SIMPLE G-MODULES WITH NORMAL CLOSURES OF MAXIMAL TORUS ORBITS

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Definition. The set of points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ is called *saturated* if
$$\mathbb{Z}_{\geqslant 0}(\mu_1, \dots, \mu_s) = \mathbb{Z}(\mu_1, \dots, \mu_s) \cap \mathbb{Q}_{\geqslant 0}(\mu_1, \dots, \mu_s),$$
where $\mathbb{Z}_{\geqslant 0}(\mu_1, \dots, \mu_s) = \{n_1\mu_1 + n_2\mu_2 + \ldots + n_s\mu_s \mid n_i \in \mathbb{Z}, n_i \geqslant 0\}$.

Definition. The set of points $\{\mu_1, \dots, \mu_s\} \subset \mathbb{Q}^n$ is called *hereditarily normal* if each of its subset is saturated.

Let us give some examples of hereditarily normal sets of points.

Example 1 (B. Sturmfels [9, 10])**.** Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{Z}^n$. The set
$$\{e_i - e_j \mid 1 \leqslant i, j \leqslant n\}$$
is hereditarily normal.

Example 2 (H. Ohsugi, T. Hibi [7] & A. Simis, W. Vasconcelos, R. Villarreal [8])**.** To a finite graph Γ with n vertices, we can associate the following finite collection $M(\Gamma)$ of vectors in the lattice $\mathbb{Z}^n$:
$$M(\Gamma) = \{e_i + e_j \mid (ij) \text{ is an edge of } \Gamma\},$$
where $e_1, e_2, \dots, e_n$ is the standard basis of $\mathbb{Z}^n$. The saturation property for this set is equivalent to the fact that for two arbitrary minimal odd cycles C and C' in Γ, either C and C' have a common vertex or there exists an edge of Γ joining a vertex of C with a vertex of C'.

Definition. A *realizable matroid* is a set of subsets in an n-element set with the following property: there exist n vectors in $\mathbb{C}^k$ such that for every subset presented in our list, and only for them, the corresponding vectors are linearly independent. A *base* of a matroid is any maximal (with respect to inclusion) subset from our list. An *incidence vector* for a set from our list is a vector of length n of 0s and 1s encoding which elements of the initial set appear in this subset.

Example 3 (N. White [11])**.** The set of incidence vectors of the bases of a realizable matroid is saturated. Geometrically it means that that for every point y in the affine cone over the classical Grassmannian $\mathrm{Gr}(k, n)$ the closure $\overline{Ty}$ is normal.

Let us pass to the geometric meaning of hereditary normality. Let T be an algebraic torus and let V be a rational T-module. We denote by $\Lambda = \Lambda(T)$ the *character lattice* of T. With respect to the T-action, the module V can be diagonalized:
$$V = \bigoplus_{\mu \in \Lambda} V_\mu, \quad \text{where} \quad V_\mu = \{v \in V \,|\, tv = \mu(t)v \quad \forall\, t \in T\}.$$
We denote by $M(V) = \{\mu \in \Lambda \,|\, V_\mu \neq 0\}$ the set of weights of V. Each nonzero vector v in V has its *weight decomposition* $v = v_{\mu_1} + \dots + v_{\mu_s}$, $v_{\mu_i} \in V_{\mu_i}$, $v_{\mu_i} \neq 0$. Denote by $M(v)$ the set $\{\mu_1, \dots, \mu_s\}$. We may generate a semigroup $\mathbb{Z}_{\geqslant 0}(\mu_1, \dots, \mu_s)$ with these weights. We may also consider a sublattice $\mathbb{Z}(\mu_1, \dots, \mu_s)$ and a rational polyhedral cone $\mathbb{Q}_{\geqslant 0}(\mu_1, \dots, \mu_s)$ in the space $\Lambda_{\mathbb{Q}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

Lemma 1 ([3, I, §1, Lemma 1])**.** *The closure $\overline{Tv}$ of the T-orbit of a vector v is normal if and only if the set $M(v)$ is saturated.*

The closure of every T-orbit is normal if and only if the set $M(V)$ is hereditarily normal.

J. Morand [6] classified all semisimple affine algebraic groups such that all T-orbit closures in the adjoint module are normal.

For projective T-actions, a combinatorial criterion can also be formulated. Let $X(v)$ be the closure of the T-orbit $T\langle v\rangle$ of a point $\langle v\rangle \in \mathbb{P}(V)$ in the projectivisation of a rational T-module V. Let $P(v)$ be the convex hull of $M(v)$ in $\Lambda_{\mathbb{Q}}$. Then $X(v)$ is normal if and only if the set $\{\mu - \mu_0 \mid \mu \in M(v)\}$ is saturated for every vertex μ_0 of the polytope $P(v)$. This and other criteria are given by J.B. Carrell and A. Kurth [2].

Let G be a connected simply connected semisimple algebraic group and T be a maximal torus in G. We have solved the following problem.

Find all pairs (G, V) of a group G as stated above and of a simple rational G-module V such that for each vector $v \in V$ the closure of its T-orbit is a normal affine algebraic variety.

Theorem 1 (K, I. Bogdanov [1], [4], [5])**.** *For the following types of simple algebraic groups and the corresponding modules, and for their dual modules, the closures of all maximal torus orbits are normal. In all other cases, the module contains a maximal torus orbit with nonnormal closure.*

Root system	*Highest weight*
$A_n, n \geqslant 1$	π_1
$A_n, n \geqslant 1$	$\pi_1 + \pi_n$ ([6, 9, 10])
A_1	$3\pi_1$
A_1	$4\pi_1$
A_2	$2\pi_1$
A_3	π_2
A_4	π_2
A_5	π_2
A_5	π_3
$B_n, n \geqslant 2$	π_1
B_2	π_2
B_2	$2\pi_2$
B_3	π_3

Root system	*Highest weight*
B_4	π_4
$C_n, n \geqslant 3$	π_1
C_3	π_2
C_4	π_2
$D_n, n \geqslant 4$	π_1
D_4	π_2
D_4	π_3
D_4	π_4
D_5	π_4
D_6	π_5
D_6	π_6
F_4	π_4
G_2	π_1

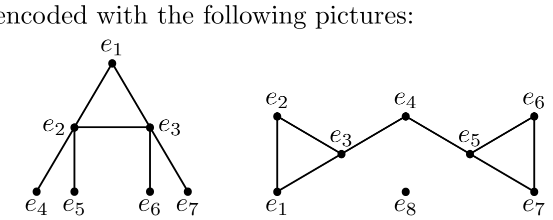
The proof of the theorem consists of several steps:

0. Guess the answer
1. Show non-saturated subsets of weights for all the modules not listed in Thm. 1
2. Prove hereditary normality for the sets of weights of all the modules from Thm. 1

1. Negative cases

Unexpectedly, this property almost never holds for the fundamental representations of $SL(n)$. It is one of the most difficult negative cases.

Example 4. Non-saturated subsets for the fundamental representations π_2 of $SL(7)$ and $SL(8)$ can be encoded with the following pictures:



The answers for $(SL(n), \pi_k)$ and $(SL(n), \pi_{n-k})$ are the same due to the fact that combinatorially the sets $M(V)$ coincide. We also use the following lemma.

Lemma 2. *A non-saturated subset in $M(V)$ for a pair $(SL(n), \pi_k)$ can be transformed into a non-saturated subset for the pair $SL(n+k), \pi_k$.*

Now we use Euclid's algorithm and reduce any pair $(SL(n), \pi_k)$ which is not mentioned in the theorem, to a "negative" one.

To deal with the other cases, we use the following simple lemma.

Lemma 3. *If the set of weights of the representation with the highest weight λ is a subset in the set of weights of the representation with the highest weight μ and we know a non-saturated subset for λ, then we can use it as a non-saturated subset for μ.*

2. Positive cases

Let us describe some methods used to prove hereditary normality.

Definition. Assume that the set of vectors $M \subset \mathbb{Q}^n$ has rank d, $d \leqslant n$, and let $L = \langle v \,|\, v \in M\rangle$ be the linear span of vectors from M. The set M is called *unimodular* if for any linearly independent vectors $v_1, \dots, v_d \in M$ the d-dimensional volume $\mathrm{vol}_d(v_1, v_2, \dots, v_d)$ has the same absolute value.

Lemma 4 ([9, Theorem 3.5])**.** *Any unimodular set is hereditarily normal.*

Example 5. For the representation $V(\lambda)$ of $SL(6)$ with the highest weight $\lambda = \pi_3$, the corresponding set $M(\lambda) = \{(\varepsilon_1, \dots, \varepsilon_6) \mid \varepsilon_i = \pm 1, \sum \varepsilon_i = 0\}$ (up to dilatation). It is unimodular, hence hereditarily normal.

Definition. More generally, we say that a subset $M \subset \mathbb{Q}^n$ of rank d is *almost unimodular* if we can choose a subset $\{v_1, v_2, \dots, v_d\} \subseteq M$ such that
$$\mathrm{vol}_d(v_1, v_2, \dots, v_d) = m,$$
and for any other vector $w \in M$ and each i the value
$$\mathrm{vol}_d(v_1, v_2, \dots, \widehat{v_i}, \dots, v_d, w)$$
is divisible by m, i.e. equals km for some $k \in \mathbb{Z}$.

In other words, M is almost unimodular if and only if the lattice $\mathbb{Z}(M)$ has a basis consisting of elements of M.

Almost unimodular sets are used to prove that certain sets of vectors are hereditarily normal. We argue by contradiction supposing that in the given set M, there exists a non-saturated subset $(v_1, \dots, v_r)$ with vector v_0 being a "hole". Then, using almost unimodularity of M, we analyze coefficients of the corresponding $\mathbb{Q}_{\geqslant 0}$-combination for v_0. If we take a "minimal" v_0, then there is a finite number of possible values for its coefficients. Then, using this data and additional information about the weight lattice, we show that $(v_1, \dots, v_r)$ is not a non-saturated subset. This implies that M is hereditarily normal.

Example 6. For the representation $V(\lambda)$ of $SO(9)$ with the highest weight $\lambda = \pi_4$ (the root system B_4), the set of weights is
$$M(V) = \{(\pm 1, \pm 1, \pm 1, \pm 1)\}$$
up to dilatation. The values of all nonzero determinants in $M(V)$ equal ± 8 and ± 16, hence it is almost unimodular, and we managed to prove that it is hereditarily normal.

Example 7. For the representation $V(\lambda)$ of Spin(10) with the highest weight $\lambda = \pi_5$ (the root system D_5), the set of weights is
$$M(V) = \{(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \,|\, \text{even number of minuses}\}$$
up to dilatation. Here the nonzero determinants can attain values 16, 32, or 48, and via some thorough analysis we proved that this set is also hereditarily normal.

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