Ordinary vs Double Schubert polynomials

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Thm There is an explicit relationship between ordinary and double Schubert polynomials

Torus equivariant cohomology of flag varieties

$G \supset T \curvearrowright G/B$: multiplication

$H_T^*(G/B;\mathbb{R})$ is freely generated by the *Schubert classes* $\mathfrak{S}_w$ ($w \in W$: Weyl gp.) over $H^*(BT;\mathbb{R}) = \mathbb{R}[t_1,\ldots,t_n]$

$$H_T^*(G/B;\mathbb{R}) \cong \bigoplus_{w\in W} \mathbb{R}[t_1,\ldots,t_n]\langle \mathfrak{S}_w \rangle$$

(1) free module

On the other hand, as an algebra over $H^*(BT;\mathbb{R})$,

$$H_T^*(G/B;\mathbb{R}) \cong \frac{\mathbb{R}[t_1,\ldots,t_n,x_1,\ldots,x_n]}{(f(x)-f(t))}$$

(f : runs all the W-invariants)

(2) coinvariant ring of W

Two different presentations

Schubert polynomials

The *double Schubert polynomials* (of 2-sets of variables) are polynomial representatives of Schubert classes in (2)

$$\mathfrak{S}_w(t;x) \in \frac{\mathbb{R}[t_1,\ldots,t_n,x_1,\ldots,x_n]}{(f(x)-f(t))}$$

Similarly, the *ordinary Schubert polynomials* (of 1-set of variables) are for the ordinary cohomology.

$$\sigma_w(x) \in \frac{\mathbb{R}[x_1,\ldots,x_n]}{(f(x))} \cong H^*(G/B;\mathbb{R})$$

A construction of σ_w is given uniformly for all Lie-types in Bernstein-Gelfand-Gelfand '73

Schunert polynomial relates the two presentations

Double ➡ Ordinary

There are two ways to obtain the ordinary Schubert polynomials from the double ones:

$$(i) \quad \sigma_w(x) = \mathfrak{S}_w(0;x)$$

$$(ii) \quad \sigma_w(x) = \frac{1}{|W|}\sum_{v\in W} \mathfrak{S}_{w^{-1}}(-x; v^{-1}(-x))$$

This is not surprising since G/B is a *GKM-manifold*, and hence the equivariant cohomology contains all the information of the ordinary cohomology

Equivariant cohomology is a finer invariant

Ordinary ➡ Double

We can also go the other way.

Let the *partition of w ∈W into i parts* be a set

$$P_i(w) = \{(w_1,w_2,\ldots,w_i) \in W^i \mid w_1 \cdot w_2 \cdots w_i = w, \quad l(w_k) > 0\ \forall k, \quad l(w_1)+\cdots+l(w_i) = l(w)\}$$

Then

$$\mathfrak{S}_w(t;x) = \sum_{i=1}^{l(w)} \sum_{(w_1,w_2,\ldots,w_i)\in P_i(w)} (-1)^i \sigma_{w_1}(t)\sigma_{w_2}(t)\cdots\sigma_{w_{i-1}}(t)\left(\sigma_{w_i}(t)-\sigma_{w_i}(x)\right)$$

In fact, ordinary cohomology recovers equivariant one

Key machinery : Characterization of Schubert classes by the localization to the fixed points and the divided difference operators