divergence for j-raw in (7.19) and noting (7.18),

$$\sum_{j=1}^{\ell} \partial_j(\det J(Du)^{[\ell]}) \cdot \delta_{ij} = \sum_{j=1}^{\ell} \partial_j\Big(\sum_{k=1}^{\ell} \widetilde{d}_{ki}\mathrm{cof}\,(J(Du)^{[\ell]})_{kj}\Big),$$

$$\sum_{k=1}^{\ell}\sum_{m=1}^{\ell} \partial_i\widetilde{d}_{km} \cdot \mathrm{cof}\,(J(Du)^{[\ell]})_{km} = \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} \partial_j\widetilde{d}_{ki} \cdot \mathrm{cof}\,(J(Du)^{[\ell]})_{kj} + \sum_{j=1}^{\ell}\sum_{k=1}^{\ell} \widetilde{d}_{ki} \cdot \partial_j\mathrm{cof}\,(J(Du)^{[\ell]})_{kj}$$

and thus we obtain

$$\sum_{k=1}^{\ell} \widetilde{d}_{ki} \cdot \sum_{j=1}^{\ell} \partial_j\mathrm{cof}\,(J(Du))_{kj} = 0.$$

Rewrite the above as

$$0 = J(Du)^{[\ell]}\mathrm{div}_{\,j}(\mathrm{cof}\,(J(Du)^{[\ell]})_{kj})$$

and multiplying the both side by $(J(Du)^{[\ell]})^{-1}$ at the point x_0 with $\det J(Du)^{[\ell]}(x_0) \neq 0$, it follows that

$$\mathrm{div}_{\,j}\mathrm{cof}\,(J(Du)^{[\ell]})_{kj} = 0.$$

❑

7.3. **Bilinear estimates.** The following bilinear estimate is well-known:

Lemma 7.3. *Let $1 \le p \le \infty$, $1 \le \sigma \le \infty$.*
If $s > 0$ then for all $f \in L^{q_2}(\mathbb{R}^n) \cap \dot{B}^s_{r_1,\sigma}(\mathbb{R}^n)$ and $g \in L^{r_2}(\mathbb{R}^n) \cap \dot{B}^s_{q_1,\sigma}(\mathbb{R}^n)$,

$$\|fg\|_{\dot{B}^s_{p,\sigma}} \le C\big(\|f\|_{\dot{B}^s_{r_1,\sigma}}\|g\|_{r_2} + \|f\|_{q_2}\|g\|_{\dot{B}^s_{q_1,\sigma}}\big), \tag{7.20}$$

where

$$\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{q_1} + \frac{1}{q_2}$$

and $C > 0$ is independent of f and g.

Proof of Lemma 7.3. Let $P_k g = \sum_{\ell=-\infty}^{k-3} \phi_\ell * g = \psi_{2^{-(k-3)}} * g$ denotes the low frequency part of the Littlewood–Paley decomposition of g. Then by the para-product decomposition of the product of f and g,

$$\begin{aligned} f \cdot g \quad &= \quad \sum_{k\in\mathbb{Z}}(\phi_k * f)(P_k g) + \sum_{k\in\mathbb{Z}}(P_k f)(\phi_k * g) + \sum_{k\in\mathbb{Z}}\sum_{|l-k|\le 2}(\phi_k * f)(\phi_l * g) \\ &\equiv \quad h_1 + h_2 + h_3. \end{aligned} \tag{7.21}$$

Since

$$\operatorname{supp} \mathcal{F}\left((\phi_k * f)(P_k g)\right) \subset \{\xi \in \mathbb{R}^n; 2^{k-2} \le |\xi| \le 2^{k+2}\},$$

we have by the Young inequality that for $\dfrac{1}{p} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$,

$$\begin{aligned} \|h_1\|_{\dot{B}^s_{p,\sigma}} \le & \left\{ \sum_{j\in\mathbb{Z}} \Big(2^{sj}\|\phi_j\|_1 \|\widetilde{\phi_j} * f\|_{r_1}\|P_{j+2}g\|_{r_2}\Big)^\sigma \right\}^{1/\sigma} \\ \le & C\|g\|_{r_2}\left\{ \sum_{j\in\mathbb{Z}} \Big(2^{sj}\|\widetilde{\phi_j} * f\|_{r_1}\Big)^\sigma \right\}^{1/\sigma} \\ \le & C\|f\|_{\dot{B}^s_{r_1,\sigma}}\|g\|_{r_2}, \end{aligned} \tag{7.22}$$

where $C = \|\mathcal{F}^{-1}\phi\|_1\|\psi\|_1$. By replacing the role of f and g with that of g and f, respectively, we see that the second term can be handled in the similar way as above. Hence there holds

$$\|h_2\|_{\dot{B}^s_{p,\sigma}} \le C\|f\|_{q_2}\|g\|_{\dot{B}^s_{q_1,\sigma}}, \quad \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{7.23}$$

To deal with the third term, we should notice that

$$\operatorname{supp} \mathcal{F}(\phi_k * f \cdot \phi_l * g) \subset \{\xi \in \mathbb{R}^n; |\xi| \le 2^{\max\{k,l\}+2}\},$$

so there holds

$$\phi_j * ((\phi_k * f)(\phi_l * g)) = 0 \quad \text{for } \max\{k, l\} \le j - 3.$$

Let r_1 and r_2 satisfy

$$\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2}, \tag{7.24}$$

then

$$
\begin{aligned}
\|h_3\|_{\dot{B}^s_{p,\sigma}} =& \Big(\sum_{j\in\mathbb{Z}} \Big(2^{sj}\Big\| \sum_{k\ge j-2} \phi_j * ((\phi_k * f)(\widetilde{\phi_k * g}))\Big\|_p\Big)^\sigma\Big)^{1/\sigma} \\
\le& \Big(\sum_{j\in\mathbb{Z}} \Big(2^{sj} \sum_{k\ge j-2} \|\phi_j\|_1 \|\phi_k * f\|_{r_1} \|\widetilde{\phi}_k * g\|_{r_2}\Big)^\sigma\Big)^{1/\sigma} \\
\le& C\Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k\ge j-2} 2^{sj} \|\phi_k * f\|_{r_1}\Big)^\sigma\Big)^{1/\sigma} \sup_{k\in\mathbb{Z}} \|\widetilde{\phi}_k * g\|_{r_2} \\
& \text{(changing } k' = k - j \text{ to see)} \\
\le& C\Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{k'\ge -2} 2^{s(k'+j)} 2^{-sk'} \|\phi_{k'+j} * f\|_{r_1}\Big)^\sigma\Big)^{1/\sigma} \sup_{k\in\mathbb{Z}} \|\widetilde{\phi}_k * g\|_{r_2} \\
\le& C \sum_{k'\ge -2} 2^{-sk'} \Big(\sum_{j\in\mathbb{Z}} \Big(2^{s(k'+j)} \|\phi_{k'+j} * f\|^\sigma_{r_1}\Big)^\sigma\Big)^{1/\sigma} \sup_{k\in\mathbb{Z}} \|\widetilde{\phi}_k * g\|_{r_2} \\
\le& C\Big(\sum_{j\in\mathbb{Z}} 2^{sj\sigma} \|\phi_j * f\|^\sigma_{r_1}\Big)^{1/\sigma} \|g\|_{r_2} \\
\le& C\|f\|_{\dot{B}^s_{r_1,\sigma}} \|g\|_{L^{r_2}},
\end{aligned}
\tag{7.25}
$$

where we use $s > 0$. The estimate (7.20) follow from (7.21), (7.22), (7.23) and (7.25).

❑

The following bilinear estimates over the whole space $\mathbb{R}^n$ are obtained by Abidi–Paicu [2] (cf. [46]).

Proposition 7.4 ([2])**.** *Let $1 \le p,\ p_1,\ p_2,\ \sigma,\ \lambda_1,\ \lambda_2 \le \infty$, $1/p \le 1/p_1 + 1/p_2$, $p_1 \le \lambda_2$, $p_2 \le \lambda_1$ and*

$$
s_1 + s_2 + n \inf\left(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}\right) > 0, \quad \frac{1}{p} \le \frac{1}{p_1} + \frac{1}{\lambda_1} \le 1, \quad \frac{1}{p} \le \frac{1}{p_2} + \frac{1}{\lambda_2} \le 1.
$$

(1) *If $s_1 + \frac{n}{\lambda_2} < \frac{n}{p_1}$ and $s_2 + \frac{n}{\lambda_1} < \frac{n}{p_2}$, then there exists $C > 0$ such that for all $f \in \dot{B}^{s_1}_{p_1,\sigma}$ and $g \in \dot{B}^{s_2}_{p_2,\infty}$, the following estimate holds*

$$
\|fg\|_{\dot{B}^{s_1+s_2-n\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right)}_{p,\sigma}} \le C\|f\|_{\dot{B}^{s_1}_{p_1,\sigma}} \|g\|_{\dot{B}^{s_2}_{p_2,\infty}}.
$$

(2) *If $s_1 + \frac{n}{\lambda_2} = \frac{n}{p_1}$ and $s_2 + \frac{n}{\lambda_1} = \frac{n}{p_2}$, then there exists $C > 0$ such that for any $f \in \dot{B}^{s_1}_{p_1,1}$ and $g \in \dot{B}^{s_2}_{p_2,1}$ the following estimate holds*

$$
\|fg\|_{\dot{B}^{s_1+s_2-n\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right)}_{p,1}} \le C\|f\|_{\dot{B}^{s_1}_{p_1,1}} \|g\|_{\dot{B}^{s_2}_{p_2,1}}. \tag{7.26}
$$

(3) *In particular, if $s_1 = -1 + n/p$, $s_2 = n/p$ and $-1 + n/p + n/p > \inf(0, n - 2n/p)$ in (2), i.e., $1 \le p < 2n$ then there exists $C > 0$ such that the following estimate holds*

$$
\|fg\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{7.27}
$$

Since Danchin–Mucha [20] treats the equations depending on the density, the restriction on the exponent p in the solution space $\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^n)$ stems from the restriction on $1 \le p < 2n$ for the above bilinear estimate (7.27). One may improve the restriction by using the divergence free - curl free structure of nonlinear terms.

The bilinear estimates as above hold for the case when the two functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ have the divergence structure condition:

$$
f \cdot D_x g = D_x(f \cdot g),
$$

where D_x denotes any combination of partial derivatives by $x = (x_1, x_2, \cdots, x_n)$ of the first order. A typical case is given by the form when f and g satisfies *divergence free-rotation free structure* as $\operatorname{div} f = 0$ and $\operatorname{rot} g = 0$.

Proposition 7.5 (Bilinear estimate under divergence structure)**.** *Let* $1 \le p < \infty$ *and* $f \in \dot{B}^{-1+n/p}_{p,1}$ *and* $g \in \dot{B}^{n/p}_{p,1}$.
(1) *If there exists* $F = F(x)$ *such that* $f \cdot g = D_x(F \cdot g)$ *with* $f = D_x F(x)$ *in the sense of distribution, where* D_x *is any combination of the first differentiation in* x. *Then*

$$\|f \cdot g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{7.28}$$

(2) *In particular with additional conditions* $\operatorname{div} f = 0$, $\operatorname{rot} g = 0$ *in the distribution sense, then*

$$\|f \cdot g\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}}. \tag{7.29}$$

Proof of Proposition 7.5. Here we show the case when $n < p < \infty$ since the other cases $1 \le p \le n$ are already proved in Proposition 7.4. The second estimate is directly obtained from the first part by observing that f, g are both vector-valued functions and satisfy $\operatorname{div} f = 0$, $\operatorname{rot} g = 0$ in the sense of distribution. Then

$$\operatorname{div}(F \wedge g) = \operatorname{rot} F \cdot g - F \cdot (\operatorname{rot} g),$$

it holds that $f \cdot g = (\operatorname{rot} F) \cdot g = \operatorname{div}(F \wedge g)$ which represents the divergence form structure.

Hence we assume that there exists a function G such that $g = D_x G$ and $f \cdot g = D_x(fG)$. Using (7.27), it follows that

$$\begin{aligned}\|fg\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} =&\|D_x(Fg)\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \le C\|Fg\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \\ \le& C\|F\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}} \le C\|f\|_{\dot{B}^{-1+\frac{n}{p}}_{p,1}} \|g\|_{\dot{B}^{\frac{n}{p}}_{p,1}}.\end{aligned}$$

In particular (7.28) and hence (7.29) holds. ❑

Proposition 7.6 (The space-time bilinear estimate)**.** *Let* $1 \le \rho \le \infty$ *and* $1 < p < 2n-1$. *Then for* $F \in \widetilde{\dot{B}^{1/2-1/(2p)}_{\rho,1}}(\mathbb{R}_+;\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1})) \cap \widetilde{L^\rho(\mathbb{R}_+};\dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$ *and* $G \in \widetilde{\dot{B}^{1/2-1/(2p)}_{\infty,1}}(\mathbb{R}_+;\dot{B}^{-1+n/p}_{p,1}(\mathbb{R}^{n-1})) \cap \widetilde{L^\infty(\mathbb{R}_+};\dot{B}^{(n-1)/p}_{p,1}(\mathbb{R}^{n-1}))$, *it holds that*

$$\begin{aligned}&\big\|F\,G\big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+};\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\ \le& C\big\|F(t)\big\|_{\widetilde{L^\rho(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\big\|G(t)\big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+};\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\ &+C\Big(\big\|F(t)\big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+};\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|F(t)\big\|_{\widetilde{L^\rho(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big)\big\|G(t)\big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))} \\ \le& C\Big(\big\|F(t)\big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+};\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|F(t)\big\|_{\widetilde{L^\rho(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big) \\ &\times\Big(\big\|G(t)\big\|_{\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+};\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}+\big\|G(t)\big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\Big),\end{aligned}$$

where the norms are defined in (5.21).

We should like to note that when $\rho = 1$, the following spaces are norm-equivalent;

$$\widetilde{\dot{B}^{\frac12-\frac{1}{2p}}_{1,1}}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})) \simeq \dot{B}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})) \simeq \dot{F}^{\frac12-\frac{1}{2p}}_{1,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})), \tag{7.30}$$

$$\widetilde{L^1(\mathbb{R}_+};\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})) \simeq L^1(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1})). \tag{7.31}$$

Proof of Proposition 7.6 . We employ the doubled Bony paraproduct decomposition in both space and time direction:

$$\begin{aligned}
\|FG\|_{\widetilde{\dot B^{\frac12-\frac1{2p}}_{\rho,1}\left(\mathbb{R}_+;\dot B^{-1+\frac np}_{p,1}(\mathbb{R}^{n-1})\right)}} \le&\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\ge 2j}2^{(\frac12-\frac1{2p})k}\Big\|\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\big(FG\big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
&+\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\le 2j}2^{(\frac12-\frac1{2p})k}\Big\|\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\big(FG\big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\equiv& I+II. \qquad (7.32)
\end{aligned}$$

The estimate for the second term of the right hand side of (7.32) is straightforward:

$$\begin{aligned}
II\le&\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\le 2j}2^{(\frac12-\frac1{2p})k}\Big\||\psi_k|\underset{(t)}{*}\big\|\phi_j\underset{(x')}{*}\big(FG\big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\le 2j}2^{(\frac12-\frac1{2p})k}\Big\|\big\|\phi_j\underset{(x')}{*}\big(FG\big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}2^{(1-\frac1p)j}\Big\|\big\|\phi_j\underset{(x')}{*}\big(FG\big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{j\in\mathbb{Z}}2^{\frac{n-1}{p}j}\Big\|\|\phi_j\|_{L^1(\mathbb{R}^{n-1})}\Big(\big\|(\widetilde\phi_j\underset{(x')}{*}F)(P_j\underset{(x')}{*}G)\big\|_{L^p(\mathbb{R}^{n-1})}+\big\|(P_j\underset{(x')}{*}F)(\widetilde\phi_j\underset{(x')}{*}G)\big\|_{L^p(\mathbb{R}^{n-1})}\\
&\qquad+\big\|\sum_{\ell\ge j-2}(\phi_\ell\underset{(x')}{*}F)(\widetilde\phi_\ell\underset{(x')}{*}G)\big\|_{L^p(\mathbb{R}^{n-1})}\Big)\Big\|_{L^\rho(\mathbb{R}_+)}\\
\le& C\sum_{j\in\mathbb{Z}}2^{\frac{n-1}{p}j}\Big\|\big\|\widetilde\phi_j\underset{(x')}{*}F\big\|_{L^p(\mathbb{R}^{n-1})}\big\|G\big\|_{L^\infty(\mathbb{R}^{n-1})}+\big\|F\big\|_{L^\infty(\mathbb{R}^{n-1})}\big\|\widetilde\phi_j\underset{(x')}{*}G\big\|_{L^p(\mathbb{R}^{n-1})}\\
&\qquad+\sum_{\ell\ge j-2}\big\|\phi_\ell\underset{(x')}{*}F\big\|_{L^\infty(\mathbb{R}^{n-1})}\big\|\widetilde\phi_\ell\underset{(x')}{*}G\big\|_{L^p(\mathbb{R}^{n-1})}\Big)\Big\|_{L^\rho(\mathbb{R}_+)}\\
\le& C\Big(\big\|F\big\|_{L^\rho(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))}\big\|G\big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}+\big\|G\big\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))}\big\|F\big\|_{\widetilde{L^\rho(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}\Big)\\
&+C\sum_{j\in\mathbb{Z}}2^{\frac{n-1}{p}j}\Big\|\sum_{\ell\ge j-2}2^{\frac{n-1}{p}\ell}\big\|\phi_\ell\underset{(x')}{*}F\big\|_{L^p(\mathbb{R}^{n-1})}\big\|\widetilde\phi_\ell\underset{(x')}{*}G\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho(\mathbb{R}_+)}\\
\le& C\Big(\big\|F\big\|_{L^\rho(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))}\big\|G\big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}+\big\|G\big\|_{L^\infty(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))}\big\|F\big\|_{\widetilde{L^\rho(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}\Big)\\
&+C\sum_{\ell\in\mathbb{Z}}2^{\frac{n-1}{p}\ell}\sum_{j\le\ell+2}2^{\frac{n-1}{p}j}\Big\|\big\|\phi_\ell\underset{(x')}{*}F\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho(\mathbb{R}_+)}\Big\|\big\|\widetilde\phi_\ell\underset{(x')}{*}G\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty(\mathbb{R}_+)}\\
\le& C\big\|F\big\|_{\widetilde{L^\rho(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}\big\|G\big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}.
\end{aligned}$$

For the estimate of I in (7.32), we employ the double Bony decomposition in both time and space regions: Set $P_m\underset{(x')}{*}F\equiv\sum_{m'\le m}\phi_{m'}\underset{(x')}{*}F$ and $Q_\ell\underset{(t)}{*}F\equiv\sum_{\ell'\le\ell}\psi_{\ell'}\underset{(t)}{*}F$. Then the Bony decomposition in space direction gives

$$\begin{aligned}
I\le&\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\ge 2j}2^{(\frac12-\frac1{2p})k}\Big\|\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}\\
&\qquad\times\Big(\sum_{m\in\mathbb{Z}}\big(\phi_m\underset{(x')}{*}F(t)\big)\cdot\big(P_m\underset{(x')}{*}G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
&+\sum_{j\in\mathbb{Z}}2^{(-1+\frac np)j}\sum_{k\ge 2j}2^{(\frac12-\frac1{2p})k}\Big\|\big\|\psi_k\underset{(t)}{*}\phi_j\underset{(x')}{*}
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times \Big(\sum_{m\in\mathbb{Z}} \big(P_m \underset{(x')}{*} F(t)\big)\cdot\big(\phi_m \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
&\quad + \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac12-\frac{1}{2p})k}\Big\|\big\|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \\
&\qquad\qquad \times \Big(\sum_{m\geq j-2} \big(\phi_m \underset{(x')}{*} F(t)\big)\cdot\big(\widetilde{\phi}_m \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
&\equiv h_1+h_2+h_3 \leq \sum_{\tau=1}^{3} h_1^\tau + \sum_{\tau=1}^{3} h_2^\tau + \sum_{\tau=1}^{3} h_3^\tau,
\end{aligned}
\tag{7.33}
$$

where h_1 can be decomposed by the Bony decomposition in time direction such as the following:

$$
\begin{aligned}
h_1 \leq &\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac12-\frac{1}{2p})k} \\
&\quad \times \Big\|\big\|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \Big(\sum_{\ell\in\mathbb{Z}} \big(\widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big)\cdot\big(Q_{\ell-2} \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
&+ \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac12-\frac{1}{2p})k} \\
&\quad \times \Big\|\big\|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \Big(\sum_{\ell\in\mathbb{Z}} \big(Q_{\ell-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big)\cdot\big(\widetilde{\psi}_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
&+ \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac12-\frac{1}{2p})k} \\
&\quad \times \Big\|\big\|\psi_k \underset{(t)}{*} \phi_j \underset{(x')}{*} \Big(\sum_{\ell\geq k-2} \big(\psi_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big)\cdot\big(\widetilde{\psi}_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\equiv &h_1^1 + h_1^2 + h_1^3.
\end{aligned}
$$

The estimates for the terms h_1^1, h_1^2 are straightforward. For instance,

$$
\begin{aligned}
h_1^2 \leq &\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac12-\frac{1}{2p})k} \\
&\quad \times \Big\||\psi_k| \underset{(t)}{*} \big\|\phi_j \underset{(x')}{*} \Big(\big(Q_{k-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big)\cdot\big(\widetilde{\psi}_k \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq &C\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac12-\frac{1}{2p})k} \\
&\quad \times \Big\|\big\|\Big(\big(Q_{k-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big)\cdot\big(\widetilde{\psi}_k \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t)\big)\Big)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq &C\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac{1}{2p})k} \sum_{2j\leq k} 2^{(-1+\frac{n}{p})j} \\
&\quad \times \Big\|\big\|Q_{k-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big\|_{L^p(\mathbb{R}^{n-1})}\big\|\widetilde{\psi}_k \underset{(t)}{*} \sum_{m\leq j-2} \phi_m \underset{(x')}{*} G(t)\big)\big\|_{L^\infty(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq &C\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac{1}{2p})k} \sum_{2m\leq k}\sum_{j\geq m+2} 2^{(-1+\frac{1}{p})j} \\
&\quad \times \Big\|\sup_{j\in\mathbb{Z}}\big\|2^{\frac{n-1}{p}j} Q_{k-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big\|_{L^p(\mathbb{R}^{n-1})}\big\|\widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} G(t)\big)\big\|_{L^\infty(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq &C\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac{1}{2p})k} \sum_{2m\leq k} 2^{(-1+\frac{1}{p})m} \\
&\quad \times \Big\|\sup_{j\in\mathbb{Z}}\big\|2^{\frac{n-1}{p}j} Q_{k-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t)\big\|_{L^p(\mathbb{R}^{n-1})}\big\|\widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} G(t)\big)\big\|_{L^\infty(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
\le & C \sup_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} 2^{\frac{n-1}{p}j} \Big\| \big\| Q_{k-2} \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
& \qquad \times \sum_{k\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})k} \sum_{2m\le k} 2^{(-1+\frac{1}{p})m} \Big\| \|\widetilde{\phi}_m\|_{p'} \big\| \widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} G(t) \big) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\le & C \big\| F(t) \big\|_{\widetilde{L^\rho_t(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,\infty}(\mathbb{R}^{n-1}))}} \big\| G(t) \big\|_{\widetilde{\dot{B}^{\frac{1}{2}-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}}.
\end{aligned}
$$

Here we need $1 < p$. While the diagonal term h_1^3 can be estimated as follows.

$$
\begin{aligned}
h_1^3 \le & \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
& \quad \times \Big\| \big\| \phi_j \underset{(x')}{*} \Big(\sum_{\ell\ge k-2} \big(\psi_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t) \big) \cdot \big(\widetilde{\psi}_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t) \big) \Big) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\le & \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
& \times \Big\| \big\| \phi_j \big\|_{L^1(\mathbb{R}^{n-1})} \sum_{\ell\ge k-2} \big\| \psi_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t) \big\|_{L^p(\mathbb{R}^{n-1})} \sup_{\ell,j} \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t) \Big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\le & C \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
& \times \sum_{\ell\ge k-2} \Big\| \big\| \psi_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \sup_{\ell,j} \Big\| \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} G(t) \Big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\le & C \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{\ell\ge 2j-2} \Big(\sum_{k\le \ell+2} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big) \Big\| \big\| \psi_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \Big\| G(t) \Big\|_{L^\infty_t(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))} \\
\le & C \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{\ell\ge 2j-2} 2^{(\frac{1}{2}-\frac{1}{2p})\ell} \Big\| \big\| \psi_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} F(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \Big\| G(t) \Big\|_{L^\infty_t(\mathbb{R}_+;L^\infty(\mathbb{R}^{n-1}))} \\
\le & C \| F(t) \|_{\widetilde{\dot{B}^{\frac{1}{2}-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \big\| G(t) \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}}.
\end{aligned}
$$

There is no restriction except $1 < p$.

For the second term h_2, we decompose by the time direction and typical term can be estimated as follows:

$$
\begin{aligned}
h_2^1 \le & \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| |\psi_k| \underset{(t)}{*} \big\| \phi_j \big\|_{L^1(\mathbb{R}^{n-1})} \big\| \widetilde{\psi}_k \underset{(t)}{*} P_{j-2} \underset{(x')}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \\
& \qquad \times \big\| Q_{k-2} \underset{(t)}{*} \phi_j \underset{(x')}{*} G(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\le & C \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \big\| \sum_{m\le j-2} \phi_m \underset{(x')}{*} \widetilde{\psi}_k \underset{(t)}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \\
& \qquad \times \big\| Q_{k-2} \underset{(t)}{*} \phi_j \underset{(x')}{*} G(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\le & C \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{m\le j-2} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \big\| \phi_m \underset{(x')}{*} \widetilde{\psi}_k \underset{(t)}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
& \qquad \times \Big\| \big\| \phi_j \underset{(x')}{*} G(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\le & C \sum_{m\in\mathbb{Z}} \sum_{j\ge m+2} 2^{(-1+\frac{1}{p})j} \sum_{k\ge 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \big\| \phi_m \underset{(x')}{*} \widetilde{\psi}_k \underset{(t)}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
& \qquad \times \sup_{j\in\mathbb{Z}} \Big\| 2^{\frac{n-1}{p}j} \big\| \phi_j \underset{(x')}{*} G(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
\leq & C \sum_{k\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})k} \sum_{m\in\mathbb{Z}} 2^{(-1+\frac{1}{p})m} 2^{\frac{n-1}{p}m} \Big\| \big\| \phi_m \underset{(x')}{*} \widetilde{\psi}_k \underset{(t)}{*} F(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
& \qquad \times \sup_{j\in\mathbb{Z}} \Big\| 2^{\frac{n-1}{p}j} \big\| \phi_j \underset{(x')}{*} G(t) \big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\leq & C \big\| F(t) \big\|_{\widetilde{\dot{B}^{\frac{1}{2}-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+;\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \big\| G(t) \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,\infty}(\mathbb{R}^{n-1}))}},
\end{aligned}
$$

where we require the restriction $1 < p$ again. The other terms h_2^2 can be treated in similar manner. Space off diagonal and time diagonal term term h_2^3 can be dominated by

$$
\begin{aligned}
h_2^3 \leq & \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
& \times \Big\| \sum_{\ell\geq k-2} \big\| \big(\psi_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} F(t)\big) \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq & \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} \sum_{\ell\geq k-2} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \big\| \psi_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq & C \sup_{\ell\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \Big\| \big\| \psi_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
& \times \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} \sum_{\ell\geq k-2} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\leq & C \sup_{\ell\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \Big\| \big\| \psi_\ell \underset{(t)}{*} P_{j-2} \underset{(x')}{*} F(t) \big\|_{L^\infty(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
& \times \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{\ell\geq 2j-2} \sum_{k\leq \ell+2} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\leq & C \big\| F(t) \big\|_{L^\rho(\mathbb{R};L^\infty(\mathbb{R}^{n-1}))} \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{\ell\geq 2j-2} 2^{(\frac{1}{2}-\frac{1}{2p})\ell} \Big\| \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_j \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
\leq & C \big\| F(t) \big\|_{\widetilde{L^\rho(\mathbb{R}_+;\dot{B}^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \Big\| G(t) \Big\|_{\widetilde{\dot{B}^{\frac{1}{2}-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+:\dot{B}^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1})}}.
\end{aligned}
$$

We estimate for the third term h_3 of right hand side in (7.33). It can be dominated by setting $1/p = 1/r + 2/p - 1$ with $r = p'$,

$$
\begin{aligned}
h_3^1 \leq & \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j} \sum_{k\geq 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
& \times \Big\| \|\phi_j\|_{L^r(\mathbb{R}^{n-1})} \Big\| \sum_{m\geq j-2} \big(\widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\big) \cdot \big(Q_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\big) \Big\|_{L^{\frac{p}{2}}(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq & C \sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p}+\frac{n-1}{r'})j} \sum_{k\geq 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \\
& \times \Big\| \sum_{m\geq j-2} \Big\| \widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t) \Big\|_{L^{r'}(\mathbb{R}^{n-1})} \Big\| Q_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq & C \sum_{m\in\mathbb{Z}} \sum_{j-2\leq m} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j} \sum_{k\geq 2j} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \Big\| \widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \\
& \qquad \times \Big\| Q_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)} \\
\leq & C \sum_{m\in\mathbb{Z}} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})m} 2^{-\frac{n-1}{p}m} \sum_{k\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})k} \Big\| \Big\| \widetilde{\psi}_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho_t(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sup_{k\in\mathbb{Z}}\sup_{m\in\mathbb{Z}} 2^{\frac{n-1}{p}m}\Big\|\Big\|Q_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
\le& C\|F\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \sup_{m\in\mathbb{Z}}\Big\|2^{\frac{n-1}{p}m}\big\|\widetilde{\phi}_m \underset{(x')}{*} G(t)\big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
\le& C\|F\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \big\|G(t)\big\|_{\widetilde{L^\infty(\mathbb{R}_+};\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))},
\end{aligned}
$$

where we used $p<2n-1$. The second term can be estimated by a very similar way:

$$
\begin{aligned}
h_3^2 \le& C\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j}\sum_{k\ge 2j} 2^{(\frac12-\frac{1}{2p})k}\\
&\times\Big\|\sum_{m\ge j-2}\Big\|Q_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|\widetilde{\psi}_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{j\in\mathbb{Z}}\sum_{m\ge j-2} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j}\sum_{k\ge 2j} 2^{(\frac12-\frac{1}{2p})k}\Big\|\Big\|Q_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\\
&\times\Big\|\widetilde{\psi}_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{m\in\mathbb{Z}} 2^{-\frac{n-1}{p}m}\sum_{j\le m+2} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j}\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac{1}{2p})k}\\
&\times\Big\|2^{\frac{n-1}{p}m}\Big\|Q_k \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\Big\|\Big\|\widetilde{\psi}_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
\le& C\sum_{m\in\mathbb{Z}} 2^{-\frac{n-1}{p}m}2^{(-1+\frac{n}{p}+\frac{n-1}{p})m}\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac{1}{2p})k}\\
&\times\Big\|2^{\frac{n-1}{p}m}\Big\|\phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\Big\|\Big\|\widetilde{\psi}_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
\le& C\sup_{m\in\mathbb{Z}} 2^{\frac{n-1}{p}m}\Big\|\Big\|\phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
&\times\sum_{k\in\mathbb{Z}} 2^{(\frac12-\frac{1}{2p})k}\sum_{m\in\mathbb{Z}} 2^{(-1+\frac{n}{p})m}\Big\|\Big\|\widetilde{\psi}_k \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty_t(\mathbb{R}_+)}\\
\le& C\|F(t)\|_{\widetilde{L^\rho(\mathbb{R}_+};\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}\big\|G(t)\big\|_{\widetilde{\dot B^{\frac12-\frac{1}{2p}}_{\infty,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,\infty}(\mathbb{R}^{n-1}))}}.
\end{aligned}
$$

This case we again need the restriction $1<p<2n-1$.

$$
\begin{aligned}
h_3^3 \le& C\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p})j}\sum_{k\ge 2j} 2^{(\frac12-\frac{1}{2p})k}\\
&\times\|\phi_j\|_{L^{p'}(\mathbb{R}^{n-1})}\Big\||\psi_k| \underset{(t)}{*}\Big\|\Big(\sum_{\ell\ge k-2}\sum_{m\ge j-2}\big(\psi_\ell \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\big)\cdot\big(\widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\big)\Big)\Big\|_{L^{\frac{p}{2}}(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{j\in\mathbb{Z}} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j}\sum_{k\ge 2j} 2^{(\frac12-\frac{1}{2p})k}\\
&\times\Big\|\sum_{\ell\ge k-2}\sum_{m\ge j-2}\Big\|\psi_\ell \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|\widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho_t(\mathbb{R}_+)}\\
\le& C\sum_{m\in\mathbb{Z}}\sum_{j\le m+2} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j}\sum_{\ell\in\mathbb{Z}}\sum_{2j\le k\le \ell+2} 2^{(\frac12-\frac{1}{2p})k}\\
&\times\Big\|\Big\|\psi_\ell \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\rho(\mathbb{R}_+)}\Big\|\Big\|\widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t)\Big\|_{L^p(\mathbb{R}^{n-1})}\Big\|_{L^\infty(\mathbb{R}_+)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C \sum_{m\in\mathbb{Z}} \sum_{j\le m+2} 2^{(-1+\frac{n}{p}+\frac{n-1}{p})j} 2^{-\frac{n-1}{p}m} \sum_{\ell\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})\ell} \\
&\qquad \times \Big\| \Big\| \psi_\ell \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho(\mathbb{R}_+)} \Big\| 2^{\frac{n-1}{p}m} \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty(\mathbb{R}_+)} \\
&\le C \sum_{m\in\mathbb{Z}} 2^{(-1+\frac{n}{p})m} \sum_{\ell\in\mathbb{Z}} 2^{(\frac{1}{2}-\frac{1}{2p})\ell} \Big\| \Big\| \psi_\ell \underset{(t)}{*} \phi_m \underset{(x')}{*} F(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\rho(\mathbb{R}_+)} \\
&\qquad\qquad \times \sup_{\ell\in\mathbb{Z}} \Big\| \sup_{m\in\mathbb{Z}} 2^{\frac{n-1}{p}m} \Big\| \widetilde{\psi}_\ell \underset{(t)}{*} \widetilde{\phi}_m \underset{(x')}{*} G(t) \Big\|_{L^p(\mathbb{R}^{n-1})} \Big\|_{L^\infty_t(\mathbb{R}_+)} \\
&\le C \big\| F(t) \big\|_{\widetilde{\dot B^{\frac{1}{2}-\frac{1}{2p}}_{\rho,1}(\mathbb{R}_+;\dot B^{-1+\frac{n}{p}}_{p,1}(\mathbb{R}^{n-1}))}} \big\| G(t) \big\|_{\widetilde{L^\infty(\mathbb{R}_+;\dot B^{\frac{n-1}{p}}_{p,1}(\mathbb{R}^{n-1}))}},
\end{aligned}
$$

where we used the condition $p < 2n-1$. The other terms h_3^1 and h_3^2 can be estimated in a similar way. ❑

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