Thom-Porteous formulas in algebraic cobordism

Thomas Hudson

advisor : Prof. Marc Levine

MSJ-SI 2012 Schubert calculus
•
International summer school and conference on Schubert calculus
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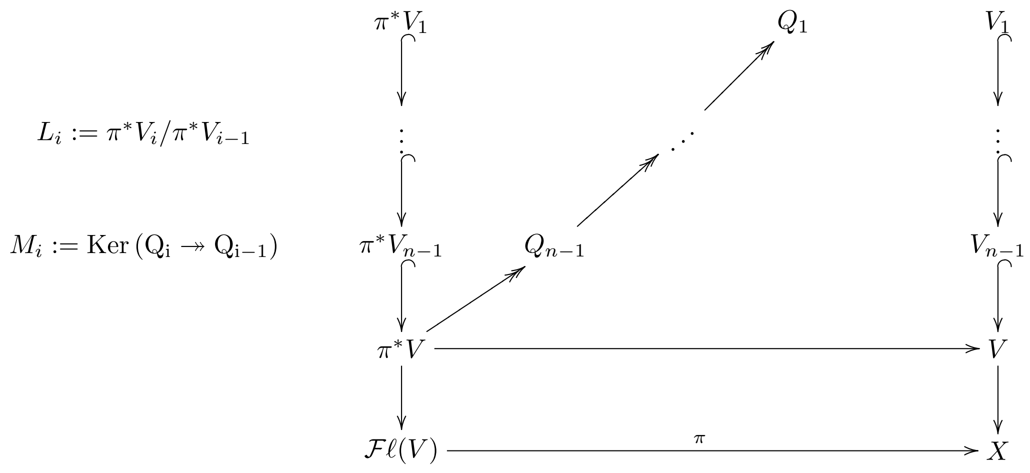
Osaka City University

Schubert varieties and Bott-Samelson resolutions

Let $X \in \mathbf{Sm}_k$ and V be a vector bundle on X endowed with a full flag of subbundles $V_\bullet = (V_1 \subset \cdots \subset V_{n-1} \subset V)$. Let us consider $\pi : \mathcal{F}\ell(V) \to X$, the full flag bundle associated to V, and the universal full flag of quotient bundles $(\pi^*V \twoheadrightarrow Q_{n-1} \twoheadrightarrow \ldots \twoheadrightarrow Q_1)$.



Definition

As a set the **Schubert variety** associated to $\omega \in S_n$ is defined as

$$\Omega_\omega := \{x \in \mathcal{F}\ell(V) \mid \mathrm{rank}(\pi^*V_i(x) \to Q_j(x)) \le r_\omega(i,j)\ \forall i,j\}$$

where the function r_ω is given by $r_\omega(i,j) := |\{k \le j \mid \omega(k) \le i\}|$.

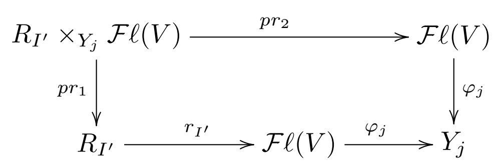
Remark.

1. For the Schubert variety associated to $\omega_0 = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}$ we have $\Omega_{\omega_0} \simeq X$. As a consequence $\Omega_{\omega_0} \in \mathbf{Sm}_k$.
2. In general a Schubert variety Ω_ω needs not to be a smooth scheme.

Definition

Let I be the l-tuple $(i_1, i_2, \ldots, i_l)$ with $i_k \in \{1, \ldots, n-1\}$. The **Bott-Samelson resolutions** $r_I : R_I \to \mathcal{F}\ell(V)$ are defined recursively:

- if $l = 0$, then $I = \emptyset$, $R_\emptyset := X$ and $r_\emptyset = i : X \to \mathcal{F}\ell(V)$ is the embedding associated to $V_\bullet$;
- if $l > 0$, then $I = (I', j)$ and by the inductive hypothesis $r_{I'} : R_{I'} \to \mathcal{F}\ell(V)$ has already been defined. One then considers the following fiber diagram



and sets $R_I := R_{I'} \times_{Y_j} \mathcal{F}\ell(V)$ and $r_I := pr_2$. Here Y_j denotes the bundle of partial flags with only the j-th level missing.

Double Schubert and Grothendieck polynomials

Definition *(Lascoux-Schützenberger [5])*

Fix $n \in \mathbb{N}$. For each $i \in \{1, \ldots, n-1\}$ the divided difference operators ∂_i and the isobaric divided difference operators π_i on $\mathbb{Z}[\mathbf{x}, \mathbf{y}]$ are defined by

$$i)\ \partial_i P = \frac{P - \sigma_i(P)}{x_i - x_{i+1}} \quad ; \quad ii)\ \pi_i P = \frac{(1 - x_{i+1})P - (1 - x_i)\sigma_i(P)}{x_i - x_{i+1}} ,$$

where σ_i is the operator exchanging x_i and x_{i+1} and 1 represents the identity operator.
For $\omega \in S_n$ the **double Schubert polynomial** $\mathfrak{S}_\omega$ and the **double Grothendieck polynomial** $\mathfrak{G}_\omega$ are defined as follows:

- if $\omega = \omega_0$ then
$$i)\ \mathfrak{S}_{\omega_0} := \prod_{i+j \le k} (x_i - y_j) \quad ; \quad ii)\ \mathfrak{G}_{\omega_0} := \prod_{i+j \le k} (x_i + y_j - x_i y_j) ;$$
- if $\omega \neq \omega_0$ then there exists an elementary transposition $s_i = (i\ i+1)$ such that $l(\omega) < l(\omega s_i)$: one then sets
$$i)\ \mathfrak{S}_\omega := \partial_i \mathfrak{S}_{\omega s_i} \quad ; \quad ii)\ \mathfrak{G}_\omega := \pi_i \mathfrak{G}_{\omega s_i} .$$

The link between Schubert varieties, Bott-Samelson resolution and double Schubert and Grothendieck polynomials is illustrated by the following two theorems:

Theorem 1. *(Fulton [2])*

Let $\omega \in S_n$ and let the l-tuple $I = (i_1, \ldots, i_l)$ correspond to a minimal length decomposition of $\omega_0\omega$, i.e. $\omega_0\omega = s_I := s_{i_1} \cdots s_{i_l}$ with minimal l. Then in the Chow ring $CH^*(\mathcal{F}\ell(V))$ one has

$$[\Omega_\omega]_{CH^*} = \mathfrak{S}_\omega(c_1(M_1), \ldots, c_1(M_n), c_1(L_1), \ldots, c_1(L_n)) = r_{I*}[R_I]_{CH^*} .$$

Theorem 2. *(Fulton-Lascoux [3])*

Let $\omega \in S_n$ and let the l-tuple $I = (i_1, \ldots, i_l)$ correspond to a minimal length decomposition of $\omega_0\omega$. Then in the Grothendieck ring of vector bundles $K^0(\mathcal{F}\ell(V))$ one has

$$[\mathcal{O}_{\Omega_\omega}]_{K^0} = \mathfrak{G}_\omega(c_1(M_1), \ldots, c_1(M_n), c_1(L_1^\vee), \ldots, c_1(L_n^\vee)) = r_{I*}[R_I]_{K^0} .$$

Since the Chow ring CH^* and the graded Grothendieck ring of vector bundles $K^0[\beta, \beta^{-1}] := K^0 \otimes_{\mathbb{Z}} \mathbb{Z}[\beta, \beta^{-1}]$ are both examples of oriented cohomology theories one may ask the following

Question:

Is there an analogue of theorems 1 and 2 for other oriented cohomology theories and in particular for algebraic cobordism?

The proof for both theorems is essentially divided into three parts:

1. compute the fundamental class of Ω_{ω_0} in $CH^*(\mathcal{F}\ell(V))$ (respectively $K^0(\mathcal{F}\ell(V))$);
2. find an explicit expression for the operators $\varphi_j^*\varphi_{j*}$ on $CH^*(\mathcal{F}\ell(V))$ (respectively $K^0(\mathcal{F}\ell(V))$) hence obtaining a description of the classes $r_{I*}[R_I]$;
3. show that for I of minimal length these classes coincide with the fundamental class of the corresponding Schubert variety.

Algebraic cobordism and oriented cohomology theories

Definition

An **oriented cohomology theory** on $\mathbf{Sm}_k$ is given by

1. An additive functor $A^* : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{R}^*$;
2. For each projective morphism $f : Y \to X$ in $\mathbf{Sm}_k$ of relative codimension d, a homomorphism of graded $A^*(X)$-modules:
$$f_* : A^*(Y) \to A^{*+d}(X) .$$

These need to satisfy the **projective bundle formula**, the **extended homotopy property** and certain compatibilities concerning the pull-back morphisms f^* and the push-forward morphisms g_*.

Proposition *(Levine-Morel [6])*

Let A^* be an oriented cohomology theory on $\mathbf{Sm}_k$. A^* admits a theory of Chern classes and for any line bundle L on $X \in \mathbf{Sm}_k$ the class $c_1(L)^n$ vanishes for n large enough. Moreover, there is a unique power series

$$F_A(u,v) = \sum_{i,j} a_{i,j} u^i v^j \in A^*(k)[[u,v]]$$

with $a_{i,j} \in A^{1-i-j}(k)$, such that, for any $X \in \mathbf{Sm}_k$ and any pair of line bundles L, M on X, we have

$$F_A(c_1(L), c_1(M)) = c_1(L \otimes M) .$$

In addition, the pair $(A^*(k), F_A)$ is a commutative formal group law of rank one with inverse $\chi_A \in A^*(k)[[u]]$.

Examples:
- For the Chow ring CH^* one has: $F_{CH}(u,v) = u + v$, $\chi_{CH}(u) = -u$;
- For $K^0[\beta, \beta^{-1}]$ one has: $F_{K^0[\beta,\beta^{-1}]}(u,v) = u + v - \beta uv$, $\chi_{K^0[\beta,\beta^{-1}]}(u) = \frac{-u}{1-\beta u}$.

Theorem 3. *(Levine-Morel [6])*

Let k be a field of characteristic 0.

1. Algebraic cobordism $X \mapsto \Omega^*(X)$ is an oriented cohomology theory on $\mathbf{Sm}_k$ and it is universal among such theories: given an oriented cohomology theory A^* on $\mathbf{Sm}_k$, there exists a unique morphism of oriented cohomology theories
$$\vartheta_{A^*} : \Omega^* \to A^* .$$
2. The canonical map
$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z} \to CH^*$$
induced by ϑ_{CH^*} is an isomorphism of oriented cohomology theories.
3. The canonical map
$$\Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta, \beta^{-1}] \to K^0[\beta, \beta^{-1}]$$
induced by $\vartheta_{K^0[\beta,\beta^{-1}]}$ is an isomorphism of oriented cohomology theories.

In [4] Hornbostel and Kiritchenko computed the classes $r_{I*}[R_I]_{\Omega^*}$ for $X = \mathrm{Spec}\, k$. In particular they addressed the problem associated to part 2 of the proof: describing explicitly the operators $\varphi_j^*\varphi_{j*}$.

Proposition

Let $\varphi : \mathbb{P}(E) \to X$ be a $\mathbb{P}^1$-bundle and $A_\varphi : \Omega^*(\mathbb{P}(E)) \to \Omega^*(\mathbb{P}(E))$ be the operator obtained from

$$\begin{array}{ccc} \Omega^*(X)[[x_1,x_2]] & \xrightarrow{A} & \Omega^*(X)[[x_1,x_2]] \\ f & \longmapsto & (1+\sigma_1)\dfrac{f}{F_\Omega(x_1, \chi_\Omega(x_2))} \end{array} ,$$

by substituting the Chern roots of E for x_1, x_2. Then $A_\varphi = \varphi^*\varphi_*$.

Main results and an application to connected K-theory

In order to complete the description of the push-forward classes of Bott-Samelson resolutions $r_{I*}[R_I]_{\Omega^*}$, one still needs to compute the fundamental class $[\Omega_{\omega_0}]_{\Omega^*}$.

Proposition

As an element of $\Omega^*(\mathcal{F}\ell(V))$ the fundamental class of the Schubert variety of highest codimension Ω_{ω_0} can be expressed as

$$[\Omega_{\omega_0}]_{\Omega^*} = \prod_{i+j \le n} F_\Omega(c_1(M_i), c_1(L_j^\vee)) .$$

Once this is known it is finally possible to describe all remaining classes. If the operators A_φ associated to the $\mathbb{P}^1$-bundle $\varphi_j : \mathcal{F}\ell(V) \to Y_j$ $j \in \{1, \ldots, n-1\}$ are denoted by A_j, one has the following result:

Theorem 4.

Let $I = (i_1, \ldots, i_n)$ be an l-tuple and let $r_I : R_I \to \mathcal{F}\ell(V)$ be the associated Bott-Samelson resolution. Then

$$r_{I*}[R_I]_{\Omega^*} = A_{i_l} \cdots A_{i_1}[\Omega_{\omega_0}]_{\Omega^*} \qquad (1)$$

An interesting application of this formula is represented by its specialization to **connected K-theory**, an oriented cohomology theory defined as $CK^* := \Omega^* \otimes_{\mathbb{L}^*} \mathbb{Z}[\beta]$ which generalizes both CH^* and $K^0[\beta, \beta^{-1}]$.

Corollary.

When specialized to CK^*, (1) recovers the double β-polynomials defined by Fomin and Kirillov in [1].

$$r_{I*}[R_I]_{CK^*} = \vartheta_{CK^*}(r_{I*}[R_I]_{\Omega^*}) = \mathfrak{H}_\omega^{(-\beta)}(c_1(M_1), \ldots, c_1(M_n), c_1(L_1^\vee), \ldots, c_1(L_n^\vee)) .$$

Definition

Fix $n \in \mathbb{N}$. For $\omega \in S_n$ the **double β polynomial** $\mathfrak{H}_\omega^{(\beta)} \in \mathbb{Z}[\beta][\mathbf{x}, \mathbf{y}]$ is defined as follows:

- if $\omega = \omega_0$ then
$$\mathfrak{H}_{\omega_0}^{(\beta)} := \prod_{i+j \le k} (x_i + y_j + \beta x_i y_j) ;$$
- if $\omega \neq \omega_0$ then there exists an elementary transposition s_i such that $l(\omega) < l(\omega s_i)$ and one sets
$$\mathfrak{H}_\omega^{(\beta)} := \phi_i \mathfrak{H}_{\omega s_i}^{(\beta)} .$$

For each $i \in \{1, \ldots, n-1\}$ the β-divided difference operator ϕ_i on $\mathbb{Z}[\beta][\mathbf{x}, \mathbf{y}]$ is defined by setting

$$\phi_i P = (1 + \sigma_i)\frac{(1 + \beta x_{i+1})P}{x_i - x_{i+1}} = \frac{(1 + \beta x_{i+1})P - (1 + \beta x_i)\sigma_i(P)}{x_i - x_{i+1}} ,$$

where σ_i is the operator exchanging x_i and x_{i+1} and 1 represents the identity operator.

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Mathematisches Institut, Universität Duisburg-Essen
thomas.hudson@stud.uni-due.de