HIGH-DIMENSIONAL ELLIPSOIDS CONVERGE TO GAUSSIAN SPACES

DAISUKE KAZUKAWA AND TAKASHI SHIOYA

Abstract. We prove the convergence of (solid) ellipsoids to a Gaussian space in Gromov's concentration/weak topology as the dimension diverges to infinity. This gives the first discovered example of an irreducible nontrivial convergent sequence in the concentration topology, where 'irreducible nontrivial' roughly means to be not constructed from Lévy families nor box convergent sequences.

1. Introduction

The study of convergence of metric measure spaces is one of central topics in geometric analysis on metric measure spaces. We refer to [10, 11, 16, 25, 26] for some celebrated works on it. This study originates in that of Gromov-Hausdorff convergence/collapsing of Riemannian manifolds, which has widely been developed and applied to solutions to many significant problems in geometry and topology. As the starting point of geometric analytic study in the collapsing theory, Fukaya [6] introduced the concept of measured Gromov-Hausdorff convergence to study the Laplacian of collapsing Riemannian manifolds. There, he discovered that not only the metric structure but also the measure structure plays an important role in the collapsing phenomena. After that, Cheeger-Colding [3–5] established the theory of Ricci limit spaces, which is nowadays widely applied in the Riemannian and Kähler geometry.

Meanwhile, Gromov [11, Chapter $3\frac{1}{2}_+$] (see also [27]) has developed a new convergence theory of metric measure spaces based on the concentration of measure phenomenon due to Lévy and V. Milman [14, 15, 17]. In Gromov's theory, he introduced two fundamental concepts of distance functions, the *observable distance function* d_{conc} and the *box distance function* $\square$, on the set, say $\mathcal{X}$, of isomorphism classes of metric measure spaces. The box distance function is nearly a metrization of measured Gromov-Hausdorff convergence (precisely the isomorphism classes are little different), while the observable distance function induces a very characteristic topology, called the *concentration topology*, which is effective in capturing the high-dimensional aspects of spaces. The concentration topology is weaker than the box topology and in particular, a measured Gromov-Hausdorff

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convergence implies a convergence in the concentration topology. He also introduced a natural compactification, say Π, of $\mathcal{X}$, with respect to the concentration topology, where the topology on Π is called the *weak topology*. The concentration topology is useful to investigate the dimension-free properties (see [7]).

The study of the concentration and weak topologies has been growing rapidly in recent years (see [7, 12, 13, 18, 20–24, 27–29]). However, it is not easy to prove the convergence of a given sequence of metric measure spaces, and there are only a few nontrivial examples of convergent sequences of metric measure spaces in the concentration and weak topologies, where 'nontrivial' means neither to be a Lévy family (i.e., convergent to a one-point space), to infinitely dissipate (see Subsection 2.6 for dissipation), nor to be box convergent. One way to construct a nontrivial convergent sequence is to take the disjoint union or the product (more generally the fibration) of trivial sequences and to perform little surgery on it (and also to repeat these procedures). We call a sequence obtained in this way a *reducible* sequence. An *irreducible* sequence is a sequence that is not reducible. In this paper, any sequence of (solid) ellipsoids has a subsequence converging to an infinite-dimensional Gaussian space in the concentration/weak topology. This provides a new family of nontrivial weak convergent sequences and especially contains the first discovered example of an irreducible nontrivial sequence that is convergent in the concentration topology.

Let us state our main results precisely. A solid ellipsoid and an ellipsoid are respectively written as

$$\mathcal{E}^n_{\{\alpha_i\}} := \{\, x \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{x_i^2}{\alpha_i^2} \le 1 \,\}, \quad \mathcal{S}^{n-1}_{\{\alpha_i\}} := \{\, x \in \mathbb{R}^n \mid \sum_{i=1}^n \frac{x_i^2}{\alpha_i^2} = 1 \,\},$$

where $\{\alpha_i\}$, $i = 1, 2, \dots, n$, is a finite sequence of positive real numbers. See Section 3 for the definition of their metric-measure structures. Denote by $E^n_{\{\alpha_i\}}$ either $\mathcal{E}^n_{\{\alpha_i\}}$ or $\mathcal{S}^{n-1}_{\{\alpha_i\}}$. Let us give a sequence $\{E^{n(j)}_{\{\alpha_{ij}\}_i}\}_j$ of (solid) ellipsoids, where $\{\alpha_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots$, is a double sequence of positive real numbers. Our problem is to determine under what condition it will converge in the concentration/weak topology and to describe its limit.

In the case where the dimension $n(j)$ is bounded for all j, the problem is easy to solve. In fact, in this case the sequence has a Hausdorff-convergent subsequence in a Euclidean space, which is also box convergent, if α_{ij} is bounded for all i and j; the sequence has an infinitely dissipating subsequence if α_{ij} is unbounded.

We set $a_{ij} := \alpha_{ij}/\sqrt{n(j) - 1}$. If $n(j)$ and $\sup_i a_{ij}$ both diverge to infinity as $j \to \infty$, then it is also easy to prove that $\{E^{n(j)}_{\{\alpha_{ij}\}}\}$ infinitely dissipates (see Proposition 3.3).

For the reasons we have mentioned above, we assume

(A0) $n(j)$ diverges to infinity as $j \to \infty$ and a_{ij} is bounded for all i and j.

We further consider the following three conditions.

(A1) $n(j)$ is monotone nondecreasing in j.
(A2) a_{ij} is monotone nonincreasing in i for each j.

(A3) a_{ij} converges to a real number, say a_i, as $j \to \infty$ for each i.

Note that (A2) and (A3) together imply that a_i is monotone nonincreasing in i.

Any sequence of (solid) ellipsoids with (A0) contains a subsequence $\{E_j\}$ such that each E_j is isomorphic to $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ for some sequence $\{a_{ij}\}$ satisfying (A0)–(A3). In fact, we have a subsequence for which the dimensions satisfy (A1). Then, exchanging the axes of coordinate provides (A2). A diagonal argument proves to have a subsequence satisfying (A3). (To be more precise, there is a subsequence $\{a_{1k_{1j}}\}$ of $\{a_{1j}\}$ convergent to a real number a_1. Then, taking subsequences iteratively, we see that $a_{ik_{ij}}$ converges to a real number a_i as $j \to \infty$ for each i, where $\{a_{i+1,k_{i+1,j}}\}_j$ is a subsequence of $\{a_{ik_{ij}}\}_j$. Replacing $\{a_{ij}\}$ to $\{a_{ik_{jj}}\}$ yields (A3).) Thus, our problem becomes to investigate the convergence of $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}_j$ satisfying (A0)–(A3).

One of our main theorems is stated as follows. Refer to Subsection 2.8 for the definition of the Gaussian space $\Gamma^\infty_{\{a_i^2\}}$.

Theorem 1.1. *Let $\{a_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots$, be a sequence of positive real numbers satisfying* (A0)–(A3). *Then, $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to the infinite-dimensional Gaussian space $\Gamma^\infty_{\{a_i^2\}}$ as $j \to \infty$. This convergence becomes a convergence in the concentration topology if and only if $\{a_i\}$ is an l^2-sequence. Moreover, this convergence becomes an asymptotic concentration* (*i.e., a d_{conc}-Cauchy sequence*) *if and only if $\{a_i\}$ converges to zero.*

Gromov presents as an exercise in [11, $3\frac{1}{2}$.57] some special cases of this theorem. For the case of round spheres and projective spaces, the theorem is formerly obtained in [27, 28], for which the convergence is only weak. Also, the weak convergence of Stiefel and flag manifolds are studied in [29].

We emphasize that convergence in the weak/concentration topology is completely different from weak convergence of measures. For instance, the Prokhorov distance between the normalized volume measure on $S^{n-1}(\sqrt{n-1})$ and the n-dimensional standard Gaussian measure on $\mathbb{R}^n$ is bounded away from zero [29], though they both converge to the infinite-dimensional standard Gaussian space in Gromov's weak topology.

As for the characterization of weak convergence of measures, we prove in Proposition 4.2 that, if $\{a_{ij}\}_i$ l^2-converges to an l^2-sequence $\{a_i\}$ as $j \to \infty$, then the measure of $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to the Gaussian measure $\gamma^\infty_{\{a_i^2\}}$ on a Hilbert space, and consequently, the weak convergence in Theorem 1.1 becomes the box convergence. Conversely, the l^2-convergence of $\{a_{ij}\}_i$ is also a necessary condition for the box convergence of the (solid) ellipsoids as is seen in the following theorem.

Theorem 1.2. *Let $\{a_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots$, be a sequence of positive real numbers satisfying* (A0)–(A3). *Then, the convergence in Theorem* 1.1

becomes a box convergence if and only if we have

$$\sum_{i=1}^{\infty} a_i^2 < +\infty \quad \textit{and} \quad \lim_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 = 0.$$

Theorems 1.1 and 1.2 together provide an example of irreducible nontrivial convergent sequence of metric measure spaces in the concentration topology, i.e., the sequence of the (solid) ellipsoids with an l^2-sequence $\{a_i\}$ and with a non-l^2-convergent $\{a_{ij}\}_i$ as $j \to \infty$.

The proof of the 'only if' part of Theorem 1.2 is highly nontrivial. If $\{a_{ij}\}_i$ does not l^2-converge, then it is easy to see that the measure of the (solid) ellipsoid in the sequence does not converge weakly in the Hilbert space. However, this is not enough to obtain the box non-convergence, because we consider the *isomorphism classes* of (solid) ellipsoids for the box convergence. For the complete proof, we need a delicate discussion using Theorem 1.1.

Let us briefly mention the outline of the proof of Theorem 1.1. For simplicity, we set $E^n := E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ and $\Gamma := \Gamma^{\infty}_{\{a_i^2\}}$. For the weak convergence, it is sufficient to show that

(1.1) the limit of E^n dominates Γ,
(1.2) Γ dominates the limit of E^n,

where, for two metric measure spaces X and Y, the space X *dominates* Y if there is a 1-Lipschitz map from X to Y preserving their measures.

(1.1) easily follows from the Maxwell-Boltzmann distribution law (Proposition 3.2).

(1.2) is much harder to prove. Let us first consider the simple case where E^n is the ball $B^n(\sqrt{n-1})$ of radius $\sqrt{n-1}$ and where $\Gamma = \Gamma^{\infty}_{\{1^2\}}$. We see that, for any fixed $0 < \theta < 1$, the n-dimensional Gaussian measure $\gamma^n_{\{1^2\}}$ and the normalized volume measure of $B^n(\theta\sqrt{n-1})$ both are very small for large n. Ignoring this small part $B^n(\theta\sqrt{n-1})$, we find a measure-preserving isotropic map, say φ, from $\Gamma^n_{\{1^2\}} \setminus B^n(\theta\sqrt{n-1})$ to the annulus $B^n(\sqrt{n-1}) \setminus B^n(\theta\sqrt{n-1})$, where we normalize their measures to be probability. Estimating the Lipschitz constant of φ, we obtain (1.2) with error. This error is estimated and we eventually obtain the required weak convergence.

We next try to apply this discussion to solid ellipsoids. We consider the distortion of the above isotropic map φ by a linear transformation determined by $\{a_{ij}\}$. However, the Lipschitz constant of the distorted isotropic map can be arbitrarily large depending on $\{a_{ij}\}$. To overcome this problem, we settle the assumptions (A0)–(A3), from which the discussion boils down to the special case where $a_i = a_N$ for all $i \geq N$ and $a_{ij} = a_i \geq a_N$ for all i, j and for a (large) number N. In fact, by (A0)–(A3), the solid ellipsoid E^n for large n and the Gaussian space Γ are both close to those in the above special case. In this special case, the Gaussian measure $\gamma^n_{\{a_i^2\}}$ and the normalized volume measure of E^n of the domain

$$\left\{\, x \in \mathbb{R}^n \setminus \{o\} \;\middle|\; \frac{|x_i|}{\|x\|} < \varepsilon \text{ for any } i = 1, \ldots, N-1 \,\right\}$$

are both almost full for large n and for any fixed $\varepsilon > 0$. On this domain, we are able to estimate the Lipschitz constant of the distorted isotropic map. With some careful error estimates, letting $\varepsilon \to 0+$ and $\theta \to 1-$, we prove the weak convergence of E^n to Γ.

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2. Preliminaries

In this section, we survey the definitions and the facts needed in this paper. We refer to [11, Chapter $3\frac{1}{2}{}_+$] and [27] for more details.

2.1. **Distance between measures.**

Definition 2.1 (Total variation distance)**.** The *total variation distance* $d_{\mathrm{TV}}(\mu,\nu)$ of two Borel probability measures μ and ν on a topological space X is defined by

$$d_{\mathrm{TV}}(\mu,\nu) := \sup_A |\, \mu(A) - \nu(A) \,| = \sup_A (\mu(A) - \nu(A)),$$

where A runs over all Borel subsets of X.

If μ and ν are both absolutely continuous with respect to a Borel measure ω on X, then

$$d_{\mathrm{TV}}(\mu,\nu) = \frac{1}{2}\int_X \left| \frac{d\mu}{d\omega} - \frac{d\nu}{d\omega} \right| \, d\omega \tag{2.1}$$

(see [30, Lemma 2.1 in Section 2.4]), where $\frac{d\mu}{d\omega}$ is the Radon-Nikodym derivative of μ with respect to ω.

Definition 2.2 (Prokhorov distance)**.** The *Prokhorov distance* $d_{\mathrm{P}}(\mu,\nu)$ between two Borel probability measures μ and ν on a metric space (X, d_X) is defined to be the infimum of $\varepsilon \geq 0$ satisfying

$$\mu(B_\varepsilon(A)) \geq \nu(A) - \varepsilon$$

for any Borel subset $A \subset X$, where $B_\varepsilon(A) := \{\, x \in X \mid d_X(x, A) \leq \varepsilon \,\}$.

The Prokhorov metric is a metrization of weak convergence of Borel probability measures on X provided that X is a separable metric space. It follows from the definitions that $d_{\mathrm{P}} \leq d_{\mathrm{TV}}$.

Definition 2.3 (Ky Fan distance)**.** Let (X,μ) be a measure space and Y a metric space. For two μ-measurable maps $f, g : X \to Y$, we define the *Ky Fan distance* $d_{\mathrm{KF}}(f,g)$ *between* f *and* g to be the infimum of $\varepsilon \geq 0$ satisfying

$$\mu(\{\, x \in X \mid d_Y(f(x), g(x)) > \varepsilon \,\}) \leq \varepsilon.$$

d_{KF} is a pseudo-metric on the set of μ-measurable maps from X to Y. It holds that $d_{\mathrm{KF}}(f,g) = 0$ if and only if $f = g$ μ-a.e. We have $d_{\mathrm{P}}(f_*\mu, g_*\mu) \leq d_{\mathrm{KF}}(f,g)$ (see [27, Lemma 1.26]), where $f_*\mu$ is the push-forward of μ by f.

Let p be a real number with $p \geq 1$, and (X, d_X) a complete separable metric space.

Definition 2.4. The *p-Wasserstein distance* between two Borel probability measures μ and ν on X is defined to be

$$W_p(\mu,\nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{X \times X} d_X(x,x')^p \, d\pi(x,x') \right)^{\frac{1}{p}} \ (\le +\infty),$$

where $\Pi(\mu,\nu)$ is the set of couplings between μ and ν, i.e., the set of Borel probability measures π on $X \times X$ such that $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for any Borel subset $A \subset X$.

Lemma 2.5 (see [31, Theorem 7.12])**.** *Let μ and μ_n, $n = 1, 2, \dots,$ be Borel probability measures on X. Then the following are equivalent to each other.*

(1) $W_p(\mu_n, \mu) \to 0$ *as* $n \to \infty$.
(2) μ_n *converges weakly to* μ *as* $n \to \infty$ *and the p-th moment of* μ_n *converges to that of* μ:

$$\lim_{n\to\infty} \int_X d_X(x_0,x)^p \, d\mu_n(x) = \int_X d_X(x_0,x)^p \, d\mu(x)$$

for some point $x_0 \in X$.

It is known that ${d_{\mathrm{P}}}^2 \le W_1$ (see [9, Theorem 2 in Section 3]). By Hölder's inequality, we have $W_p \le W_q$ for any $1 \le p \le q$.

2.2. mm-Isomorphism and Lipschitz order.

Definition 2.6 (mm-Space)**.** Let (X, d_X) be a complete separable metric space and μ_X a Borel probability measure on X. We call the triple (X, d_X, μ_X) an *mm-space*. We sometimes say that X is an mm-space, in which case the metric and the Borel measure of X are respectively indicated by d_X and μ_X.

Definition 2.7 (mm-Isomorphism)**.** Two mm-spaces X and Y are said to be *mm-isomorphic* to each other if there exists an isometry $f : \operatorname{supp} \mu_X \to \operatorname{supp} \mu_Y$ with $f_*\mu_X = \mu_Y$, where $\operatorname{supp} \mu_X$ is the support of μ_X. Such an isometry f is called an *mm-isomorphism*. Denote by $\mathcal{X}$ the set of mm-isomorphism classes of mm-spaces.

Note that X is mm-isomorphic to $(\operatorname{supp} \mu_X, d_X, \mu_X)$.

We assume that an mm-space X satisfies

$$X = \operatorname{supp} \mu_X$$

unless otherwise stated.

Definition 2.8 (Lipschitz order)**.** Let X and Y be two mm-spaces. We say that X (*Lipschitz*) *dominates* Y and write $Y \prec X$ if there exists a 1-Lipschitz map $f : X \to Y$ satisfying $f_*\mu_X = \mu_Y$. We call the relation $\prec$ on $\mathcal{X}$ the *Lipschitz order*.

The Lipschitz order $\prec$ is a partial order relation on $\mathcal{X}$ (see [27, Proposition 2.11]).

2.3. **Observable diameter.** The observable diameter is one of the most fundamental invariants of an mm-space up to mm-isomorphism.

Definition 2.9 (Partial and observable diameter)**.** Let X be an mm-space and let $\kappa > 0$. We define the *κ-partial diameter* $\operatorname{diam}(X; 1-\kappa) = \operatorname{diam}(\mu_X; 1-\kappa)$ of X to be the infimum of the diameter of A, where $A \subset X$ runs over all Borel subsets with $\mu_X(A) \geq 1-\kappa$. Denote by $\mathcal{L}ip_1(X)$ the set of 1-Lipschitz continuous real-valued functions on X. We define the *(κ-)observable diameter of X* by

$$\begin{aligned} \operatorname{ObsDiam}(X; -\kappa) &:= \sup_{f \in \mathcal{L}ip_1(X)} \operatorname{diam}(f_*\mu_X; 1-\kappa), \\ \operatorname{ObsDiam}(X) &:= \inf_{\kappa > 0} \max\{\operatorname{ObsDiam}(X; -\kappa), \kappa\}. \end{aligned}$$

It is easy to see that the (κ-)observable diameter is monotone nondecreasing with respect to the Lipschitz order relation.

2.4. **Box distance and observable distance.**

Definition 2.10 (Parameter)**.** Let $I := [\,0, 1\,)$ and let X be an mm-space. A map $\varphi : I \to X$ is called a *parameter of X* if φ is a Borel measurable map with $\varphi_*\mathcal{L}^1 = \mu_X$, where $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure on I.

It is known that any mm-space has a parameter (see [27, Lemma 4.2]).

Definition 2.11 (Box distance)**.** We define the *box distance $\square(X, Y)$ between two mm-spaces X and Y* to be the infimum of $\varepsilon \geq 0$ satisfying that there exist parameters $\varphi : I \to X$, $\psi : I \to Y$, and a Borel subset $\tilde{I} \subset I$ such that

$$\mathcal{L}^1(\tilde{I}) \geq 1 - \varepsilon \quad \text{and} \quad |\,\varphi^* d_X(s, t) - \psi^* d_Y(s, t)\,| \leq \varepsilon$$

for any $s, t \in \tilde{I}$, where $\varphi^* d_X(s, t) := d_X(\varphi(s), \varphi(t))$ for $s, t \in I$.

The box metric $\square$ is a complete separable metric on $\mathcal{X}$ (see [27, Theorem 4.14 and Proposition 4.25]).

Definition 2.12 (ε-mm-isomorphism)**.** Let ε be a nonnegative real number. A map $f : X \to Y$ between two mm-spaces X and Y is called an *ε-mm-isomorphism* if there exists a Borel subset $\tilde{X} \subset X$ such that

(i) $\mu_X(\tilde{X}) \geq 1 - \varepsilon$,
(ii) $|\,d_X(x, x') - d_Y(f(x), f(x'))\,| \leq \varepsilon$ for any $x, x' \in \tilde{X}$,
(iii) $d_{\mathrm{P}}(f_*\mu_X, \mu_Y) \leq \varepsilon$.

We call the set $\tilde{X}$ a *nonexceptional domain* of f.

Lemma 2.13 (see [27, Lemma 4.22])**.** *Let X and Y be two mm-spaces and let $\varepsilon \geq 0$.*

(1) *If there exists an ε-mm-isomorphism from X to Y, then $\square(X, Y) \leq 3\varepsilon$.*
(2) *If $\square(X, Y) \leq \varepsilon$, then there exists a 3ε-mm-isomorphism from X to Y.*

Definition 2.14 (Observable distance)**.** For any parameter φ of X, we set

$$\varphi^*\mathcal{L}ip_1(X) := \{\ f \circ \varphi \mid f \in \mathcal{L}ip_1(X)\ \}.$$

Define the *observable distance* $d_{\mathrm{conc}}(X,Y)$ *between two mm-spaces* X *and* Y by

$$d_{\mathrm{conc}}(X,Y) := \inf_{\varphi,\psi} d_{\mathrm{H}}(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(Y)),$$

where $\varphi : I \to X$ and $\psi : I \to Y$ run over all parameters of X and Y, respectively, and where d_{H} is the Hausdorff metric with respect to the Ky Fan metric for the one-dimensional Lebesgue measure on I.

d_{conc} is a metric on $\mathcal{X}$ (see [27, Theorem 5.13]) and

$$d_{\mathrm{conc}} \le \square$$

holds (see [27, Proposition 5.5]).

2.5. **Pyramid.**

Definition 2.15 (Pyramid)**.** A subset $\mathcal{P} \subset \mathcal{X}$ is called a *pyramid* if it satisfies the following (i)–(iii).

- (i) If $X \in \mathcal{P}$ and if $Y \prec X$, then $Y \in \mathcal{P}$.
- (ii) For any two mm-spaces $X, X' \in \mathcal{P}$, there exists an mm-space $Y \in \mathcal{P}$ such that $X \prec Y$ and $X' \prec Y$.
- (iii) $\mathcal{P}$ is nonempty and box closed.

We denote the set of pyramids by Π. Note that Gromov's definition of a pyramid is by (i) and (ii) only. (iii) is added in [27] for the Hausdorff property of Π.

For an mm-space X we define

$$\mathcal{P}X := \{\ X' \in \mathcal{X} \mid X' \prec X\ \},$$

which is a pyramid (where the closedness of $\mathcal{P}X$ follows from [27, Theorem 4.35]). We call $\mathcal{P}X$ the *pyramid associated with* X.

We observe that $X \prec Y$ if and only if $\mathcal{P}X \subset \mathcal{P}Y$. It is trivial that $\mathcal{X}$ is a pyramid.

We have a metric, denoted by ρ, on Π, for which we omit to state the definition (see [27, Definition 6.21] for the detail). We say that a sequence of pyramids *converges weakly* to a pyramid if it converges with respect to ρ. We have the following.

1. (1) The map $\iota : \mathcal{X} \ni X \mapsto \mathcal{P}X \in \Pi$ is a 1-Lipschitz topological embedding map with respect to d_{conc} and ρ (see [27, Theorem 6.23]).
2. (2) Π is ρ-compact (see [27, Theorem 6.22]).
3. (3) $\iota(\mathcal{X})$ is ρ-dense in Π (see [27, Lemma 7.14]).

In particular, (Π, ρ) is a compactification of $(\mathcal{X}, d_{\mathrm{conc}})$. We say that a sequence of mm-spaces *converges weakly* to a pyramid if the associated pyramid converges weakly. Note that we identify X with $\mathcal{P}X$ in Section 1.

For an mm-space X, a pyramid $\mathcal{P}$, and $t > 0$, we define

$$tX := (X, t\,d_X, \mu_X) \quad \text{and} \quad t\mathcal{P} := \{\ tX \mid X \in \mathcal{P}\ \}.$$

We see $\mathcal{P}\,tX = t\,\mathcal{P}X$. It is easy to see that $t\mathcal{P}$ is continuous in t with respect to ρ.

From [27, Theorem 6.25, Propositions 5.5 and 4.12] we have the following.

Proposition 2.16. *For any two Borel probability measures μ and ν on a complete separable metric space X, we have*

$$\begin{aligned}\rho(\mathcal{P}(X,\mu),\mathcal{P}(X,\nu)) &\le d_{\mathrm{conc}}((X,\mu),(X,\nu)) \le \square((X,\mu),(X,\nu)) \\ &\le 2\,d_{\mathrm{P}}(\mu,\nu) \le 2\,d_{\mathrm{TV}}(\mu,\nu).\end{aligned}$$

2.6. **Dissipation.** Dissipation is the opposite notion to concentration. We omit to state the definition of the infinite dissipation (see [27, Definition 8.1] for the definition). Instead, we state the following proposition. Let $\{X_n\}$, $n = 1, 2, \dots$, be a sequence of mm-spaces.

Proposition 2.17 (see [27, Proposition 8.5(2)])**.** *The sequence $\{X_n\}$ infinitely dissipates if and only if $\mathcal{P}X_n$ converges weakly to $\mathcal{X}$ as $n \to \infty$.*

An easy discussion using [22, Lemma 6.6] leads to the following.

Proposition 2.18. *The following are equivalent to each other.*

(1) *The κ-observable diameter* $\mathrm{ObsDiam}(X_n; -\kappa)$ *diverges to infinity as $n \to \infty$ for any $\kappa \in (\,0, 1\,)$.*
(2) *$\{X_n\}$ infinitely dissipates.*

2.7. **Asymptotic concentration.** We say that a sequence of mm-spaces *asymptotically concentrates* if it is a d_{conc}-Cauchy sequence. It is known that any asymptotically concentrating sequence converges weakly to a pyramid (see [27, Proposition 7.2]). A pyramid $\mathcal{P}$ is said to be *concentrated* if $\{(\mathcal{L}ip_1(X)/\sim, d_{\mathrm{KF}})\}_{X\in\mathcal{P}}$ is precompact with respect to the Gromov-Hausdorff distance, where $f \sim g$ holds if $f - g$ is constant. The following is derived from [27, Corollary 7.24 and Theorem 7.25].

Theorem 2.19. *Let $\mathcal{P}$ be a pyramid. The following are equivalent to each other.*

(1) *$\mathcal{P}$ is concentrated.*
(2) *There exists a sequence of mm-spaces asymptotically concentrating to $\mathcal{P}$.*
(3) *If a sequence of mm-spaces converges weakly to $\mathcal{P}$, then it asymptotically concentrates.*

2.8. **Gaussian space.** Let $\{a_i\}$, $i = 1, 2, \dots, n$, be a finite sequence of nonnegative real numbers. The product

$$\gamma^n_{\{a_i^2\}} := \bigotimes_{i=1}^n \gamma^1_{a_i^2}$$

of the one-dimensional centered Gaussian measure $\gamma^1_{a_i^2}$ of variance a_i^2 is an n-dimensional centered Gaussian measure on $\mathbb{R}^n$, where we agree that $\gamma^1_{0^2}$ is the Dirac measure at 0, and $\gamma^n_{\{a_i^2\}}$ is possibly degenerate. We call the mm-space $\Gamma^n_{\{a_i^2\}} := (\mathbb{R}^n, \|\cdot\|, \gamma^n_{\{a_i^2\}})$ the *n-dimensional Gaussian space with variance $\{a_i^2\}$.* Note that, for any Gaussian measure γ on $\mathbb{R}^n$, the mm-space $(\mathbb{R}^n, \|\cdot\|, \gamma)$ is mm-isomorphic to $\Gamma^n_{\{a_i^2\}}$, where a_i^2 are the eigenvalues of the covariance matrix of γ.

We now take an infinite sequence $\{a_i\}$, $i = 1, 2, \dots$, of nonnegative real numbers. For $1 \le k \le n$, we denote by $\pi^n_k : \mathbb{R}^n \to \mathbb{R}^k$ the natural projection, i.e.,

$$\pi^n_k(x_1, x_2, \dots, x_n) := (x_1, x_2, \dots, x_k), \quad (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

Since the projection $\pi^n_{n-1} : \Gamma^n_{\{a_i^2\}} \to \Gamma^{n-1}_{\{a_i^2\}}$ is 1-Lipschitz continuous and measure-preserving for any $n \ge 2$, the Gaussian space $\Gamma^n_{\{a_i^2\}}$ is monotone nondecreasing in n with respect to the Lipschitz order, so that, as $n \to \infty$, the associated pyramid $\mathcal{P}\Gamma^n_{\{a_i^2\}}$ converges weakly to the $\square$-closure of $\bigcup_{n=1}^\infty \mathcal{P}\Gamma^n_{\{a_i^2\}}$, denoted by $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$. We call $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ the *virtual Gaussian space with variance* $\{a_i^2\}$. We remark that the infinite product measure

$$\gamma^\infty_{\{a_i^2\}} := \bigotimes_{i=1}^\infty \gamma^1_{a_i^2}$$

is a Borel probability measure on $\mathbb{R}^\infty$ with respect to the product topology, but is not necessarily Borel with respect to the l^2-norm. Only in the case where

$$\sum_{i=1}^\infty a_i^2 < +\infty, \tag{2.2}$$

the measure $\gamma^\infty_{\{a_i^2\}}$ is a Borel measure with respect to the l^2-norm $\|\cdot\|$ which is supported in the separable Hilbert space $H := \{\, x \in \mathbb{R}^\infty \mid \|x\| < +\infty \,\}$ (see [1, §2.3]), and consequently, $\Gamma^\infty_{\{a_i^2\}} = (H, \|\cdot\|, \gamma^\infty_{\{a_i^2\}})$ is an mm-space. In the case of (2.2), the variance of $\gamma^\infty_{\{a_i^2\}}$ satisfies

$$\int_H \|x\|^2 \, d\gamma^\infty_{\{a_i^2\}}(x) = \sum_{i=1}^\infty a_i^2.$$

3. Weak convergence of ellipsoids

In this section we prove Theorem 1.1. We also prove the convergence of Gaussian spaces as a corollary to the theorem.

Let $\{\alpha_i\}$, $i = 1, 2, \dots, n$, be a sequence of positive real numbers. The n-dimensional solid ellipsoid $\mathcal{E}^n$ and the $(n-1)$-dimensional ellipsoid $\mathcal{S}^{n-1}$ (defined in Section 1) are respectively obtained as the image of the closed unit ball $B^n(1)$ and the unit sphere $S^{n-1}(1)$ in $\mathbb{R}^n$ by the linear isomorphism $L^n_{\{\alpha_i\}} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$L^n_{\{\alpha_i\}}(x) := (\alpha_1 x_1, \dots, \alpha_n x_n), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

We assume that the n-dimensional solid ellipsoid $\mathcal{E}^n_{\{\alpha_i\}}$ is equipped with the restriction of the Euclidean distance function and with the normalized Lebesgue measure $\epsilon^n_{\{\alpha_i\}} := \widetilde{\mathcal{L}^n|_{\mathcal{E}^n_{\{\alpha_i\}}}}$, where $\widetilde{\mu} := \mu(X)^{-1}\mu$ is the *normalization* of a finite measure μ on a space X and $\mathcal{L}^n$ the n-dimensional Lebesgue measure on $\mathbb{R}^n$. The $(n-1)$-dimensional ellipsoid $\mathcal{S}^{n-1}_{\{\alpha_i\}}$ is assumed to be equipped with the restriction of the Euclidean distance function and with the push-forward $\sigma^{n-1}_{\{\alpha_i\}} := (L^n_{\{\alpha_i\}})_* \sigma^{n-1}$ of the normalized volume measure σ^{n-1} on the unit sphere $S^{n-1}(1)$ in $\mathbb{R}^n$.

Throughout this paper, let $(E^n_{\{\alpha_i\}}, e^n_{\{\alpha_i\}})$ *be either*

$$(\mathcal{E}^n_{\{\alpha_i\}}, \epsilon^n_{\{\alpha_i\}}) \quad or \quad (\mathcal{S}^{n-1}_{\{\alpha_i\}}, \sigma^{n-1}_{\{\alpha_i\}})$$

for any $n \geq 2$ *and* $\{\alpha_i\}$. The measure $e^n_{\{\alpha_i\}}$ is sometimes considered as a Borel measure on $\mathbb{R}^n$, supported on $E^n_{\{\alpha_i\}}$.

Lemma 3.1. *Let* $\{\alpha_i\}$ *and* $\{\beta_i\}$, $i = 1, 2, \dots, n$, *be two sequences of positive real numbers. If* $\alpha_i \leq \beta_i$ *for all* $i = 1, 2, \dots, n$, *then* $E^n_{\{\alpha_i\}}$ *is dominated by* $E^n_{\{\beta_i\}}$.

Proof. The map $L^n_{\{\alpha_i/\beta_i\}} : E^n_{\{\beta_i\}} \to E^n_{\{\alpha_i\}}$ is 1-Lipschitz continuous and preserves their measures. □

Proposition 3.2 (Maxwell-Boltzmann distribution law). *Let* k *be a positive integer and let* $\{a_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots$, *be a sequence of positive real numbers such that, as* $j \to \infty$, $n(j)$ *is divergent to infinity and* a_{ij} *converges to a nonnegative real number* a_i *for each* i *with* $1 \leq i \leq k$. *Then,* $(\pi^{n(j)}_k)_* e^n_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ *converges weakly to* $\gamma^k_{\{a_i^2\}}$ *as* $j \to \infty$, *where* π^n_k *is defined in Subsection* 2.8.

Proof. In the following we write $n(j)$ as n. The ordinary Maxwell-Boltzmann distribution law (see [27, Proposition 2.1]) states that $(\pi^n_k)_*\sigma^{n-1}_{\{\sqrt{n-1}\}}$ converges weakly to $\gamma^k_{\{1^2\}}$ as $j \to \infty$. In the same way of proof, we see that $(\pi^n_k)_*\epsilon^n_{\{\sqrt{n-1}\}}$ converges weakly to $\gamma^k_{\{1^2\}}$ as $j \to \infty$, namely

$$(\pi^n_k)_* e^n_{\{\sqrt{n-1}\}} \to \gamma^k_{\{1^2\}} \text{ weakly as } j \to \infty. \tag{3.1}$$

Since

$$\begin{aligned}(L^n_{\{a_{ij}\}})^{-1}((\pi^n_k)^{-1}(A)) &= (L^n_{\{a_{ij}\}})^{-1}(A \times \mathbb{R}^{n-k}) \\ &= (L^k_{\{a_{ij}\}})^{-1}(A) \times \mathbb{R}^{n-k} = (\pi^n_k)^{-1}((L^k_{\{a_{ij}\}})^{-1}(A))\end{aligned}$$

for any Borel set $A \subset \mathbb{R}^k$, we have

$$(\pi^n_k)_* e^n_{\{\sqrt{n-1}\,a_{ij}\}} = (\pi^n_k)_*(L^n_{\{a_{ij}\}})_* e^n_{\{\sqrt{n-1}\}} = (L^k_{\{a_{ij}\}})_*(\pi^n_k)_* e^n_{\{\sqrt{n-1}\}}. \tag{3.2}$$

Let $f : \mathbb{R}^k \to \mathbb{R}$ be any continuous function with compact support. By the uniform continuity of f, we observe that $f \circ L^k_{\{a_{ij}\}}$ converges uniformly to $f \circ L^k_{\{a_i\}}$ as $j \to \infty$ and hence, by (3.1),

$$\begin{aligned}\int_{\mathbb{R}^k} f\, d(L^k_{\{a_{ij}\}})_*(\pi^n_k)_* e^n_{\{\sqrt{n-1}\}} &= \int_{\mathbb{R}^k} f \circ L^k_{\{a_{ij}\}}\, d(\pi^n_k)_* e^n_{\{\sqrt{n-1}\}} \\ \xrightarrow{j\to\infty} \int_{\mathbb{R}^k} f \circ L^k_{\{a_i\}}\, d\gamma^k_{\{1^2\}} &= \int_{\mathbb{R}^k} f\, d(L^k_{\{a_i\}})_*\gamma^k_{\{1^2\}},\end{aligned}$$

which implies that (3.2) converges weakly to $(L^k_{\{a_i\}})_*\gamma^k_{\{1^2\}} = \gamma^k_{\{a_i^2\}}$. This completes the proof. □

Proposition 3.3. *Let* $\{a_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots$, *be a sequence of positive real numbers, where* $\{n(j)\}$, $j = 1, 2, \dots$, *is a sequence of positive integers divergent to infinity. If* $\sup_i a_{ij}$ *diverges to infinity as* $j \to \infty$, *then* $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$ *infinitely dissipates.*

Proof. Assume that $\sup_i a_{ij}$ diverges to infinity as $j \to \infty$. Exchanging the coordinates, we assume that a_{1j} diverges to infinity as $j \to \infty$. We take any positive real number a and fix it. Let $\hat{a}_{ij} := \min\{a_{ij}, a\}$. Note that $\hat{a}_{1j} = a$ for all sufficiently large j. By Lemma 3.1, the 1-Lipschitz continuity of $\pi_1^{n(j)}$, and the Maxwell-Boltzmann distribution law (Proposition 3.2), we have

$$\begin{aligned}
&\liminf_{j\to\infty} \mathrm{ObsDiam}(E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}; -\kappa)\\
&\geq \liminf_{j\to\infty} \mathrm{ObsDiam}(E^{n(j)}_{\{\sqrt{n(j)-1}\,\hat{a}_{ij}\}_i}; -\kappa)\\
&\geq \lim_{j\to\infty} \mathrm{diam}((\pi_1^{n(j)})_* e^{n(j)}_{\{\sqrt{n(j)-1}\,\hat{a}_{ij}\}_i}; 1-\kappa)\\
&= \mathrm{diam}(\gamma^1_{a^2}; 1-\kappa),
\end{aligned}$$

which diverges to infinity as $a \to \infty$. Proposition 2.18 leads us to the dissipation property for $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$. $\square$

Let $\{a_i\}$, $i = 1, 2, \ldots, n$, be a sequence of positive real numbers and let $L := L^n_{\{a_i\}}$. We remark that

$$\epsilon^n_{\{\sqrt{n-1}\,a_i\}} = L_*\epsilon^n_{\sqrt{n-1}}, \qquad \sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}} = L_*\sigma^n_{\sqrt{n-1}}, \quad \text{and} \quad \gamma^n_{\{a_i^2\}} = L_*\gamma^n_{\{1^2\}},$$

where $\epsilon^n_{\sqrt{n-1}} := \epsilon^n_{\{\sqrt{n-1}\}}$ and $\sigma^n_{\sqrt{n-1}} := \sigma^n_{\{\sqrt{n-1}\}}$. Let us construct a transport map from $\gamma^n_{\{a_i^2\}}$ to $\epsilon^n_{\{\sqrt{n-1}\,a_i\}}$. For $r \geq 0$ we determine a real number $R = R(r)$ in such a way that $0 \leq R \leq \sqrt{n-1}$ and $\gamma^n_{\{1^2\}}(B_r(o)) = \epsilon^n_{\sqrt{n-1}}(B_R(o))$. It holds that

$$R = (n-1)^{\frac{1}{2}} \left(\frac{1}{I_{n-1}} \int_0^r t^{n-1} e^{-\frac{t^2}{2}}\, dt \right)^{\frac{1}{n}}, \qquad I_m := \int_0^\infty t^m e^{-\frac{t^2}{2}}\, dt.$$

Note that R is strictly monotone increasing in r. Define an isotropic map $\bar{\varphi} : \mathbb{R}^n \to \mathcal{E}^n_{\sqrt{n-1}}$ by

$$\bar{\varphi}(x) := \frac{R(\|x\|)}{\|x\|} x, \qquad x \in \mathbb{R}^n.$$

We remark that

$$\bar{\varphi}_*\gamma^n_{\{1^2\}} = \epsilon^n_{\sqrt{n-1}}. \tag{3.3}$$

Let $r := r(x) := \|L^{-1}(x)\|$. We define

$$\varphi^{\mathcal{E}} := L \circ \bar{\varphi} \circ L^{-1} : \mathbb{R}^n \to \mathcal{E}^n_{\{\sqrt{n-1}\,a_i\}}.$$

The map $\varphi^{\mathcal{E}}$ is a transport map from $\gamma^n_{\{a_i^2\}}$ to $\epsilon^n_{\{\sqrt{n-1}\,a_i\}}$, i.e.,

$$\varphi^{\mathcal{E}}_*\gamma^n_{\{a_i^2\}} = \epsilon^n_{\{\sqrt{n-1}\,a_i\}}. \tag{3.4}$$

It holds that $\varphi^{\mathcal{E}}(x) = \frac{R}{r}x$ if $x \neq o$. We denote by $\varphi^{\mathcal{S}} : \mathbb{R}^n \setminus \{o\} \to \mathcal{S}^{n-1}_{\{\sqrt{n-1}\,a_i\}}$ the central projection with center o, i.e.,

$$\varphi^{\mathcal{S}}(x) := \frac{\sqrt{n-1}}{r} x, \qquad x \in \mathbb{R}^n \setminus \{o\},$$

which is a transport map from $\gamma^n_{\{a_i^2\}}$ to $\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}$, i.e.,

$$\varphi^{\mathcal{S}}_*\gamma^n_{\{a_i^2\}} = \sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}. \tag{3.5}$$

For an integer N with $1 \le N \le n$ and for $\varepsilon > 0$, we define

$$D^n_{N,\varepsilon} := \{\, x \in \mathbb{R}^n \setminus \{o\} \mid \frac{|x_j|}{\|x\|} < \varepsilon \text{ for any } j = 1, \dots, N-1 \,\}.$$

For $0 < \theta < 1$, let

$$F^n_\theta := \{\, x \in \mathbb{R}^n \mid \|L^{-1}(x)\| \ge \theta\sqrt{n} \,\}.$$

Note that

$$L^{-1}(F^n_\theta) = \{\, x \in \mathbb{R}^n \mid \|x\| \ge \theta\sqrt{n} \,\}. \tag{3.6}$$

Lemma 3.4. *We assume that*

(i) $a_i \ge a$ *for any* $i = 1, 2, \dots, n$,
(ii) $a_i = a$ *for any* i *with* $N \le i \le n$ *and for a positive integer* N *with* $N \le n$.

Then, there exists a universal positive real number C *such that, for any two real numbers* θ *and* ε *with* $0 < \theta < 1$ *and* $0 < \varepsilon \le 1/N$, *the operator norms of the differentials of* $\varphi^{\mathcal{E}}$ *and* $\varphi^{\mathcal{S}}$ *satisfy*

$$\|d\varphi^{\mathcal{E}}_x\| \le \frac{\sqrt{1 + CN\varepsilon}}{\theta} \quad \text{and} \quad \|d\varphi^{\mathcal{S}}_x\| \le \frac{\sqrt{1 + CN\varepsilon}}{\theta}$$

for any $x \in D^n_{N,\varepsilon} \cap F^n_\theta$.

Proof. Let $x \in D^n_{N,\varepsilon} \cap F^n_\theta$ be any point. We first estimate $\|d\varphi^{\mathcal{E}}_x\|$. Take any unit vector $v \in \mathbb{R}^n$. We see that

$$\begin{aligned}
\|d\varphi^{\mathcal{E}}_x(v)\|^2 &= \sum_{j=1}^n \left(\frac{\partial}{\partial r}\left(\frac{R}{r}\right) \frac{\partial r}{\partial x_j} \langle x, v\rangle + \frac{R}{r} v_j \right)^2 \\
&= \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(\frac{R}{r}\right)\right)^2 \langle x, v\rangle^2 \sum_{j=1}^n \frac{x_j^2}{a_j^4} + 2\frac{R}{r^2}\frac{\partial}{\partial r}\left(\frac{R}{r}\right) \langle x, v\rangle \sum_{j=1}^n \frac{v_j x_j}{a_j^2} + \frac{R^2}{r^2}.
\end{aligned}$$

It follows from (i), (ii), and $x \in D^n_{N,\varepsilon}$ that

$$\frac{a^2 r^2}{\|x\|^2} = 1 + \sum_{j=1}^{N-1} \left(\frac{a^2}{a_j^2} - 1 \right) \frac{x_j^2}{\|x\|^2} = 1 + O(N\varepsilon^2)$$

and so

$$\frac{ar}{\|x\|} = 1 + O(N\varepsilon^2), \qquad \frac{\|x\|}{ar} = 1 + O(N\varepsilon^2).$$

We also have

$$\frac{a^4}{\|x\|^2}\sum_{j=1}^{n}\frac{x_j^2}{a_j^4} = 1+\sum_{j=1}^{N-1}\left(\frac{a^4}{a_j^4}-1\right)\frac{x_j^2}{\|x\|^2} = 1+O(N\varepsilon^2),$$
$$\frac{a^2}{\|x\|}\sum_{j=1}^{n}\frac{v_j x_j}{a_j^2} = \sum_{j=1}^{n}\frac{v_j x_j}{\|x\|}+\sum_{j=1}^{N-1}\left(\frac{a^2}{a_j^2}-1\right)\frac{v_j x_j}{\|x\|} = \frac{\langle x, v\rangle}{\|x\|}+O(N\varepsilon).$$

By these formulas, setting $t := \langle x, v\rangle/\|x\|$ and $g := r\frac{\partial}{\partial r}\left(\frac{R}{r}\right)$, we have

$$\begin{aligned}\|d\varphi_x^{\mathcal{E}}(v)\|^2 &= t^2 g^2(1+O(N\varepsilon^2)) + \frac{2t^2 Rg}{r}(1+O(N\varepsilon^2)) \\ &\quad + \frac{2tRg}{r}O(N\varepsilon)+\frac{R^2}{r^2}.\end{aligned} \tag{3.7}$$

We are going to estimate g. Letting $f(r) := \int_0^r t^{n-1}e^{-\frac{t^2}{2}}\,dt$, we have

$$\frac{\partial R}{\partial r} = \sqrt{n-1}\,n^{-1}I_{n-1}^{-\frac{1}{n}}f(r)^{\frac{1}{n}-1}r^{n-1}e^{-\frac{r^2}{2}} \le n^{-\frac{1}{2}}f(r)^{-1}r^{n-1}e^{-\frac{r^2}{2}},$$

which together with $f(r) \ge e^{-\frac{r^2}{2}}\int_0^r t^{n-1}dt = r^n e^{-\frac{r^2}{2}}/n$ and $r \ge \theta\sqrt{n}$ yields

$$0 \le \frac{\partial R}{\partial r} \le \frac{\sqrt{n}}{r} \le \frac{1}{\theta}.$$

Since $R \le \sqrt{n-1}$ and $r \ge \theta\sqrt{n}$, we have $0 \le R/r < 1/\theta$. Therefore,

$$|g| = \left|\frac{\partial R}{\partial r}-\frac{R}{r}\right| \le \frac{1}{\theta}.$$

Thus, (3.7) is reduced to

$$\begin{aligned}\|d\varphi_x^{\mathcal{E}}(v)\|^2 &= t^2g^2+\frac{2t^2Rg}{r}+\frac{R^2}{r^2}+O(\theta^{-2}N\varepsilon) \\ &= t^2\left(\frac{\partial R}{\partial r}\right)^2+(1-t^2)\frac{R^2}{r^2}+O(\theta^{-2}N\varepsilon) \le \theta^{-2}+O(\theta^{-2}N\varepsilon).\end{aligned}$$

This completes the required estimate of $\|d\varphi_x^{\mathcal{E}}(v)\|$.

If we replace R with $\sqrt{n-1}$, then $\varphi^{\mathcal{E}}$ becomes $\varphi^{\mathcal{S}}$ and the above formulas are all true also for $\varphi^{\mathcal{S}}$. This completes the proof. □

We now give an infinite sequence $\{a_i\}$, $i = 1, 2, \dots,$ of positive real numbers and a positive real number a. Consider the following two conditions.

(a1) $a_i \ge a$ for any i.
(a2) $a_i = a$ for any $i \ge N$ and for a positive integer N.

Lemma 3.5. *If we assume* (a2)*, then, for any real numbers* $0 < \theta < 1$ *and* $\varepsilon > 0$*, we have*

$$\lim_{n\to\infty} e^n_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon}) = 1, \tag{1}$$

$$\lim_{n\to\infty} \gamma^n_{\{a_i^2\}}(D^n_{N,\varepsilon} \cap F^n_\theta) = 1, \tag{2}$$

$$\lim_{n\to\infty} \epsilon^n_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon} \cap \varphi^{\mathcal{E}}(F^n_\theta)) = 1. \tag{3}$$

Proof. The injectivity of $\varphi^{\mathcal{E}}$ and (3.4) imply $\epsilon^n_{\{\sqrt{n-1}\,a_i\}}(\varphi^{\mathcal{E}}(F^n_\theta)) = \gamma^n_{\{a_i^2\}}(F^n_\theta)$. Lemma [27, Lemma 7.41] tells us that $\gamma^n_{\{a_i^2\}}(F^n_\theta) = \gamma^n_{\{1^2\}}(L^{-1}(F^n_\theta))$ tends to 1 as $n \to \infty$. Since $\epsilon^n_{\sqrt{n-1}}$, $\sigma^{n-1}_{\sqrt{n-1}}$, and $\gamma^n_{\{1^2\}}$ are rotationally symmetric, and since $L^{-1}(D^n_{N,\varepsilon})$ is scale-invariant, we see that

$$\epsilon^n_{\sqrt{n-1}}(L^{-1}(D^n_{N,\varepsilon})) = \sigma^{n-1}_{\sqrt{n-1}}(L^{-1}(D^n_{N,\varepsilon})) = \gamma^n_{\{1^2\}}(L^{-1}(D^n_{N,\varepsilon})),$$

so that

$$\epsilon^n_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon}) = \sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon}) = \gamma^n_{\{a_i^2\}}(D^n_{N,\varepsilon}).$$

The Maxwell-Boltzmann distribution law (Proposition 3.2) leads us that $\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon})$ converges to 1 as $n \to \infty$, where we note that (a2) implies (A3) and that (A0) is satisfied clearly. This completes the proof. $\square$

Lemma 3.6. *Assume* (a1) *and* (a2)*. If a subsequence of* $\{\mathcal{P}E^n_{\{\sqrt{n-1}\,a_i\}}\}_n$ *converges weakly to a pyramid* $\mathcal{P}_\infty$ *as* $n \to \infty$*, then*

$$\mathcal{P}_\infty \subset \mathcal{P}\Gamma^\infty_{\{a_i^2\}}.$$

Proof. Take any real number ε with $0 < \varepsilon < 1/N$ and fix it. Let $\theta := 1/\sqrt{1 + CN\varepsilon}$, where C is the constant in Lemma 3.4. Note that θ satisfies $0 < \theta < 1$ and tends to 1 as $\varepsilon \to 0+$.

We first prove the lemma for $(E^n_{\{\sqrt{n-1}\,a_i\}}, e^n_{\{\sqrt{n-1}\,a_i\}}) = (\mathcal{E}^n_{\{\sqrt{n-1}\,a_i\}}, \epsilon^n_{\{\sqrt{n-1}\,a_i\}})$. By Lemma 3.4, the map $\varphi^{\mathcal{E}}$ is θ^{-2}-Lipschitz continuous on $D^n_{N,\varepsilon} \cap F^n_\theta$. We remark that $\varphi^{\mathcal{E}}$ is injective and satisfies $(\varphi^{\mathcal{E}})^{-1}(D^n_{N,\varepsilon}) = D^n_{N,\varepsilon}$. For any Borel subset $A \subset \mathbb{R}^n$, we see from (3.4) that

$$\begin{aligned}
\varphi^{\mathcal{E}}_*(\widetilde{\gamma^n_{\{a_i^2\}}|_{D^n_{N,\varepsilon}\cap F^n_\theta}})(A) &= \frac{\gamma^n_{\{a_i^2\}}(D^n_{N,\varepsilon} \cap F^n_\theta \cap (\varphi^{\mathcal{E}})^{-1}(A))}{\gamma^n_{\{a_i^2\}}(D^n_{N,\varepsilon} \cap F^n_\theta)} \\
&= \frac{\gamma^n_{\{a_i^2\}}((\varphi^{\mathcal{E}})^{-1}(D^n_{N,\varepsilon} \cap \varphi^{\mathcal{E}}(F^n_\theta) \cap A))}{\gamma^n_{\{a_i^2\}}((\varphi^{\mathcal{E}})^{-1}(D^n_{N,\varepsilon} \cap \varphi^{\mathcal{E}}(F^n_\theta)))} = \frac{\epsilon^n_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon} \cap \varphi^{\mathcal{E}}(F^n_\theta) \cap A)}{\epsilon^n_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon} \cap \varphi^{\mathcal{E}}(F^n_\theta))} \\
&= \widetilde{\epsilon^n_{\{\sqrt{n-1}\,a_i\}}|_{D^n_{N,\varepsilon}\cap\varphi^{\mathcal{E}}(F^n_\theta)}}(A).
\end{aligned}$$

Thus, the θ^2-scale change $\theta^2 X_n$ of the mm-space

$$X_n := (\mathbb{R}^n, \|\cdot\|, \widetilde{\epsilon^n_{\{\sqrt{n-1}\,a_i\}}|_{D^n_{N,\varepsilon}\cap\varphi^{\mathcal{E}}(F^n_\theta)}})$$

is dominated by $Y_n := (\mathbb{R}^n, \|\cdot\|, \widetilde{\gamma^n_{\{a_i^2\}}|_{D^n_{N,\varepsilon}\cap F^n_\theta}})$ and so $\theta^2\mathcal{P}X_n = \mathcal{P}\theta^2 X_n \subset \mathcal{P}Y_n$ for any n. Combining Lemma 3.5 with Proposition 2.16, we see that, as $n \to \infty$,

$$\rho(\theta^2\mathcal{P}X_n, \mathcal{P}\mathcal{E}^n_{\{\sqrt{n-1}\,a_i\}}) \le 2\, d_{\mathrm{TV}}(\widetilde{\epsilon^n_{\{\sqrt{n-1}\,a_i\}}|_{D^n_{N,\varepsilon}\cap\varphi^{\mathcal{E}}(F^n_\theta)}}, \epsilon^n_{\{\sqrt{n-1}\,a_i\}}) \to 0,$$

$$\rho(\mathcal{P}Y_n, \mathcal{P}\Gamma^n_{\{a_i^2\}}) \le 2\, d_{\mathrm{TV}}(\widetilde{\gamma^n_{\{a_i^2\}}|_{D^n_{N,\varepsilon}\cap F^n_\theta}}, \gamma^n_{\{a_i^2\}}) \to 0.$$

Therefore, $\theta^2\mathcal{P}_\infty$ is contained in $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$. As $\varepsilon \to 0+$, we have $\theta \to 1$ and $\theta^2\mathcal{P}_\infty \to \mathcal{P}_\infty$. This completes the proof in the case where $(E^n_{\{\sqrt{n-1}\,a_i\}}, e^n_{\{\sqrt{n-1}\,a_i\}}) = (\mathcal{E}^n_{\{\sqrt{n-1}\,a_i\}}, \epsilon^n_{\{\sqrt{n-1}\,a_i\}})$.

We prove the lemma for $(E^n_{\{\sqrt{n-1}\,a_i\}}, e^n_{\{\sqrt{n-1}\,a_i\}}) = (\mathcal{S}^{n-1}_{\{\sqrt{n-1}\,a_i\}}, \sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}})$. For any Borel subset $A \subset \mathcal{S}^{n-1}_{\{\sqrt{n-1}\,a_i\}}$, we have

$$\begin{aligned}\varphi^{\mathcal{S}}_*(\widetilde{\gamma^n_{\{a_i^2\}}|_{D^n_{N,\varepsilon}\cap F^n_\theta}})(A) &= \frac{\gamma^n_{\{a_i^2\}}(D^n_{N,\varepsilon}\cap F^n_\theta \cap (\varphi^{\mathcal{S}})^{-1}(A))}{\gamma^n_{\{a_i^2\}}(D^n_{N,\varepsilon}\cap F^n_\theta)}\\ &= \frac{\gamma^n_{\{1^2\}}(L^{-1}(D^n_{N,\varepsilon})\cap L^{-1}(F^n_\theta)\cap L^{-1}((\varphi^{\mathcal{S}})^{-1}(A)))}{\gamma^n_{\{1^2\}}(L^{-1}(D^n_{N,\varepsilon})\cap L^{-1}(F^n_\theta))}.\end{aligned}$$

Here, $L^{-1}(D^n_{N,\varepsilon})$ is scale-invariant, i.e., a cone and $L^{-1}((\varphi^{\mathcal{S}})^{-1}(A))$ is a cone generated by $L^{-1}(A)$. Thus, by (3.6) and the rotational symmetry of $\gamma^n_{\{1^2\}}$, the above is equal to

$$\frac{\sigma^{n-1}_{\sqrt{n-1}}(L^{-1}(D^n_{N,\varepsilon})\cap L^{-1}(A))}{\sigma^{n-1}_{\sqrt{n-1}}(L^{-1}(D^n_{N,\varepsilon}))} = \frac{\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon}\cap A)}{\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}(D^n_{N,\varepsilon})} = \widetilde{\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}|_{D^n_{N,\varepsilon}}}(A).$$

Since $\varphi^{\mathcal{S}}$ is θ^{-2}-Lipschitz continuous on $D^n_{N,\varepsilon}\cap F^n_\theta$, the θ^2-scale change $\theta^2 X'_n$ of the mm-space

$$X'_n := (\mathbb{R}^n, \|\cdot\|, \widetilde{\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}|_{D^n_{N,\varepsilon}}})$$

is dominated by $Y_n := (\mathbb{R}^n, \|\cdot\|, \widetilde{\gamma^n_{\{a_i^2\}}|_{D^n_{N,\varepsilon}\cap F^n_\theta}})$. The rest of the proof is exactly in the same way as before. This completes the proof. $\square$

Lemma 3.7. *If we assume* (a2), *then* $\mathcal{P}E^n_{\{\sqrt{n-1}\,a_i\}}$ *converges weakly to* $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ *as* $n \to \infty$.

Proof. Assume (a2) and suppose that $\mathcal{P}E^n_{\{\sqrt{n-1}\,a_i\}}$ does not converge weakly to $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ as $n \to \infty$. Then, there is a subsequence $\{n(j)\}$ of $\{n\}$ such that $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,a_i\}}$ converges weakly to a pyramid $\mathcal{P}_\infty$ different from $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$.

The Maxwell-Boltzmann distribution law tells us that the push-forward measure $\nu^k_{n(j)} := (\pi^{n(j)}_k)_* e^{n(j)}_{\{\sqrt{n(j)-1}\,a_i\}}$ converges weakly to $\gamma^k_{\{a_i^2\}}$ as $j \to \infty$ for any k, so that $(\mathbb{R}^k, \|\cdot\|, \nu^k_{n(j)})$ box converges to $\Gamma^k_{\{a_i^2\}}$. Since $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_i\}}$ dominates $(\mathbb{R}^k, \|\cdot\|, \nu^k_{n(j)})$, the limit pyramid $\mathcal{P}_\infty$ contains $\Gamma^k_{\{a_i^2\}}$ for any k. This proves

$$\mathcal{P}_\infty \supset \mathcal{P}\Gamma^\infty_{\{a_i^2\}}. \tag{3.8}$$

Let $\hat{a}_i := \max\{a_i, a\}$. It follows from $a_i \le \hat{a}_i$ that $E^n_{\{\sqrt{n-1}\,a_i\}}$ is dominated by $E^n_{\{\sqrt{n-1}\,\hat{a}_i\}}$, which implies $\mathcal{P}E^n_{\{\sqrt{n-1}\,a_i\}} \subset \mathcal{P}E^n_{\{\sqrt{n-1}\,\hat{a}_i\}}$ for any n. By applying Lemma 3.6, the limit of any weakly convergent sequence of $\{\mathcal{P}E^n_{\{\sqrt{n-1}\,\hat{a}_i\}}\}_n$ is contained in $\mathcal{P}\Gamma^\infty_{\{\hat{a}_i^2\}}$. Therefore, $\mathcal{P}_\infty$ is contained in $\mathcal{P}\Gamma^\infty_{\{\hat{a}_i^2\}}$. Denote by l the number of i's with $a_i < a$. For any $k \ge N$, we consider the projection from $\Gamma^{k+l}_{\{a_i^2\}}$ to $\Gamma^k_{\{\hat{a}_i^2\}}$ dropping the axes x_i with $a_i < a$, which is 1-Lipschitz continuous and preserves their measures. This shows that $\Gamma^{k+l}_{\{a_i^2\}}$ dominates $\Gamma^k_{\{\hat{a}_i^2\}}$, and so $\mathcal{P}\Gamma^\infty_{\{a_i^2\}} \supset \mathcal{P}\Gamma^\infty_{\{\hat{a}_i^2\}}$. We thus obtain

$$\mathcal{P}_\infty \subset \mathcal{P}\Gamma^\infty_{\{a_i^2\}}. \tag{3.9}$$

Combining (3.8) and (3.9) yields $\mathcal{P}_\infty = \mathcal{P}\Gamma^\infty_{\{a_i^2\}}$, which is a contradiction. This completes the proof. □

Lemma 3.8. *Let $\{a_{ij}\}$ satisfy* (A0)–(A3). *If $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to a pyramid $\mathcal{P}_\infty$ as $j \to \infty$, then*

$$\mathcal{P}_\infty \supset \mathcal{P}\Gamma^\infty_{\{a_i^2\}}.$$

Proof. Note that the sequence $\{a_i\}$ is monotone nonincreasing. Put $i_0 := \sup\{\, i \mid a_i > 0 \,\}$ $(\le \infty)$. We see $a_{i_0} > 0$ if $i_0 < \infty$. The Maxwell-Boltzmann distribution law proves that $\nu^k_{n(j)} := (\pi^{n(j)}_k)_* e^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to $\gamma^k_{\{a_i^2\}}$ as $j \to \infty$ for each finite k with $1 \le k \le i_0$. The ellipsoid $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ dominates $(\mathbb{R}^k, \|\cdot\|, \nu^k_{n(j)})$, which converges to $\Gamma^k_{\{a_i^2\}}$, so that $\Gamma^k_{\{a_i^2\}}$ belongs to $\mathcal{P}_\infty$. Since $\Gamma^k_{\{a_i^2\}}$ for any $k \ge i_0$ is mm-isomorphic to $\Gamma^{i_0}_{\{a_i^2\}}$ provided $i_0 < \infty$, we obtain the lemma. □

Lemma 3.9. *Let $\{a_{ij}\}$ satisfy* (A0)–(A3). *If $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to a pyramid $\mathcal{P}_\infty$ as $j \to \infty$, then*

$$\mathcal{P}_\infty \subset \mathcal{P}\Gamma^\infty_{\{a_i^2\}}.$$

Proof. Since $\{a_i\}$ is monotone nonincreasing, it converges to a nonnegative real number, say a_∞.

We first assume that $a_\infty > 0$. We see that $a_i > 0$ for any i. For any $\varepsilon > 0$ there is a number $I(\varepsilon)$ such that

$$a_i \le (1+\varepsilon)a_\infty \qquad \text{for any } i \ge I(\varepsilon). \tag{3.10}$$

Also, there is a number $J(\varepsilon)$ such that

$$a_{ij} \le a_i + a_\infty\varepsilon \qquad \text{for any } i \le I(\varepsilon) \text{ and } j \ge J(\varepsilon). \tag{3.11}$$

By the monotonicity of a_{ij} in i, (3.10), and (3.11), we have

$$a_{ij} \le a_{I(\varepsilon),j} \le a_{I(\varepsilon)} + a_\infty\varepsilon \le (1+2\varepsilon)a_\infty \text{ for any } i \ge I(\varepsilon) \text{ and } j \ge J(\varepsilon). \tag{3.12}$$

It follows from (3.11) and $a_\infty \le a_i$ that

$$a_{ij} \le a_i + a_\infty\varepsilon \le (1+\varepsilon)a_i \quad \text{for any } i \le I(\varepsilon) \text{ and } j \ge J(\varepsilon). \tag{3.13}$$

Let

$$b_{\varepsilon,i} := \begin{cases} a_i & \text{if } i \le I(\varepsilon), \\ a_\infty & \text{if } i > I(\varepsilon). \end{cases}$$

By (3.12) and (3.13), for any i and $j \ge J(\varepsilon)$, we see that $a_{ij} \le (1+2\varepsilon)b_{\varepsilon,i}$ and so $E^{n(j)}_{\{a_{ij}\}_i} \prec E^{n(j)}_{\{(1+2\varepsilon)b_{\varepsilon,i}\}} = (1+2\varepsilon)E^{n(j)}_{\{b_{\varepsilon,i}\}}$. Lemma 3.7 implies that $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,b_{\varepsilon,i}\}}$ converges weakly to $\mathcal{P}\Gamma^\infty_{\{b^2_{\varepsilon,i}\}}$ as $j \to \infty$. Therefore, $\mathcal{P}_\infty$ is contained in $(1+2\varepsilon)\mathcal{P}\Gamma^\infty_{\{b^2_{\varepsilon,i}\}}$ for any $\varepsilon > 0$. Since $b_{\varepsilon,i} \le a_i$, we see that $\mathcal{P}_\infty$ is contained in $(1+2\varepsilon)\mathcal{P}\Gamma^\infty_{\{a^2_i\}}$ for any $\varepsilon > 0$. This proves the lemma in this case.

We next assume $a_\infty = 0$. For any $\varepsilon > 0$ there is a number $I(\varepsilon)$ such that

$$a_i < \varepsilon \qquad \text{for any } i \ge I(\varepsilon). \tag{3.14}$$

We may assume that $I(\varepsilon) = i_0 + 1$ if $i_0 < \infty$, where $i_0 := \sup\{\, i \mid a_i > 0 \,\}$. Also, there is a number $J(\varepsilon)$ such that

$$a_{I(\varepsilon),j} < a_{I(\varepsilon)} + \varepsilon \qquad \text{for any } j \ge J(\varepsilon); \tag{3.15}$$

$$a_{ij} < (1+\varepsilon)a_i \qquad \text{for any } i < I(\varepsilon) \text{ and } j \ge J(\varepsilon). \tag{3.16}$$

It follows from (3.14) and (3.15) that

$$a_{ij} \le a_{I(\varepsilon),j} < a_{I(\varepsilon)} + \varepsilon < 2\varepsilon \quad \text{for any } i \ge I(\varepsilon) \text{ and } j \ge J(\varepsilon). \tag{3.17}$$

Let

$$b_{\varepsilon,i} := \begin{cases} (1+\varepsilon)a_i & \text{if } i < I(\varepsilon), \\ 2\varepsilon & \text{if } i \ge I(\varepsilon). \end{cases}$$

From (3.16) and (3.17), we have $a_{ij} < b_{\varepsilon,i}$ for any i and $j \ge J(\varepsilon)$, and so $E^{n(j)}_{\{a_{ij}\}_i} \prec E^{n(j)}_{\{b_{\varepsilon,i}\}}$ for $j \ge J(\varepsilon)$. Lemma 3.7 implies that $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,b_{\varepsilon,i}\}}$ converges weakly to $\mathcal{P}\Gamma^\infty_{\{b^2_{\varepsilon,i}\}}$ as $j \to \infty$. Therefore, $\mathcal{P}_\infty$ is contained in $\mathcal{P}\Gamma^\infty_{\{b^2_{\varepsilon,i}\}}$ for any $\varepsilon > 0$. Let k be any number with $k \ge I(\varepsilon)$. The Gaussian space $\Gamma^k_{\{b^2_{\varepsilon,i}\}}$ is mm-isomorphic to the l^2-product of $\Gamma^{I(\varepsilon)-1}_{\{(1+\varepsilon)^2 a^2_i\}}$ and $\Gamma^{k-I(\varepsilon)+1}_{\{(2\varepsilon)^2\}}$. It follows from the Gaussian isoperimetry that

$$\operatorname{ObsDiam}(\Gamma^{k-I(\varepsilon)+1}_{\{(2\varepsilon)^2\}}) = \inf_{\kappa>0} \max\{2\varepsilon \operatorname{diam}(\gamma^1_{1^2}; 1-\kappa), \kappa\} =: \tau(\varepsilon),$$

which tends to zero as $\varepsilon \to 0+$. If $\tau(\varepsilon) < 1/2$, then, by [27, Proposition 7.32],

$$\rho(\mathcal{P}\Gamma^k_{\{b^2_{\varepsilon,i}\}}, \mathcal{P}\Gamma^{I(\varepsilon)-1}_{\{(1+\varepsilon)^2 a^2_i\}}) \le d_{\mathrm{conc}}(\Gamma^k_{\{b^2_{\varepsilon,i}\}}, \Gamma^{I(\varepsilon)-1}_{\{(1+\varepsilon)^2 a^2_i\}}) \le \tau(\varepsilon).$$

Taking the limit as $k \to \infty$ yields

$$\rho(\mathcal{P}\Gamma^\infty_{\{b^2_{\varepsilon,i}\}}, \mathcal{P}\Gamma^{I(\varepsilon)-1}_{\{(1+\varepsilon)^2 a^2_i\}}) \le \tau(\varepsilon).$$

There is a sequence $\{\varepsilon(l)\}$, $l = 1, 2, \dots$, of positive real numbers tending to zero such that $\mathcal{P}\Gamma^\infty_{\{b^2_{\varepsilon(l),i}\}}$ converges weakly to a pyramid $\mathcal{P}'_\infty$ as $l \to \infty$. $\mathcal{P}'_\infty$ contains $\mathcal{P}_\infty$ and $\mathcal{P}\Gamma^{I(\varepsilon(l))-1}_{\{(1+\varepsilon(l))^2 a^2_i\}}$ converges weakly to $\mathcal{P}'_\infty$ as $l \to \infty$. Since $\mathcal{P}\Gamma^{I(\varepsilon(l))-1}_{\{(1+\varepsilon(l))^2 a^2_i\}}$ is contained in $\mathcal{P}\Gamma^\infty_{\{(1+\varepsilon(l))^2 a^2_i\}}$ and since $\mathcal{P}\Gamma^\infty_{\{(1+\varepsilon(l))^2 a^2_i\}} = (1+\varepsilon(l))\mathcal{P}\Gamma^\infty_{\{a^2_i\}}$ converges weakly to $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ as $l \to \infty$, the pyramid $\mathcal{P}'_\infty$ is contained in $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$, so that $\mathcal{P}_\infty$ is contained in $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$. This completes the proof. □

Proof of Theorem 1.1. Suppose that $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ does not converge weakly to $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ as $j \to \infty$. Then, taking a subsequence of $\{j\}$ we may assume that $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to a pyramid $\mathcal{P}_\infty$ different from $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$, which contradicts Lemmas 3.8 and 3.9. Thus, $\mathcal{P}E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ converges weakly to $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ as $j \to \infty$.

As is mentioned in Subsection 2.8, the infinite-dimensional Gaussian space $\Gamma^\infty_{\{a_i^2\}}$ is well-defined as an mm-space if and only if $\{a_i\}$ is an l^2-sequence, only in which case the above sequence of (solid) ellipsoids becomes a convergent sequence in the concentration topology.

Assume that a_i converges to zero as $i \to \infty$. It is well-known that the Ornstein-Uhlenbeck operator (or the drifted Laplacian) on $\Gamma^1_{a^2}$ has compact resolvent and spectrum $\{ka^{-2} \mid k = 0, 1, 2 \dots\}$ (see [19]). Thus, the same proof as in [27, Corollary 7.35] yields that $\Gamma^n_{\{a_i^2\}}$ asymptotically (spectrally) concentrates to $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$.

Conversely, we assume that a_i is bounded away from zero and set $\underline{a} := \inf_i a_i$. Applying [27, Proposition 7.37] yields that $\mathcal{P}\Gamma^\infty_{\{\underline{a}^2\}}$ is not concentrated. Since $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ contains $\mathcal{P}\Gamma^\infty_{\{\underline{a}^2\}}$, the pyramid $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ is not concentrated, which implies that $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$ does not asymptotically concentrate (see Theorem 2.19).

This completes the proof of the theorem. □

Let us next consider the convergence of the Gaussian spaces.

Proposition 3.10. *Let $\{a_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots$, be a sequence of nonnegative real numbers. If $\sup_i a_{ij}$ diverges to infinity as $j \to \infty$, then $\Gamma^{n(j)}_{\{a_{ij}^2\}}$ infinitely dissipates.*

Proof. Exchanging the coordinates, we assume that a_{1j} diverges to infinity as $j \to \infty$. Since $\Gamma^1_{a_{1j}^2}$ is dominated by $\Gamma^{n(j)}_{\{a_{ij}^2\}}$, we have

$$\mathrm{ObsDiam}(\Gamma^{n(j)}_{\{a_{ij}^2\}}; -\kappa) \geq \mathrm{diam}(\Gamma^1_{a_{1j}^2}; 1-\kappa) \to \infty \quad \text{as } j \to \infty.$$

This together with Proposition 2.18 completes the proof. □

In a similar way as in the proof of Theorem 1.1, we obtain the following.

Corollary 3.11. *Let $\{a_{ij}\}$ satisfy* (A0)–(A3)*. Then, $\Gamma^{n(j)}_{\{a_{ij}^2\}}$ converges weakly to $\mathcal{P}\Gamma^\infty_{\{a_i^2\}}$ as $j \to \infty$. This convergence becomes a convergence in the concentration topology if and only if $\{a_i\}$ is an l^2-sequence. Moreover, this convergence becomes an asymptotic concentration if and only if $\{a_i\}$ converges to zero.*

Proof. Suppose that $\Gamma^{n(j)}_{\{a_{ij}^2\}}$ does not converge weakly to $\mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$ as $j \to \infty$. Then there is a subsequence of $\{\mathcal{P}\Gamma^{n(j)}_{\{a_{ij}^2\}}\}_j$ that converges weakly to a pyramid $\mathcal{P}_\infty$ different from $\mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$. We write such a subsequence by the same notation $\{\mathcal{P}\Gamma^{n(j)}_{\{a_{ij}^2\}}\}_j$.

Since $\Gamma^k_{\{a_{ij}^2\}}$ is dominated by $\Gamma^{n(j)}_{\{a_{ij}^2\}}$ for $k \le n(j)$ and $\Gamma^k_{\{a_{ij}^2\}}$ converges weakly to $\Gamma^k_{\{a_i^2\}}$ as $j \to \infty$, we see that $\Gamma^k_{\{a_i^2\}}$ belongs to $\mathcal{P}_\infty$ for any k, so that

$$\mathcal{P}_\infty \supset \mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}.$$

We prove $\mathcal{P}_\infty \subset \mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$ in the case of $a_\infty > 0$, where $a_\infty := \lim_{i\to\infty} a_i$. Under $a_\infty > 0$, the same discussion as in the proof of Lemma 3.8 proves that there are two numbers $I(\varepsilon)$ and $J(\varepsilon)$ for any $\varepsilon > 0$ such that (3.12) and (3.13) both hold. We therefore see that, for any k and $j \ge J(\varepsilon)$, $\Gamma^k_{\{a_{ij}^2\}}$ is dominated by $(1+\varepsilon)\Gamma^k_{\{a_i^2\}}$, and so $\mathcal{P}\Gamma^{n(j)}_{\{a_{ij}^2\}_i} \subset (1+\varepsilon)\mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$. This proves $\mathcal{P}_\infty \subset \mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$.

We next prove $\mathcal{P}_\infty \subset \mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$ in the case of $a_\infty = 0$. Let $b_{\varepsilon,i}$ be as in Lemma 3.9. The discussion in the proof of Lemma 3.9 yields that $a_{ij} < b_{\varepsilon,i}$ for any i and for every sufficiently large j, which implies $\mathcal{P}_\infty \subset \mathcal{P}\Gamma^{\infty}_{\{b_{\varepsilon,i}^2\}}$. We obtain $\mathcal{P}_\infty \subset \mathcal{P}\Gamma^{\infty}_{\{a_i^2\}}$ in the same way as in the proof of Lemma 3.9. The weak convergence of $\Gamma^{n(j)}_{\{a_{ij}^2\}}$ to $\mathcal{P}\Gamma^{\infty}_{\{b_{\varepsilon,i}^2\}}$ has been proved.

The rest is identical to the proof of Theorem 1.1. This completes the proof. $\square$

4. Box convergence of ellipsoids

The main purpose of this section is to prove Theorem 1.2.

Let us first prove the weak convergence of $e^n_{\{\sqrt{n-1}\,a_{ij}\}}$ if $\{a_{ij}\}$ l^2-converges.

Lemma 4.1. *Let $\mathcal{A}$ be a family of sequences of positive real numbers such that $\mathcal{A}$ is bounded in ℓ^2. Then we have*

$$\lim_{n\to\infty} \sup_{\{a_i\}\in\mathcal{A}} d_{\mathrm{P}}(\epsilon^n_{\{\sqrt{n-1}\,a_i\}}, \gamma^n_{\{a_i^2\}}) = 0, \tag{1}$$

$$\limsup_{n\to\infty} \sup_{\{a_i\}\in\mathcal{A}} W_2(\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}, \gamma^n_{\{a_i^2\}})^2 \le \sqrt{2} \sup_{\{a_i\}\in\mathcal{A}} \sum_{i=k+1}^{\infty} a_i^2 \tag{2}$$

for any positive integer k.

Proof. We prove (1). Let $r(x) := \|L^{-1}(x)\|$ as in Section 3. Take any real number θ with $0 < \theta < 1$ and fix it. Let us consider the normalization of the measures $\epsilon^n_{\{\sqrt{n-1}\,a_i\}}|_{r^{-1}([\theta\sqrt{n-1},\sqrt{n-1}])}$ and $\gamma^n_{\{a_i^2\}}|_{r^{-1}([\theta\sqrt{n-1},\theta^{-1}\sqrt{n-1}])}$, which we denote by ϵ^n_θ and γ^n_θ, respectively. Set

$$v_{\theta,n} := \epsilon^n_{\sqrt{n-1}}(\{\, x \in \mathbb{R}^n \mid \theta\sqrt{n-1} \le \|x\| \le \sqrt{n-1}\,\}),$$
$$w_{\theta,n} := \gamma^n_{\{1^2\}}(\{\, x \in \mathbb{R}^n \mid \theta\sqrt{n-1} \le \|x\| \le \theta^{-1}\sqrt{n-1}\,\}).$$

We remark that

$$\begin{aligned}
v_{\theta,n} &= \epsilon^n_{\{\sqrt{n-1}\,a_i\}}(r^{-1}([\,\theta\sqrt{n-1}, \sqrt{n-1}\,])),\\
w_{\theta,n} &= \gamma^n_{\{a_i^2\}}(r^{-1}([\,\theta\sqrt{n-1}, \theta^{-1}\sqrt{n-1}\,])),\\
\lim_{n\to\infty} v_{\theta,n} &= \lim_{n\to\infty} w_{\theta,n} = 1.
\end{aligned}$$

It then holds that

$$d_{\mathrm{P}}(\epsilon^n_{\{\sqrt{n-1}\,a_i\}}, \epsilon^n_\theta) \le d_{\mathrm{TV}}(\epsilon^n_{\{\sqrt{n-1}\,a_i\}}, \epsilon^n_\theta) = 1 - v_{\theta,n}, \tag{4.1}$$

$$d_{\mathrm{P}}(\gamma^n_{\{a_i^2\}}, \gamma^n_\theta) \le d_{\mathrm{TV}}(\gamma^n_{\{a_i^2\}}, \gamma^n_\theta) = 1 - w_{\theta,n}, \tag{4.2}$$

where the right equality in (4.1) follows from (2.1) and

$$\frac{d\epsilon^n_\theta}{d\epsilon^n_{\{\sqrt{n-1}\,a_i\}}} = \begin{cases} \frac{1}{v_{\theta,n}} & \text{on } r^{-1}([\,\theta\sqrt{n-1}, \sqrt{n-1}\,]),\\ 0 & \text{on } \mathcal{E}^n_{\{\sqrt{n-1}\,a_i\}} \setminus r^{-1}([\,\theta\sqrt{n-1}, \sqrt{n-1}\,]). \end{cases}$$

(4.2) is obtained in the same way.

To estimate $d_{\mathrm{P}}(\epsilon^n_\theta, \gamma^n_\theta)$, we define a transport map, say ψ, from γ^n_θ to ϵ^n_θ in the same manner as for $\varphi^{\mathcal{E}}$ in Section 3, which is expressed as

$$\psi(x) = \frac{\tilde{R}}{r}x, \quad x \in r^{-1}([\,\theta\sqrt{n-1}, \theta^{-1}\sqrt{n-1}\,]),$$

where $\tilde{R}$ is the function of variable $r \in [\,\theta\sqrt{n-1}, \theta^{-1}\sqrt{n-1}\,]$ defined by

$$\theta\sqrt{n-1} \le \tilde{R} \le \sqrt{n-1} \quad \text{and} \quad \gamma^n_\theta(B_r(o)) = \epsilon^n_\theta(B_{\tilde{R}}(o)).$$

It holds that

$$\tilde{R}^n = \underline{R}^n + (\overline{R}^n - \underline{R}^n)\frac{\int_{\underline{r}}^{r} t^{n-1}e^{-\frac{1}{2}t^2}\,dt}{\int_{\underline{r}}^{\overline{r}} t^{n-1}e^{-\frac{1}{2}t^2}\,dt},$$

where $\underline{r} := \underline{R} := \theta\sqrt{n-1}$, $\overline{r} := \theta^{-1}\sqrt{n-1}$, and $\overline{R} := \sqrt{n-1}$. Looking at the ranges of r and $\tilde{R}$, we have $\theta^2 \le \tilde{R}/r \le \theta^{-1}$, which implies

$$\left(\frac{\tilde{R}}{r} - 1\right)^2 \le \max\{(1-\theta^2)^2, (\theta^{-1}-1)^2\} = (1-\theta^2)^2$$

if θ is sufficiently close to 1. Then we have

$$\begin{aligned}
W_2(\epsilon^n_\theta, \gamma^n_\theta)^2 &\le \int_{\mathbb{R}^n} \|\,\psi(x) - x\,\|^2\,d\gamma^n_\theta(x) = \int_{\mathbb{R}^n}\left(\frac{\tilde{R}}{r} - 1\right)^2 \|x\|^2\,d\gamma^n_\theta(x)\\
&\le (\theta^2-1)^2\int_{\mathbb{R}^n}\|x\|^2\,d\gamma^n_\theta(x)\\
&\le \frac{(\theta^2-1)^2}{\gamma^n_{\{a_i^2\}}(r^{-1}([\,\theta\sqrt{n-1}, \theta^{-1}\sqrt{n-1}\,]))}\int_{\mathbb{R}^n}\|x\|^2\,d\gamma^n_{\{a_i^2\}}(x)\\
&= \frac{(\theta^2-1)^2}{w_{\theta,n}}\sum_{i=1}^n a_i^2,
\end{aligned}$$

which together with (4.1) and (4.2) implies

$$d_{\mathrm{P}}(\epsilon^n_{\{\sqrt{n-1}\,a_i\}}, \gamma^n_{\{a_i^2\}}) \le 2 - v_{\theta,n} - w_{\theta,n} + \left(\frac{(\theta^2-1)^2}{w_{\theta,n}} \sum_{i=1}^n a_i^2 \right)^{\frac{1}{4}}$$

and hence

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{\{a_i\}\in\mathcal{A}} d_{\mathrm{P}}(\epsilon^n_{\{\sqrt{n-1}\,a_i\}}, \gamma^n_{\{a_i^2\}}) \\ &\le \left((\theta^2-1)^2 \sup_{\{a_i\}\in\mathcal{A}} \sum_{i=1}^\infty a_i^2 \right)^{\frac{1}{4}} \to 0 \quad \text{as } \theta \to 1+. \end{aligned}$$

This proves (1).

We prove (2). Using the transport map $\varphi^{\mathcal{S}}$ from $\gamma^n_{\{a_i^2\}}$ to $\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}$ in Section 3, we have

$$\begin{aligned} W_2(\sigma^{n-1}_{\{\sqrt{n-1}\,a_i\}}, \gamma^n_{\{a_i^2\}})^2 &\le \int_{\mathbb{R}^n} \left\| \, z - \varphi^{\mathcal{S}}(z) \, \right\|^2 \, d\gamma^n_{\{a_i^2\}}(z) \\ &= \int_{S^{n-1}(1)} \sum_{i=1}^n a_i^2 x_i^2 \, d\sigma^{n-1}(x) \cdot \frac{1}{I_{n-1}} \int_0^\infty (r - \sqrt{n-1})^2 r^{n-1} e^{-r^2/2} \, dr, \end{aligned}$$

where $I_m := \int_0^\infty t^m e^{-t^2/2} \, dt$. We see in the proof of [27, Lemma 7.41] that $r^m e^{-r^2/2} \le m^{m/2} e^{-m/2} e^{-(r-\sqrt{m})^2/2}$ and also that $I_m \sim \sqrt{\pi}(m-1)^{m/2} e^{-(m-1)/2}$. Therefore,

$$\begin{aligned} &\frac{1}{I_{n-1}} \int_0^\infty (r-\sqrt{n-1})^2 r^{n-1} e^{-r^2/2} \, dr \le \frac{1}{I_{n-1}} \sqrt{2\pi} (n-1)^{(n-1)/2} e^{-(n-1)/2} \\ &\sim \frac{\sqrt{2} e^{-1/2}}{(1 - 1/(n-1))^{(n-1)/2}} \longrightarrow \sqrt{2} \text{ as } n \to \infty. \end{aligned}$$

For any $\varepsilon > 0$ and k with $1 \le k \le n-1$, let $S^{n-1}_{k,\varepsilon} := \{\, x \in S^{n-1}(1) \mid |x_i| < \varepsilon \text{ for } i = 1, 2, \dots, k \,\}$. Then,

$$\int_{S^{n-1}(1)\setminus S^{n-1}_{k,\varepsilon}} \sum_{i=1}^n a_i^2 x_i^2 \, d\sigma^{n-1}(x) \le \sigma^{n-1}(S^{n-1}(1) \setminus S^{n-1}_{k,\varepsilon}) \sum_{i=1}^n a_i^2,$$

$$\int_{S^{n-1}_{k,\varepsilon}} \sum_{i=1}^n a_i^2 x_i^2 \, d\sigma^{n-1}(x) \le \varepsilon^2 \sum_{i=1}^k a_i^2 + \sum_{i=k+1}^n a_i^2,$$

which imply

$$\begin{aligned} &\sup_{\{a_i\}\in\mathcal{A}} \int_{S^{n-1}(1)} \sum_{i=1}^n a_i^2 x_i^2 \, d\sigma^{n-1}(x) \\ &\le (\sigma^{n-1}(S^{n-1}(1) \setminus S^{n-1}_{k,\varepsilon}) + \varepsilon^2) \sup_{\{a_i\}\in\mathcal{A}} \sum_{i=1}^\infty a_i^2 + \sup_{\{a_i\}\in\mathcal{A}} \sum_{i=k+1}^\infty a_i^2. \end{aligned}$$

Since $\sigma^{n-1}(S^{n-1}(1)\setminus S^{n-1}_{k,\varepsilon})$ tends to zero as $n \to \infty$, we obtain (2). This completes the proof. $\square$

Proposition 4.2. *Let $\{a_{ij}\}$, $i = 1,2,\dots,n(j)$, $j = 1,2,\dots,$ be a sequence of positive real numbers, where $\{n(j)\}$, $j = 1,2,\dots,$ is a sequence of positive integers divergent to infinity. Let $\{a_i\}$, $i = 1,2,\dots,$ be an l^2-sequence of nonnegative real numbers. We assume*

$$\lim_{j\to\infty} \sum_{i=1}^{n(j)} (a_{ij} - a_i)^2 = 0.$$

Then we have

1. $\epsilon^{n(j)}_{\{\sqrt{n(j)-1}\, a_{ij}\}_i}$ *converges weakly to* $\gamma^{\infty}_{\{a_i^2\}_i}$ *as* $j \to \infty$;
2. $\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\, a_{ij}\}_i}$ *converges to* $\gamma^{\infty}_{\{a_i^2\}_i}$ *in the* 2*-Wasserstein metric as* $j \to \infty$.

In particular, $E^{n(j)}_{\{\sqrt{n(j)-1}\, a_{ij}\}_i}$ *box converges to* $\Gamma^{\infty}_{\{a_i^2\}_i}$ *as* $j \to \infty$.

Proof. We first prove (2). We set $a_{ij} := 0$ for $i \geq n(j)+1$. Lemma 4.1(2) implies

$$\begin{aligned}
&\limsup_{j\to\infty} W_2(\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\, a_{ij}\}_i}, \gamma^{n(j)}_{\{a_{ij}^2\}_i})^2 \\
&\leq \sqrt{2}\, \limsup_{j\to\infty} \sum_{i=k+1}^{\infty} a_{ij}^2 = \sqrt{2} \sum_{i=k+1}^{\infty} a_i^2 \longrightarrow 0 \text{ as } k \to \infty.
\end{aligned}$$

Gelbrich's formula [8] tells us that

$$W_2(\gamma^{n(j)}_{\{a_{ij}^2\}_i}, \gamma^{\infty}_{\{a_i^2\}})^2 = \sum_{i=1}^{\infty} (a_{ij} - a_i)^2 \longrightarrow 0 \text{ as } j \to \infty.$$

By a triangle inequality, we obtain (2).

(1) is proved in the same way by using Lemma 4.1(1) and by remarking ${d_{\mathrm{P}}}^2 \leq W_2$. This completes the proof. $\square$

Lemma 4.3. *Let $\{b_{ij}\}$, $i = 1,2,\dots,n(j)$, $j = 1,2,\dots,$ be a sequence of positive real numbers, where $\{n(j)\}$, $j = 1,2,\dots,$ is a sequence of positive integers divergent to infinity. If $\sum_{i=1}^{n(j)} b_{ij}^2$ converges to a positive real number as $j \to \infty$, then $\{e^{n(j)}_{\{\sqrt{n(j)-1}\, b_{ij}\}}\}$ has no subsequence converging weakly to the Dirac measure δ_o at the origin o in H, where we embed the (solid) ellipsoids $E^{n(j)}_{\{\sqrt{n(j)-1}\, b_{ij}\}} \subset \mathbb{R}^{n(j)}$ into the Hilbert space H naturally and consider $e^{n(j)}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}$ as Borel probability measures on H.*

Proof. Applying Lemma 4.1(1) yields that

$$\lim_{j\to\infty} d_{\mathrm{P}}(\epsilon^{n(j)}_{\{\sqrt{n(j)-1}\, b_{ij}\}}, \gamma^{n(j)}_{\{b_{ij}^2\}}) = 0.$$

We prove that $\{\gamma^{n(j)}_{\{b_{ij}^2\}}\}$ have no subsequence converging weakly to δ_o. Suppose that $d_{\mathrm{P}}(\gamma^{n(j)}_{\{b_{ij}^2\}}, \delta_o) \le \varepsilon_j$ for some $\varepsilon_j \to 0$ as $j \to \infty$. Then we have

$$\gamma^{n(j)}_{\{b_{ij}^2\}}(\mathbb{R}^n \setminus B_{\varepsilon_j}(o)) \le \varepsilon_j.$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|^2 \, d\gamma^{n(j)}_{\{b_{ij}^2\}}(x) &\le \varepsilon_j^2 + \int_{\mathbb{R}^n \setminus B_{\varepsilon_j}(o)} \|x\|^2 \, d\gamma^{n(j)}_{\{b_{ij}^2\}}(x) \\ &\le \varepsilon_j^2 + \gamma^{n(j)}_{\{b_{ij}^2\}}(\mathbb{R}^n \setminus B_{\varepsilon_j}(o))^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \|x\|^4 \, d\gamma^{n(j)}_{\{b_{ij}^2\}}(x) \right)^{\frac{1}{2}} \\ &\le \varepsilon_j^2 + 2c\sqrt{\varepsilon_j} \int_{\mathbb{R}^n} \|x\|^2 \, d\gamma^{n(j)}_{\{b_{ij}^2\}}(x),\end{aligned}$$

where the last inequality follows from the reverse Hölder inequality for Gaussian measures; there is an absolute constant $c > 0$ such that

$$\left(\int_{\mathbb{R}^n} \|x\|^q \, d\gamma(x) \right)^{\frac{1}{q}} \le c \frac{q}{p} \left(\int_{\mathbb{R}^n} \|x\|^p \, d\gamma(x) \right)^{\frac{1}{p}}$$

for any $q \ge p \ge 1$ and for any Gaussian measure γ on $\mathbb{R}^n$ (see [2, Theorem 2.4.6]). Hence

$$\lim_{j\to\infty} \sum_{i=1}^{n(j)} b_{ij}^2 = \lim_{j\to\infty} \int_{\mathbb{R}^n} \|x\|^2 \, d\gamma^{n(j)}_{\{b_{ij}^2\}}(x) = 0,$$

which is a contradiction. Therefore $\{\gamma^{n(j)}_{\{b_{ij}^2\}}\}$ does not have a subsequence converging weakly to δ_o and neither does $\{\epsilon^{n(j)}_{\{\sqrt{n(j)-1}\, b_{ij}\}}\}$.

We next prove the lemma for $\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}}$. It holds that

$$\int_{\mathcal{S}^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}} \|y\|^2 \, d\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}(y) = \left(\sum_{i=1}^{n(j)} b_{ij}^2 \right) (n(j)-1) \int_{S^{n-1}(1)} x_1^2 \, d\sigma^{n(j)-1}(x).$$

It follows from the Maxwell-Boltzmann distribution law that

$$\lim_{j\to\infty} (n(j)-1) \int_{S^{n-1}(1)} x_1^2 \, d\sigma^{n(j)-1}(x) = 1.$$

We therefore have

$$\lim_{j\to\infty} \int_{\mathcal{S}^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}} \|y\|^2 \, d\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}(y) = \lim_{j\to\infty} \sum_{i=1}^{n(j)} b_{ij}^2 > 0.$$

On the other hand, the reverse Hölder inequality on a sphere (see [2, Remarks 2.4.7]) yields

$$(1 - 2c\sqrt{\varepsilon_j}) \int_{\mathcal{S}^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}} \|y\|^2 \, d\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\, b_{ij}\}_i}(y) \le \varepsilon_j^2$$

if $d_{\mathrm{P}}(\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}, \delta_o) \le \varepsilon_j$, in the same way as above. These contradict each other. Thus $\{\sigma^{n(j)-1}_{\{\sqrt{n(j)-1}\,b_{ij}\}}\}$ have no subsequence converging weakly to δ_o. This completes the proof of the lemma. $\square$

For the proof of Theorem 1.2, we need the following lemma, which is the special case of Theorem 1.2 where the limit is a one-point mm-space.

Lemma 4.4. *Let $\{a_{ij}\}$, $i = 1, 2, \dots, n(j)$, $j = 1, 2, \dots,$ be a sequence of positive real numbers, where $\{n(j)\}$, $j = 1, 2, \dots,$ is a sequence of positive integers divergent to infinity. We assume that*

(i) $$\lim_{j\to\infty} a_{ij} = 0 \quad \textit{for any } i,$$

(ii) $$\liminf_{j\to\infty} \sum_{i=1}^{n(j)} a_{ij}^2 > 0.$$

Then, there exists no box convergent subsequence of $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$.

Proof. Let $\{a_{ij}\}$ be a sequence as in the assumption of the theorem. Sorting $\{a_{ij}\}$ in ascending order in i, we may assume that a_{ij} is monotone nonincreasing in i for each j. We suppose that $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$ has a box convergent subsequence, for which we use the same notation. Then, by (i) and Theorem 1.1, the box limit of $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$ is mm-isomorphic to a one-point mm-space. We set

$$A_j := \left(\sum_{i=1}^{n(j)} a_{ij}^2\right)^{\frac{1}{2}} \quad \text{and} \quad b_{ij} := \frac{a_{ij}}{\max\{A_j, 1\}}.$$

Since $b_{ij} \le a_{ij}$, we see that $E^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}$ is dominated by $E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}$, so that $E^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}$ box converges to a one-point mm-space as $j \to \infty$. We remark that

$$\liminf_{j\to\infty} \sum_{i=1}^{n(j)} b_{ij}^2 > 0 \quad \text{and} \quad \sum_{i=1}^{n(j)} b_{ij}^2 \le 1.$$

Taking a subsequence again, we assume that $\sum_{i=1}^{n(j)} b_{ij}^2$ converges to a positive real number as $j \to \infty$. Applying Lemma 4.3 yields that $\{e^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}}\}$ has no subsequence converging weakly to δ_o in H. Since $E^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}$ box converges to a one-point mm-space, say $*$, as $j \to \infty$, Lemma 2.13 implies that there is a sequence of ε_j-mm-isomorphisms $f_j : E^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i} \to *$ with $\varepsilon_j \to 0+$ as $j \to \infty$. A nonexceptional domain of f_j has $e^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}}$-measure at least $1 - \varepsilon_j$ and diameter at most ε_j. There is a closed metric ball $B_j \subset H$ of radius ε_j

that contains the nonexceptional domain of f_j. Note that $e^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}(B_j) \geq 1-\varepsilon_j \to 1$ as $j \to \infty$. If B_j were to contain the origin o of H for infinitely many j, then a subsequence of $\{e^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}\}$ would converge weakly to δ_o, which is a contradiction. Thus, all but finitely many B_j do not contain the origin of H, and B_j do not intersect $-B_j$ for any such B_j. Since $e^{n(j)}_{\{\sqrt{n(j)-1}\,b_{ij}\}_i}$ is centrally symmetric with respect to the origin, we see that $e^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}(-B_j) = e^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}(B_j) \geq 1-\varepsilon_j$, which is a contradiction if j is large enough. This completes the proof. □

Lemma 4.5. *Let $\{X_n\}$, $n = 1, 2, \dots$, be a box convergent sequence of mm-spaces and $\{Y_n\}$, $n = 1, 2, \dots$, a sequence of mm-spaces with $Y_n \prec X_n$. Then, $\{Y_n\}$ has a box convergent subsequence.*

Proof. The lemma follows from [27, Lemma 4.28 (1) and (3)]. □

Proof of Theorem 1.2. We assume (A0)–(A3).

The 'if' part follows from Proposition 4.2.

We prove the 'only if' part. Suppose that $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$ is box convergent and that $\{a_{ij}\}_i$ does not l^2-converge to $\{a_i\}$ as $j \to \infty$. We first prove that $\{a_i\}$ is an l^2-sequence. This is because, if not, then, by Theorem 1.1, the weak limit of $\{E^{n(j)}_{\{\sqrt{n(j)-1}\,a_{ij}\}_i}\}$ is not an mm-space, which is a contradiction to the box convergence. Replacing $\{a_{ij}\}_i$ with a subsequence with respect to the index j, we assume that $\lim_{j\to\infty} \sum_{i=1}^{n(j)} a_{ij}^2$ exists in $[\,0, +\infty\,]$. We prove

$$\lim_{j\to\infty} \sum_{i=1}^{n(j)} a_{ij}^2 > \sum_{i=1}^{\infty} a_i^2. \tag{4.3}$$

In fact, if the left-hand side of (4.3) is infinity, then this is clear. If not, the Banach-Alaoglu theorem tells us the existence of an l^2-weakly convergent subsequence of $\{a_{ij}\}$. Since $\{a_{ij}\}_i$ does not converge to $\{a_i\}$ l^2-strongly as $j \to \infty$, we obtain (4.3).

Take a real number ε_0 in such a way that

$$0 < \varepsilon_0 < \lim_{j\to\infty} \sum_{i=1}^{n(j)} a_{ij}^2 - \sum_{i=1}^{\infty} a_i^2.$$

Setting

$$a_{ijk} := \begin{cases} a_{kj} & \text{if } i \leq k, \\ a_{ij} & \text{if } i \geq k+1, \end{cases}$$

we have

$$\lim_{j\to\infty}\sum_{i=1}^{n(j)} a_{ijk}^2 = \lim_{j\to\infty}\left(\sum_{i=1}^{k} a_{ijk}^2 + \sum_{i=k+1}^{n(j)} a_{ijk}^2\right) = ka_k^2 + \lim_{j\to\infty}\sum_{i=k+1}^{n(j)} a_{ij}^2$$
$$= ka_k^2 + \lim_{j\to\infty}\sum_{i=1}^{n(j)} a_{ij}^2 - \sum_{i=1}^{k} a_i^2 > \varepsilon_0.$$

Thus, for any positive integer k there is $j(k)$ such that

$$\sum_{i=1}^{n(j(k))} a_{ij(k)k}^2 > \varepsilon_0 \quad \text{and} \quad |\, a_{kj(k)} - a_k \,| < \frac{1}{k}.$$

Letting $b_{ik} := a_{ij(k)k}$, we observe the following.

- $b_{ik} \le a_{ij(k)}$ for any i and k.
- b_{ik} is monotone nonincreasing in i for each k.
- $b_{1k} = a_{1j(k)k} = a_{kj(k)} < a_k + 1/k \to 0$ as $k \to \infty$.
- $\sum_{i=1}^{n(j(k))} b_{ik}^2 > \varepsilon_0 > 0$ for any k.

Consider $E_k := E^{n(j(k))}_{\{\sqrt{n(j(k))-1}\, b_{ik}\}_i}$. It follows from Lemma 3.1 that E_k is dominated by $E^{n(j(k))}_{\{\sqrt{n(j(k))-1}\, a_{ij(k)}\}_i}$ for any k and so Lemma 4.5 implies that $\{E_k\}$ has a box convergent subsequence. However, Lemma 4.4 proves that $\{E_k\}$ has no box convergent subsequence, which is a contradiction. This completes the proof. $\square$

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Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, JAPAN
Email address: kazukawa@math.kyushu-u.ac.jp

Mathematical Institute, Tohoku University, Sendai 980-8578, JAPAN
Email address: shioya@math.tohoku.ac.jp