Hessenberg varieties and Chromatic symmetric functions

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Hessenberg varieties

A *Hessenberg variety* $\mathsf{X}_{\mathfrak{h}}$ is the collection of complete flags $V_1 \subset V_2 \subset \cdots \subset V_n$ which satisfy

$$\mathsf{X}.V_i \subset V_{\mathfrak{h}(i)}$$

for X a $n \times n$ regular semisimple matrix and a non-decreasing function

$$\mathfrak{h} : \{1, \cdots, n\} \longrightarrow \{1, \cdots, n\}$$

such that $\mathfrak{h}(i) \geq i$.

The *Young subgroup* of $\mathsf{X}_{\mathfrak{h}}$ is the group

$$\mathfrak{S}_{\lambda_{\mathfrak{h}}} := \langle (ij) \in S_n \mid i < j \text{ and } \mathfrak{h}(i) \geq j \rangle .$$

GKM Theory

The (torus) equivariant cohomology of $\mathsf{X}_{\mathfrak{h}}$ has the presentation [4]

$$H_T^*(\mathsf{X}_{\mathfrak{h}}) := \left\{ \begin{array}{l} \mathcal{P} : \mathfrak{S}_n \longmapsto \mathbb{C}[t_1, \cdots, t_n] \mid \\ u = v(ij) \text{ with } i < j \text{ and } \mathfrak{h}(i) \geq j \\ \text{then } \mathcal{P}_u - \mathcal{P}_v \in \langle t_{v(i)} - t_{v(j)} \rangle \end{array} \right\} .$$

- $\mathfrak{S}_n$ acts on $H_T^*(\mathsf{X}_{\mathfrak{h}})$ (and hence $H^*(\mathsf{X}_{\mathfrak{h}})$) under the rule
$$w \cdot \mathcal{P}(u) = w * \mathcal{P}(w^{-1}u)$$
where $w*$ is the action of $\mathfrak{S}_n$ on $\mathbb{C}[t_1, \cdots, t_n]$.
- *Question 1:* Can we identify these representations?
- *Question 2:* Are they permutation representations?

Unifying Conjecture [3]

Fix the function $\mathfrak{h}$ and let $\mathrm{inc}(\mathfrak{h})$ be the incomparibilty graph of $\mathfrak{h}$ as a semi-order. Then

$$\mathrm{ch} H^*(\mathsf{X}_{\mathfrak{h}}) = \omega X_{\mathrm{inc}(\mathfrak{h})}$$

where ch is the Frobenius characteristic and ω is the involution $e_\lambda \longmapsto h_\lambda$.

Theorem

We say $\mathsf{X}_{\mathfrak{h}}$ is *parabolic* if for every $(ij) \in S_{\mathfrak{h}}$, then $i < j$ and $h(i) \geq j$.

Parabolic Hessenberg varieties [4]

Let $\mathsf{X}_{\mathfrak{h}}$ be a parabolic Hessenberg variety. Then

$$H_T^*(\mathsf{X}_{\mathfrak{h}}) \cong |\mathfrak{S}_{\mathfrak{h}}| M^{\lambda_{\mathfrak{h}}}$$

as a $\mathbb{C}[t_1, \cdots, t_n][\mathfrak{S}_n]$-module.
Further, as a $\mathbb{C}[\mathfrak{S}_n]$-module we have

$$H^*(\mathsf{X}_{\mathfrak{h}}) \cong |\mathfrak{S}_{\mathfrak{h}}| M^{\lambda_{\mathfrak{h}}}$$

and under the Frobenius characteristic we get

$$\mathrm{ch} H^*(\mathsf{X}_{\mathfrak{h}}) = |\mathfrak{S}_{\mathfrak{h}}| h_{\lambda_{\mathfrak{h}}}.$$

Lastly, this decomposition confirms the "unifying conjecture"

$$\mathrm{ch} H^*(\mathsf{X}_{\mathfrak{h}}) = \omega X_{\mathrm{inc}(\mathfrak{h})}.$$

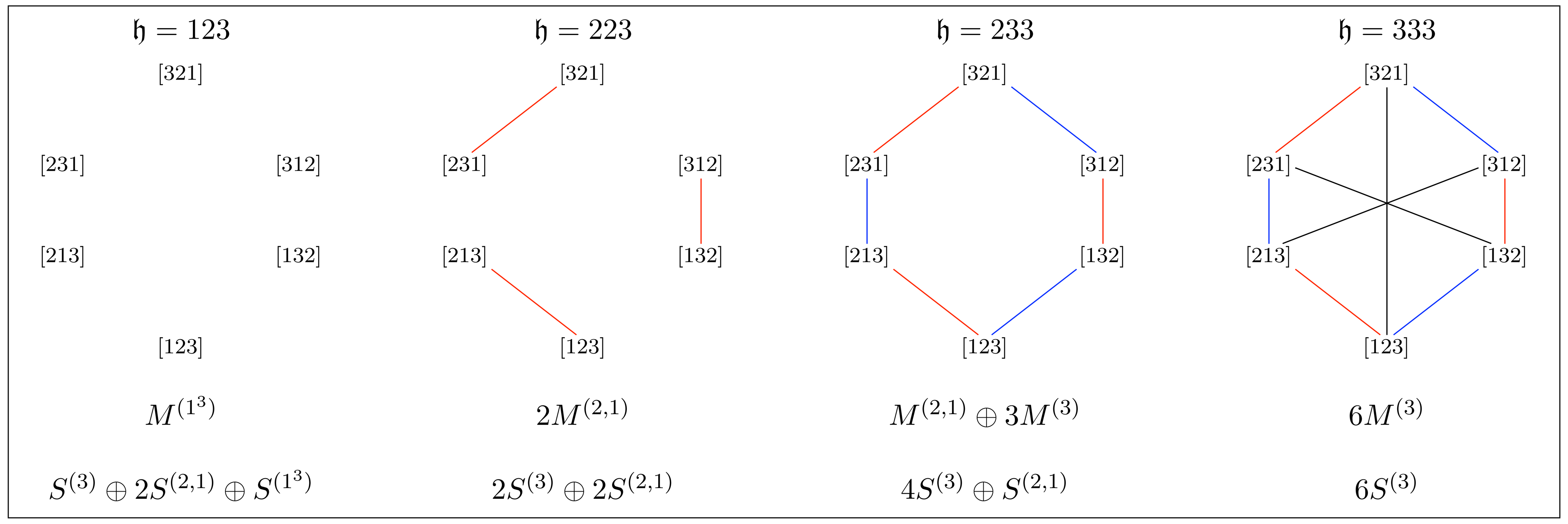

Figure: Hessenberg representations

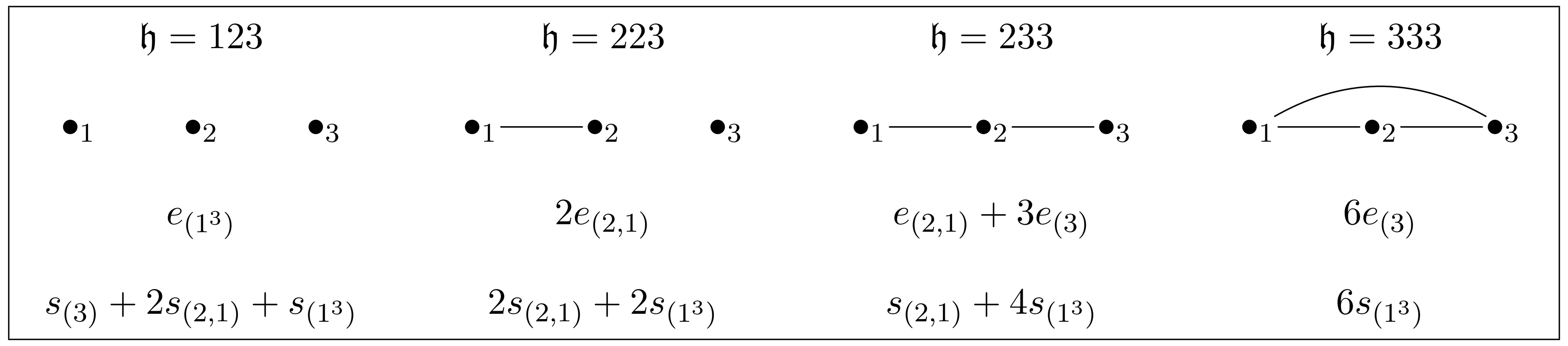

Figure: Chromatic symmetric functions

Chromatic symmetric functions

Let G be a finite simple graph with vertices $V = \{1, \cdots, n\}$ and edges $E := \{ij \mid i, j \in V\}$. A *proper coloring* of G is a map $\kappa : V \longrightarrow \mathbb{N}$ such that whenever $uv \in E$ then $\kappa(u) \neq \kappa(v)$.
Given a proper coloring let

$$x_\kappa = x_{\kappa(1)} \cdots x_{\kappa(n)}.$$

Stanley [2] defined the *chromatic symmetric function* of G to be

$$X_G(x_1, x_2, \cdots) := \sum_\kappa x_\kappa .$$

Incomparability graphs

Let $(\mathfrak{p}, \preceq)$ be a poset. The incomparability graph of $\mathfrak{p}$, has vertices $\mathfrak{p}$ and edges $\{ij \mid i \npreceq j \text{ and} j \npreceq i\}$.

For the function $\mathfrak{h}$, we can define a poset, called a *semi-order*, by the rule

$$i \prec j \Longleftrightarrow \mathfrak{h}(i) < j.$$

Semi-orders are the posets which are both $[3] + [1]$-free and $[2] + [2]$-free. The chromatic symmetric function of their incomparability graphs are of principal interest in[1],[2], and [3].

Stanley-Stembridge Conjecture

Stanley-Stembridge conjecture [2]

Let $\mathrm{inc}(\mathfrak{h})$ be the incomparability graph of a semi-order. The expansion of $X_{\mathrm{inc}(\mathfrak{h})}$ in the e_λ-basis has non-negitive coefficients (e-positive).

Gasharov's theorem [1]

Let $\mathrm{inc}(\mathfrak{h})$ be the incomparability graph of a semi-order. Then $X_{\mathrm{inc}(\mathfrak{h})}$ is *Schur*-positive, i.e. the Frobenius characteristic of a representation of $\mathfrak{S}_n$.

References

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[4] N. Teff. " Representations on Hessenberg Varieties and Young's Rule " DMTCS Proceedings [Online], 0 11 Jun 2011

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