The ABC's of Affine Grassmannians

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Summary of results

1. We give a new Pieri rule for k-Schur functions.
2. We introduce a new combinatorial structure called ABC's
3. We prove that the weight generating function of ABC's are representatives for the cohomology classes of the affine Grassmannian.
4. We conjecture a statistic on ABC's for the k-Schur expansion for Hall-Littlewood polynomials (and prove for k large).

Inspiration

Pieri rule for Schur functions

$$h_\ell\, s_\mu = \sum_{\lambda = \mu + hor\ \ell\text{-strip}} s_\lambda$$

Gives rise to tableaux

$$h_{\mu_1} h_{\mu_2} \cdots h_{\mu_\ell}\, s_\emptyset = \sum_\lambda (\#\ \text{tableaux})\, s_\lambda$$

Duality $\langle h_\mu, m_\lambda\rangle = \langle s_\mu, s_\lambda\rangle = \delta_{\lambda\mu}$ gives Schur weight generating function

$$s_\lambda = \sum_\mu (\#\ \text{tableaux})\, m_\mu = \sum_{\text{tableau } T} x^T$$

Tableaux are central to counting problems including:

Computing intersections of Schubert varieties in the Grassmannian

$$s_\lambda\, s_\mu = \sum (\#\ \text{yamanouchi tableaux})\, s_\nu$$

Computing Hall-Littlewood polynomials in terms of Schur functions

$$H_\mu[X;t] = \sum_{\text{tableau}T} t^{charge(T)} s_{shape(T)}$$

We use this approach to study related highlights in k-Schur theory:

$$s^k_\lambda\, s^k_\mu = \sum (\text{Gromov-Witten invariants/WZW fusion})\, s^k_\nu$$

$$H_\mu = \sum (\text{some positive polynomial})\, s^k_\lambda$$

$\{\mathfrak{S}^k_\lambda\}$ = cohomology classes for the affine Grassmannian

Pieri rule for k-Schur functions

Original rule uses weak order on affine Weyl group $\tilde{A}$

Weak Cover: $\rho \lessdot \gamma$ when $\rho \subset \gamma$ are $k+1$-cores

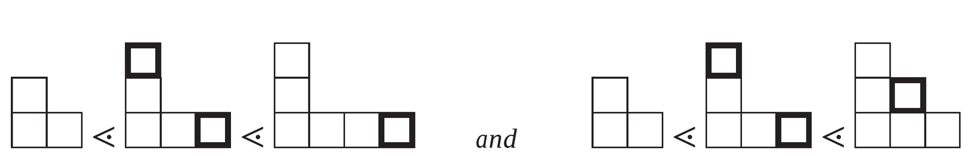
$deg(\gamma) = deg(\rho) + 1$ *where $deg(\gamma)$ =# cells of γ with hook-length $\leqslant k$.*

γ/ρ has less than 2 cells in each row and column.

Weak ℓ-Strip: The skew shape ν/λ is a *weak ℓ-strip* if ν/λ is a horizontal strip and there is a weak saturated chain of cores

$$\lambda = \gamma^0 \lessdot \gamma^1 \lessdot \cdots \lessdot \gamma^\ell = \nu\,.$$

Example: *When $k=3$, the skews $(4,1,1)/(2,1)$ and $(3,2,1)/(2,1)$ are weak 2-strips given*



k-Pieri Rule for k-Schurs For $k+1$-core λ and $0 < \ell \leqslant k$,

$$h_\ell\, s^{(k)}_\lambda = \sum_{\nu/\lambda \text{ is a weak } \ell\text{-strip}} s^{(k)}_\nu\,.$$

Example:

$$h_2\, s^3_{2,1} = s^3_{4,1,1} + s^3_{3,2,1}\,.$$

Pieri rule for dual k-Schur functions

Rule uses strong order on affine Weyl group $\tilde{A}$

Strong Cover: $\rho \lessdot_B \gamma$ when $\rho \subseteq \gamma$ are k+1-cores with $deg(\gamma) = deg(\rho) + 1$.

Strong ℓ-Strip: A *strong ℓ-strip* from $k+1$-core λ to $k+1$-core γ is a strong saturated chain

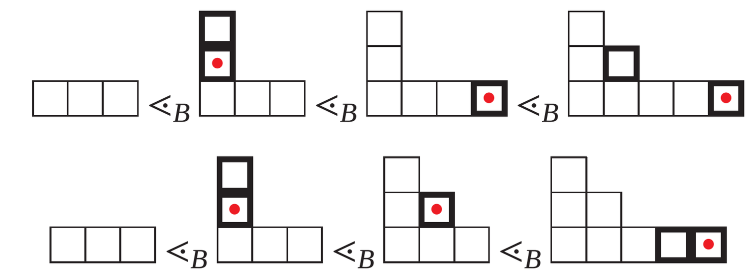
$$\lambda = \gamma^0 \lessdot_B \gamma^1 \lessdot_B \ldots \lessdot_B \gamma^\ell = \gamma$$

together with a content vector $c = (c_1 < c_2 < \cdots < c_\ell)$ where c_i is the content of the head of a ribbon in γ^i/γ^{i-1}.

Example: *When $k=3$, the strong 3-strips from (3) to $(5,2,1)$ are*

$c = (-1, 3, 4)$

$c = (-1, 0, 4)$



k-Pieri Rule for dual k-Schurs: For $0 < \ell \leqslant k$ and $k+1$-core λ,

$$h_\ell \mathfrak{S}^{(k)}_\lambda = \sum_\gamma d_\gamma \mathfrak{S}^{(k)}_\gamma,$$

where d_γ is the number of strong ℓ-strips from λ to γ.

Note by the previous example, $d_{(5,2,1)} = 2$ when $k=3$.

New Pieri rule for k-Schur functions

This Pieri rule for k-Schur functions uses strong rather than weak order

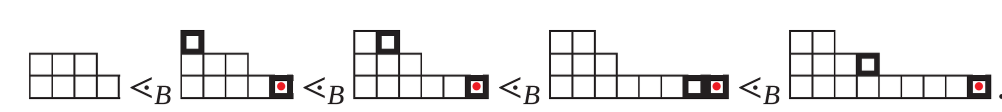
Theorem: For any $0 \leqslant \ell \leqslant k$ and $k+1$-core λ,

$$h_\ell\, s^{(k)}_\lambda = \sum_{(k+\lambda_1,\lambda)/\nu \text{ is a bottom strong } k-\ell\text{-strip}} s^{(k)}_\nu\,,$$

where γ/ν is a *bottom strong ℓ-strip* if it is a horizontal strip and there is a strong ℓ-strip from ν to γ whose content vector $(c_1, \ldots, c_\ell)$ satisfies $c_1 \geqslant \nu_1$.

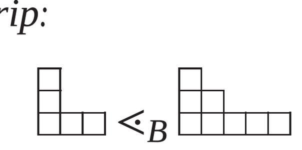
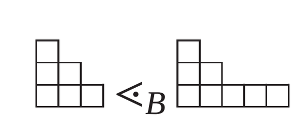
Example: *The skew shape $(9,4,2)/(4,3)$ of 6-cores is a bottom strong 4-strip since*

$c = (4, 5, 7, 8)$



Note by the previous example that the skew shape $(5,2,1)/(3)$ is not a bottom strong 3-strip.

Example: *To compute $h_2 s^{(3)}_{2,1}$ instead using this new rule requires finding all ν where $(5,2,1)/\nu$ is a bottom strong 1-strip:*

$c = (4)$  and $c = (4)$ 

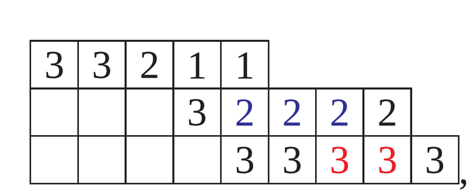
Affine Bruhat Counter-tableaux

An *affine Bruhat counter-tableau* (or *ABC*) A of k-weight α is a skew counter-tableau filling where, for all $1 \leqslant i \leqslant \ell(\alpha)$, letting $\lambda^{(i)}$ be the top i rows of A restricted to letters larger than i,

$$(k + \lambda^{(i-1)}_1, \lambda^{(i-1)})/\lambda^{(i)}$$

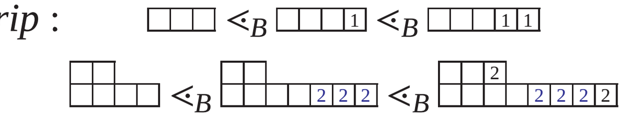
is a bottom strong $(k - \alpha_i)$-strip filled with letter i. Note, we consider empty cells to contain ∞.

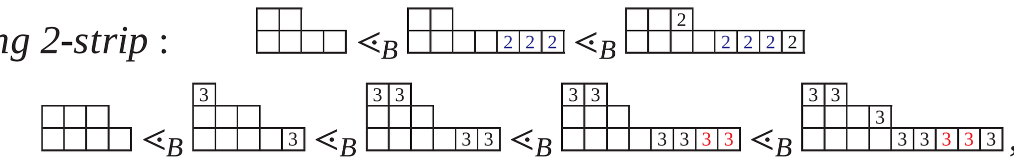
Example: *An ABC of 5-weight $(3,3,1)$ and inner shape (4,3) is*



since

bottom strong 2-strip :

bottom strong 2-strip : 

bottom strong 4-strip : 

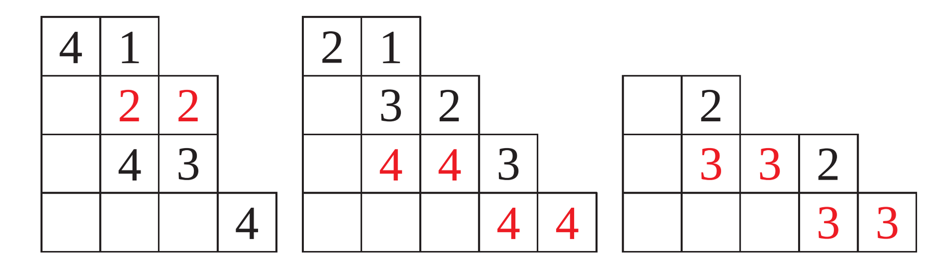
Applications of ABC's

ABC's give a new characterization for the representatives $\{\mathfrak{S}^k_\lambda\}_{\lambda\ k+1-core}$ of cohomology classes of the type A affine Grassmannian.

Theorem:

$$\mathfrak{S}^{(k)}_\lambda = \sum_{\text{A: ABC of inner shape } \lambda} x^{k\text{-weight}(A)}$$

Example

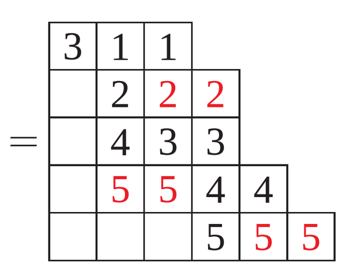
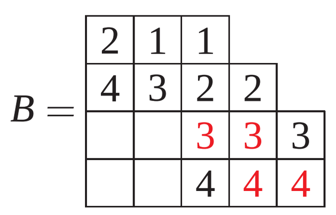


$$\Longrightarrow \quad \mathfrak{S}^{(2)}_{3,1,1} = 2m_{1,1,1,1} + m_{2,1,1}$$

We give a statistic on ABC's that characterizes Hall-Littlewood polynomials and conjecturally gives their k-Schur expansion.

Offset: For $i > 1$, an i-ribbon R in an ABC A is an *offset* if there is an identical ribbon in a lower row and a hook of length k separates their heads.

Example: *Consider the ABCs A of 3-weight 1^5 and B of 3-weight 1^4:*

$A =$  $B =$ 

A has only one offset: 5 5 *in the second row from the bottom and B has no offsets.*

Spin Statistic: Let A be an ABC of k-weight 1^n and define

$$spin^k(A) = \#offsets(A) + \sum_i i\,\chi_A(i),$$

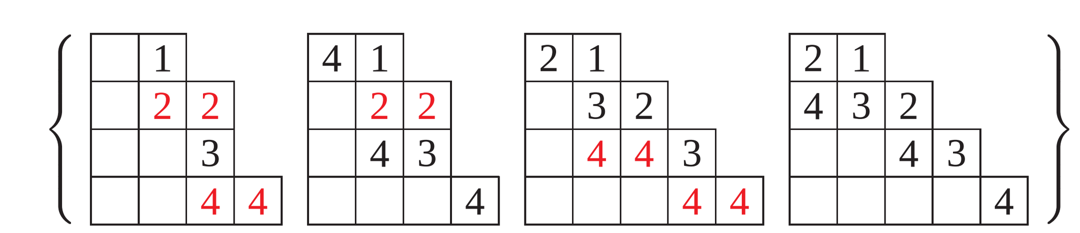
where $\chi_A(i)$ is one when row i has a non-offset 2-ribbon which is not east of a non-offset 2-ribbon in row $i+1$, and zero otherwise.

In the previous example, $spin(A) = 1+5$ and $spin(B) = 0+2$.

Conjecture:

$$H_{1^n}[X;t] = \sum_{\substack{k\text{-weight}(A)=1^n \\ \text{inner shape}(A)=\lambda}} t^{spin^k(A)}\, s^{(k)}_\lambda[X;t].$$

Example: *The set of all ABC's of 2-weight 1^4 are*



which gives the expansion of $H_{1^4}[X,t]$ in terms of 2-Schur functions:

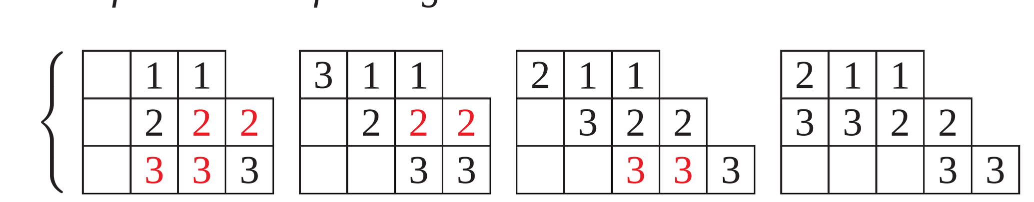
$$H_{1^4}[X;t] = t^4 s^{(2)}_{(2,2,1,1)} + t^3 s^{(2)}_{(3,1,1)} + t^2 s^{(2)}_{(3,1,1)} + s^{(2)}_{(4,2)}\,.$$

When $k = n$, an ABC never has any offsets.

Theorem: Hall-Littlewood polynomials can be characterized by

$$H_{1^n}[X;t] = \sum_{\substack{ABC \text{ of } n\text{-weight } 1^n \\ \text{inner shape}(A)=\lambda}} t^{\sum i\chi_A(i)}\, s_\lambda\,.$$

Example: *The set of all ABC's of 3-weight 1^3 are*



The respective values of $\sum_i \chi_A(i)$ are $\{3, 2, 1, 0\}$. We thus have that

$$H_{1^3}[X;t] = t^3 s_{(1,1,1)} + (t^2+t)\, s_{(2,1)} + s_{(3)}\,.$$