Poincaré Polynomials of Singular Schubert Varieties

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Abstract

We have defined a new polynomial which predicts the dimensions of the intersection cohomology groups of Schubert varieties in certain cases, including some non-rationally smooth cases.

The Ordinary Poincaré Polynomial

The Poincaré polynomial of a complex algebraic variety X is given by

$$\mathcal{P}_X(q) = \sum_{i \geq 0} \dim_{\mathbb{C}}(H^i(X))q^i$$

where $H^i(X)$ is the singular homology of X, viewed in its analytic topology. If X_w is a Schubert variety, then its Poincaré polynomial can be described combinatorially by the formula

$$\mathcal{P}_w(q) = \sum_{x \leq w} q^{\ell(x)}$$

where the sum is over all elements $x \leq w$ in the Bruhat-Chevalley order on W.

The Intersection Cohomology Poincaré Polynomial

The Poincaré polynomial for the full intersection cohomology for X is defined to be

$$\mathcal{I}_X(q) = \sum_{i \geq 0} \dim_{\mathbb{C}}(IH^i(X))q^i.$$

As was the case for the ordinary Poincaré polynomial $\mathcal{P}_X(q)$, when X_w is a Schubert variety the intersection cohomology Poincaré polynomial has a combinatorial description, described as follows. For $w \in W$ and $x \leq w$ in the Bruhat-Chevalley order on W, let $P_{x,w}$ denote the Kazhdan-Lusztig polynomial indexed by x and w (see [5]). The Poincaré polynomial for the full intersection cohomology for X_w is then given by

$$\mathcal{I}_w(q) = \sum_{u \leq w} P_{u,w}(q)q^{\ell(u)}.$$

It has been shown that X is rationally smooth if and only if the ordinary cohomology groups $H^*(X)$ coincide with the intersection cohomology groups $IH^*(X)$ [4]. In other words, we have $\mathcal{I}_w = \mathcal{P}_w$ if and only if X_w is a rationally smooth Schubert variety.

Pattern Containment

An element $w \in S_n$ **contains the pattern** $v \in S_k$ if w contains a subword of length k whose entries are in the same relative order as the entries of v.

Ex. $541623 \in S_6$ contains 3412 and avoids 4231.

Billey and Braden have extended the notion of pattern containment to general Weyl groups [2].

Many geometric properties of a Schubert Variety X_w are equivalent to combinatorial statements about patterns.

- For $w \in S_n$, X_w is rationally smooth iff w avoids 3412 and 4231 [6].
- See [1] for the lists of patterns which are avoided precisely when X_w is smooth/rationally smooth for Weyl groups of types B, D and E.

A Factorization of $\mathcal{P}_w(q)$

For an element $w \in S_n$, a value $r \in [n]$ is a **record position** of w if $w(r) > \max\{w(i) : 1 \leq i < r\}$. For $i \in [n]$, let r and r' be the record positions of w such that $r \leq i < r'$ and there are no other record positions s of w such that $r < s < r'$. Define

$$e_i := \#\{j : r \leq j < i, w(j) > w(i)\} \\ +\#\{k : r' \leq k \leq n, w(k) < w(i)\}.$$

If w avoids the patterns 3412 and 4231, then the ordinary Poincaré polynomial for w factors as

$$\mathcal{P}_w(q) = \prod_{i=1}^{n} [e_i + 1]_q$$

where $[a+1]_q := q^a + q^{a-1} + \cdots + q + 1$ [3].

The Inversion Polynomial

Main Goal:
to combinatorially define a new polynomial which will coincide with $\mathcal{I}_w$.

Let $N(w)$ denote the collection of all positive roots sent negative by w. Say a set $S \subset N(w)$ is $N(w)$-closed if whenever $\alpha, \beta \in S$, we have $\alpha + \beta \in S$. Define $\mathcal{N}(w)$ to be the collection of all sets $S \in N(w)$ such that both S and $N(w) \setminus S$ are $N(w)$-closed. The **inversion polynomial** for w is then defined to be

$$\mathcal{N}_w(q) = \sum_{S \in \mathcal{N}(w)} q^{|S|}.$$

Results

Theorem 1. *Suppose $w \in S_n$ avoids 3412 and 4231. Then $\mathcal{N}_w$ coincides with $\mathcal{P}_w(q)$ (and thus $\mathcal{I}_w$).*

Theorem 2. *Suppose $w \in S_n$ avoids 4231, 45312, 45213, and 35412. Assume w contains the pattern 3412. There exists in w a particular 3412 pattern $XYZW$ where Y and Z are adjacent. Let s be the simple reflection which switches Y and Z. Then ws also avoids the required four patterns, and*

$$\mathcal{N}_w(q) = (q+1)\mathcal{N}_{ws}(q).$$

Theorem 3. *Let w and s be as above. Suppose further that for any $x, xs \leq ws$, at least one of x, xs corresponds to a smooth point in the Schubert variety X_{ws}. Then*

$$\mathcal{I}_w(q) = (q+1)\mathcal{I}_{ws}(q).$$

We can then inductively show that under these conditions, we have $\mathcal{N}_w = \mathcal{I}_w$.

Conjecture. *Suppose $w \in S_n$. Then $\mathcal{N}_w(q) = \mathcal{I}_w(q)$ iff w avoids the patterns 4231 and 45312.*

This conjecture has been tested in S_n for $n \leq 8$.

A Worked Example

Let $w = 541623 \in S_6$. For $s = s_4$, we have $ws = 541263$. By the factorization formula, we have

$$\mathcal{P}_{ws}(q) = [1+1]_q^2[2+1]_q^3.$$

Since ws is rationally smooth, we have $\mathcal{N}_{ws}(q) = \mathcal{P}_{ws}(q) = \mathcal{I}_{ws}(q)$ by Theorem 1. By Theorem 2, we have

$$\mathcal{N}_w(q) = (q+1)\mathcal{N}_{ws}(q) = (q+1)\mathcal{P}_{ws}(q) = [1+1]_q^3[2+1]_q^3$$

and by Theorem 3, we have

$$\mathcal{I}_w(q) = (q+1)\mathcal{I}_{ws}(q) = (q+1)\mathcal{P}_{ws}(q) = [1+1]_q^3[2+1]_q^3$$

so we can see that $\mathcal{N}_w(q) = \mathcal{I}_w(q)$. Since w contains instances of the pattern 3412, we know that this polynomial is not equal to the ordinary Poincaré polynomial. Indeed, we have

$$\mathcal{P}_w(q) = (q+1)^2(q^2+q+1)^2(q^3+2q^2+q+1).$$

History

The definition of $\mathcal{N}(w)$ was motivated in part by Tymoczko's work in [9], where an analog of $\mathcal{N}(w)$ is used to compute the ordinary Poincaré polynomial of regular nilpotent Hessenberg varieties, and by Sommers and Tymoczko's work in [8], where the factorization of the Poincaré polynomials for regular Hessenberg varieties was studied. The latter paper is analogous to work done by Oh, Postnikov, and Yoo, who used combinatorial and graphical methods to create a similarly defined polynomial $\mathcal{R}_w(q)$ which coincides with $\mathcal{P}_w$ when $w \in S_n$ is rationally smooth [7].

References

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