Natural Bases for the Cohomology of Regular Nilpotent Hessenberg Varieties

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1 Schubert Classes and Billey's Formula

The set of Schubert classes $\{\sigma_w\}$ form a basis for the equivariant cohomology of the flag variety. An equivariant cohomology class in $H_T^*(GL_n(\mathbb{C})/B)$ can be thought of as a collection of $n!$ elements of $\mathbb{C}[t_1,\ldots,t_n]$ each of which corresponds to an element $v \in S_n$. Billey identified these elements by giving an explicit formula for the localizations $\sigma_w(v)$ of the equivariant Schubert class σ_w at each permutation flag vB.

Billey's Formula. Fix a reduced word for $v = s_{b_1}s_{b_2}\cdots s_{b_{\ell(v)}}$ and let

$$\mathbf{r}(\mathbf{i},v) = s_{b_1}s_{b_2}\cdots s_{b_{i-1}}(t_{b_i} - t_{b_i+1}).$$

Built from these terms, which are polynomials of degree 1, Billey's formula

$$\sigma_w(v) = \sum_{w=s_{b_{j_1}}s_{b_{j_2}}\cdots s_{b_{j_{\ell(w)}}}} \prod_{i=1}^{\ell(w)} \mathbf{r}(\mathbf{j_i},v)$$

gives the localization of σ_w at the permutation flag vB. This is a polynomial of degree $\ell(w)$ in n variables with non-negative integer coefficients.

2 Hessenberg Varieties

We work with a subvariety $\mathcal{X}_\mathcal{H}$ of the full flag variety GL_n/B called a **regular nilpotent Hessenberg variety**. The variety $\mathcal{X}_\mathcal{H}$ is determined by a shape $\mathcal{H}$ which is a subspace of $n \times n$ matrices containing the upper triangular matrices and which has a staircase boundary. Explicitly the matrix basis units $E_{(i,i)} \in \mathcal{H}$ for all i, and if $E_{(i,j)} \in \mathcal{H}$ then so are $E_{(k,j)}$ for $k < i$ and $E_{(i,l)}$ for $l > j$.

Given such a shape $\mathcal{H}$

- the variety $\mathcal{X}_\mathcal{H} = \{gB \in GL_n/B : g^{-1}Xg \in \mathcal{H}\}$ where X is the regular nilpotent $n \times n$ matrix (since X is fixed, it is suppressed in the notation),
- the set of coset representatives is $V_\mathcal{H} = \{v \in S_n : v^{-1}Xv \in \mathcal{H}\}$.

Example. In A_3 we have

$$X = \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} *&*&*&*\\ 0&*&*&*\\ 0&*&*&*\\ 0&0&0&* \end{bmatrix}, \quad \text{and } V_\mathcal{H} = \{1, s_2\}$$

Remark. The subvariety $\mathcal{X}_\mathcal{H}$ is not T-fixed, rather it is acted on by a one dimensional sub-torus S.

3 Main Conjecture

Harada and Tymoczko conjectured that a basis for the equivariant cohomology $H_S^*(\mathcal{X}_\mathcal{H})$ can be obtained by taking the equivariant Schubert classes σ_w for $w \in W_\mathcal{H} = \{w \in \mathcal{H}\}$ localized at the points $v \in V_\mathcal{H}$.

The conjecture has been proven for certain shapes $\mathcal{H}$ including

Springer Variety

$$\begin{bmatrix} *&*&*&*&*\\ 0&*&*&*&*\\ 0&0&*&*&*\\ 0&0&0&*&*\\ 0&0&0&0&* \end{bmatrix}$$

Peterson Variety

$$\begin{bmatrix} *&*&*&*&*\\ *&*&*&*&*\\ 0&*&*&*&*\\ 0&0&*&*&*\\ 0&0&0&*&* \end{bmatrix}$$

Harada-Tymoczko

Modified Peterson Variety

$$\begin{bmatrix} *&*&*&*&*\\ *&*&*&*&*\\ *&*&*&*&*\\ 0&0&*&*&*\\ 0&0&0&*&* \end{bmatrix}$$

Bayegan-Harada

The conjecture is trivially true for the Springer variety. The modified Peterson variety contains the Peterson variety and allows non-zero entries in $(3,1)$.

4 Computer Testing the Conjecture

In order to prove that the classes $\sigma_w(v)_{v\in V_\mathcal{H}}$ such that $w \in W_\mathcal{H}$ give a basis of $H_S^*(\mathcal{X}_\mathcal{H})$, I have been working with the matrices

$$A_\mathcal{H} = (\sigma_w(v))_{\substack{w\in W_\mathcal{H}\\ v\in V_\mathcal{H}}} \text{ and } \tilde{A}_\mathcal{H}$$

where $\sigma_w(v)$ is Billey's formula and $\tilde{A}_\mathcal{H}$ is the image of $A_\mathcal{H}$ under the map $t_i \mapsto -it$. Harada and Tymoczko have shown that to prove the conjecture holds for a shape $\mathcal{H}$, it suffices to show that $\tilde{A}_\mathcal{H}$ has linearly independent columns.

Using a program I wrote in Sage, I can compute the matrix $\tilde{A}_\mathcal{H}$ and determine whether $\det \tilde{A}_\mathcal{H} = 0$. If $\det \tilde{A}_\mathcal{H} \neq 0$ then the conjecture holds for that shape $\mathcal{H}$.

Example. In A_4 consider the following

$$\mathcal{H} = \begin{bmatrix} *&*&*&*&*\\ *&*&*&*&*\\ 0&0&*&*&*\\ 0&0&*&*&*\\ 0&0&0&*&* \end{bmatrix}, \text{ and } \quad V_\mathcal{H} = \left\{ \begin{matrix} 1, s_3, s_3s_4s_3, s_4, \\ s_1, s_1s_3, s_1s_3s_4s_3, s_1s_4 \end{matrix} \right\}, \quad W_\mathcal{H} = \left\{ \begin{matrix} 1, s_3, s_3s_4, s_4, \\ s_1, s_1s_3, s_1s_3s_4, s_1s_4 \end{matrix} \right\}$$

The program calculates

$$\tilde{A}_\mathcal{H} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 1&t&0&0&0&0&0&0\\ 1&2t&2t^2&2t&0&0&0&0\\ 1&0&0&t&0&0&0&0\\ 1&0&0&0&t&0&0&0\\ 1&t&0&0&t&t^2&0&0\\ 1&2t&2t^2&2t&t&2t^2&2t^3&2t^2\\ 1&0&0&t&t&0&0&t^2 \end{bmatrix} \text{ which has } \det \tilde{A}_\mathcal{H} = 4t^{12} \neq 0.$$

This implies that the conjecture is true in this case.

Using my program I verified that the conjecture holds for all regular nilpotent Hessenberg varieties when working in types A_m for $m < 5$.

5 The Relationship between $\mathcal{H}$ and $\mathcal{H}^\perp$

Given the regular nilpotent Hessenberg variety $\mathcal{X}_\mathcal{H}$, it is natural to compare it with $\mathcal{X}_{\mathcal{H}^\perp}$ where the shape $\mathcal{H}^\perp$ is obtained from flipping the shape $\mathcal{H}$ along its antidiagonal.

Example.

$$\mathcal{H} = \begin{bmatrix} *&*&*&*&*\\ *&*&*&*&*\\ 0&0&*&*&*\\ 0&0&*&*&*\\ 0&0&0&*&* \end{bmatrix}, \quad \mathcal{H}^\perp = \begin{bmatrix} *&*&*&*&*\\ *&*&*&*&*\\ 0&*&*&*&*\\ 0&0&0&*&*\\ 0&0&0&*&* \end{bmatrix}$$

Theorem (D). *The coset representatives are related by*

$$V_{\mathcal{H}^\perp} = w_0 V_\mathcal{H} w_0.$$

The proof of this theorem gives rise to the geometric map

$$\begin{aligned} GL_n(\mathbb{C})/B &\to GL_n(\mathbb{C})/B \\ gB &\mapsto w_0(g^\top)^{-1}w_0B \end{aligned}$$

which restricts to a homeomorphism between regular nilpotent Hessenberg varieties $\mathcal{X}_\mathcal{H}$ and $\mathcal{X}_{\mathcal{H}^\perp}$. Even though it is not evident from Billey's formula, this gives the surprising conclusion that $H_S^*(\mathcal{X}_\mathcal{H}) \cong H_S^*(\mathcal{X}_{\mathcal{H}^\perp})$.

6 Decomposable Hessenberg Varieties

A regular nilpotent Hessenberg variety $\mathcal{X}_\mathcal{H}$ is said to be **decomposable** if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

Example.

$$\mathcal{H} = \begin{bmatrix} *&*&*&*&*\\ *&*&*&*&*\\ 0&0&*&*&*\\ 0&0&*&*&*\\ 0&0&0&*&* \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1 & \begin{matrix} *&*&*\\ *&*&* \end{matrix} \\ \begin{matrix} 0&0\\ 0&0\\ 0&0 \end{matrix} & \mathcal{H}_2 \end{bmatrix}$$

Theorem (D). *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ then $A_\mathcal{H} = A_{\mathcal{H}_1} \otimes A_{\mathcal{H}_2}$ and therefore has linearly independent columns if and only if both $A_{\mathcal{H}_1}$ and $A_{\mathcal{H}_2}$ do as well.*

The proof relies on the claims:

- If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ then $W_\mathcal{H} = W_{\mathcal{H}_1} \oplus W_{\mathcal{H}_2}$ and $V_\mathcal{H} = V_{\mathcal{H}_1} \oplus V_{\mathcal{H}_2}$.
  Example.
  $$V_{\mathcal{H}_1} = \{1, s_1\} \qquad V_{\mathcal{H}_2} = \{1, s_3, s_3s_4s_3, s_4\}$$
  and
  $$W_{\mathcal{H}_1} = \{1, s_1\} \qquad W_{\mathcal{H}_2} = \{1, s_3, s_3s_4, s_4\}$$
- If $w = w_1 \oplus w_2$ for $w_1 \in W_{\mathcal{H}_1}$ and $w_2 \in W_{\mathcal{H}_2}$ and $v = v_1 \oplus v_2$ for $v_1 \in V_{\mathcal{H}_1}$ and $v_2 \in V_{\mathcal{H}_2}$ then Billey's formula has the property that $\sigma_w(v) = \sigma_{w_1}(v_1)\sigma_{w_2}(v_2)$.
  Example. Using
  $$s_1 \in W_{\mathcal{H}_1} \quad s_3s_4 \in W_{\mathcal{H}_2}$$
  $$s_1 \in V_{\mathcal{H}_1} \quad s_3s_4s_3 \in V_{\mathcal{H}_2}$$
  $$\sigma_{s_1s_3s_4}(s_1s_3s_4s_3) = (t_1 - t_2)(t_3 - t_4)(t_3 - t_5) = \sigma_{s_1}(s_1)\sigma_{s_3s_4}(s_3s_4s_3)$$
- These show that if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ then $A_\mathcal{H} = A_{\mathcal{H}_1} \otimes A_{\mathcal{H}_2}$.
  Example.
  $$\tilde{A}_\mathcal{H} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 1&t&0&0&0&0&0&0\\ 1&2t&2t^2&2t&0&0&0&0\\ 1&0&0&t&0&0&0&0\\ 1&0&0&0&t&0&0&0\\ 1&t&0&0&t&t^2&0&0\\ 1&2t&2t^2&2t&t&2t^2&2t^3&2t^2\\ 1&0&0&t&t&0&0&t^2 \end{bmatrix} = \overset{\tilde{A}_{\mathcal{H}_1}}{\begin{bmatrix} 1&0\\ 1&t \end{bmatrix}} \otimes \overset{\tilde{A}_{\mathcal{H}_2}}{\begin{bmatrix} 1&0&0&0\\ 1&t&0&0\\ 1&2t&2t^2&2t\\ 1&0&0&t \end{bmatrix}}$$

7 References

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