Coxeter groups, palindromic Poincaré polynomials and triangle group avoidance

by

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1 Coxeter groups and Poincaré polynomials

Let W be a Coxeter group with finite reflection set S. By definition, W is the group generated by S subject to the relations $st^{m_{st}} = e$ where $m_{st} \in \{1, 2, 3, \ldots, \infty\}$ and

$$m_{st} = 1 \Leftrightarrow s = t.$$

If ℓ and $\leq$ denote the length function and Bruhat order on W, then for any $w \in W$ we define the Poincaré polynomial

$$P_w(q) := \sum_{x \leq w} q^{\ell(x)}.$$

If W is crystallographic (i.e. $m_{st} \in \{1, 2, 3, 4, 6, \infty\}$), then W is the Weyl group of some Kac-Moody group G. Each element $w \in W$ indexes a Schubert variety $X_w \subseteq G/B$. Topologically we have

$$P_w(q) = \sum_{i \geq 0} \dim H^{\frac{i}{2}}(X_w, \mathbb{C})\, q^i.$$

Question 1. *When is $P_w(q)$ a palindromic polynomial?*

A degree ℓ polynomial $\sum_i a_i\, q^i$ is palindromic if $a_i = a_{\ell - i}\ \forall\, i$. For crystallographic W, X_w is rationally smooth if and only if $P_w(q)$ is palindromic [4]. If W is a simply laced Weyl group of finite type, then rationally smooth can be replaced by smooth.

For finite Weyl groups, the answer to Question 1 is well understood. In particular, palindromic Poincaré polynomials can be characterized using permutation pattern-avoidance in classical types and using root system avoidance in all types [1, 3, 5]. The characterization using permutation pattern avoidance has been extended to the affine type A case as well [2].

Combinatorial answers to Question 1 for general Coxeter groups are unknown. We introduce a family of Coxeter groups (mostly) outside the finite and affine cases for which it is possible to determine if $P_w(q)$ is palindromic by calculating a few of its coefficients.

2 Triangle group avoidance

A *triangle group* is a Coxeter group with $|S| = 3$. If $S = \{r, s, t\}$, then a triangle group is completely characterized by the triple (m_{rs}, m_{rt}, m_{st}). We will denote a triangle group by its corresponding triple.

Definition 1. A Coxeter group W *contains the triangle* (a, b, c) if there exists a subset $\{r, s, t\} \subseteq S$ such that $(a, b, c) = (m_{rs}, m_{rt}, m_{st})$.

If S contains no such subset, then W *avoids the triangle* (a, b, c).

Definition 2. A polynomial $\sum_{i=0}^{\ell} a_i\, q^i$ is *k-palindromic* if $a_i = a_{\ell - i}\ \forall\, i \leq k$.

Consider the set of triangle groups: $\mathrm{HQ} := \{(2, b, c) \mid b, c \geq 3 \text{ and } b < \infty\}$.

Theorem 1. *(R-Slofstra) Suppose W avoids all triangles in the set HQ.*

For any $w \in W$, if $P_w(q)$ is 4-palindromic, then $P_w(q)$ is palindromic.

Furthermore, suppose W avoids triangles $(3, 3, c)$ where $3 < c < \infty$. Then every 2-palindromic $P_w(q)$ is palindromic.

3 Parabolic factorizations

Theorem 1 follows from a factorization theorem of Poincaré polynomials given that W avoids triangles in HQ and that $P_w(q)$ is 2-palindromic. For any subset $J \subseteq S$ we define the parabolic subgroup $W_J \subseteq W$ generated by J. Let W^J denote the set of minimal length representatives of the cosets $W_J \backslash W$.

For any subset J there is a unique *parabolic factorization* $w = uv$ where $u \in W_J, v \in W^J$ and $\ell(w) = \ell(u) + \ell(v)$. Let $[e, w]$ denote the set of elements less than or equal to w.

The following proposition is due to Billey and Postnikov in [3, Theorem 6.4].

Proposition 1. *For any parabolic factorization $w = uv$, we have that u is maximal in $W_J \cap [e, w]$ if and only if*

$$P_w(q) = P_u(q) \cdot P_v^J(q)$$

where $P_v^J(q) := \sum_{x \in [e,v] \cap W^J} q^{\ell(x)}$. We call any such factorization BP.

Theorem 2. *(R-Slofstra) Suppose that W avoids all triangle groups in HQ. Let $w \in W$ be 2-palindromic and fix a parabolic factorization $w = uv$ such that*

$$|S \cap [e, w]| = |S \cap [e, u]| + 1.$$

Then $w = uv$ is a BP-factorization where $|S \cap [e, v]| \leq 3$.

Moreover, if $S \cap [e, v] = \{r, s, t\}$, then $3 \leq m_{rs} \leq \infty, 3 \leq m_{st} < \infty$ with one of the following:

1. $v = tr\underbrace{stst\ldots}_{m_{st}-1}$ *where $\{r, s, t\}$ generates the triangle $(3, m_{rs}, m_{st})$.*
2. $v = rstr\underbrace{stst\ldots}_{m_{st}-1}$ *where $\{r, s, t\}$ generates the triangle $(3, 3, m_{st})$.*
3. $v = strstr\cdots$ *where $\{r, s, t\}$ generates the triangle $(3, 3, 3)$ and $\ell(v)$ is even.*

If w is 2-palindromic, then u is also 2-palindromic. Hence Theorem 2 can be applied inductively to factor $P_w(q)$. Define the q-integer

$$[k]_q := 1 + q + \cdots + q^{k-1}.$$

Corollary 1. *Suppose $w = uv \in W$ satisfies the conditions in Theorem 2. Then $P_v^J(q)$ equals one of the following polynomials.*

1. $[\ell(v) + 1]_q$.
2. $[\ell(v) + 1]_q + q^2[\ell(v) - 3]_q$.
3. $[\ell(v) + 1]_q + q^2[\ell(v) - 3]_q + q^4[\ell(v) - 6]_q$.
4. $\sum_{i=0}^{k} q^{2i}[\ell(v) - 4i + 1]_q$ *with* $k = \left\lfloor \frac{\ell(v)}{4} \right\rfloor$.

Observe that all the polynomials listed are palindromic except the third which is 3-palindromic but not 4-palindromic. This third polynomial corresponds to part 2 of Theorem 2 which proves Theorem 1.

Example 1. Consider the Coxeter group W with $S = \{s_1, s_2, s_3, s_4\}$ defined by $m_{st} = 3\ \forall\, s, t \in S$. Let $w = s_1 s_2 s_1 s_3 s_2 s_1 s_3 s_2 s_1 s_4$. Then w is 2-palindromic with factorization:

$$w = \underbrace{(s_1)}_{v_1}\underbrace{(s_2 s_1)}_{v_2}\underbrace{(s_3 s_2 s_1 s_3 s_2 s_1)}_{v_3}\underbrace{(s_4)}_{v_4}.$$

The corresponding Poincaré polynomial factorization is

$$P_w(q) = [2]_q[3]_q([5]_q + q^2[1]_q)[2]_q = (1+q)(1+q+q^2)(1+q+2q^2+q^3+q^4)(1+q).$$

4 Enumeration results

One consequence of Theorem 2 is that the number of elements with palindromic Poincaré polynomials is finite if W avoids triangles in HQ and $(3, 3, 3)$.

We can explicitly enumerate the number of palindromic elements in uniform Coxeter groups. For any positive integers m, n let $W(m, n)$ denote the Coxeter group with $|S| = n$ and $m_{s,t} = m\ \forall\, s, t \in S$. Define the generating series

$$\Phi_m(q, t) := \sum_{n, k \geq 0} P_{n,k} \frac{q^k t^n}{n!}$$

where $P_{n,k}$ denotes the number of palindromic $w \in W(m, n)$ of length k.

Corollary 2. *For any $m \geq 4$, the series $\Phi_m(q, t) = \dfrac{\exp(t)}{1 - \phi_m(q, t)}$ where*

$$\phi_m(q, t) = \frac{2qt - 3q^m t^2 - q^{m+2}[m-3]_q t^3}{2 - 2q^2 t([m-2]_q + q^{m-3})}.$$

Example 2. The expansion of $\Phi_4(q, t) - 1$ is:

$$(1{+}q)\, t + (1{+}2q{+}2q^2{+}2q^3{+}q^4)\, \frac{t^2}{2} + (1{+}3q{+}6q^2{+}12q^3{+}15q^4{+}12q^5{+}12q^6{+}6q^7)\, \frac{t^3}{6} + O(t^4).$$

The following table lists the number of palindromic elements in $W(m, n)$.

$m \diagdown n$	1	2	3	4	5	6	7
4	2	8	67	893	15596	330082	8165963
5	2	10	115	2057	47356	1314292	42584795
6	2	12	175	3893	110436	3768982	150113447
7	2	14	247	6545	219956	8884312	418725119
8	2	16	331	10157	393916	18351562	997538291

References

[1] S. Billey. Pattern avoidance and rational smoothness of Schubert varieties. *Adv. Math.*, 139(1):141–156, 1998.

[2] S. Billey and A. Crites. Rational smoothness and affine Schubert varieties of type A. In *23rd ICFPSAC, 2011*, Discrete Math. Theor. Comput. Sci. Proc., AO, pages 171–181.

[3] S. Billey and A. Postnikov. Smoothness of Schubert varieties via patterns in root subsystems. *Adv. in Appl. Math.*, 34(3):447–466, 2005.

[4] J. B. Carrell. The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties. In *Algebraic groups and their generalizations: classical methods (Uni. Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 53–61. AMS, Providence, RI, 1994.

[5] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$. *Proc. Indian Acad. Sci. Math. Sci.*, 100(1):45–52, 1990.